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Physical Picture of Electron Spin †

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Abstract: Pauli established the standard view that the spin of the electron was a completely abstract non-classical angular momentum that could not be thought of as the rotation of anything. Here, we give a pedagogical presentation of old work by Belifante (1939), recently updated by Ohanian (1986), which shows that, contrary to Pauli's edict, the spin of the electron can be viewed as the rotational angular momentum in the wave field of the electron.

Keywords: spin angular momentum; electron structure; Dirac field

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1. Introduction

The standard view of the spin of the electron is that it is an internal angular momentum which cannot be pictured as a tiny rotating ball or in fact as the rotation of anything [1,2]. If one were to picture the spin of the electron as coming from the rotation of a spherical shell of radius of the electron is 10^{-15} m (this is an upper limit). We know that the electron the speed would be bigger. It is just an assumption if we take a less value, it would be very inconsistent to the special relativity. And the mass of the electron is 10^{-31} kg, which is equal to the mass of the proton, and having an angular momentum of $\hbar/2$ then one would find that points on the surface of this spherical shell would need to move a ~100 times of the speed of light, which is impossible. Based on the fact it violates the Special Relativity, Pauli said that the picture of the electron spin was not violable.

However, quite early in the formulation of quantum theory, Belinfante [3] and Gordon [4] argued that one could interpret the spin as a rotating angular momentum coming from a rotating energy-momentum in the Dirac field the describes the electron.

Here, we review the arguments of Belinfante and Gordon, as well as more recent work by Ohanian [5] which shows in detail how the spin of the electron can be seen as a rotation of energy-momentum in the Dirac wave field. This shows that electron spin is exactly of the same character as any other angular momentum, rather than some mysterious quantum property. We start by showing how this viewpoint can be applied to the electromagnetic field to get the spin of the photon and then we move to the Dirac field to obtain the spin of the electron.

2. Spin of the Electromagnetic Field

In this section we obtain the spin of the photon using the same method we will use in the next section to obtain the spin of the electron. The reason for this is that this shows the connection between and commonality of the spin coming from Maxwell's equations and the Dirac equation.

Firstly, I worked on the momentum density in the electromagnetic field [6] is given by

$$\vec{G} = \frac{\vec{E} \times \vec{B}}{\mu_0 c^2} \tag{1}$$

Now taking $\vec{B} = \nabla \times \vec{\mathrm{A}}$. Substitute this term into the above equation, we get

$$\vec{G} = \frac{\vec{E} \times \left(\nabla \times \vec{\mathrm{A}}\right)}{\mu_0 c^2} \quad (2)$$

using the double cross product formula, we have

$$\vec{\boldsymbol{G}} = \frac{\left[\boldsymbol{E}^n \nabla\, \boldsymbol{A}^n - \left(\vec{\boldsymbol{E}}.\nabla\right)\vec{\boldsymbol{A}}\right]}{\boldsymbol{\mu_0 c^2}} \quad (3)$$

From Equation (2), we obtain the angular momentum density by taking the cross product of the momentum density with x as $\vec{\mathcal{J}} = \frac{\vec{x}\times(\vec{E}\times\vec{B})}{\mu_0 c^2}$.

On integration the momentum density, we can obtain the net angular momentum, it can split into two terms

$$\vec{\boldsymbol{J}} = \frac{\mathbf{1}}{\boldsymbol{\mu_0 c^2}} \int \boldsymbol{x} \times \left(\vec{\boldsymbol{E}}^n \nabla \vec{\boldsymbol{A}}^n\right) \boldsymbol{d^3 x} + \frac{\mathbf{1}}{\boldsymbol{\mu_0 c^2}} \int \boldsymbol{x} \times \left[-\left(\vec{\boldsymbol{E}}.\nabla\right)\vec{\boldsymbol{A}}\right] \boldsymbol{d^3 x} \quad (4)$$

Using an integration by parts, with $\nabla \,.\, \vec{E} = 0$, then it becomes as,

$$\vec{\boldsymbol{J}} = \frac{\mathbf{1}}{\boldsymbol{\mu_0 c^2}} \int \boldsymbol{x} \times \left(\vec{\boldsymbol{E}}^n \nabla \vec{\boldsymbol{A}}^n\right) \boldsymbol{d^3 x} + \frac{\mathbf{1}}{\boldsymbol{\mu_0 c^2}} \int (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{A}})\, \boldsymbol{d^3 x} \quad (5)$$

The first term in above equation represents the orbital angular momentum and the second term represents the spin.

To justify this interpretation, consider a circularly polarized plane wave with vector potential.

$$\vec{A} = (\hat{x} \pm i\hat{y}) \left(\frac{iE_0}{\omega}\right) e^{i\omega(t - \frac{z}{c})}$$

The time-averaged values of the integrals from Equation (3) are:

$$\vec{L} = \frac{1}{2\mu_0 c^2} \int Re\,(x \times (E^n \nabla A^{*n}) d^3 x)$$

Now, from the above equation calculating the middle term in x and y components (n = 1, 2 leads to x and y components correspondingly) because there is no z component in the above equation.

$(E^x \nabla A^{*x}) = \frac{E_0{}^2}{c}\, \hat{z}$, and $(E^y \nabla A^{*y}) = \frac{E_0{}^2}{c}\, \hat{z}$, therefore, $E^n \nabla A^{*n} = \frac{2E_0{}^2}{c}\, \hat{z}$.

Substitute in above equation, we get,

$$\vec{\boldsymbol{L}} = \frac{\mathbf{1}}{\boldsymbol{\mu_0 c^3}} \int \vec{\boldsymbol{x}} \times \left(\hat{\boldsymbol{z}} \boldsymbol{E_0}^{\mathbf{2}}\right) \boldsymbol{d^3 x} \quad (6)$$

Now

$$\boldsymbol{S} = \frac{\mathbf{1}}{\boldsymbol{2\mu_0 c^2}} \int \boldsymbol{Re}\, (\boldsymbol{E}^n \nabla \boldsymbol{A}^{*n}) \boldsymbol{d^3 x} \quad (7)$$

$$\mathbf{E} = (\hat{\boldsymbol{x}} \pm \boldsymbol{i}\hat{\boldsymbol{y}})\, \boldsymbol{E_0} \boldsymbol{e}^{\boldsymbol{i\omega(t - \frac{z}{c})}} (\boldsymbol{e}^{\boldsymbol{\alpha}})$$

Here, the electric field comes from the solution of the wave equation for the Electromagnetic waves in the homogeneous medium. Assume e^{α} as exponential term and $\alpha$is the vector.

Now, consider $E \times A^*$; we get $E \times A^* = \pm \frac{2E_0{}^2}{\omega}\, \hat{z}$.

Therefore,

$$S = \pm \frac{\boldsymbol{E_0}^2}{\boldsymbol{\mu_0 c^2 \omega}} \int \boldsymbol{d^3 x \hat{z}}$$

The first of these expressions is polarization independent, and is exactly what we expect for the plane wave's orbital angular momentum. Because, the second expression is not affected by the polarization. We must identify it as the spin. But, the individual integrals in equation are not gauge invariant.

The energy density in the wave equation is

$$\boldsymbol{U} = \frac{\boldsymbol{1}}{\boldsymbol{2\mu_0 c^2}} \int \boldsymbol{Re}\ (\boldsymbol{E}.\ \boldsymbol{E}^*)\boldsymbol{d^3x} \tag{8}$$

Here, $E = (\hat{x}\ \pm i\hat{y})\ E_0 e^{i\omega(t-\frac{z}{c})}$ and $E^* = (\hat{x}\ \mp\ i\hat{y})\ E_0 e^{-i\omega(t-\frac{z}{c})}$.

Now, consider the real part of the energy density in Equation (6), then we get $\boldsymbol{E}.\boldsymbol{E}^* = \boldsymbol{E_0}^2 +\ \boldsymbol{E_0}^2$ i.e., $\boldsymbol{Re}(E.\boldsymbol{E}^*) = 2\boldsymbol{E_0}^2$.

Then, the energy density equation becomes as

$$\boldsymbol{U} = \frac{\boldsymbol{1}}{\boldsymbol{\mu_0 c^2}} \int \boldsymbol{E_0^2}\ \boldsymbol{d^3x} \tag{9}$$

From spin and energy density equations, we have $\frac{S_z}{U} = \pm\frac{1}{\omega}$.

Since the energy is one quantum U = $\hbar\omega$, then the spin will be $S_z = \pm\hbar$.

3. Dirac Field Angular Momentum

We now perform a similar calculation to the one done in Section 2 for the electromagnetic field, but now for the Dirac field [7]. First the momentum density of the Dirac field is given by

$$\vec{\boldsymbol{G}} = \frac{\hbar}{\boldsymbol{4i}}\left(\boldsymbol{\psi}^\dagger\nabla\boldsymbol{\psi} - \boldsymbol{\psi}^\dagger\vec{\boldsymbol{\alpha}}\partial_t\boldsymbol{\psi}\right) + \boldsymbol{hc}$$

where *hc* stands for Hermitian conjugate, ψ is the Dirac spinor field, and $\vec{\alpha}$ are Dirac matrices may be giving a cite to the standard form of the alpha. The time derivative in above equation can be eliminated by means of Dirac equation

$$\frac{1}{c}\frac{\partial\alpha}{\partial t} = \left(-\vec{\alpha}.\nabla + \frac{mc^2}{i\hbar}\ \beta\right)\psi$$

It gives,

$$\vec{G} = \left(\frac{\hbar}{4i}\right)\left(\psi^\dagger\nabla\psi + \psi^\dagger\vec{\alpha}\left(\vec{\alpha}.\nabla\right)\psi\right) + hc$$

Since the commutation relations for $\boldsymbol{\alpha}$, after doing some calculations, we get

$$\vec{G} = \left(\frac{\hbar}{2i}\right)\left(\psi^\dagger\nabla\psi - \left(\nabla\psi^\dagger\right)\psi\right) + \frac{\hbar}{4}\ \nabla\times\left(\psi^\dagger\sigma\psi\right)$$

where $\sigma_1 = -i\alpha_2\alpha_3,\ \sigma_2 = -i\alpha_3\alpha_1,\ \sigma_3 = -i\alpha_1\alpha_2$.

Here, α is a 4×4 matrix with diagonal are zeros and the spinors [8].

The first term in above equation is the translational motion of the electron, whereas the second term is the circulating flow of energy.

For example, consider the Gaussian packet, $\psi = \left(\pi d^2\right)^{-\frac{3}{4}} e^{-\left(\frac{1}{2}\right)\frac{r}{d}^2}\omega^{|}(0)$. It defines an electron spin with zero expectation value of the momentum. It represents the Gaussian function, which means that the probability density of finding the particle. It refers to the spinor wave function for an electron.

From above equation, in the first term if we substitute the gaussian function $\psi^\dagger \nabla \psi - (\nabla \psi^\dagger)\psi$ should get cancel. So, the first term is zero and the second term is,

$$\vec{G} = \frac{\hbar}{4}\left(\frac{1}{\pi d^2}\right)^{3/2} \frac{e}{d^2}^{-\frac{r^2}{d^2}}(-2y\,\hat{x} + 2x\hat{y}).$$

In case of an electromagnetic wave, the angular momentum rises due to a circulating flow of energy. Then, the angular momentum is the spin of an electron. Therefore, the net angular momentum is,

$$\vec{J} = \int \frac{\hbar}{2i}\boldsymbol{x} \times \left[\boldsymbol{\psi}^\dagger \nabla \boldsymbol{\psi} - \left(\nabla \boldsymbol{\psi}^\dagger\right)\boldsymbol{\psi}\right] \boldsymbol{d}^3\boldsymbol{x} + \int \frac{\hbar}{4}\boldsymbol{x} \times \left[\nabla \times \left(\boldsymbol{\psi}^\dagger \sigma\right)\boldsymbol{\psi}\right] \boldsymbol{d}^3\boldsymbol{x} \tag{10}$$

Using the triple cross product in the second term can be expanded into two dot products and then integrate both of these terms by parts. This gives,

$$\vec{J} = \frac{\hbar}{2i}\int \boldsymbol{x} \times \left[\boldsymbol{\psi}^\dagger \nabla \boldsymbol{\psi} - \left(\nabla \boldsymbol{\psi}^\dagger\right)\boldsymbol{\psi}\right] \boldsymbol{d}^3\boldsymbol{x} + \frac{\hbar}{2}\int \boldsymbol{\psi}^\dagger \vec{\sigma}\, \boldsymbol{\psi}\, \boldsymbol{d}^3\boldsymbol{x} \tag{11}$$

Here, the spin is the second term, and the first term is the orbital angular momentum.

From the spin in above equation, the expectation value of the quantum mechanical operator $\vec{\sigma}$, the operator representing the spin must be

$$\text{Sop} = \frac{\hbar}{2}$$

It yields the value $\pm\frac{\hbar}{2}$ for the integral S_z.

4. Summary and Conclusions

As stated in previous sections, the spin is fundamentally a quantum mechanical feature. The type of wave is less important in these calculations than the fact that the spin is ultimately a wave feature. The spin of a classical wave is a continuous macroscopic quantity, whereas the quantum spin is quantized and represented by a quantum mechanical operator. This is the fundamental distinction between the spins of the two types of waves. Because the spin of a quantum mechanical particle has a fixed magnitude, it is impossible to reach the classical limit.

We calculated the spin of the photon starting with Maxwell's equations in the second section and then calculated the spin of the electron in the third section.

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