physical sciences forum

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Proceeding Paper

The Bound of the Non-Commutative Parameter Based on Gravitational Measurements †

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† Presented at the 2nd Electronic Conference on Universe, 16 February–2 March 2023; Available online: https://ecu2023.sciforum.net/.

Abstract: In this paper, we investigate the four classical tests of general relativity in the non-commutative (NC) gauge theory of gravity. Using the Seiberg–Witten (SW) map and the star product, we calculate the deformed metric components $\hat{g}_{\mu\nu}(r,\theta)$ of the Schwarzschild black hole (SBH). The use of this deformed metric enables us to calculate the gravitational periastron advance of mercury, the red shift, the deflection of light, and time delays in the NC spacetime. Our results for the NC prediction of the gravitational deflection of light and time delays show a newer behavior than the classical one. As an application, we use a typical primordial black hole to estimate the NC parameter θ, where our results show $\theta^{phy} \approx 10^{-34}$ m for the gravitational red shift, the deflection of light, and time delays at the final stage of inflation, and $\theta^{phy} \approx 10^{-31}$ m for the gravitational periastron advance of some planets from our solar system.

Keywords: non-commutative gauge field theory; gauge field gravity; gravitational measurements

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Citation: Touati, A.; Zaim, S. The Bound of the Non-Commutative Parameter Based on Gravitational Measurements. *Phys. Sci. Forum* **2023**, *7*, 54. https://doi.org/10.3390/ECU2023-14061

Academic Editor: Lorenzo Iorio

Published: 18 February 2023

1. Introduction

General relativity (GR) is considered one of the major scientific discoveries at the beginning of the 20th century; it describes an excellent relativistic description of gravity, which is one of the fundamental interactions that describe all phenomena in nature at the macroscopic scale. This theory was successful due to the prediction of experimental results in the first three tests, proposed by Albert Einstein in 1915, i.e., the periastron advance of Mercury's orbit, the deflection of light, and the red shift [1]. Later, in 1964, I. Shapiro discovered and observed the time delay due to the presence of massive objects, which became another successful test of GR, also known as the fourth classical test of GR [2].

However, this theory is still unable to describe gravity signals at the quantum scale; this problem led to the emergence of many new ideas. One of them adopts the same concept of quantum mechanics in quantization concerning the relations of commutation between the observables, known as $\left[\hat{x}_i, \hat{p}_j\right] = -i\hbar\delta_{ij}$. In this theory, the coordinates of spacetime $\hat{x}_\mu$ are considered a non-commutative observation, subject to the commutation relation between the coordinates themselves, namely:

$$\left[\hat{x}_\mu, \hat{x}_\nu\right] = i\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is an anti-symmetric real matrix of the NC parameter, which describes the fundamental cell discretization within spacetime, while the general idea of the NC geometry suggests that the quantization of spacetime can lead to the quantization of gravity.

Moreover, in this theory, the scalar product between two arbitrary functions $f(x)$ and $g(x)$ changes to the star product.

$$(f * g)(x) = f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x). \tag{2}$$

Recently, there has been a lot of research in determining a lower bound of the NC parameter and studies on quantum gravity effects, with several approaches [3–21]. Our aim is to estimate a lower bound on the NC parameter using the four classical experimental tests of GR inspired by the NC geometry based on the gauge theory of gravity and compared to the other results obtained by another approach of NC geometry [12,16]. In this study, we give the NC corrections based on the four classical predictions of GR in the NC gauge theory of gravity. Firstly, we obtain the NC periastron advance of orbit and we choose some planets within our solar system as examples for the numerical value of θ; for the deflection of light, the red shift, and the time delay, we use data of a typical primordial black hole at the early universe, and we use the scale factor to obtain the physical distance measured at any time [12,16]. Our results show that the NC property of spacetime appears before the Planck scale.

In this paper, we discuss the bound in the NC parameter in the NC gauge theory of gravity using the four classical tests of GR. A brief review of the NC gauge theory of gravity for the SBH metric is presented in Section 2. The NC parameter for different experimental tests of GR in NC spacetime is estimated and discussed in Section 3. In Section 4, we present our conclusions.

2. Non-Commutative Corrections for the Schwarzschild Black Hole

In our previous studies [21], we used the tetrad formalism and both the star product and the SW map [22] to construct the NC gauge theory for a static metric with spherical symmetry. A perturbation form for the SW map can also be used to describe the deformed tetrad fields $\hat{e}^a_\mu$ as a development in the power of θ up to the second order, which can be obtained by following the same approach mentioned in Ref. [23]:

$$\begin{gathered}\hat{e}^a_\mu = e^a_\mu - \tfrac{i}{4}\theta^{\upsilon\rho}\left[\omega^{ab}_\upsilon \partial_\rho e^d_\mu + \left(\partial_\rho \omega^{ac}_\mu + R^{ac}_{\rho\mu}\right)e^d_\upsilon\right]\eta_{cd} + \tfrac{1}{32}\theta^{\upsilon\rho}\theta^{\lambda\tau}\left[2\{R_{\tau\upsilon}, R_{\mu\rho}\}^{ab} e^c_\lambda - \omega^{ab}_\lambda\left(D_\rho R^{cd}_{\tau\upsilon} + \right.\right. \\ \left. \partial_\rho R^{cd}_{\tau\upsilon}\right)e^m_\upsilon \eta_{dm} - \{\omega_\nu(D_\rho R_{\tau\nu} + \partial_\rho R_{\tau\nu})\}^{ab} e^c_\lambda - \partial_\tau\{\omega_{\nu,}(\partial_\rho\omega_\mu + R_{\rho\mu})\}^{ab} e^c_\lambda - \omega^{ab}_\lambda\left(\omega^{cd}_\upsilon \partial_\rho e^m_\mu + \right. \\ \left.\left(\partial_\rho\omega^{cd}_\mu + R^{cd}_{\rho\mu}\right)e^m_\nu\right)\eta_{dm} + 2\partial_\upsilon \omega^{ab}_\lambda \partial_\rho \partial_\tau e^c_\mu - 2\partial_\rho\left(\partial_\tau \omega^{ab}_\mu + R^{ab}_{\tau\mu}\right)\partial_\upsilon e^c_\lambda - \{\omega_{\upsilon,}(\partial_\rho\omega_\lambda + R_{\rho\lambda})\}^{ab}\partial_\tau e^c_\mu - \\ \left.\left(\partial_\tau\omega^{ab}_\mu + R^{ab}_{\tau\mu}\right)\left(\omega^{cd}_\nu \partial_\rho e^m_\mu + \left(\partial_\rho \omega^{cd}_\lambda + R^{cd}_{\rho\lambda}\right)e^m_\nu \eta_{dm}\right)\right]\eta_{bc,},\end{gathered} \tag{3}$$

where $\hat{e}^a_\mu$ and ω^{ab}_μ are the tetrad field and the spin connection (gauge field):

$$\begin{gathered}\{\alpha, \beta\}^{ab} = \left(\alpha^{ac}\beta^{db} + \beta^{ac}\alpha^{db}\right)\eta_{cd},\ [\alpha, \beta]^{ab} = \left(\alpha^{ac}\beta^{db} - \beta^{ac}\alpha^{db}\right)\eta_{cd} \\ D_\mu R^{ab}_{\rho\sigma} = \partial_\mu R^{ab}_{\rho\sigma} + \left(\omega^{ac}_\mu R^{db}_{\rho\sigma} + \omega^{bc}_\mu R^{da}_{\rho\sigma}\right).\end{gathered} \tag{4}$$

The deformed metric can be written as:

$$\hat{g}_{\mu\nu} = \frac{1}{2}\left(\hat{e}^a_\mu * \hat{e}^{b\dagger}_\nu + \hat{e}^a_\nu * \hat{e}^{b\dagger}_\mu\right)\eta_{ab}. \tag{5}$$

For the SBH solution, we choose the following tetrad fields:

$$\begin{gathered}e^0_\mu = \left(\sqrt{1-\tfrac{2m}{r}}, 0, 0, 0\right),\ e^1_\mu = \left(0, \frac{1}{\sqrt{1-\frac{2m}{r}}} \sin\theta\cos\phi\ ,\ r\cos\theta\cos\phi\ ,\ -r\sin\theta\sin\phi\right), \\ e^2_\mu = \left(0, \frac{1}{\sqrt{1-\frac{2m}{r}}}\sin\theta\sin\phi\ ,\ r\cos\theta\sin\phi\ ,\ r\sin\theta\cos\phi\right),\ e^3_\mu = \left(0, \frac{1}{\sqrt{1-\frac{2m}{r}}}\cos\theta,\ -r\sin\theta, 0\right).\end{gathered} \tag{6}$$

The deformed tetrad fields are calculated in Ref. [21]; thus, we follow the same step to compute the deformed metric components of SBH in the equatorial plane $\theta = \frac{\pi}{2}$,

$$-\hat{g}_{00} = \left(1 - \frac{2m}{r}\right) + \left\{\frac{m\left(88m^2 + mr\left(-77 + 15\sqrt{1-\frac{2m}{r}}\right) - 8r^2\left(-2 + \sqrt{1-\frac{2m}{r}}\right)\right)}{16r^4(r-2m)}\right\}\theta^2 + O(\theta^4), \tag{7}$$

$$\hat{g}_{11} = \frac{1}{(1-\frac{2m}{r})} - \left\{ \frac{m\left(12m^2 + mr\left(-14+\sqrt{1-\frac{2m}{r}}\right) - r^2\left(5+\sqrt{1-\frac{2m}{r}}\right)\right)}{8r^2(r-2m)^3} \right\}\theta^2 + O(\theta^4), \tag{8}$$

$$\hat{g}_{22} = r^2 - \left\{ \frac{m\left(m\left(10-6\sqrt{1-\frac{2m}{r}}\right) - \frac{8m^2}{r} + r\left(-3+5\sqrt{1-\frac{2m}{r}}\right)\right)}{16(r-2m)^2} \right\}\theta^2 + O(\theta^4), \tag{9}$$

$$\hat{g}_{33} = r^2 - \left\{ \frac{5}{8} - \frac{3}{8}\sqrt{1-\frac{2m}{r}} + \frac{m\left(-17+\frac{5}{\sqrt{1-\frac{2m}{r}}}\right)}{16r} + \frac{m^2\sqrt{1-\frac{2m}{r}}}{(r-2m)^2} \right\}\theta^2 + O(\theta^4), \tag{10}$$

where $m = GM$ denotes the mass of the SBH. It is clear that in the limit of $\theta \to 0$, we can obtain the commutative SBH solution.

3. Experimental Test of GR in NC Spacetime

In this section, we present the NC corrections to the four classical tests of GR, using the deformed SBH metric as the background.

3.1. Gravitational Periastron Advance

In our previous study [21], we derive the expression of the angle deviation after one revolution in the NC SBH metric (7)–(10) using the perturbation form of the geodesic equation outlined in Ref. [24]. Then, we find:

$$\Delta\phi = \frac{6\pi GM}{c^2\alpha(1-e^2)} + \pi\theta^2\left\{ \frac{\left({E_0}^2 - m_0^2c^4\right)}{2GM\alpha(1-e^2)} + \frac{6\left(m_0^2c^2 - (E_0/c)^2\right)}{\alpha^2(1-e^2)^2} + \frac{m_0^2c^2}{2\alpha^2(1-e^2)^2} \right\}, \tag{11}$$

where α, e denote the major semi-axis and the eccentricity of the movement, respectively. For a numerical application, we choose the problem of Mercury's orbit, where the NC parameter is in the order:

$$\theta^{phy} = \sqrt{\hbar\theta} \approx 5.7876 \times 10^{-31} \text{ m}. \tag{12}$$

Here, the NC parameter θ^{phy} is very small for the solar system, which means that our solar system is very sensitive to the NC parameter. For the other planets, the lower bound on θ^{phy} is shown in Table 1.

Table 1. Some observable values of orbital precession for different planets of our solar system are shown in column 2. The prediction of the orbital precession in general relativity is given in column 3, and we give the lower bound for the non-commutative parameter θ^{phy} in the final column.

Planet	$\Delta\phi^{\text{obs}}(\frac{\text{arc-sec}}{\text{centry}})$	$\Delta\phi^{\text{GR}}(\frac{\text{arc-sec}}{\text{centry}})$	**L.b of θ^{phy} ($\times 10^{-31}$ m)**
Mercury	42.9800 ± 0.0020	42.9805	≤ 05.7876
Venus	8.6247 ± 0.0005	8.6283	≤ 04.5239
Earth	3.8387 ± 0.0004	3.8399	≤ 04.0976

The experimental data can be found in Refs. [25,26].

As shown in Table 1, the lower bound of θ^{phy} is in the same order for the planet's orbit of our solar system $\theta^{phy} \sim 10^{-31}$ m.

3.2. Deflection of Light

Another successful experimental test of GR is the gravitational deflection of light, as predicted by Albert Einstein in his theory of gravity, where the light is deflected from its original path when it passes near a strong gravitational field. The formula that describes this phenomenon is given by [1], and the NC expression can be read as:

$$\Delta\hat{\phi} = 2\int_b^\infty \frac{1}{r\sqrt{\hat{g}_{00}(r)}}\left(\frac{r^2}{b^2}\left|\frac{\hat{g}_{00}(b)}{\hat{g}_{00}(r)}\right| - 1\right)dr - \pi. \tag{13}$$

This integral can be computed after expanding our expression on the first order in m/r and we stop to the second order in θ, with the following calculations:

$$\Delta\hat{\phi} = \frac{4GM}{c^2 b} - \frac{5GM}{6c^2 b^3}\theta^2. \tag{14}$$

As we can see, the first term represents the GR prediction and the second term represents the NC corrections to the gravitational deflection of light, where this correction should be smaller than the accuracy of the measurements [27]. To estimate θ for this phenomenon, we use the radius $(r \sim b \approx 1.5 \times 10^{-3}\text{ m})$ and the mass $(GM \sim 5 \times 10^{-4}\text{ m})$ of a typical micro black hole, so we obtain

$$\theta^{phy} = \sqrt{\alpha^2\theta^2} \leq 5.7 \times 10^{-34}\text{ m}, \tag{15}$$

where α is the scale factor at the end of inflation. We then multiply our results by α^2 because we use the space–space (θ^{ij}) NC parameter [12] to obtain physical results. It is worth noting that the NC geometry can change the behavior of the angle deviation; as we see when $b \to 0$, the NC term is dominant and the behavior of $\Delta\hat{\phi} \propto -\frac{1}{b^3}$ becomes negative.

3.3. Gravitational Red Shift

The third succus experimental test of GR, i.e., the gravitational red shift, where the spectral of light shifts due to gravity, is given in [1], and this NC form can be computed using the NC-deformed metric (7):

$$\hat{z} = \sqrt{\left|\frac{\hat{g}_{00}(r_2)}{\hat{g}_{00}(r_1)}\right|} - 1. \tag{16}$$

For an asymptotic observer $r_2 \to \infty$, the measured red shift for the NC SBH is given by $\hat{z}$:

$$\hat{z} = z\left(1 - \left(\frac{z+1}{z}\right)\left[\frac{\left(88GM^2 + GMr_1\left(-77 + 15\sqrt{1-\frac{2GM}{r_1}}\right) - 8r_1^2\left(-2+\sqrt{1-\frac{2GM}{r_1}}\right)\right)}{32r^3(r_1 - 2GM)^2}\right]\theta^2\right), \tag{17}$$

where $z = \left(\left(1 - \frac{2GM}{r_1}\right)^{-1/2} - 1\right)$ is the red shift that is predicted by GR. We use the same data of micro-black holes with accurate measurements [28], and we then obtain the bound on the θ^{phy} parameter:

$$\theta^{phy} = \sqrt{\alpha^2\theta^2} \leq 2.09 \times 10^{-34}\text{ m}. \tag{18}$$

3.4. Time Delay (Shapiro Effect)

The fourth successful classical test of GR is discovered by I. Shapiro [1,2], also called the gravitational time delay and the Shapiro effect, and is a phenomenon which studies the time that is necessary for a radar signal to be emitted from one point to another one, travel near a massive object, and return to the emitting point. Radar traveling can also be supposed from point

$$\Delta\hat{t} = 2\Big[\hat{t}(r_1, b) + \hat{t}(r_2, b) - \sqrt{b - r_1} - \sqrt{b - r_2}\Big], \tag{19}$$

where

$$\hat{t}(r, b) = \int_b^r \frac{1}{\hat{g}_{00}(r')}\left(1 - \frac{b^2\hat{g}_{00}(r')}{r'^2\hat{g}_{00}(b)}\right)dr', \tag{20}$$

we expand our expression on the first order in m and stop at the second order in θ. After some calculations, we obtain:

$$\hat{t}(r, b) = \sqrt{r^2 - b^2} + 2GMln\left(\frac{r + \sqrt{r^2 - b^2}}{b}\right) + GM\left(\frac{r - b}{r + b}\right)^{\frac{1}{2}} - \frac{GM(3b - 4r)\sqrt{r^2 - b^2}}{4b^2 r(r + b)}\theta^2. \tag{21}$$

We consider that $r_1 \ll b$ and $r_2 \ll b$, and we obtain the full expression of the time delay in the NC spacetime:

$$\Delta\hat{t} \approx 4GM\left[ln\left(\frac{4r_1 r_2}{b^2}\right) + 1\right] - \frac{4GM}{b^2}\theta^2. \tag{22}$$

The same can be noted in the behavior of the time delay in the NC spacetime, when $b \to 0$, and the dominant NC correction term and the behavior of $\Delta\hat{\phi} \propto -\frac{1}{b^2}$ become negative, meaning that the NC geometry removes the divergent behaviors.

For a numerical application, we take the same micro black hole data and the ratio $\frac{4r_1 r_2}{b^2}$ is considered in the same order as in our solar system scale, with accurate measurements [29]. Thus, we obtain:

$$\theta^{phy} = \sqrt{\alpha^2\theta^2} \le 2.57 \times 10^{-34}\ \text{m}. \tag{23}$$

4. Conclusions

This paper investigates the four classical tests of GR in the NC spacetime. As a background for our calculation, we use a deformed SBH metric via the NC geometry using the SW maps and the star product.

As a first step, we obtain a correction to the periastron advance of mercury up to the second order in θ [21], where our results show that the NC parameter is close to the Planck scale $\theta^{phy} \sim 10^{-31}$ m, and the reach of this application to other planets of our solar system shows that θ^{phy} is of the same order and acts as a fundamental constant of the solar system. Then, we compute the correction to the light deflection, the red shift, and the time delay in the NC spacetime. For application purposes, we choose the data of a microscopic black hole at the early universe.

Our results show that the experimental test uses a radio wave or a light bound on the NC parameter in the order of $\theta^{phy} \sim 10^{-34}$ m, where our results are smaller than those obtained in Refs. [12,16] as a different approach is used in this study. However, for the orbital motion of a massive particle (planet), the bound on θ is in the order of $\theta^{phy} \sim 10^{-31}$ m, and is remarkable so that our result is close to the one obtained using the classical mechanics in NC flat spacetime, as mentioned in Refs. [4,5]. This result indicates that the macroscopic system is very sensitive to the NC parameter. It is important to mention that through the study of black holes and thermodynamics in NC spacetime, the bound on the NC parameter $\sqrt{\theta}$ has been obtained in some papers, e.g., [6–9], and is expected to be $\sqrt{\theta} \sim 10^{-1}.l_p$ using the point-like structure for the matter with the NC Gaussian distribution. However, in this study and our privious works [21,30], we show that the lower bound of θ^{phy} is limited before the Planck scale in the range of (10^{-31} m–10^{-34} m), where we use the four classical tests of GR, confirming that the NC property of spacetime appears before the Planck scale.

The use of the NC gauge theory of gravity allows us to obtain good results on the bound of the NC parameter. This theory needs more attention, and so it may be beneficial to describe quantum gravity in the future.

Author Contributions: Conceptualization, A.T. and S.Z.; methodology, A.T.; software, A.T.; validation, S.Z.; investigation, A.T.; resources, A.T. and S.Z.; writing—original draft preparation, A.T.; writing—review and editing, A.T.; visualization, A.T.; supervision, S.Z.; project administration, S.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This study is supported by project B00L02UN050120230003, Univ. Batna 1, Algeria.

Conflicts of Interest: The authors declare no conflict of interest.

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