physical sciences forum

Proceeding Paper

Construction of Discrete Symmetries Using the Pauli Algebra Form of the Dirac Equation †

Avraham Nofech

Department of Mathematics and Statistics, MacEwan University, 10700 104 Ave NW, Edmonton, AB T5J 4S2, Canada; anofech@gmail.com

† Presented at the 2nd Electronic Conference on Universe, 16 February–2 March 2023; Available online: https://ecu2023.sciforum.net/.

Abstract: Two equations whose variables take values in the Pauli algebra of complex quaternions are shown to be equivalent to the standard Dirac equation and its Hermitian conjugate taken together. They are transformed one into the other by an outer automorphism of the Pauli algebra. Given a solution to the Dirac equation, a new solution is obtained by multiplying it on the right by one of the 16 matrices of the Pauli group. This defines a homomorphism from the Pauli group into the group of discrete symmetries, whose kernel is a cyclic group of order four. The group of discrete symmetries is shown to be the Klein four-group consisting of four elements: the identity Id; the charge conjugation symmetry C; the mass inversion symmetry M; and their composition in either order, CM = MC. The mass inversion symmetry inverts the sign of the mass, leaving the electric charge unchanged. The outer "bar-star" automorphism is identified with the parity operation, resulting in proof of CPT = M or, equivalently, CPTM = Identity.

Keywords: discrete symmetry; charge conjugation; mass inversion; negative mass; group of symmetries; Dirac equation; quaternionic Dirac equation; Pauli group; Pauli algebra

Citation: Nofech, A. Construction of Discrete Symmetries Using the Pauli Algebra Form of the Dirac Equation. *Phys. Sci. Forum* **2023**, *7*, 41. https://doi.org/10.3390/ECU2023-14054

Academic Editor: Gerald Cleaver

Published: 17 February 2023

1. Introduction

A group of discrete symmetries of a fermion field is constructed using a Pauli algebra form of the Dirac equation. In this form, the variable is not a four-component spinor as in the standard Dirac equation but rather a function of spacetime variables that takes values in the Pauli algebra $\mathrm{Cl}_{1,2}(\mathbb{R}) \cong M_2(\mathbb{C})$. There is an outer automorphism of the Pauli algebra $z \to \bar{z}^*$, which fixes the subalgebra of real quaternions $\mathbb{H}$ and is the complex quaternionic conjugation. This "bar-star" homomorphism transforms one equation of (9) into the other.

Altogether, this amounts to eight scalar equations. The four in the left column are the same as in the standard Dirac equation, and the four in the right column are the same as in the Hermitian conjugate of the standard Dirac equation. Hence, the two equations taken together are equivalent to the Dirac equation and its Hermitian conjugate taken together.

However, there is a difference when considering transformations of a fermion field: multiplying the Pauli algebra spinor on the right by an arbitrary constant 2×2 matrix commutes with the differentials that act on the left. Multiplying the Pauli algebra spinor ψ on the right by each one of the 16 matrices of the basic Pauli group P_1, we obtain a homomorphism from P_1 into a group of discrete symmetries of the fermion field.

It turns out that the kernel of this homomorphism is a cyclic group of order 4. When factoring out the kernel, a Klein four-group of symmetries is obtained, as shown in the exact sequence:

$$\{1\} \to C_4 \to P_1 \to C_2 \times C_2 \to \{1\} \quad (1)$$

The generator of the cyclic group is $i\sigma_3$

$$(i\sigma_3)^2 = -1 \quad (i\sigma_3)^3 = -i\sigma_3 \quad (i\sigma_3)^4 = 1 \quad (2)$$

The group of discrete symmetries consists of four elements:

$$C_2 \times C_2 = \{Id,\ C,\ M,\ CM\} \tag{3}$$

where Id is the identity, C is the charge conjugation, M is the mass inversion, and they commute. The charge conjugation reverses the charges and leaves the mass without change, while mass inversion reverses the sign of mass and leaves the charges without change.

The parity and the time inversion symmetries are not obtained by the right action of Pauli group. The parity operation is identified with the bar-star automorphism, and the time inversion is parity followed by right multiplication by $-i\sigma_2$.

It is shown that $CPTM = Id$, or equivalently $CPT = M$, and the equality is exact.

The standard Pauli–Dirac representation is used throughout for four-component spinors. Transformations in four-component form are compared with textbooks [1,2].

2. The Pauli Algebra Form of the Dirac Equation

The purpose of this section is a quick reminder of the construction described in [3] and a correction of the error there. The main idea is to complete the four-component spinor with a right column, so that the additional four scalar equations will be exactly the four scalar equations of the Hermitian conjugate Dirac equation. However, the new four equations need a change of sign, which is obtained by multiplying the mass term on the right by σ_3. This matrix at the right of the equation corrects the error of [3]. Letters are used instead of Ψ_μ.

$$\begin{bmatrix} \partial_0 & 0 & \partial_3 & -i\partial_2+\partial_1 \\ 0 & \partial_0 & i\partial_2+\partial_1 & -\partial_3 \\ -\partial_3 & i\partial_2-\partial_1 & -\partial_0 & 0 \\ -i\partial_2-\partial_1 & \partial_3 & 0 & -\partial_0 \end{bmatrix} \begin{bmatrix} a & b^* \\ b & -a^* \\ c & -d^* \\ d & c^* \end{bmatrix} = -im \begin{bmatrix} a & b^* \\ b & -a^* \\ c & -d^* \\ d & c^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

Now, the bottom square becomes a real quaternionic function, and the top square an imaginary quaternionic one, i.e., multiplied by the imaginary unit, and we can rewrite in two by two blocks:

$$\begin{bmatrix} \partial_0 & \sigma\cdot\nabla \\ -\sigma\cdot\nabla & -\partial_0 \end{bmatrix} \begin{bmatrix} iv \\ u \end{bmatrix} = -im \begin{bmatrix} iv \\ u \end{bmatrix} [\sigma_3]\, u, v \in \mathbb{H} \tag{5}$$

Multiplying out, we obtain:

$$\partial_0\, iv + \sigma\cdot\nabla\, u \quad = \quad -m\, iv i\sigma_3 \tag{6}$$

$$\partial_0\, u + \sigma\cdot\nabla\, iv \quad = \quad m\, u i\sigma_3 \tag{7}$$

add and subtract the equations, and introduce the notation (the bar-star outer automorphism reverses the sign of the vector summand and also takes the Hermitian conjugate):

$$\psi := u + iv \quad \overline{\psi}^* = u - iv \tag{8}$$

This results in the complex quaternionic Dirac equation and its bar-star image:

$$(\partial_0 + \sigma\cdot\nabla\,)\psi \quad = \quad m\,\overline{\psi}^* i\sigma_3 \tag{9}$$

$$(\partial_0 - \sigma\cdot\nabla\,)\overline{\psi}^* \quad = \quad m\ \psi i\sigma_3 \tag{10}$$

These two equations are transformed one into the other by the bar-star automorphism.

Proposition 1. *When equipped with multiplication as below, the completed two-column four-component spinors form an algebra isomorphic to the Pauli algebra. In this algebra,* *reversing the sign of the upper square corresponds to the bar-star automorphism, which is the complex quaternionic conjugation.*

$$\begin{bmatrix} iv_1 \\ u_1 \end{bmatrix} \begin{bmatrix} iv_2 \\ u_2 \end{bmatrix} := \begin{bmatrix} u_1 iv_2 + iv_1 u_2 \\ u_1 u_2 - v_1 v_2 \end{bmatrix} \tag{11}$$

Proof. This is the complex quaternionic multiplication corresponding to $\psi_1\psi_2$. □

3. The Short Exact Sequence Connecting the Pauli Group with the Group of Symmetries

Proposition 2. *Multiplying the Pauli algebra spinor on the right by one of the 16 matrices of the basic Pauli group P_1 defines a homomorphism from the Pauli group into the group of discrete symmetries, where discrete symmetries are considered equivalent if they only differ by a phase factor. The kernel of this homomorphism is a cyclic group of order four generated by $i\sigma_3$, and the group of discrete symmetries is the Klein four-group consisting of identity Id, the charge conjugation symmetry C, the mass inversion symmetry M, and their composition in either order $CM = MC$.*

Proof. We use (11) to calculate the translation of $\psi(i\sigma_3)$ into the four-component language:

$$\psi(i\sigma_3) \leftrightarrow \Psi \begin{bmatrix} 0 \\ i\sigma_3 \end{bmatrix} = \begin{bmatrix} iv \\ u \end{bmatrix} \begin{bmatrix} 0 \\ i\sigma_3 \end{bmatrix} = \begin{bmatrix} iv(i\sigma_3) \\ u(i\sigma_3) \end{bmatrix} = \begin{bmatrix} a & b^* \\ b & -a^* \\ c & -d^* \\ d & c^* \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} ia & -ib^* \\ ib & ia^* \\ ic & id^* \\ id & -ic^* \end{bmatrix} \tag{12}$$

Now, we leave only the left column to go back to the standard four-component form, so as to confirm that this transformation differs from identity by only a phase factor $i = e^{i\frac{\pi}{2}}$. □

4. The Charge Conjugation and the Mass Inversion Symmetries

Next, we consider the elements of P_1 that generate the charge conjugation and the mass inversion symmetries. First, the charge conjugation symmetry $\psi \to \psi_c$

When including the interaction term, the sign in the right column, which comes from the Hermitian conjugate equation, is the opposite of the sign in the left column, exactly as in the mass term. To correct this, the σ_3 factor appears at the end.

The equation including the EM interaction term:

$$(\,\partial_0 + \sigma\cdot\nabla\,)\psi + iq\,A^\mu\sigma_\mu\psi\sigma_3 = im\,\overline{\psi}^*\sigma_3 \tag{13}$$

Now, multiply both sides of the equation by σ_1 on the right:

$$(\,\partial_0 + \sigma\cdot\nabla\,)\psi\sigma_1 + i(-q)\,A^\mu\sigma_\mu\psi\sigma_1\sigma_3 = im\,\overline{\psi}^*(-\sigma_1)\sigma_3 \tag{14}$$

Note that $\overline{\psi}_c^* = \overline{(\psi\sigma_1)}^* = -\psi\sigma_1$

We obtain the equation for ψ_c:

$$(\,\partial_0 + \sigma\cdot\nabla\,)\psi_c + i(-q)\,A^\mu\sigma_\mu\psi_c\sigma_3 = im\,\overline{\psi_c}^*\sigma_3 \tag{15}$$

We see that the mass remains the same; however, the charge switched signs.

Now, we define $\psi_m := \psi\sigma_3$. Multiply both sides of the equation by σ_3:

$$(\,\partial_0 + \sigma\cdot\nabla\,)\psi\sigma_3 + iq\,A^\mu\sigma_\mu\psi\sigma_3\sigma_3 = -im\,\overline{\psi}^*(-\sigma_3)\sigma_3 \tag{16}$$

Write the equation again for the modified ψ_m:

$$(\,\partial_0 + \sigma\cdot\nabla\,)\psi_m + iq\,A^\mu\sigma_\mu\psi_m\sigma_3 = i(-m)\,\overline{\psi_m}^*\sigma_3 \tag{17}$$

Now, the mass switched signs; however, the charge remained the same.

Let us now find the matrices of the transformations C and M in the four-component language. Start with $C : \psi \to \psi_c = \psi\sigma_1$

Represent the multiplication as the multiplication of two-column four-component spinors:

$$\begin{bmatrix} iv \\ u \end{bmatrix}\begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} = \begin{bmatrix} u\sigma_1 \\ iv\sigma_1 \end{bmatrix} = \begin{bmatrix} c & -d^* \\ d & c^* \\ a & b^* \\ b & -a^* \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -d^* & c \\ c^* & d \\ b^* & a \\ -a^* & b \end{bmatrix} \tag{18}$$

Now, we leave only the left column and find its transformation matrix:

$$\Psi_c = \begin{bmatrix} -d^* \\ c^* \\ b^* \\ -a^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = -i\gamma^2\Psi^* \tag{19}$$

This is the charge conjugation matrix that can be found in textbooks.

Note that it acts on the left unlike σ_1, which acts on the right. The right action on Pauli algebra spinors commutes with differentials that act on the left, but the left action of four-component spinors generally does not commute with differentials, which also act on the left.

Next, we find the four-component matrix for the mass inversion M:

$$\begin{bmatrix} iv \\ u \end{bmatrix}\begin{bmatrix} \sigma_3 \\ 0 \end{bmatrix} = \begin{bmatrix} u\sigma_3 \\ iv\sigma_3 \end{bmatrix} = \begin{bmatrix} c & -d^* \\ d & c^* \\ a & b^* \\ b & -a^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} c & d^* \\ d & -c^* \\ a & -b^* \\ b & a^* \end{bmatrix} \tag{20}$$

Then, as before we leave only the left column:

$$\begin{bmatrix} c \\ d \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \Psi_m = \gamma^5\Psi \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \gamma^5\begin{bmatrix} \chi \\ \eta \end{bmatrix} = \begin{bmatrix} \eta \\ \chi \end{bmatrix} \tag{21}$$

5. Identifying Pauli Algebra Expressions for Parity and Time-Reversal Symmetries

The parity and time-reversal transformations are not part of the Klein four-group of symmetries generated by the right multiplication of the spinor by elements of the Pauli group.

We begin by identifying the parity transformation:

Proposition 3. *The parity transformation is the bar-star automorphism:* $\psi_p = \overline{\psi}^*$

Proof. Rewrite the second equation of (9) in this new notation. Then, introduce the notation $\sigma\cdot\overline{\nabla} := -\sigma\cdot\nabla$, i.e., reverse the spatial derivatives:

$$(\partial_0 - \sigma\cdot\nabla)\psi_p = m\ \overline{\psi_p}^* \quad i\sigma_3(\partial_0 + \sigma\cdot\overline{\nabla})\psi_p = m\ \overline{\psi_p}^* i\sigma_3 \tag{22}$$

ψ_p now satisfies the first equation of (9) with spatial derivatives reversed. To confirm it is indeed the parity transformation we calculate its translation to the four-component language. The bar-star is the complex quaternionic conjugation and reverses the sign of iv, the top square. So, $\Psi_p = -\gamma^0\,\Psi$, which matches the $S_P = \pm\gamma^0$ found in textbooks. □

Next, we identify the time-reversal operation:

Proposition 4.

$$\psi_t \; = \; \overline{\psi}^*(-i\sigma_2) = \psi_p(-i\sigma_2) \tag{23}$$

Proof. Begin with the equation for parity and multiply it on the right by $-i\sigma_2$. Then, reverse the signs so that only the time derivative changes its sign.

$$(\partial_0 - \sigma\cdot\nabla\,)\psi_p(-i\sigma_2) = m\;\overline{\psi_p}^* i\sigma_3(-i\sigma_2) \tag{24}$$

$$(\partial_0 - \sigma\cdot\nabla\,)\psi_p(-i\sigma_2) = -m\;\overline{\psi_p(-i\sigma_2)}^* \quad i\sigma_3\left(\overline{\partial_0} + \sigma\cdot\nabla\,\right)\psi_t = m\;\overline{\psi_t}^* i\sigma_3 \tag{25}$$

We confirm by calculating the translation into the four-component language and comparing with the known matrix S_t:

$$\begin{bmatrix} -iv \\ u \end{bmatrix}\begin{bmatrix} 0 \\ -i\sigma_2 \end{bmatrix} = \begin{bmatrix} -iv \\ u \end{bmatrix}\begin{bmatrix} -i\sigma_2 \end{bmatrix} = \begin{bmatrix} -a & -b^* \\ -b & a^* \\ c & -d^* \\ d & c^* \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -b^* & a \\ a^* & b \\ -d^* & -c \\ c^* & -d \end{bmatrix} \tag{26}$$

Now, we leave only the first column and write down the matrix, which obtains it from Ψ^*:

$$\begin{bmatrix} -b^* \\ a^* \\ -d^* \\ c^* \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = -\gamma^1\gamma^3\Psi^* \tag{27}$$

□

6. The Proof of CPTM Invariance

Proposition 5.

$$CPTM \; = \; Id \tag{28}$$

Proof. The Pauli algebra proof:

$$\psi \to \psi_c = \psi\sigma_1 \quad \psi_c \to \psi_{cp} = \overline{\psi_c}^* = -\overline{\psi}^*\sigma_1 \tag{29}$$

$$\psi_{cp} \to \psi_{cpt} = \overline{(\psi_{cp})}^*(-i\sigma_2) = -i\psi\sigma_1\sigma_2 = \psi\sigma_3 \tag{30}$$

$$\psi_{cpt} \to \psi_{cptm} = \psi_{cpt}\sigma_3 = \psi\sigma_3\sigma_3 = \psi \tag{31}$$

□

Proof. The four-component proof:

$$\Psi \to \Psi_c = -i\gamma^2\Psi^* \qquad \Psi_c \to \Psi_{cp} = -\gamma^0\Psi_c = i\gamma^0\gamma^2\Psi^* \tag{32}$$

$$\Psi_{cp} \to \Psi_{cpt} = -\gamma^1\gamma^3\Psi^*_{cp} = -i\gamma^1\gamma^3\gamma^0\gamma^2\Psi = i\gamma^0\gamma^1\gamma^2\gamma^3\Psi = \gamma^5\Psi \tag{33}$$

$$\Psi_{cpt} \to \Psi_{cptm} = \gamma^5\Psi_{cpt} = \gamma^5\gamma^5\Psi = \Psi \tag{34}$$

□

7. Relation to Other Work

Negative mass and the mass inversion symmetry are considered in [4,5], though their approaches are different from the one used here.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Peng, J.C. Physics 570 Lecture Notes. Available online: https://courses.physics.illinois.edu/phys570/fa2018/note.html (accessed on 30 November 2022).
2. Schwartz, M.D. *Quantum Field Theory and the Standard Model*; Cambridge University Press: Cambridge, UK, 2014.
3. Nofech, A. Biquaternionic Dirac equation predicts zero mass for Majorana fermions. *Symmetry* **2020**, *12*, 1144. [CrossRef]
4. Debergh, N.; Petit, J.P.; D'Agostini, G. On evidence for negative energies and masses in the Dirac equation through a unitary time-reversal operator. *J. Phys. Commun.* **2018**, *2*, 115012. [CrossRef]
5. Marsch, E.; Narita, Y. CPTM Symmetry for the Dirac Equation and Its Extended Version Based on the Vector Representation of the Lorentz Group. *Front. Phys.* **2021**, *9*, 249. [CrossRef]