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Correlation Coefficient of Complex Interval-Valued Intuitionistic Fuzzy Sets and Their Applications in Pattern Recognition

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Abstract

Complex interval-valued fuzzy sets are powerful tools for representing uncertainty and periodicity that occur in many real-life problems. They not only take both periodicity semantics and uncertainty into account but also describe the information using interval-valued membership grades, which gives experts more freedom to solve complex real-life problems effectively. Complex interval-valued fuzzy sets have been successfully employed many times in medical diagnosis and pattern recognition problems. In this article, two novel methods for computing the correlation coefficient and weighted correlation coefficient of complex interval-valued fuzzy sets are proposed. The methods employed fuzzy statistical parameters such as mean, variance, and covariance of complex interval-valued fuzzy sets. Several important mathematical properties are rigorously established. In order to establish the efficacy and implementation of the methods, a real-life application related to pattern recognition for mineral identification is discussed in detail. Furthermore, based on the proposed correlation and weighted correlation, a classification algorithm is developed. The time and space complexities of the proposed algorithm are analyzed. The proposed algorithm is validated through experiments conducted on two real benchmark datasets. The results demonstrate that the proposed approaches are more reliable and accurate than several existing approaches by achieving classification accuracies exceeding 98%.

Keywords: intuitionistic fuzzy set; interval-valued fuzzy set; interval-valued intuitionistic fuzzy set; complex fuzzy set; complex interval-valued intuitionistic fuzzy set; mean of CIVIFS; covariance of CIVIFS

1. Introduction

Decision-making is an essential component of almost every domain of human knowledge. Real-life decision-making is usually associated with uncertainty, imprecision, vagueness, and ambiguity; therefore, the classical crisp set framework fails to model such situations adequately. Zadeh [1] introduced the concept of fuzzy set theory by associating every element of a set with a degree of membership between 0 and 1. Atanassov [2] introduced the concept of intuitionistic fuzzy set (IFS) by incorporating an independent nonmembership value with the membership value, and in [3], the authors generalized IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs) by representing both membership and nonmembership degrees as intervals. Since then, the fuzzy set theory has been extensively used to represent uncertainty in numerous real-life problems of pattern recognition, classification, decision science, etc. [4–11].

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Although the aforementioned extensions of fuzzy sets [2,3] successfully model the uncertain information, they cannot adequately model the periodicity and phase-dependent phenomena. To address the issue, fuzzy sets were extended to the complex plane [12–15]. In [16], the authors introduced the concept of complex fuzzy sets (CFSs) by extending the membership function on the unit circle of the complex plane. This extension represents amplitude and phase information simultaneously, making CFSs especially suitable for modeling time-dependent, periodic, and cyclic phenomena. Subsequently, CFSs have been widely used in astronomy, geology, communication systems, optics, signal processing, and many problems of science, engineering, and technology [16,17]. The idea of complex intuitionistic fuzzy sets (CIFs) was introduced by Tamir et al. [18]. Later, [19] introduced the concept of complex interval-valued intuitionistic fuzzy set (CIVIFS) and [20] demonstrated its various real-life applications. The CIVIFS provides a more expressive framework for representing complex and high-level uncertain information. Moreover, many aggregation operators in the CIVIFS environment and their applications in healthcare and decision-making problems have been reported in [21].

One of the most basic statistical concepts is correlation analysis, which is used to quantify both the strength and the direction of the association between variables. It has been extended to various fuzzy environments [22–35] to address many problems in pattern recognition, classification, clustering, decision-making, and medical diagnosis, and numerous formulae for correlation coefficients have been proposed. In [33], the authors proposed a statistical formulation of the correlation coefficient for IFSs, while later studies extended the concept to IVIFSs [34,35]. Subsequently, many researchers have investigated several correlation and similarity frameworks to extend the idea to complex fuzzy environments [36–40]. Chiang and Lin [41] proposed a statistical correlation model for fuzzy sets using sample means and covariance. Furthermore, statistical correlation formulae based on sample means have been proposed for IFSs in [42].

In spite of the substantial developments, a couple of vital challenges still remain. The majority of existing correlation models are derived from distance or similarity measures rather than from a strong statistical framework consisting of the mean, variance, covariance, etc. Moreover, most of the existing formulae quantify only the strength of association and not the direction since the values lie in the interval [0, 1]. Furthermore, numerous real-life applications, such as medical diagnosis, profit-loss analysis, mining engineering, etc., require the detection of both positive and negative associations for effective decision-making. In weighted correlation framework, the challenge is determining the attribute weights. In [43], the authors proposed a fuzzy analytic hierarchy process-based decision-making model for healthcare systems by combining the preferences and judgements of multiple expert decision-makers. In [44], the authors introduced the surface fuzzy analytic hierarchy process by integrating differential geometry with fuzzy logic. In [45], the authors proposed a hybrid decision-making framework spherical fuzzy analytic hierarchy process combining partial least squares structural equation modeling and artificial neural networks to predict COVID-19 vaccination intention. However, the aforementioned approaches involve human preferences and subjective judgments to determine the attribute weights. Though these approaches are highly suitable for decision support systems, they are inadequate for classification and pattern recognition problems where minimal expert involvement is highly recommended.

Motivated by these observations, this paper develops a covariance-based statistical framework for CIVIFSs. Unlike the existing methods, the proposed framework is derived from statistical parameters such as the mean, weighted mean, covariance, weighted covariance, variance, and weighted variance rather than directly from similarity or distance measures. Furthermore, the proposed framework is data-driven with minimal expert intervention where the attribute weights are determined from their standard deviations. In the present approach, the statistical parameters are first established for CIVIFSs. Based on these statistical parameters, ordinary and weighted correlation coefficients are developed whose values lie in the interval $[-1, 1]$. Thus, the proposed formulae extend classical statistical interpretation of both the strength and the direction of association to the CIVIFS environment. Several important mathematical properties of the proposed correlation coefficients are rigorously established. Finally, an efficient correlation-based classification algorithm is developed and validated on the Iris dataset [46,47] and the Wisconsin Breast Cancer dataset [48]. Experimental results demonstrate that the proposed methods outperform several existing approaches reported in the literature [20,49–53].

The remainder of this paper is organized as follows. Section 2 reviews the related literature and presents the necessary preliminaries. Section 3 introduces the proposed statistical framework and develops the ordinary and weighted correlation coefficients together with their mathematical properties. Section 4 presents the proposed classification algorithm and numerical examples. Section 5 discusses the experimental results. Finally, Section 6 concludes the paper and discusses the conclusions, limitations and directions for future research.

2. Related Literature Review and Preliminaries

2.1. Related Literature Review

Since Zadeh's [1] introduction of fuzzy sets, several studies have been conducted on distance similarity, correlation, and other measures under various fuzzy environments. Similarity measures for IFSs [2] and their applications [4–6] have been widely used in medical diagnosis, pattern recognition, decision-making, clustering, etc. [7–11]. Statistical approaches to formulating the correlation coefficients for IFSs [22–32] and their fuzzy extensions have received extensive attention from researchers. Based on covariance, Hung [33] proposed a statistical framework for correlation analysis of IFSs. Subsequently, the approach has been extended to IVIFS [3] and applied in real-life problems [34,35]. Similar approaches for fuzzy sets and IFSs based on sample means and covariance have also been proposed [41,42].

Similarity and correlation measures have received attention from researchers after the introduction of complex fuzzy structures. The concept of CFSs was introduced by Ramot et al. [16] by extending the membership functions to the unit circle in the complex plane. Subsequently, several studies were conducted not only on the theoretical aspects but also their applications in many branches of human knowledge [12–17]. The concept of CIFSs was introduced by Tamir et al. [20], along with the rigorously established mathematical properties. Later, a study was conducted on the aggregation operators and applications to healthcare and decision-making problems [21].

After the introduction CIVIFSs [18], many researchers explored successfully modelling uncertainty involving both interval-valued and phase-dependent information. It has been demonstrated in practice that CIVIFS model provides a more realistic view of complex uncertain systems than conventional fuzzy models [19]. Several measures have also been proposed for complex fuzzy environments. In [36–39], the authors investigated a couple of correlation coefficients and similarity measures for pattern recognition and decision-making problems. Later, similar approaches were extended to interval-valued Pythagorean fuzzy hypersoft sets [40]. In [43], the authors proposed a fuzzy analytic hierarchy process-based decision-making model for the healthcare system by combining the preferences and judgements of multiple expert decision-makers. In [44], the authors introduced the surface fuzzy analytic hierarchy process by integrating differential geometry with fuzzy logic. In [45], the authors proposed a hybrid decision-making framework spherical fuzzy analytic hierarchy process combining partial least squares structural equation modeling and artificial neural networks to predict COVID-19 vaccination intention. In [54], the authors proposed a generalized subtraction operation and a generalized division operation for IVIFSs by minimizing the hamming distance. In [55], the authors used membership, nonmembership, global maximum, and global minimum differences to construct an efficient similarity measure for IFSs for software quality evaluation and face recognition. In [56], the authors introduced an integrated framework based on principal component analysis and the fuzzy analytical hierarchy process to evaluate and rank the enhancement of 21st-century proficiency through higher education.

Despite these substantial progresses, three important limitations still exist. First, several existing correlation measures are derived directly from either distance or similarity measures rather than from a rigorous statistical framework. Second, many existing correlation formulae measure only the strength of the association, as they produce values in the interval [0, 1]. Third, some of the approaches are not suitable for pattern recognition and classification problems, as they involve human preferences and subjective judgments to determine the attribute weights. Furthermore, statistical parameters such as weighted mean, weighted covariance, and weighted variance have not been systematically developed for CIVIFSs.

To address the aforementioned research gaps, a covariance-based statistical framework for CIVIFSs is proposed in this paper that uses statistical parameters such as mean, weighted mean, covariance, weighted covariance, variance, and weighted variance. Subsequently, ordinary and weighted correlation coefficients are developed based on the aforementioned statistical parameters with values in the interval $[-1, 1]$. Thus, the formulae preserve the classical statistical interpretation of both the strength and direction of association while extending it to the CIVIFS environment. This unified framework is further validated through rigorous theoretical analysis and applications to benchmark pattern recognition problems [46–53].

2.2. Preliminaries

Definition 1 ([2])**.** *An intuitionistic fuzzy set (IFS) A defined on X is characterized by*

$$A = \{(x,\ \mu_A(x), \nu_A(x)),\ x \in X\} \tag{1}$$

where $\mu_A(x) \in [0,1]$ *and* $\nu_A(x) \in [0,1]$ *are the membership and nonmembership values of x on A, respectively, such that* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for all* $x \in X$*. Also,* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *is the degree of hesitation of x to A.*

Definition 2 ([3])**.** *Let X be a non-empty finite set. An interval-valued intuitionistic fuzzy set (IVIFS) A in X can be represented as*

$$A = \left\{\left(x, \left[\mu_A^-(x),\ \mu_A^+(x)\right], \left[\nu_A^-(x),\ \nu_A^+(x)\right]; x \in X\right\}\right. \tag{2}$$

where $\left[\mu_A^-(x),\ \mu_A^+(x)\right] \subseteq [0,\ 1]$ *and* $\left[\nu_A^-(x),\ \nu_A^+(x)\right] \subseteq [0,1]$ *and* $\mu_A^+(x) + \nu_A^+(x) \leq 1$.

Definition 3 ([16,17])**.** *A complex fuzzy set (CFS) A defined on X is characterized by a membership function* $\mu_A(x)$ *that assigns a complex-valued membership grade in A to each element* $x \in X$*. By definition, all the membership values lie within the unit circle in the complex plane and are expressed as* $\mu_A(x) = r_A(x)e^{i\theta_A(x)}$*, where both* $r_A(x)$ *and* $\theta_A(x)$ *are real-valued and* $0 \leq r_A(x) \leq 1$ *and* $0 \leq \theta_A(x) \leq 2\pi$*. Therefore, a CFS A can be represented as*

$$A = \{(x,\ \mu_A(x));\ x \in X\} = \left\{\left(x;\ r_A(x) \cdot e^{2\pi i\theta_A(x)}\right);\ x \in X\right\}, \text{where } i = \sqrt{-1}. \tag{3}$$

Obviously, $A \subset U \times D^2$, where $D^2 = \{r_A(x)\cdot e^{2\pi i\theta_A(x)} | r_A(x) \in [0,\ 1]$ and $\theta_A(x) \in [0, 1]\}$.

Definition 4 ([16]). *Let A be a CFS defined on X. The complement of A, denoted by A^c, is defined as*

$$A = \left\{ \left(x;\ (1 - r_A(x))\cdot e^{2\pi i\theta_A(x)} \right);\ x \in X \right\}, \text{ where } i = \sqrt{-1}. \tag{4}$$

Definition 5 ([18,19]). *Let X be a universe of discourse. A complex interval-valued intuitionistic fuzzy set (CIVIFS) defined on X is expressed as*

$$A = \left\{ \left(x,\ [\mu_A^-(x),\ \mu_A^+(x)], [\nu_A^-(x),\ \nu_A^+(x)] \right); x \in X \right\} \tag{5}$$

where $\mu_A^-(x)$, $\mu_A^+(x)$, and $\nu_A^-(x)$, $\nu_A^+(x)$ represent the lower and upper bound degrees of the membership and nonmembership values, respectively. They are defined as $\mu_A^-(x) = z_1^- = r_A^-(x)e^{2\pi i\theta_{r_A}^-(x)}$, $\mu_A^+(x) = z_1^+ = r_A^+(x)e^{2\pi i\theta_{r_A}^+(x)}$ such that $|z_1^-| \leq |z_1^+|$, while $\nu_A^-(x) = z_2^- = \kappa_A^-(x)e^{2\pi i\theta_{\kappa_A}^-(x)}$, $\nu_A^+(x) = z_2^+ = \kappa_A^+(x)e^{2\pi i\theta_{\kappa_A}^+(x)}$ such that $|z_2^-| \leq |z_2^+|$. The amplitudes $r_A^-, r_A^+, \kappa_A^-, \kappa_A^+ \in [0,\ 1]$ satisfy the inequalities $r_A^- \leq r_A^+, \kappa_A^- \leq \kappa_A^+$ and $r_A^+ + \kappa_A^+ \leq 1$ for all $x \in X$. In contrast, the phase terms $\theta_{r_A}^-$, $\theta_{r_A}^+$, $\theta_{\kappa_A}^-$, and $\theta_{\kappa_A}^+$ are real numbers in the interval [0, 1]. They satisfy the inequalities $\theta_{r_A}^- \leq \theta_{r_A}^+$, $\theta_{\kappa_A}^- \leq \theta_{\kappa_A}^+$, and $\theta_{r_A}^+ + \theta_{\kappa_A}^+ \leq 1$. Therefore, mathematically, a CIVIFS can be expressed as

$$A = \left\{ \left(x,\ [r_A^-(x),\ r_A^+(x)]e^{2\pi i[\theta_{r_A}^-,\ \theta_{r_A}^+]}, [\kappa_A^-(x),\ k_A^+(x)]e^{2\pi i[\theta_{\kappa_A}^-,\ \theta_{\kappa_A}^+]} \right); x \in X \right\} \tag{6}$$

Definition 6 ([1,11]). *The complement of CIVIFS A, denoted by A^c, is defined as*

$$A = \left\{ \left(x,\ [\kappa_A^-(x),\ k_A^+(x)]e^{2\pi i[\theta_{\kappa_A}^-,\ \theta_{\kappa_A}^+]},\ [r_A^-(x),\ r_A^+(x)]e^{2\pi i[\theta_{r_A}^-,\ \theta_{r_A}^+]} \right); x \in X \right\} \tag{7}$$

3. Proposed Method

The proposed formula is derived from statistical parameters rather than directly from the distance or similarity measures. Since covariance provides a framework for quantifying the joint variation between two CIVIFSs, normalizing it with the corresponding variances of the CIVIFSs yields a formula for a correlation coefficient that lies in the interval $[-1, 1]$. This formulation preserves the classical statistical interpretations of both strength and direction while extending it to the CIVIFS environment. Moreover, it can also accommodate the attribute-weights, leading to a unified framework for both ordinary and weighted formulae for correlation coefficients with clear mathematical interpretation and practical applicability. The proposed framework is presented as follows.

Definition 7 (Correlation Coefficient of CIVIFSs). *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse and*

$$A = \left\{ \left(x_i,\ [r_A^-(x_i),\ r_A^+(x_i)]e^{2\pi i[\theta_{r_A}^-(x_i),\ \theta_{r_A}^+(x_i)]}, [\kappa_A^-(x_i),\ \kappa_A^+(x_i)]e^{2\pi i[\theta_{\kappa_A}^-(x_i),\ \theta_{\kappa_A}^+(x_i)]} \right); x_i \in X \right\}$$

$$B = \left\{ \left(x_i,\ [r_B^-(x_i),\ r_B^+(x_i)]e^{2\pi i[\theta_{r_B}^-(x_i),\ \theta_{r_B}^+(x_i)]}, [\kappa_B^-(x_i),\ \kappa_B^+(x_i)]e^{2\pi i[\theta_{\kappa_B}^-(x_i),\ \theta_{\kappa_B}^+(x_i)]} \right); x_i \in X \right\}$$

be two CIVIFSs over X, such that $\lambda_A^-(x_i) = r_A^-(x_i) - \kappa_A^-(x_i)$, $\lambda_A^+(x_i) = r_A^+(x_i) - \kappa_A^+(x_i)$, $\alpha_A^-(x_i) = \theta_{r_A}^-(x_i) - \theta_{\kappa_A}^-(x_i)$, $\alpha_A^+(x_i) = \theta_{r_A}^+(x_i) - \theta_{\kappa_A}^+(x_i)$, $\lambda_B^-(x_i) = r_B^-(x_i) - \kappa_B^-(x_i)$,

$\lambda_B^+(x_i) = r_B^+(x_i) - \kappa_B^+(x_i)$, $\alpha_B^-(x_i) = \theta_{r_B}^-(x_i) - \theta_{\kappa_B}^-(x_i)$ and $\alpha_B^+(x_i) = \theta_{r_B}^+(x_i) - \theta_{\kappa_B}^+(x_i)$. Then, the correlation coefficient $C(A,B)$ between A and B can be expressed as follows:

$$C(A,B) = \frac{cov(A,B)}{\sqrt{V(A)\cdot V(B)}} \tag{8}$$

where

$$cov(A,\ B) = \frac{\left[\sum_{i=1}^{n}\left\{\frac{\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}}{2}+\frac{\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}}{2}+\frac{\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}}{2}+\frac{\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}}{2}\right\}\cdot\left\{\frac{\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}}{2}+\frac{\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}}{2}+\frac{\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}}{2}+\frac{\alpha_B^+(x_i)-\overline{\alpha_B^+(x)}}{2}\right\}\right]}{n-1}$$

is the covariance of A and B.

$$V(A) = \frac{\sum_{i=1}^{n}\left\{\frac{\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}}{2}+\frac{\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}}{2}+\frac{\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}}{2}+\frac{\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}}{2}\right\}^2}{n-1}$$ and $$V(B) = \frac{\sum_{i=1}^{n}\left\{\frac{\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}}{2}+\frac{\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}}{2}+\frac{\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}}{2}+\frac{\alpha_B^+(x_i)-\overline{\alpha_B^+(x)}}{2}\right\}^2}{n-1}$$ are the variances of A and B, respectively. And $\overline{\lambda_A^-(x)} = \frac{\sum_{i=1}^{n}\lambda_A^-(x_i)}{n}$, $\overline{\lambda_A^+(x)} = \frac{\sum_{i=1}^{n}\lambda_A^+(x_i)}{n}$, $\overline{\alpha_A^-(x)} = \frac{\sum_{i=1}^{n}\alpha_A^-(x_i)}{n}$, $\overline{\alpha_A^+(x)} = \frac{\sum_{i=1}^{n}\alpha_A^+(x_i)}{n}$, $\overline{\lambda_B^-(x)} = \frac{\sum_{i=1}^{n}\lambda_B^-(x_i)}{n}$, $\overline{\lambda_B^+(x)} = \frac{\sum_{i=1}^{n}\lambda_B^+(x_i)}{n}$, $\overline{\alpha_B^-(x)} = \frac{\sum_{i=1}^{n}\alpha_B^-(x_i)}{n}$, $\overline{\alpha_B^+(x)} = \frac{\sum_{i=}^{n},\alpha_B^+(x_i)}{n}$.

Therefore, $C(A,B)$ can be expressed as

$$C(A,B) = \frac{\frac{\left[\sum_{i=1}^{n}\begin{Bmatrix}\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}+\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}+\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}+\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}\end{Bmatrix}\begin{Bmatrix}\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}+\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}+\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}+\ \alpha_B^+(x_i)-\overline{\alpha_B^+(x)}\end{Bmatrix}\right]}{n-1}}{\sqrt{\frac{\sum_{i=1}^{n}\left\{\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}+\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}+\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}+\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}\right\}^2}{n-1}\frac{\sum_{i=1}^{n}\left\{\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}+\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}+\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}+\ \alpha_B^+(x_i)-\overline{\alpha_B^+(x)}\right\}^2}{n-1}}}$$
$$= \frac{\sum_{i=1}^{n}\begin{Bmatrix}\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}+\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}+\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}+\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}\end{Bmatrix}\begin{Bmatrix}\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}+\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}+\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}+\ \alpha_B^+(x_i)-\overline{\alpha_B^+(x)}\end{Bmatrix}}{\sqrt{\sum_{i=1}^{n}\left\{\lambda_A^-(x_i)-\overline{\lambda_A^-(x)}+\lambda_A^+(x_i)-\overline{\lambda_A^+(x)}+\alpha_A^-(x_i)-\overline{\alpha_A^-(x)}+\alpha_A^+(x_i)-\overline{\alpha_A^+(x)}\right\}^2\sum_{i=1}^{n}\left\{\lambda_B^-(x_i)-\overline{\lambda_B^-(x)}+\lambda_B^+(x_i)-\overline{\lambda_B^+(x)}+\alpha_B^-(x_i)-\overline{\alpha_B^-(x)}+\ \alpha_B^+(x_i)-\overline{\alpha_B^+(x)}\right\}^2}} \tag{9}$$

Theorem 1. *If A and B are two CIVIFSs defined over universe discourse X, then the correlation coefficient $C(A,B)$ satisfies the following properties:*

(i) *If A = B, then $C(A,B) = \pm 1$;*
(ii) *$C(A,B) = C(B,A)$;*
(iii) *$-1 \le C(A,B) \le 1$.*

Proof. Let

$$A = \left\{\left(x_i,\ [r_A^-(x_i),\ r_A^+(x_i)]e^{2\pi i[\theta_{r_A}^-(x_i),\ \theta_{r_A}^+(x_i)]},\ [\kappa_A^-(x_i),\ \kappa_A^+(x_i)]e^{2\pi i[\theta_{\kappa_A}^-(x_i),\ \theta_{\kappa_A}^+(x_i)]}\right); x_i \in X\right\}$$

$$B = \left\{\left(x_i,\ [r_B^-(x_i),\ r_B^+(x_i)]e^{2\pi i[\theta_{r_B}^-(x_i),\ \theta_{r_B}^+(x_i)]},\ [\kappa_B^-(x_i),\ \kappa_B^+(x_i)]e^{2\pi i[\theta_{\kappa_B}^-(x_i),\ \theta_{\kappa_B}^+(x_i)]}\right); x_i \in X\right\}$$

be two CIVIFSs over $X = \{x_1, x_2, \ldots, x_n\}$, such that $\lambda_A^-(x_i) = r_A^-(x_i) - \kappa_A^-(x_i)$, $\lambda_A^+(x_i) = r_A^+(x_i) - \kappa_A^+(x_i)$, $\alpha_A^-(x_i) = \theta_{r_A}^-(x_i) - \theta_{\kappa_A}^-(x_i)$, $\alpha_A^+(x_i) = \theta_{r_A}^+(x_i) - \theta_{\kappa_A}^+(x_i)$, $\lambda_B^-(x_i) = r_B^-(x_i) -$

$\kappa_B^-(x_i)$, $\lambda_B^+(x_i) = r_B^+(x_i) - \kappa_B^+(x_i)$, $\alpha_B^-(x_i) = \theta_{r_B}^-(x_i) - \theta_{\kappa_B}^-(x_i)$ and $\alpha_B^+(x_i) = \theta_{r_B}^+(x_i) - \theta_{\kappa_B}^+(x_i)$. Then, the correlation coefficient $C(A,B)$ of A and B can be expressed as

$$C(A,B) = \frac{cov(A,B)}{\sqrt{V(A)\cdot V(B)}}$$

$$= \frac{\sum_{i=1}^{n} \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \end{Bmatrix} \begin{Bmatrix} \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}}{\sqrt{\sum_{i=1}^{n} \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2 \sum_{i=1}^{n} \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2}}$$

Properties (i) and (ii) follow directly from Definition 7.

To prove the property (iii), let us proceed as follows:

By using the Cauchy–Schwarz inequality,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \tag{10}$$

Putting $a_i = \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}$ and $b_i = \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}$ in (10), we get

$$\sum_{i=1}^{n} \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \end{Bmatrix} \begin{Bmatrix} \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}$$

$$\leq \left(\sum_{i=1}^{n} \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2 \right)^{1/2} \left(\sum_{i=1}^{n} \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2 \right)^{1/2}$$

$$\Rightarrow \sum_{i=1}^{n} \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \end{Bmatrix} \begin{Bmatrix} \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}$$

$$\leq \sqrt{\sum_{i=1}^{n} \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2 \sum_{i=1}^{n} \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2}$$

$$\Rightarrow \left| \frac{\sum_{i=1}^{n} \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \end{Bmatrix} \begin{Bmatrix} \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}}{\sqrt{\sum_{i=1}^{n} \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2 \sum_{i=1}^{n} \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2}} \right| \leq 1$$

$$\Rightarrow |C(A,B)| \leq 1 \text{ [using (9)]}.$$

$$\Rightarrow -1 \leq C(A,B) \leq 1$$

This completes the proof. □

Definition 8 (Weighted Correlation Coefficient of CIVIFSs)**.** *Let* $X = \{x_1, x_2, \ldots, x_n\}$ *be a finite universe of discourse and* $W = \{w_1, w_2, \ldots, w_n\}$ *be the weight vector associated with* x_i; $\{i = 1, 2, \ldots n\}$, *such that* $\sum_{i=1}^{n} w_i = 1$ *and* $w_i \geq 0 \ \forall\ x_i \in X$. *Also, let*

$$A = \left\{ \left(x_i,\ [r_A^-(x_i),\ r_A^+(x_i)] e^{2\pi i [\theta_{r_A}^-(x_i),\ \theta_{r_A}^+(x_i)]}, [\kappa_A^-(x_i),\ k_A^+(x_i)] e^{2\pi i [\theta_{\kappa_A}^-(x_i),\ \theta_{\kappa_A}^+(x_i)]} \right); x_i \in X \right\}$$

$$B = \left\{ \left(x_i,\ [r_B^-(x_i),\ r_B^+(x_i)] e^{2\pi i [\theta_{r_B}^-(x_i),\ \theta_{r_B}^+(x_i)]}, [\kappa_B^-(x_i),\ k_B^+(x_i)] e^{2\pi i [\theta_{\kappa_B}^-(x_i),\ \theta_{\kappa_B}^+(x_i)]} \right); x_i \in X \right\}$$

be two CIVIFSs over X, such that $\lambda_A^-(x_i) = r_A^-(x_i) - \kappa_A^-(x_i)$, $\lambda_A^+(x_i) = r_A^+(x_i) - \kappa_A^+(x_i)$, $\alpha_A^-(x_i) = \theta_{r_A}^-(x_i) - \theta_{\kappa_A}^-(x_i)$, $\alpha_A^+(x_i) = \theta_{r_A}^+(x_i) - \theta_{\kappa_A}^+(x_i)$, $\lambda_B^-(x_i) = r_B^-(x_i) - \kappa_B^-(x_i)$, $\lambda_B^+(x_i) = r_B^+(x_i) - \kappa_B^+(x_i)$, $\alpha_B^-(x_i) = \theta_{r_B}^-(x_i) - \theta_{\kappa_B}^-(x_i)$ *and* $\alpha_B^+(x_i) = \theta_{r_B}^+(x_i) - \theta_{\kappa_B}^+(x_i)$. *Then, the weighted correlation coefficient* $C(A,B)$ *between A and B can be expressed as follows:*

$$WC(A,B) = \frac{Wcov(A,B)}{\sqrt{WV(A)\cdot WV(B)}} \tag{11}$$

where $Wcov(A,\,A) = \dfrac{\left[\sum_{i=1}^{n} .w_i \begin{Bmatrix} \frac{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)}}{2} + \frac{\lambda_A^+(x_i) - \overline{\lambda_A^+(x)}}{2} + \frac{\alpha_A^-(x_i) - \overline{\alpha_A^-(x)}}{2} + \frac{\alpha_A^+(x_i) - \overline{\alpha_A^+(x)}}{2} \end{Bmatrix} \times \begin{Bmatrix} \frac{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)}}{2} + \frac{\lambda_B^+(x_i) - \overline{\lambda_B^+(x)}}{2} + \frac{\alpha_B^-(x_i) - \overline{\alpha_B^-(x)}}{2} + \frac{\alpha_B^+(x_i) - \overline{\alpha_B^+(x)}}{2} \end{Bmatrix}\right]}{n-1}$ *is the weighted covariance of A and B.*

$WV(A) = \dfrac{\sum_{i=1}^{n} w_i \left\{ \frac{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)}}{2} + \frac{\lambda_A^+(x_i) - \overline{\lambda_A^+(x)}}{2} + \frac{\alpha_A^-(x_i) - \overline{\alpha_A^-(x)}}{2} + \frac{\alpha_A^+(x_i) - \overline{\alpha_A^+(x)}}{2} \right\}^2}{n-1}$ *and* $V(B) = \dfrac{\sum_{i=1}^{n} w_i \left\{ \frac{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)}}{2} + \frac{\lambda_B^+(x_i) - \overline{\lambda_B^+(x)}}{2} + \frac{\alpha_B^-(x_i) - \overline{\alpha_B^-(x)}}{2} + \frac{\alpha_B^+(x_i) - \overline{\alpha_B^+(x)}}{2} \right\}^2}{n-1}$ *are the weighted variances of A and B, respectively. And* $\overline{\lambda_A^-(x)} = \frac{\sum_{i=1}^{n} w_i \lambda_A^-(x_i)}{\sum_{i=1}^{n} w_i} = \sum_{i=1}^{n} w_i \lambda_A^-(x_i)$, $\overline{\lambda_A^+(x)} = \sum_{i=1}^{n} w_i \lambda_A^+(x_i)$, $\overline{\alpha_A^-(x)} = \sum_{i=1}^{n} w_i \alpha_A^-(x_i)$, $\overline{\alpha_A^+(x)} = \sum_{i=1}^{n} w_i \alpha_A^+(x_i)$, $\overline{\lambda_B^-(x)} = \sum_{i=1}^{n} w_i \lambda_B^-(x_i)$, $\overline{\lambda_B^+(x)} = \sum_{i=1}^{n} w_i \lambda_B^+(x_i)$, $\overline{\alpha_B^-(x)} = \sum_{i=1}^{n} w_i \alpha_B^-(x_i)$, $\overline{\alpha_B^+(x)} = \sum_{i=1}^{n} w_i \alpha_B^+(x_i)$.

Therefore, using (11), $WC(A,B)$ can be expressed as

$$\begin{aligned} &WC(A,B) \\ &= \frac{\dfrac{\left[\sum_{i=1}^{n} w_i \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \\ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}\right]}{n-1}}{\sqrt{\dfrac{\sum_{i=1}^{n} w_i \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2}{n-1} \dfrac{\sum_{i=1}^{n} w_i \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2}{n-1}}} \\ &= \frac{\sum_{i=1}^{n} w_i \begin{Bmatrix} \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \\ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \end{Bmatrix}}{\sqrt{\sum_{i=1}^{n} w_i \left\{ \lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)} \right\}^2 \sum_{i=1}^{n} w_i \left\{ \lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)} \right\}^2}} \end{aligned} \tag{12}$$

Theorem 2. *If A and B are two CIVIFSs defined over universe discourse X, then the weighted correlation coefficient* $WC(A,B)$ *satisfies the following properties:*

(i) *If* $A = B$, *then* $WC(A,B) = \pm 1$;
(ii) $WC(A,B) = WC(B,A)$;
(iii) $-1 \le WC(A,B) \le 1$.

Proof. The proof follows immediately from Theorem 1 using weighted covariance and variances. □

Theorem 3. *If the weights associated with each element of the universe discourse are equal, then the weighted correlation coefficient formula reduces to the formula of correlation coefficient.*

Proof. If $w_1 = w_2 = \ldots = w_\text{n} = w$ (say) are the weights associated with the elements of X (universe of discourse), then the weighted correlation coefficient between two CIVIFSs A and B is given by

$$\begin{aligned}
WC(A,B) &= \frac{\sum_{i=1}^{n} w_i \begin{array}{l}\left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\} \\ \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}\end{array}}{\sqrt{\begin{array}{l}\sum_{i=1}^{n} w_i \left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\}^2 \\ \sum_{i=1}^{n} w_i \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}^2\end{array}}} \\
&= \frac{\sum_{i=1}^{n} w \begin{array}{l}\left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\} \\ \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}\end{array}}{\sqrt{\begin{array}{l}\sum_{i=1}^{n} w \left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\}^2 \\ \sum_{i=1}^{n} w \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}^2\end{array}}} \\
&= \frac{w\sum_{i=1}^{n} \begin{array}{l}\left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\} \\ \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}\end{array}}{w\sqrt{\begin{array}{l}\sum_{i=1}^{n} \left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\}^2 \\ \sum_{i=1}^{n} \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}^2\end{array}}} \\
&= \frac{\sum_{i=1}^{n} \begin{array}{l}\left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\} \\ \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}\end{array}}{\sqrt{\begin{array}{l}\sum_{i=1}^{n} \left\{\lambda_A^-(x_i) - \overline{\lambda_A^-(x)} + \lambda_A^+(x_i) - \overline{\lambda_A^+(x)} + \alpha_A^-(x_i) - \overline{\alpha_A^-(x)} + \alpha_A^+(x_i) - \overline{\alpha_A^+(x)}\right\}^2 \\ \sum_{i=1}^{n} \left\{\lambda_B^-(x_i) - \overline{\lambda_B^-(x)} + \lambda_B^+(x_i) - \overline{\lambda_B^+(x)} + \alpha_B^-(x_i) - \overline{\alpha_B^-(x)} + \alpha_B^+(x_i) - \overline{\alpha_B^+(x)}\right\}^2\end{array}}} \\
&= C(A,B) \ [\text{using } (9)].
\end{aligned}$$

□

Theorem 4. *If A and B are two CIVIFSs defined over universe discourse X, and A^* is the CIVIFS obtained by translation, then $C(A^*,B) = C(A,B)$.*

Proof. Let

$$A = \left\{\left(x_i,\ \left[r_A^-(x_i),\ r_A^+(x_i)\right]e^{2\pi i[\theta_{r_A}^-(x_i),\ \theta_{r_A}^+(x_i)]},\ \left[\kappa_A^-(x_i),\ k_A^+(x_i)\right]e^{2\pi i[\theta_{\kappa_A}^-(x_i),\ \theta_{\kappa_A}^+(x_i)]}\right); x_i \in X\right\}$$

$$B = \left\{\left(x_i,\ \left[r_B^-(x_i),\ r_B^+(x_i)\right]e^{2\pi i[\theta_{r_B}^-(x_i),\ \theta_{r_B}^+(x_i)]},\ \left[\kappa_B^-(x_i),\ k_B^+(x_i)\right]e^{2\pi i[\theta_{\kappa_B}^-(x_i),\ \theta_{\kappa_B}^+(x_i)]}\right); x_i \in X\right\}$$

be two CIVIFSs over $X = \{x_1, x_2, \ldots, x_\text{n}\}$. Suppose A^* is a CIVIFS obtained by translation, such that

$$A^* = \left\{\left(x_i,\ \left[r_A^-(x_i)+c_1,\ r_A^+(x_i)+c_1\right]e^{2\pi i[\theta_{r_A}^-(x_i)+c_2,\ \theta_{r_A}^+(x_i)+c_2]},\ \left[\kappa_A^-(x_i)+c_3,\ k_A^+(x_i)+c_3\right]e^{2\pi i[\theta_{\kappa_A}^-(x_i)+c_4,\ \theta_{\kappa_A}^+(x_i)+c_4]}\right); x_i \in X\right\}$$

where $c_1,\ c_2,\ c_3,\ c_4 \in R$ are constants and the translated amplitudes and phases of the membership and nonmembership functions satisfying the conditions imposed by CIVIFS.

□

Now,

$$\lambda^-_{A^*}(x_i) = (r^-_A(x_i) - \kappa^-_A(x_i)) + (c_1 - c_3) = \lambda^-_A(x_i) + (c_1 - c_3)$$

$$\lambda^+_{A^*}(x_i) = (r^+_A(x_i) - \kappa^+_A(x_i)) + (c_1 - c_3) = \lambda^+_A(x_i) + (c_1 - c_3)$$

$$\alpha^-_{A^*}(x_i) = (\theta^-_{r_A}(x_i) - \theta^-_{\kappa_A}(x_i)) + (c_2 - c_4) = \alpha^-_A(x_i) + (c_2 - c_4)$$

$$\alpha^+_{A^*}(x_i) = (\theta^+_{r_A}(x_i) - \theta^+_{\kappa_A}(x_i)) + (c_2 - c_4) = \alpha^+_A(x_i) + (c_2 - c_4)$$

$$\overline{\lambda^-_{A^*}(x)} = \frac{\sum_{i=1}^n \lambda^-_{A^*}(x_i)}{n} = \frac{\sum_{i=1}^n (\lambda^-_A(x_i) + (c_1 - c_3))}{n} = \overline{\lambda^-_A(x)} + (c_1 - c_3)$$

Similarly,

$$\overline{\lambda^+_{A^*}(x)} = \overline{\lambda^+_A(x)} + + (c_1 - c_3)$$

$$\overline{\alpha^-_{A^*}(x)} = \overline{\alpha^-_A(x)} + (c_2 - c_4), \overline{\alpha^+_{A^*}(x)} = \overline{\alpha^-_A(x)} + (c_2 - c_4)$$

Using (9), we obtain

$$C(A^*, B) = \frac{\sum_{i=1}^n \begin{array}{c} \left\{\lambda^-_{A^*}(x_i) - \overline{\lambda^-_{A^*}(x)} + \lambda^+_{A^*}(x_i) - \overline{\lambda^+_{A^*}(x)} + \alpha^-_{A^*}(x_i) - \overline{\alpha^-_{A^*}(x)} + \alpha^+_{A^*}(x_i) - \overline{\alpha^+_{A^*}(x)}\right\} \\ \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\} \end{array}}{\sqrt{\begin{array}{c} \sum_{i=1}^n \left\{\lambda^-_{A^*}(x_i) - \overline{\lambda^-_{A^*}(x)} + \lambda^+_{A^*}(x_i) - \overline{\lambda^+_{A^*}(x)} + \alpha^-_{A^*}(x_i) - \overline{\alpha^-_{A^*}(x)} + \alpha^+_{A^*}(x_i) - \overline{\alpha^+_{A^*}(x)}\right\}^2 \\ \sum_{i=1}^n \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\}^2 \end{array}}}$$

$$= \frac{\begin{array}{c} \sum_{i=1}^n \left\{ \begin{array}{l} \lambda^-_A(x_i) + (c_1 - c_3) - \left(\overline{\lambda^-_A(x)} + (c_1 - c_3)\right) + \lambda^+_A(x_i) + (c_1 - c_3) - \left(\overline{\lambda^+_A(x)} + (c_1 - c_3)\right) \\ +\alpha^-_A(x_i) + (c_2 - c_4) - \left(\overline{\alpha^-_A(x)} + (c_2 - c_4)\right) + \alpha^+_A(x_i) + (c_2 - c_4) - \left(\overline{\alpha^+_A(x)} + (c_2 - c_4)\right) \end{array} \right\} \\ \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\} \end{array}}{\sqrt{\begin{array}{c} \sum_{i=1}^n \left\{ \begin{array}{l} \lambda^-_A(x_i) + (c_1 - c_3) - \left(\overline{\lambda^-_A(x)} + (c_1 - c_3)\right) + \lambda^+_A(x_i) + (c_1 - c_3) - \left(\overline{\lambda^+_A(x)} + (c_1 - c_3)\right) \\ +\alpha^-_A(x_i) + (c_2 - c_4) - \left(\overline{\alpha^-_A(x)} + (c_2 - c_4)\right) + \alpha^+_A(x_i) + (c_2 - c_4) - \left(\overline{\alpha^+_A(x)} + (c_2 - c_4)\right) \end{array} \right\}^2 \\ \sum_{i=1}^n \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\}^2 \end{array}}}$$

$$= \frac{\sum_{i=1}^n \begin{array}{c} \left\{\lambda^-_A(x_i) - \overline{\lambda^-_A(x)} + \lambda^+_A(x_i) - \overline{\lambda^+_A(x)} + \alpha^-_A(x_i) - \overline{\alpha^-_A(x)} + \alpha^+_A(x_i) - \overline{\alpha^+_A(x)}\right\} \\ \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\} \end{array}}{\sqrt{\begin{array}{c} \sum_{i=1}^n \left\{\lambda^-_A(x_i) - \overline{\lambda^-_A(x)} + \lambda^+_A(x_i) - \overline{\lambda^+_A(x)} + \alpha^-_A(x_i) - \overline{\alpha^-_A(x)} + \alpha^+_A(x_i) - \overline{\alpha^+_A(x)}\right\}^2 \\ \sum_{i=1}^n \left\{\lambda^-_B(x_i) - \overline{\lambda^-_B(x)} + \lambda^+_B(x_i) - \overline{\lambda^+_B(x)} + \alpha^-_B(x_i) - \overline{\alpha^-_B(x)} + \ \alpha^+_B(x_i) - \overline{\alpha^+_B(x)}\right\}^2 \end{array}}}$$

$$C(A, B)$$

Theorem 5. (Correlation coefficient between a CIVIFS and its complement)**.** *The correlation coefficient between a* ***CIVIFS*** *A and its complement* A^C *is*$= -1$.

Proof. Let $A = \left\{\left(x_i,\ [r^-_A(x_i),\ r^+_A(x_i)]e^{2\pi i[\theta^-_{r_A}(x_i),\ \theta^+_{r_A}(x_i)]}, [\kappa^-_A(x_i),\ \kappa^+_A(x_i)]e^{2\pi i[\theta^-_{\kappa_A}(x_i),\ \theta^+_{\kappa_A}(x_i)]}\right); x_i \in X\right\}$ be a CIVIFS over X. Then, $\lambda^-_A(x_i) = r^-_A(x_i) - \kappa^-_A(x_i)$, $\lambda^+_A(x_i) = r^+_A(x_i) - \kappa^+_A(x_i)$, $\alpha^-_A(x_i) = \theta^-_{r_A}(x_i) - \theta^-_{\kappa_A}(x_i)$, $\alpha^+_A(x_i) = \theta^+_{r_A}(x_i) - \theta^+_{\kappa_A}(x_i)$. Also, the complement of A is given by

$$A^c = \left\{\left(x_i,\ [\kappa^-_A(x_i),\ \kappa^+_A(x_i)]e^{2\pi i[\theta^-_{\kappa_A}(x_i),\ \theta^+_{\kappa_A}(x_i)]},\ [r^-_A(x_i),\ r^+_A(x_i)]e^{2\pi i[\theta^-_{r_A}(x_i),\ \theta^+_{r_A}(x_i)]}\right); x_i \in X\right\}$$

such that $\lambda^{-}_{A^c}(x_i) = \kappa^{-}_{A}(x_i) - r^{-}_{A}(x_i) = -\lambda^{-}_{A}(x_i)$, $\lambda^{+}_{A^c}(x_i) = \kappa^{+}_{A}(x_i) - r^{+}_{A}(x_i) = -\lambda^{+}_{A}(x_i)$, $\alpha^{-}_{A^c}(x_i) = \theta^{-}_{\kappa_A}(x_i) - \theta^{-}_{r_A}(x_i) = -\alpha^{-}_{A}(x_i)$, $\alpha^{+}_{A^c}(x_i) = \theta^{+}_{\kappa_A}(x_i) - \theta^{+}_{r_A}(x_i) = -\alpha^{+}_{A}(x_i)$. The correlation coefficient $C(A, A^c)$ between A and A^c can be expressed using (9) as follows:

$$
\begin{aligned}
C(A,B) &= \frac{\sum_{i=1}^{n} \begin{matrix}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\} \\ \left\{\lambda^{-}_{A^c}(x_i) - \overline{\lambda^{-}_{A^c}(x)} + \lambda^{+}_{A^c}(x_i) - \overline{\lambda^{+}_{A^c}(x)} + \alpha^{-}_{A^c}(x_i) - \overline{\alpha^{-}_{A^c}(x)} + \alpha^{+}_{A^c}(x_i) - \overline{\alpha^{+}_{A^c}(x)}\right\}\end{matrix}}{\sqrt{\begin{matrix}\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2 \\ \sum_{i=1}^{n}\left\{\lambda^{-}_{A^c}(x_i) - \overline{\lambda^{-}_{A^c}(x)} + \lambda^{+}_{A^c}(x_i) - \overline{\lambda^{+}_{A^c}(x)} + \alpha^{-}_{A^c}(x_i) - \overline{\alpha^{-}_{A^c}(x)} + \alpha^{+}_{A^c}(x_i) - \overline{\alpha^{+}_{A^c}(x)}\right\}^2\end{matrix}}} \\
&= \frac{\sum_{i=1}^{n} \begin{matrix}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\} \\ \left\{-\lambda^{-}_{A}(x_i) + \overline{\lambda^{-}_{A}(x)} - \lambda^{+}_{A}(x_i) + \overline{\lambda^{+}_{A}(x)} - \alpha^{-}_{A}(x_i) + \overline{\alpha^{-}_{A^c}(x)} - \alpha^{+}_{A}(x_i) + \overline{\alpha^{+}_{A}(x)}\right\}\end{matrix}}{\sqrt{\begin{matrix}\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2 \\ \sum_{i=1}^{n}\left\{-\lambda^{-}_{A}(x_i) + \overline{\lambda^{-}_{A}(x)} - \lambda^{+}_{A}(x_i) + \overline{\lambda^{+}_{A}(x)} - \alpha^{-}_{A}(x_i) + \overline{\alpha^{-}_{A^c}(x)} - \alpha^{+}_{A}(x_i) + \overline{\alpha^{+}_{A}(x)}\right\}^2\end{matrix}}} \\
&= \frac{-\sum_{i=1}^{n} \begin{matrix}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\} \\ \left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A^c}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}\end{matrix}}{\sqrt{\begin{matrix}\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2 \\ \sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A^c}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2\end{matrix}}} \\
&= \frac{-\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2}{\sqrt{\left[\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2\right]^2}} \\
&= \frac{-\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2}{\sum_{i=1}^{n}\left\{\lambda^{-}_{A}(x_i) - \overline{\lambda^{-}_{A}(x)} + \lambda^{+}_{A}(x_i) - \overline{\lambda^{+}_{A}(x)} + \alpha^{-}_{A}(x_i) - \overline{\alpha^{-}_{A}(x)} + \alpha^{+}_{A}(x_i) - \overline{\alpha^{+}_{A}(x)}\right\}^2} \\
&= -1
\end{aligned}
$$

Hence, the correlation coefficient between a CIVIFS and its complement is -1. □

Remark 1. *The proposed correlation coefficient attains its maximal value for identical CIVIFSs and minimum value for complementary CIVIFSs.*

4. Application in Sentiment Analysis

The process of identifying patterns in data or features that yield information about a particular system or dataset is known as pattern recognition. Pattern recognition is the art of finding patterns using machine learning techniques. Machine learning and artificial intelligence are closely related to pattern recognition. The applications of machine learning in pattern recognition are limited by uncertainties and imprecisions. The advent of CIVIFSs in machine learning has aided in curbing uncertainties in pattern recognition. In this section, we propose the construction of a family of CIVIFSs from raw data, along with the algorithm and flowchart for a pattern recognition problem using the correlation coefficient of CIVIFSs defined in (9).

4.1. Construction Complex Interval-Valued Intuitionistic Fuzzy Patterns

The proposed approach requires each pattern to be represented as a CIVIFS. The raw data associated with the rare earth elements (REEs) are first converted into CIVIFSs as follows:

Suppose $X = \{x_1,\ x_2, \ldots, x_n\}$ is the universe of discourse, where each x_j denotes the jth attribute. In this study, the universe of discourse is given by

$$X = \{\text{density}(x_1),\ \text{melting point}(x_2), \text{conductivity and conductance}(x_3)\}$$

For each REE, the corresponding physical measurements are collected from standard mineral databases or domain experts. Let $V = (v_{ij})_{m\times n}$ denote the raw data matrix, where v_{ij} is the observed value of the jth attribute for the ith REE. Since the attributes have different values computed at different scales, the raw data are normalized into [0, 1] using the Formula

$$\alpha_{ij} = \frac{v_{ij} - \min\limits_{1\le i\le m} v_{ij}}{\max\limits_{1\le i\le m} v_{ij} - \min\limits_{1\le i\le m} v_{ij}} \tag{13}$$

Obviously, $\alpha_{ij}\ \epsilon\ [0,1]$.

An interval is assigned around the normalized value obtained using (13) for incorporating measurement uncertainty.

Let $\varepsilon_j > 0$ be the uncertainty associated with the jth attribute. Then, the amplitude terms of the interval-valued complex membership function can be obtained as

$$\left.\begin{array}{l} r_{ij}^- = max\{0,\ \alpha_{ij} - \varepsilon_j\} \\ r_{ij}^+ = mix\{1,\ \alpha_{ij} + \varepsilon_j\} \end{array}\right\} \tag{14}$$

where $r_{ij} = \left[r_{ij}^-,\ r_{ij}^+\right]$.

Similarly, the amplitude terms of the interval-valued complex nonmembership function can be obtained as

$$\left.\begin{array}{l} \kappa_{ij}^- = max\left\{0,\ 1 - r_{ij}^-\right\} \\ \kappa_{ij}^+ = mix\left\{1,\ 1 - r_{ij}^+\right\} \end{array}\right\} \tag{15}$$

where $\kappa_{ij} = \left[\kappa_{ij}^-,\ \kappa_{ij}^+\right]$ such that $r_{ij}^+ + \kappa_{ij}^+ \le 1$.

The phase term characterizes the periodicity associated with each attribute. Suppose $\beta_{ij}\ \epsilon\ [0,1]$ is the normalized periodic component corresponding to the jth attribute. Let $\delta_j > 0$ be the uncertainty associated with the phase component. Then, $\theta_{r_{ij}}^- = max\{0, \beta_{ij} -\ \delta_j\}$, $\theta_{r_{ij}}^+ = min\{1, \beta_{ij} + \delta_j\}$, and hence, $\theta_{r_{ij}} = [\theta_{r_{ij}}^-,\ \theta_{r_{ij}}^+]$.

Therefore, the complex interval-valued membership function is represented by

$$\left[r_{ij}^-,\ r_{r_{ij}}^+\right] e^{2\pi i[\theta_{r_{ij}}^-,\ \theta_{r_{ij}}^+]} \tag{16}$$

Similarly, the complex interval-valued nonmembership function is represented by

$$\left[\kappa_{ij}^-,\ \kappa_{ij}^+\right] e^{2\pi i[\theta_{\kappa_{ij}}^-,\ \theta_{\kappa_{ij}}^-]} \tag{17}$$

Thus, the ith REE is represented by the CIVIFS as follows:

$$A_i = \left\{ \left(\begin{array}{c} x_j,\ \left[r_{ij}^-(x_j),\ r_{r_{ij}}^+(x_j)\right] e^{2\pi i[\theta_{r_{ij}}^-\ (x_j),\ \theta_{r_{ij}}^+(x_j)]}, \\ \left[\kappa_{ij}^-(x_j),\ \kappa_{ij}^+(x_j)\right] e^{2\pi i[\theta_{\kappa_{ij}}^-(x_j),\ \theta_{\kappa_{ij}}^-(x_j)]} \end{array} \right); j = 1,\ 2, \ldots, n \right\} \tag{18}$$

such that $0 \le r_{ij}^- \le r_{r_{ij}}^+ \le 1$, $0 \le \kappa_{ij}^- \le \kappa_{ij}^+ \le 1$, $r_{ij}^- + \kappa_{ij}^+ \le 1$, and $0 \le \theta_{r_{ij}}^-,\ \ \theta_{r_{ij}}^+, \theta_{\kappa_{ij}}^-,\ \ \theta_{\kappa_{ij}}^- \le 1$ for all $i = 1,\ 2, \ldots,\ m,\ j = 1,\ 2,\ \ldots,\ n$. The CIVIFS representation of (18) associated with each REE will be the input to the proposed pattern recognition algorithm.

4.2. Algorithm of Pattern Recognition

The algorithm for pattern recognition is described below:

Suppose there are m alternatives, expressed as CIVIFSs P_i (i = 1, 2,. . ., m) on a feature space X. Let U be a sample alternative represented as CIVIFS, which is to be assigned to any P_i. Then, compute

$$C(P_i, U) = \max\{C(P_1, U), C(P_2, U), \ldots C(P_m, U)$$

where $C(P_i, U)$ states the grade of the correlation index between P_i and U.

Similarly, compute

$$WC(P_i, U) = \max\{WC(P_1, U), WC(P_2, U), \ldots WC(P_m, U)$$

where $WC(A_i, B)$ states the grade of the weighed correlation index between P_i and U.

If such a value exists, then we assert that the alternative U is associated with the corresponding P_i, where i = 1, 2, . . ., m.

The steps of the algorithm are given as follows:

Step 1. Using the domain expert's knowledge or standard mineral databases, express each of the known patterns with the help of CIVIFSs {$P_{(i)}$, for i = 1, 2, . . ., m}, defined in terms of features associated with the patterns.

Step 2. Using the domain expert's knowledge or standard mineral databases, express the unknown pattern with a CIVIFS U, defined in terms of the. same features associated with the known patterns.

Step 3. Compute the variances (or weighted variances) of all the patterns (known as well as unknown) using the formula given in Section 3.

Step 4. Compute the covariance (or weighted covariance) of all the known patterns with the unknown patterns.

Step 5. Compute the correlation coefficient $C\left(P_{(i)},\ U\right)$ (or weighted correlation coefficient $WC\left(P_{(i)},\ U\right)$) for i = 1, 2, . . ., n using the formula (9) or (12).

Step 6. if $C\left(P_{(i)}, U\right) \geq C\left(P_{(i)}, U\right)$ (or $WC\left(P_{(i)}, U\right) \geq WC\left(P_{(i)}, U\right)$), $i \neq j$, then the pattern U belongs to the category of pattern $P_{(i)}$.

The flowchart of the proposed algorithm is given in Figure 1 below.

4.3. Complexity Analysis

4.3.1. Time Complexity Analysis

Each CIVIFS contains lower membership, upper membership, lower nonmembership, and upper nonmembership, together with their phase components in each of the n attributes. The computational complexity of construction of each CIVIFS representing a known pattern is $O(k_1 n)$ where $k_1 =$ sum of the constant cost of computing the lower nonmembership and upper nonmembership, together their phase components. Therefore, the computational complexity of all the m known patterns is $O(k_1 mn)$. Computation of the mean and variance require passing over all the n attributes. Therefore, the time complexity for computing the mean and variance is $O(nk_2 + nk_3)$, where k_1 = cost of computing the mean for each attribute in each pattern and k_2 = cost of computing the variance for each attribute in each pattern. Thus, the cost of computing the means and variances of all the m CIVIFSs is $O((nk_2 + nk_3)m)$. Similarly, the cost of the computing correlation between all the m known with the unknown pattern is $O(mnk_4)$, where k_4 = the corresponding constant cost associated with the correlation coefficients. Finding $\max_i C(P_i,\ U)$ requires scanning all patterns; therefore, its complexity is $O(m)$. Therefore, the overall complexity of the proposed algorithm in Section 4.2 is

$$O(k_1 mn + (nk_2 + nk_3)m + mnk_4 + m) = O(mn)$$

The algorithm runs linearly with respect to the number of known patterns, as well as the number of attributes.

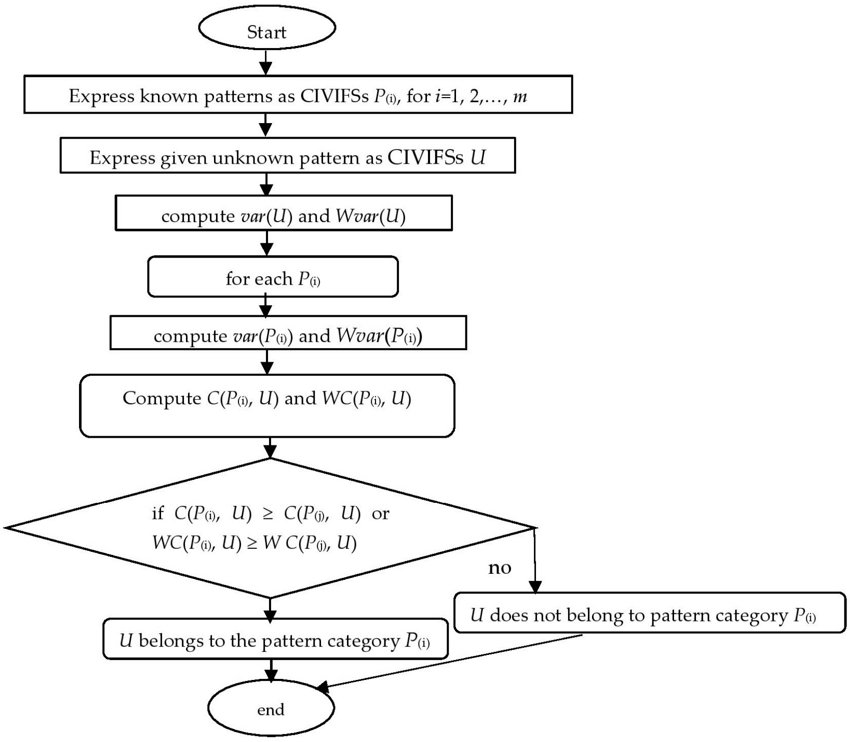

Figure 1. Flowchart of the proposed pattern recognition algorithm.

4.3.2. Space Complexity Analysis

Each CIVIFS stores four intervals together with four phase values for every attributes. Therefore, storage requires per pattern $O(n)$. Hence, the storage required for all the m patterns is $O(mn)$. Therefore, the overall space complexity of the proposed algorithm in Section 4.2 is $O(mn)$.

We present a numeric example to illustrate the potential uses of the proposed method.

Example 1. *In this example, a solution to a pattern recognition problem related to the mining industry is discussed. Rare earth elements (REEs) consist of scandium, yttrium, and 15 lanthanides namely lanthanum (La), cerium (Ce), praseodymium (Pr), neodymium (Nd), promethium (Pm), samarium (Sm), europium (Eu), gadolinium (Gd), terbium (Tb), dysprosium (Dy), holmium (Ho), erbium (Er), thulium (Tm), ytterbium (Yb), and lutetium (Lu), with their principal sources being bastnaesite, xenotime, loparite, and monazite. Several rare minerals contain thorium and uranium invariably and cerium abundantly.*

Experts are aware that there are 17 patterns of REEs, including scandium (P_1), yttrium (P_2), lanthanum (P_3), cerium (P_4), praseodymium (P_5), neodymium (P_6), promethium (P_7), samarium (P_8), europium (P_9), gadolinium (P_{10}), terbium (P_{11}), dysprosium (P_{12}), holmium (P_{13}), erbium (P_{14}), thulium (P_{15}), ytterbium (P_{16}) and lutetium (P_{17}), which are already known. Now, the objective is to identify a new-found pattern (U) and categorize it as any of the REEs. For this purpose, the known patterns are reanalyzed and expressed as

CIVIFSs over the universe of discourse X = {density (x_1), melting point (x_2), conductivity and conductance (x_3)}. The CIVIFSs representing three types of REEs are given in Table 1 below.:

Table 1. CIVIFSs representing the known patterns of REEs.

	x_1	x_2	x_3
P_1	$[0.10,\ 0.4]e^{i2\pi[0.2,\ 0.5]}$ $[0.20,\ 0.5]e^{i2\pi[0.1,\ 0.4]}$	$[0.10,\ 0.3]e^{i2\pi[0.2,\ 0.3]}$ $[0.10,\ 0.5]e^{i2\pi[0.2,\ 0.4]}$	$[0.20,\ 0.3]e^{i2\pi[0.1,\ 0.4]}$ $[0.10,\ 0.6]e^{i2\pi[0.05,\ 0.3]}$
P_2	$[0.05,\ 0.4]e^{i2\pi[0.02,\ 0.5]}$ $[0.02,\ 0.5]e^{i2\pi[0.01,\ 0.4]}$	$[0.01,\ 0.4]e^{i2\pi[0.2,\ 0.35]}$ $[0.01,\ 0.55]e^{i2\pi[0.2,\ 0.45]}$	$[0.22,\ 0.32]e^{i2\pi[0.21,\ 0.42]}$ $[0.11,\ 0.64]e^{i2\pi[0.12,\ 0.33]}$
P_3	$[0.11,\ 0.41]e^{i2\pi[0.12,\ 0.45]}$ $[0.21,\ 0.52]e^{i2\pi[0.12,\ 0.34]}$	$[0.01,\ 0.34]e^{i2\pi[0.21,\ 0.32]}$ $[0.01,\ 0.59]e^{i2\pi[0.02,\ 0.40]}$	$[0.21,\ 0.30]e^{i2\pi[0.23,\ 0.40]}$ $[0.21,\ 0.60]e^{i2\pi[0.11,\ 0.30]}$
P_4	$[0.01,\ 0.44]e^{i2\pi[0.2,\ 0.55]}$ $[0.02,\ 0.55]e^{i2\pi[0.1,\ 0.44]}$	$[0.11,\ 0.36]e^{i2\pi[0.24,\ 0.34]}$ $[0.21,\ 0.53]e^{i2\pi[0.25,\ 0.44]}$	$[0.22,\ 0.32]e^{i2\pi[0.26,\ 0.44]}$ $[0.31,\ 0.62]e^{i2\pi[0.15,\ 0.33]}$
P_5	$[0.21,\ 0.42]e^{i2\pi[0.12,\ 0.51]}$ $[0.12,\ 0.52]e^{i2\pi[0.11,\ 0.48]}$	$[0.11,\ 0.33]e^{i2\pi[0.22,\ 0.34]}$ $[0.11,\ 0.52]e^{i2\pi[0.23,\ 0.43]}$	$[0.12,\ 0.31]e^{i2\pi[0.22,\ 0.45]}$ $[0.11,\ 0.61]e^{i2\pi[0.15,\ 0.35]}$
P_6	$[0.15,\ 0.44]e^{i2\pi[0.22,\ 0.45]}$ $[0.02,\ 0.51]e^{i2\pi[0.11,\ 0.41]}$	$[0.21,\ 0.42]e^{i2\pi[0.21,\ 0.33]}$ $[0.41,\ 0.56]e^{i2\pi[0.12,\ 0.43]}$	$[0.21,\ 0.36]e^{i2\pi[0.22,\ 0.43]}$ $[0.12,\ 0.63]e^{i2\pi[0.13,\ 0.34]}$
P_7	$[0.12,\ 0.42]e^{i2\pi[0.13,\ 0.46]}$ $[0.21,\ 0.52]e^{i2\pi[0.12,\ 0.33]}$	$[0.11,\ 0.34]e^{i2\pi[0.20,\ 0.33]}$ $[0.21,\ 0.58]e^{i2\pi[0.10,\ 0.41]}$	$[0.22,\ 0.31]e^{i2\pi[0.21,\ 0.41]}$ $[0.24,\ 0.61]e^{i2\pi[0.11,\ 0.31]}$
P_8	$[0.11,\ 0.45]e^{i2\pi[0.2,\ 0.55]}$ $[0.02,\ 0.55]e^{i2\pi[0.1,\ 0.44],}$	$[0.11,\ 0.36]e^{i2\pi[0.24,\ 0.34]}$ $[0.21,\ 0.53]e^{i2\pi[0.25,\ 0.44]}$	$[0.22,\ 0.32]e^{i2\pi[0.26,\ 0.44]}$ $[0.31,\ 0.62]e^{i2\pi[0.15,\ 0.33]}$
P_9	$[0.20,\ 0.43]e^{i2\pi[0.11,\ 0.50]}$ $[0.11,\ 0.53]e^{i2\pi[0.12,\ 0.49]}$	$[0.12,\ 0.32]e^{i2\pi[0.20,\ 0.33]}$ $[0.10,\ 0.51]e^{i2\pi[0.21,\ 0.44]}$	$[0.13,\ 0.32]e^{i2\pi[0.20,\ 0.49]}$ $[0.13,\ 0.62]e^{i2\pi[0.15,\ 0.36]}$
P_{10}	$[0.01,\ 0.42]e^{i2\pi[0.21,\ 0.50]}$ $[0.02,\ 0.55]e^{i2\pi[0.13,\ 0.45]}$	$[0.01,\ 0.38]e^{i2\pi[0.21,\ 0.30]}$ $[0.01,\ 0.53]e^{i2\pi[0.02,\ 0.40]}$	$[0.20,\ 0.30]e^{i2\pi[0.22,\ 0.42]}$ $[0.01,\ 0.61]e^{i2\pi[0.11,\ 0.31]}$
P_{11}	$[0.06,\ 0.40]e^{i2\pi[0.02,\ 0.51]}$ $[0.02,\ 0.50]e^{i2\pi[0.01,\ 0.40]}$	$[0.01,\ 0.41]e^{i2\pi[0.21,\ 0.35]}$ $[0.01,\ 0.55]e^{i2\pi[0.2,\ 0.45]}$	$[0.22,\ 0.33]e^{i2\pi[0.21,\ 0.43]}$ $[0.10,\ 0.64]e^{i2\pi[0.12,\ 0.33]}$
P_{12}	$[0.10,\ 0.41]e^{i2\pi[0.12,\ 0.45]}$ $[0.20,\ 0.52]e^{i2\pi[0.12,\ 0.35]}$	$[0.01,\ 0.35]e^{i2\pi[0.20,\ 0.30]}$ $[0.01,\ 0.58]e^{i2\pi[0.02,\ 0.42]}$	$[0.21,\ 0.31]e^{i2\pi[0.21,\ 0.4]}$ $[0.21,\ 0.61]e^{i2\pi[0.12,\ 0.3]}$
P_{13}	$[0.01,\ 0.44]e^{i2\pi[0.2,\ 0.55]}$ $[0.02,\ 0.55]e^{i2\pi[0.1,\ 0.44]}$	$[0.11,\ 0.36]e^{i2\pi[0.24,\ 0.34]}$ $[0.21,\ 0.53]e^{i2\pi[0.25,\ 0.44]}$	$[0.22,\ 0.32]e^{i2\pi[0.26,\ 0.44]}$ $[0.31,\ 0.62]e^{i2\pi[0.15,\ 0.33]}$
P_{14}	$[0.20,\ 0.42]e^{i2\pi[0.12,\ 0.51]}$ $[0.12,\ 0.52]e^{i2\pi[0.12,\ 0.48]}$	$[0.13,\ 0.33]e^{i2\pi[0.22,\ 0.35]}$ $[0.11,\ 0.53]e^{i2\pi[0.20,\ 0.43]}$	$[0.12,\ 0.31]e^{i2\pi[0.22,\ 0.45]}$ $[0.11,\ 0.61]e^{i2\pi[0.15,\ 0.35]}$
P_{15}	$[0.15,\ 0.44]e^{i2\pi[0.22,\ 0.45]}$ $[0.02,\ 0.51]e^{i2\pi[0.11,\ 0.41]}$	$[0.23,\ 0.42]e^{i2\pi[0.21,\ 0.33]}$ $[0.41,\ 0.56]e^{i2\pi[0.13,\ 0.43]}$	$[0.21,\ 0.36]e^{i2\pi[0.22,\ 0.43]}$ $[0.12,\ 0.63]e^{i2\pi[0.13,\ 0.34]}$
P_{16}	$[0.12,\ 0.42]e^{i2\pi[0.10,\ 0.46]}$ $[0.21,\ 0.52]e^{i2\pi[0.12,\ 0.33]}$	$[0.10,\ 0.34]e^{i2\pi[0.20,\ 0.33]}$ $[0.21,\ 0.58]e^{i2\pi[0.11,\ 0.41]}$	$[0.20,\ 0.31]e^{i2\pi[0.20,\ 0.41]}$ $[0.24,\ 0.61]e^{i2\pi[0.11,\ 0.31]}$
P_{17}	$[0.12,\ 0.43]e^{i2\pi[0.21,\ 0.55]}$ $[0.21,\ 0.53]e^{i2\pi[0.11,\ 0.44]}$	$[0.11,\ 0.35]e^{i2\pi[0.23,\ 0.36]}$ $[0.22,\ 0.53]e^{i2\pi[0.22,\ 0.54]}$	$[0.21,\ 0.33]e^{i2\pi[0.25,\ 0.45]}$ $[0.32,\ 0.61]e^{i2\pi[0.13,\ 0.34]}$

Suppose a mining firm discovers a new type of REE while working on a site. Experts analyze it completely, taking into account its density, melting point, conductivity and conductance and giving their opinion about its pattern. Based on these expert opinions, the REE can be represented by a CIVIFS over the set X as follows:

$$U = \left\{ \begin{array}{c} \left(\begin{array}{l} x_1,\ [0.10,\ 0.41]e^{i2\pi[0.12,\ \ 0.45]} \\ \qquad [0.20,\ 0.52]e^{i2\pi[0.12,\ \ 0.35]} \end{array} \right), \left(\begin{array}{l} x_2,\ [0.23,\ 0.56]e^{i2\pi[0.21,\ \ 0.33]} \\ \qquad [0.41,\ 0.42]e^{i2\pi[0.13,\ \ 0.43]} \end{array} \right), \\ \left(\begin{array}{l} x_3,\ [0.11,\ 0.57]e^{i2\pi[0.21,\ \ 0.42]} \\ \qquad [0.22,\ 0.43]e^{i2\pi[0.12,\ \ 0.33]} \end{array} \right) \end{array} \right\}$$

Let us consider the weights of the attributes x_1, x_2, and x_3 as 0.4, 0.3, and 0.3, respectively. The correlation coefficients and weighted correlation coefficients of U with P_i (i = 1, 2, . . ., 17) are given in Table 2 below.

Table 2. Correlation and weighted correlation coefficient between known and unknown REEs.

	$C(P_i,U)$	$WC(P_i,U)$
P_1 (scandium)	0.219866	−0.15141
P_2 (yttrium)	0.183512	0.054588
P_3 (lanthanides)	−0.361587	0.604712
P_4 (lanthanum)	−0.20561	−0.149114
P_5 (cerium)	−0.09950	0.7013955
P_6 (praseodymium)	0.117064	−0.304837
P_7 (neodymium)	0.341608	0.035566
P_8 promethium	0.559843	−0.368478
P_9 (samarium)	−0.022649	0.0247794
P_{10} (europium)	0.989319	−0.215561
P_{11} (gadolinium)	0.179622	0.090799
P_{12} (terbium)	−0.321039	0.1858862
P_{13} (dysprosium)	−0.102167	−0.3371937
P_{14} (erbium)	−0.132021	−0.7242431
P_{15} (thulium)	−0.013604	0.2528868
P_{16} (ytterbium)	0.114424	0.0248879

The following conclusions can be drawn from Table 2. It can be observed that the correlation coefficients between the unknown pattern, representing a new type of REE, and the known REEs, are in the following order.

$$\begin{aligned} C(P_{10},\ U) = 0.989319 > C(P_8,\ U) = 0.559843 > C(P_7,\ U) = 0.341608 > C(P_1,\ U) \\ = 0.219866 > C(P_2,\ U) = 0.183512 > C(P_{11},\ U) = 0.179622 \\ > C(P_6,\ U) = 0.117064 > C(P_{16},\ U) = 0.114424 > C(P_{17},\ U) \\ = -0.002396 > C(P_{15},\ U) = -0.013604 > C(P_9,\ U) = -0.022649 \\ > C(P_5,\ U) = -0.09950 > C(P_{13},\ U) = -0.102167 > C(P_{14},\ U) \\ = -0.132021 > C(P_4,\ U) = -0.20561 > C(P_3,\ U) = -0.20561 \\ > C(P_{12},\ U) = -0.321039 \end{aligned}$$

It can be observed from the results that the correlation coefficient between P_{10} and U is the highest among all considered patterns. Therefore, U is most likely to be of category P_{10}. In other words, the unknown REE U is most likely to be europium.

Furthermore, the weighted correlation coefficients are in the following order.

$$\begin{aligned} WC(P_5,\, U) = 0.7013955 > WC(P_3,\, U) = 0.604712 > WC(P_{15},\, U) = 0.2528868 \\ > WC(P_{12},\, U) = 0.1858862 > WC(P_{11},\, U) = 0.090799 > WC(P_2,\, U) \\ = 0.054588 > WC(P_7,\, U) = 0.035566 > WC(P_{16},\, U) = 0.0248879 \\ > WC(P_9,\, U) = 0.0247794 > WC(P_{17},\, U) = -0.0892388 > WC(P_4,\, U) \\ = -0.149114 > WC(P_1,\, U) = -0.15141 > WC(P_{10},\, U) = -0.215561 \\ > WC(P_6,\, U) = -0.304837 > WC(P_{13},\, U) = -0.3371937 \\ > WC(P_8,\, U) = -0.368478 > C(P_{14},\, U) = -0.7242431 \end{aligned}$$

It was found that the weighted correlation coefficient between P_5 and U is the highest among all considered patterns. Therefore, the unknown REE may belong to cerium.

It can be observed that the correlation coefficient formula and the weighted correlation formula give different results. Since the correlation coefficient is a special case of the weighted correlation formula with equally weighted attributes, it can be concluded that the formula is sensitive to the weights of attributes. In other words, a slight change in the weights leads to a big change in the results; only the corresponding CIVIFSs representing REEs have to be redefined in terms of all the attributes.

In this example, the density, melting point, conductivity and conductance of the REE are considered for the detection of an unknown pattern. However, if additional properties such as color, crystal structure, boiling point, electrical properties, thermal expansion, elastic properties, magnetic properties, chemical properties, and compound etc., of the unknown pattern are also considered, then the proposed formulae will produce better results, and hence, the pattern recognition will be more accurate.

5. Experimental Study and Discussions

To validate the efficacy of the proposed approach, let us use two real-life datasets, namely Iris [46,47] and the Breast Cancer Wisconsin dataset [48]. The datasets are briefly described in the following Table 3.

Table 3. Characteristics of datasets.

Dataset	No. of Samples	No. of Attributes	No. of Classes	Feature Type	Data Characteristics
Iris [46,47]	150	4 (numeric)	3	Continuous	Multivariate
Breast Cancer Wisconsin [48]	569	30 (real-valued)	2	Continuous	Multivariate

The Iris dataset [46,47] has class labels: Iris setosa (50), Iris versicolor (50), and Iris virginica (50). For the weighted correlation coefficient of CIVIFSs, it is important to assign a suitable weight to each attribute based on its relative importance. There are several ways to assign weights to the attributes. Since the aforementioned datasets [46–48] consist of continuous numeric attributes, a reasonable approach is to assign weights according to the variability of each attribute. An attribute with larger variability usually contains greater discriminate information, and hence, greater assigned weights. The variability is determined using either variance or standard deviation. In this study, the standard deviation was used to determine the weights of the attributes. Therefore, the weights were computed using the formula

$$w_j = \frac{s_j}{\sum_{i=1}^{n} s_j}$$

where s_j denotes the standard deviation of the jth attribute.

The proposed pattern recognition algorithm was implemented on a standard machine using MATLAB R2022a. The outcomes were recorded in terms of accuracy, precision, recall, F1-score, confusion matrix and runtimes. The obtained results are represented diagrammatically in Figures 2–5.

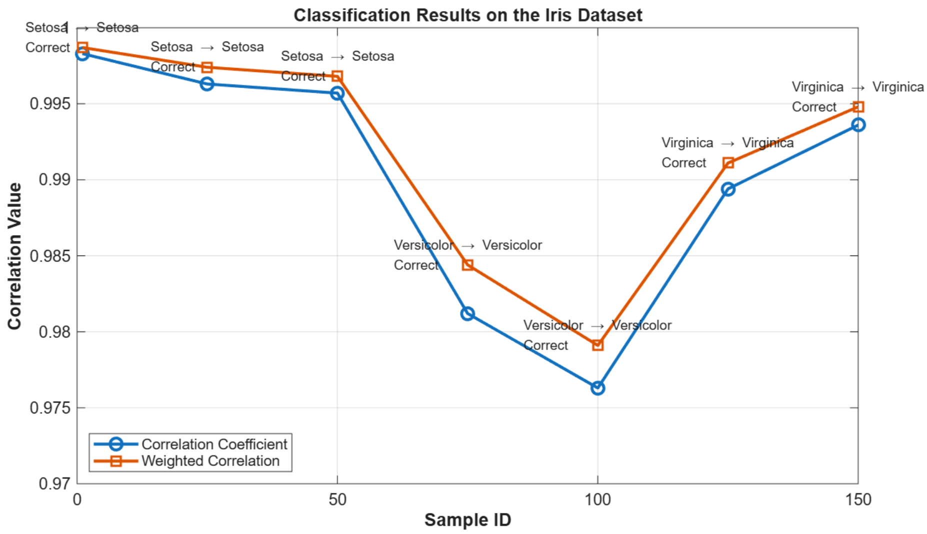

Figure 2. Classification results (actual versus predicted classes) of representative Iris samples using the proposed CIVIFS-based correlation measures.

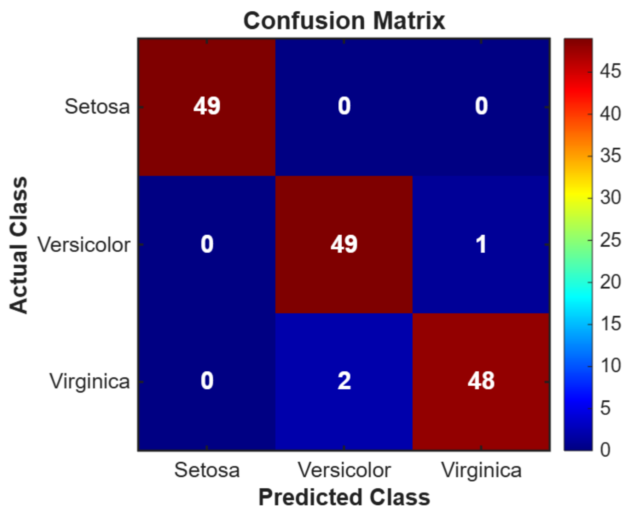

Figure 3. Confusion matrix of the proposed CIVIFS-based classification on the Iris dataset.

Figures 2–4 portray the performances and a comparative assessment of the proposed approach with conventional approaches, like the Euclidean distance-based approach [49], Cosine similarity-based approach [50], Pearson correlation coefficient-based approach [51], the existing CIVIFS-based approach [20], K-Nearest Neighbor (KNN) [52], Support Vector Machine (SVM) [53], Random Forest (RF) [53], the proposed correlation coefficient-based approach, and the proposed weighted correlation coefficient-based approach. The results demonstrate the superiority of the proposed approaches by achieving classification accuracies of 98.01 and 98.66, respectively. They also attain the highest overall performances in all performance measures. The performance improvement can be attributed to the ability of the proposed approaches to exploit the interval-valued representations of both amplitude and phase information of CIVIFSs. Moreover, by incorporating the relative importance of different attributes, the proposed weighted correlation coefficient-based approach increases the discrimination between similar classes.

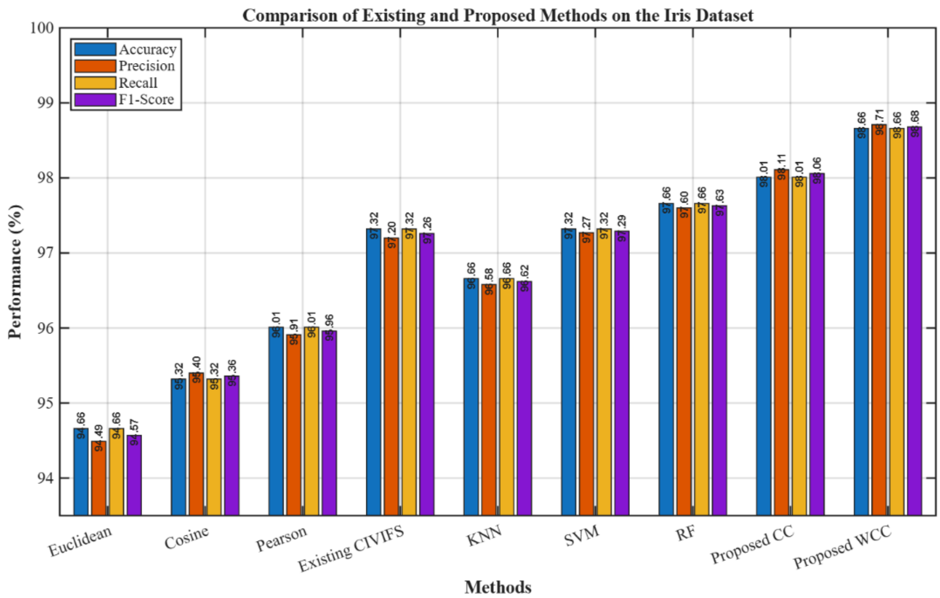

Figure 4. Comparison of the performances of the proposed and existing methods on the Iris dataset.

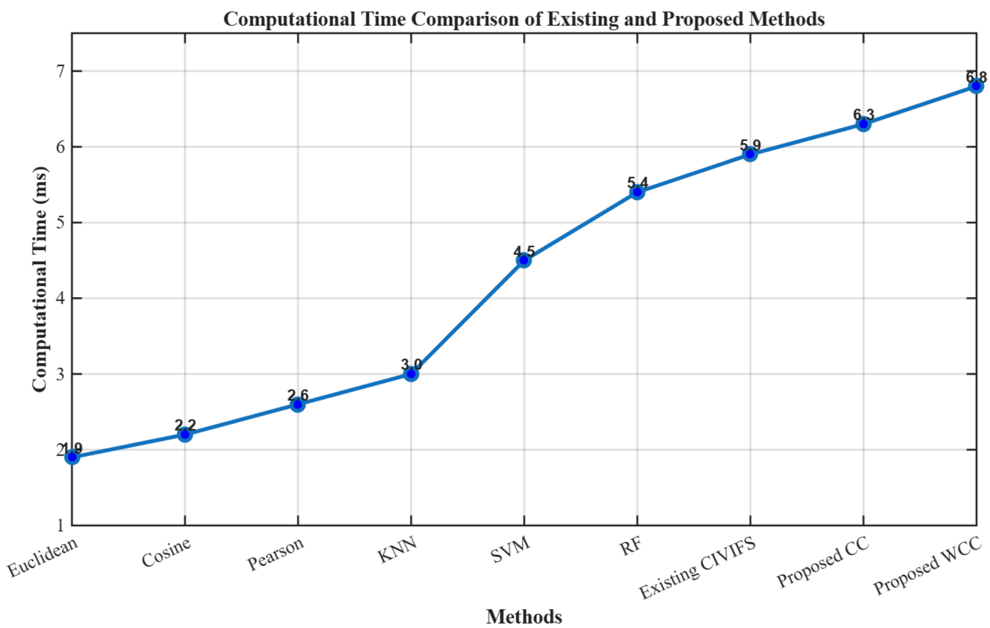

Figure 5. Comparative analysis of time complexity.

Figure 5 demonstrates that the execution times of the proposed approaches are moderately higher than that of the existing approaches. However, the overall execution times remain below 7 ms. The moderate increment is due to fact that the proposed approach requires additional processing of interval-valued amplitude and interval-valued phases, whereas the proposed weighted approach requires additional time of processing of the weights associated with the attributes. Furthermore, the computed runtimes of the proposed approaches are consistent with their theoretical time complexities computed in Section 4.3. This demonstrates their practical usefulness in the pattern recognition domains.

Similar experimental studies have been conducted on the Breast Cancer Wisconsin dataset [47], and the results are presented in Figures 6–9.

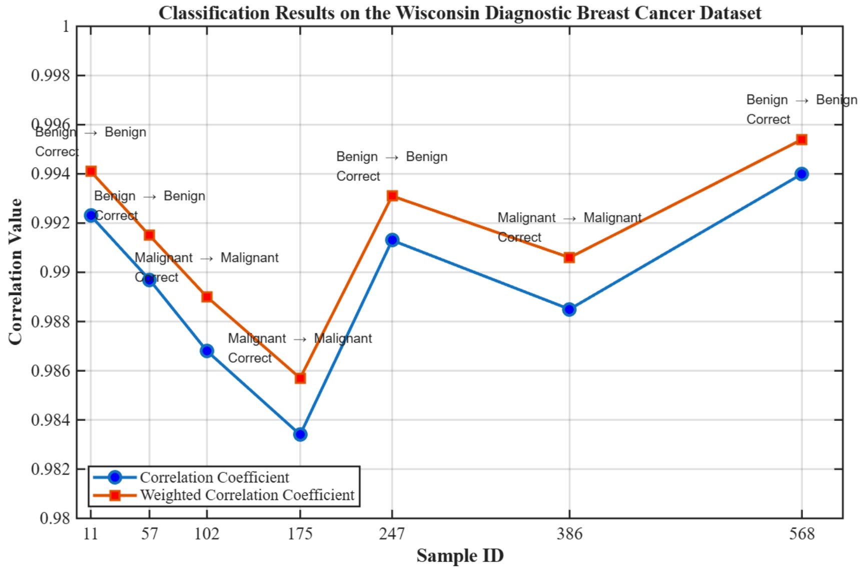

Figure 6. Classification (actual → predicted class) results of representative Breast Cancer Wisconsin dataset samples using the proposed CIVIFS-based correlation measures.

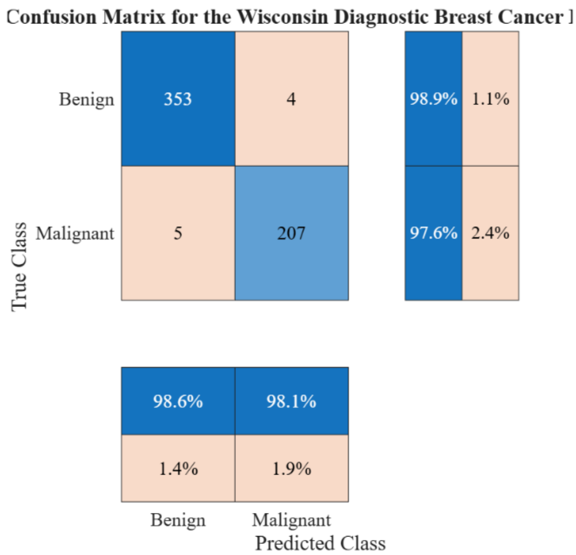

Figure 7. Confusion matrix of the proposed CIVIFS-based classification on the Breast Cancer Wisconsin dataset.

The efficacy of the proposed approaches was further validated using the benchmark dataset [48] widely used in medical diagnosis. Figures 7–9 demonstrate that they attain excellent classification performance in correctly detecting the majority of benign and malignant cancer cases. Figure 8 indicates that only a small portion of the samples were misclassified, resulting in an overall classification accuracy of 98.76%. The weighted correlation coefficient approach gives the best accuracy performance of 99.11%, indicating that the attribute's weight improves the discrimination between benign and malignant cases. Compared with several conventional and state-of-art approaches [20,49–53], the proposed approaches were found to be more consistent in almost all performance metrics while maintaining a moderate increment in the computational complexity, as evident in Figures 8 and 9. In fact, the computed runtimes are consistent with the theoretical time complexities of the proposed approaches. Thus, the proposed CIVIFS-based approaches successfully address both interval-valued uncertainty and phase information, making them most appropriate for high-dimensional problems in the medical domains.

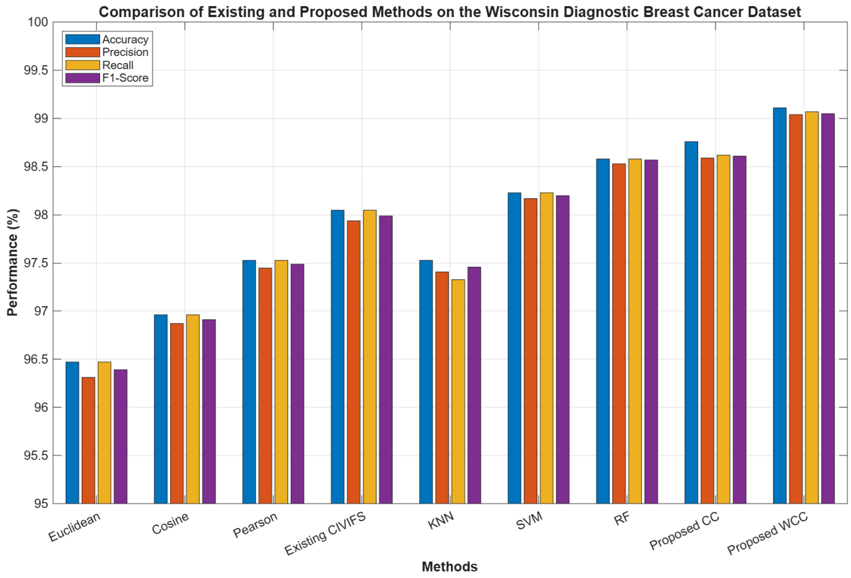

Figure 8. Comparison of the performance of the proposed and existing methods on the Breast Cancer Wisconsin dataset.

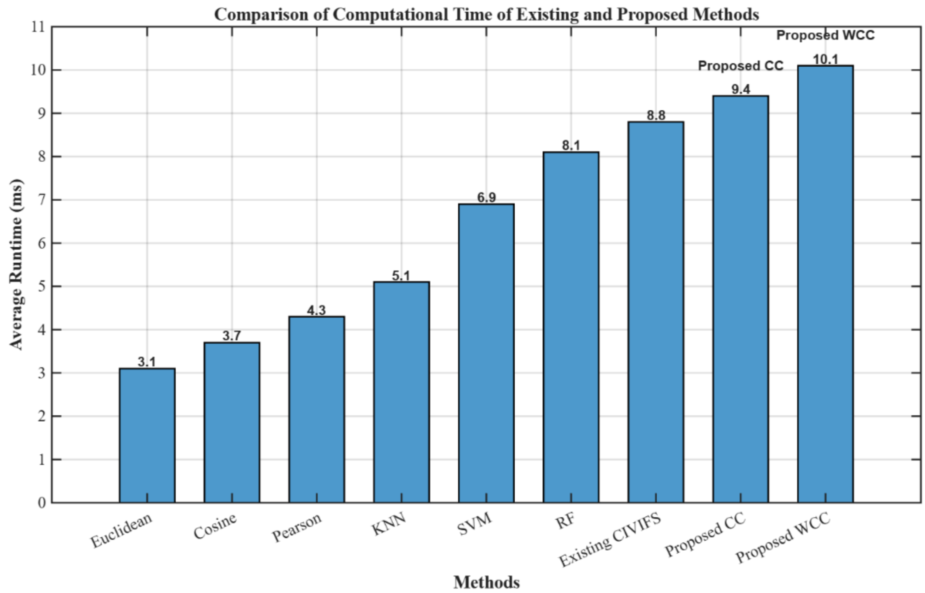

Figure 9. Comparative analysis of time complexity.

Therefore, the proposed approaches were found to be more accurate and reliable than some of the well-known existing approaches while preserving computational efficacy. This demonstrates their usefulness in pattern recognition applications involving complex-valued uncertain information.

The first dataset validates the proposed correlation coefficient on a multi-class pattern recognition problem, while the second dataset demonstrates its real-life applicability in high-dimensional problems. The use of both datasets demonstrates strong evidence of the proposed approach's robustness, scalability, and practical application across different domains.

6. Conclusions, Limitations and Directions for Future Research

6.1. Conclusions

In this article, two new approaches for calculating the correlation coefficient and weighted correlation coefficient for CIVIFSs were proposed. The proposed correlation measure was defined in terms of mean and variance of the CIVIFSs over a universe discourse consisting of attributes. However, its weighted counterpart was defined in terms of weighted versions of the aforementioned statistical parameters, incorporating the relative importance of attributes by a normalized weight vector. Theoretically, some fundamental properties of the proposed correlation measures were proven, demonstrating its usefulness in real-life problems. It is worth mentioning that the values of the proposed correlation and weighted correlation measures lie in the interval $[-1, 1]$. It is also worth mentioning that the weighted correlation coefficient formula reduces to the correlation coefficient formula when the weights of the elements are assumed to be the same. Thus, the correlation coefficient is a special case of the weighted correlation coefficient. Furthermore, a real-life example of pattern recognition of an unknown REE is discussed in detail to showcase the efficacy of the proposed methods.

To demonstrate their practical utilities, a pattern recognition framework was developed. The framework includes a procedure of constructing CIVIFS from raw data, a classification algorithm for pattern recognition, and analysis of its time and space complexities. Based on the analysis, it was found that the algorithm is efficient and scalable for many application domains. The efficacies of the proposed approaches have been validated by an experimental study conducted on the Iris [46,47] and Wisconsin Diagnostic Breast Cancer [48] datasets. The results of the experimental study indicate that the proposed approaches attain better classification accuracies than the conventional methods, such as Euclidean distance [49], cosine similarity [50], Pearson correlation [51], KNN [52], SVM [53], RF [53], and the existing CIVIFS-based correlation measure [20]. In fact, the weighted correlation coefficient-based approach achieves the best performance by achieving the highest accuracy, precision, recall, and F1-score while maintaining modest increments in time complexity.

Existing distance-based approaches define similarity measures with the help of distance between fuzzy objects. These approaches are simple, efficient, and easy to implement. However, the current study identifies several limitations of the aforementioned approaches. The approaches produce value in the interval [0, 1] with no statistical interpretation. Therefore, they cannot discriminate the positive and negative relationship, and they ignore the covariance structures of the fuzzy sets. These methods are less suitable for applications where statistical interpretation is essential.

Existing similarity measures for CIVIFSs compute similarity between CIVIFSs directly. They possess good discrimination capacity and are suitable for decision-making applications. Since they are not derived statistically, they cannot explain covariance, and they also fail to determine the direction of associations, as their values lies in the interval [0, 1]. Additionally, the existing CIVIFS correlation-based methods are similarity- or distance-based rather than covariance-based. Their weighted extensions generally rely on the domain expert's subjective judgement to determine the attribute's weight, making them unsuitable for classification problems.

The proposed approaches address all the shortcomings of the aforementioned approaches by introducing statistical parameters, including the mean, weighted mean, variance, weighted variance, covariance, and weighted covariance within the CIVIFS framework. The proposed formulae are analogous to the classical Pearson correlation and appropriately extended to the CIVIFS framework. A notable contribution of this work is that the proposed correlation coefficients take the values in the interval $[-1, 1]$, enabling them to capture both the strength and the direction of the relationship between objects represented by CIVIFSs. Moreover, the weight of each attribute is computed automatically from the standard deviation of the attribute, demonstrating that the weighted correlation formula is data-driven rather than expert-driven. Consequently, the proposed framework is particularly suitable for pattern recognition and classification problems. The paper also rigorously establishes several fundamental mathematical properties of the proposed correlation coefficient for CIVIFSs. The effectiveness of the proposed framework is validated on two well-known benchmark datasets, and its performance is compared with several existing methods. Experimental results demonstrate that the proposed weighted correlation-based method achieves the highest classification accuracy among the compared methods while incurring only a modest increase in execution time.

6.2. Limitations and Directions for Future Research

Although the proposed approaches have been validated on two benchmark datasets, their applications are not restricted. They can be applied to a broader range of uncertainty-driven problems. Further validation with larger benchmark datasets will be pursued in the future.

It has been also found that the formulae are quite sensitive to the weights of the attributes. Even slight changes in the weights might change the results significantly. Since no formal sensitivity analysis was conducted, studying the robustness of the proposed approach under different weight assignment strategies can be an important direction of future research.

Again, the proposed models assume that the uncertainty information can be efficiently represented by CIVIFSs, and their performance can be affected if a more appropriate representation can be provided.

Finally, although the proposed algorithm runs in polynomial time, a simpler distance- or similarity-based approach can be preferable for large applications.

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