

Article

Symmetries and Bäcklund Transformations for the Modified Veronese Web Equation

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Abstract

This paper investigates recursion operators and nonlocal symmetry structures for the modified Veronese web equation. The novelty of the work lies in the explicit construction of a direct recursion operator and its inverse in the tangent-covering framework. Starting from a compatible linear covering with a spectral parameter, we derive both operators and interpret them as auto-Bäcklund transformations for the corresponding linearized equation. We also determine the contact symmetry algebra and compute the action of the two recursion operators on its infinitesimal generators. In particular, the inverse recursion operator produces shadows of nonlocal symmetries associated with conservation-law coverings. These results provide a concrete recursive mechanism for the symmetry space of the modified Veronese web equation and clarify its covering-based nonlocal geometric structure.

Keywords: symmetries of differential equations; recursion operators; differential coverings

1. Introduction

In the present paper, recursion operators are understood as mappings between symmetry characteristics. Instead of representing them as formal integro-differential operators, we use the covering interpretation, where a recursion operator is written as a compatible first-order system for two characteristics. This formulation is particularly suitable for multidimensional dispersionless equations, since spectral parameters and nonlocal variables naturally appear through differential coverings [1–6]. Recent developments in symbolic computation, cotangent coverings, full-fledged nonlocal symmetries, and multidimensional recursion operators further support this viewpoint [7–12].

The modified Veronese web equation considered below belongs to the class of Veronese-web-type dispersionless equations. Such equations are connected with three-dimensional web structures, hydrodynamic-type systems, dispersionless Lax representations, and Einstein–Weyl geometry [13–16]. For the purpose of this paper, the most relevant aspect of this geometric background is the existence of spectral-parameter coverings. These coverings provide the analytic framework in which the recursion operators and their actions on symmetry characteristics will be constructed.

The modified Veronese web equation has appeared in several related contexts. Nonlocal symmetries and differential coverings for Lax-integrable linearly degenerate equations were studied in [17–19]. For the Veronese web equation, recursion operators and their action on nonlocal shadows were investigated in [20]. Higher symmetries of the cotangent covering for the modified Veronese web equation were later considered in [21]. Other



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constructions of recursion operators and coverings for multidimensional Lax-integrable equations can be found in [12,22–29]. In contrast to these works, the present paper derives an explicit direct–inverse pair of recursion operators for the modified Veronese web equation in tangent-covering form and then computes their action on the contact symmetry algebra.

The purpose of this paper is to fill this gap. We consider the modified Veronese web equation

$$u_{ty} - u_t u_{xy} + u_y u_{tx} = 0, \tag{1}$$

where x and y are spatial variables and t is the temporal variable. Starting from a suitable linear covering, we derive a direct recursion operator and its inverse. Both operators are interpreted as auto-Bäcklund transformations of the tangent covering. We also compute the contact symmetry algebra of Equation (1), describe the action of the recursion operators on its infinitesimal generators, and obtain shadows of nonlocal symmetries associated with conservation-law coverings. These results complement the existing studies on Veronese-web-type equations by describing how the covering structure acts on local symmetry characteristics and produces nonlocal symmetry data. The scope of the paper is restricted to the covering, tangent-covering, and symmetry-shadow aspects of the modified Veronese web equation.

The paper is organized as follows. Section 2 recalls the basic notions of jet spaces, symmetries, differential coverings, and tangent coverings that are needed for the construction. Section 3 applies the linear-covering and tangent-covering approach to Equation (1) and constructs the direct and inverse recursion operators. Section 4 computes the contact symmetry algebra, studies the action of the recursion operators on its generators, and presents illustrative examples and nonlocal symmetry shadows. Section 5 summarizes the results and discusses possible further developments.

2. Preliminaries

We fix the notation used in the sequel. Standard notions from the geometry of differential equations, including jet spaces, total derivatives, symmetries, differential coverings, and tangent coverings, are used in the sense of [3,4,7,30–33]. For completeness, we recall the notation and formulas that are needed for the subsequent construction.

Let

$$\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad \pi : (x^1, \dots, x^n, u^1, \dots, u^m) \mapsto (x^1, \dots, x^n),$$

be the trivial bundle over \mathbb{R}^n . Its infinite jet bundle is denoted by $J^\infty(\pi)$. We use $(x^i, u^\alpha, u_I^\alpha)$ as local coordinates on $J^\infty(\pi)$, where $I = (i_1, \dots, i_n)$ is a multi-index. For a local section $f : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ of π , the corresponding infinite prolongation $j_\infty(f) : \mathbb{R}^n \rightarrow J^\infty(\pi)$ is characterized by

$$u_I^\alpha(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1 + \dots + i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}.$$

As usual, we write $u^\alpha = u_{(0, \dots, 0)}^\alpha$. In the particular case $n = 3$ and $m = 1$, we put $x^1 = t, x^2 = x, x^3 = y$, and use the notation $u_{i,j,k}^1 = u_{t \dots t x \dots x y \dots y}$, where the variables t, x , and y occur i, j , and k times, respectively.

The total derivative operators on $J^\infty(\pi)$ are denoted by

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u_{I+1_k}^\alpha \frac{\partial}{\partial u_I^\alpha}, \quad k \in \{1, \dots, n\}.$$

Here $I + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$ if $I = (i_1, \dots, i_k, \dots, i_n)$. These operators commute on $J^\infty(\pi)$, that is, $[D_{x^i}, D_{x^j}] = 0$.

Given a vector-valued smooth function

$$\varphi : J^\infty(\pi) \rightarrow \mathbb{R}^m,$$

its associated evolutionary vector field is defined by

$$E_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u_I^\alpha}, \tag{2}$$

where $D_I = D_{(i_1, \dots, i_n)} = D_{x^1}^{i_1} \circ \dots \circ D_{x^n}^{i_n}$.

Consider a system of partial differential equations

$$F_r(x^i, u_I^\alpha) = 0, \quad \#I \leq s, \quad r \in \{1, \dots, R\},$$

of order $s \geq 1$, with $R \geq 1$. It determines the infinite prolongation

$$\mathcal{E} = \{(x^i, u_I^\alpha) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_I^\alpha)) = 0, \#K \geq 0\}$$

as a submanifold of $J^\infty(\pi)$. A function $\varphi : J^\infty(\pi) \rightarrow \mathbb{R}^m$ is called the generator of an infinitesimal symmetry of \mathcal{E} if $E_\varphi(F) = 0$ holds on \mathcal{E} .

Equivalently, the symmetry generator $\varphi = (\varphi^1, \dots, \varphi^m)$ satisfies the linearized equation

$$\ell_{\mathcal{E}}(\varphi) = 0. \tag{3}$$

Here $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$, and the linearization of $F = (F_1, \dots, F_R)$ is given componentwise by

$$(\ell_F(\varphi))_r = \sum_{\alpha=1}^m \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} D_I(\varphi^\alpha), \quad r = 1, \dots, R.$$

The set of all solutions of Equation (3) is denoted by $\text{sym}(\mathcal{E})$. Its contact symmetry subspace is

$$\text{sym}_0(\mathcal{E}) = \text{sym}(\mathcal{E}) \cap C^\infty(J^1(\pi), \mathbb{R}^m).$$

We now recall the notion of a differential covering. Let $\mathcal{W} = \mathbb{R}^\infty$ have coordinates $w^s, s \in \mathbb{N} \cup \{0\}$, and write $w = (w^0, w^1, w^2, \dots)$ for the collection of all nonlocal variables. Locally, an infinite-dimensional differential covering over \mathcal{E} is the trivial bundle $\tau : J^\infty(\pi) \times \mathcal{W} \rightarrow J^\infty(\pi)$ together with the extended total derivatives

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^\infty T_k^s(x^i, u_I^\alpha, w) \frac{\partial}{\partial w^s}, \tag{4}$$

which satisfy the commutativity condition

$$[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0, \quad i \neq j,$$

whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. The derivatives of the nonlocal variables are introduced by $w_{x^k}^s = \tilde{D}_{x^k}(w^s)$, and this gives the covering equations

$$w_{x^k}^s = T_k^s(x^i, u_I^\alpha, w). \tag{5}$$

The compatibility of this overdetermined system follows from the above commutativity condition on \mathcal{E} .

Let \tilde{E}_φ denote the operator obtained from Equation (2) after replacing each D_{x^k} by \tilde{D}_{x^k} . A function $\varphi \in C^\infty(\mathcal{E} \times \mathcal{W}, \mathbb{R}^m)$ is called a shadow of a nonlocal symmetry associated with the covering τ , or simply a τ -shadow, if

$$\tilde{E}_\varphi(F) = 0 \tag{6}$$

is a consequence of the equations $D_K(F_r) = 0, r = 1, \dots, R$, and the covering Equation (5). A genuine nonlocal symmetry corresponding to τ is represented by the vector field

$$\tilde{E}_{\varphi,A} = \tilde{E}_\varphi + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w^s}, \tag{7}$$

where $A^s \in C^\infty(\mathcal{E} \times \mathcal{W})$. In addition to (6), the components A^s have to satisfy

$$\tilde{D}_{x^k}(A^s) = \tilde{E}_{\varphi,A}(T_k^s) \tag{8}$$

for the functions T_k^s appearing in (4); see [7,30–32].

Remark 1. A τ -shadow does not necessarily lift to a τ -symmetry, because Equation (8) may obstruct the existence of the functions A^s in Equation (7). Nevertheless, for every τ -shadow φ , one can construct a suitable covering τ_φ in which there exists a nonlocal τ_φ -symmetry whose projection coincides with the given shadow φ ; see [7,30–32].

We also recall the interpretation of recursion operators used in this work. In the classical formulation, a recursion operator \mathcal{R} for \mathcal{E} is an \mathbb{R} -linear mapping that sends each local symmetry, or each shadow of a nonlocal symmetry, to another object of the same type. For nonlinearizable systems, however, recursion operators often cannot be treated merely as ordinary linear operators on the symmetry space. A more appropriate viewpoint is to regard them as auto-Bäcklund transformations for the linearized, or tangent, covering of the equation. This interpretation will be adopted below. It is consistent with the classical treatment of nonlocal symmetries and recursion operators [2,34], the covering approach to dispersionless systems [5,6], and recent work on cotangent coverings and full-fledged nonlocal symmetries [9,10].

The tangent covering of the equation \mathcal{E} is constructed as follows [3,4,7]. Take the trivial bundle $\sigma : J^\infty(\pi) \times \mathcal{Q} \rightarrow J^\infty(\pi)$, where the fiber \mathcal{Q} has coordinates $q_I^\alpha, \#I \geq 0$. On this bundle, introduce the extended total derivatives

$$\hat{D}_{x^k} = D_{x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m q_{I+1^k}^\alpha \frac{\partial}{\partial q_I^\alpha}.$$

For $\hat{D}_I = \hat{D}_{x^1}^{i_1} \circ \dots \circ \hat{D}_{x^{i_n}}^{i_n}$, define the lifted linearization by

$$(\hat{\ell}_F(q))_r = \sum_{\alpha=1}^m \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} \hat{D}_I(q^\alpha), \quad r = 1, \dots, R.$$

Then set

$$\mathcal{J}(\mathcal{E}) = \left\{ (x^i, u_I^\alpha, q_I^\alpha) \in J^\infty(\pi) \times \mathcal{Q} \mid D_K(F_r) = 0, \hat{D}_K((\hat{\ell}_F(q))_r) = 0, \#K \geq 0, r = 1, \dots, R \right\}.$$

The tangent covering is obtained by restricting σ to $\mathcal{J}(\mathcal{E})$. Its extended total derivatives are denoted by

$$\tilde{D}_{x^k} = \hat{D}_{x^k} |_{\mathcal{J}(\mathcal{E})}.$$

A symmetry generator of \mathcal{E} can then be identified with a section of this tangent covering satisfying the corresponding linearized equation.

Example 1. For the modified Veronese web equation, we write Equation (1) in the form

$$u_{ty} - u_t u_{xy} + u_y u_{tx} = 0.$$

The corresponding linearization is

$$\ell_F(\varphi) = D_t D_y(\varphi) - u_{xy} D_t(\varphi) - u_t D_x D_y(\varphi) + u_{tx} D_y(\varphi) + u_y D_t D_x(\varphi),$$

and its lifted version on the tangent covering is

$$\hat{\ell}_F(q) = q_{(1,0,1)} - u_{xy} q_{(1,0,0)} - u_t q_{(0,1,1)} + u_{tx} q_{(0,0,1)} + u_y q_{(1,1,0)}.$$

The fiber coordinates of the tangent covering can be chosen as $q_{(1,j,0)}$ and $q_{(0,j,1)}$.

The extended total derivatives in the tangent covering are given below. Here D_x^j denotes the j -fold composition of the total derivative D_x , with D_x^0 being the identity operator.

$$\begin{cases} \hat{D}_t = D_t + \sum_{j=0}^{\infty} q_{(2,j,0)} \frac{\partial}{\partial q_{(1,j,0)}} + \sum_{j=0}^{\infty} D_x^j (u_{xy} q_{(1,0,0)} + u_t q_{(0,1,1)} - u_{tx} q_{(0,0,1)} - u_y q_{(1,1,0)}) \frac{\partial}{\partial q_{(0,j,1)}}, \\ \hat{D}_x = D_x + \sum_{j=0}^{\infty} q_{(1,j+1,0)} \frac{\partial}{\partial q_{(1,j,0)}} + \sum_{j=0}^{\infty} q_{(0,j+1,1)} \frac{\partial}{\partial q_{(0,j,1)}}, \\ \hat{D}_y = D_y + \sum_{j=0}^{\infty} D_x^j (u_{xy} q_{(1,0,0)} + u_t q_{(0,1,1)} - u_{tx} q_{(0,0,1)} - u_y q_{(1,1,0)}) \frac{\partial}{\partial q_{(1,j,0)}} + \sum_{j=0}^{\infty} q_{(0,j,2)} \frac{\partial}{\partial q_{(0,j,1)}}. \end{cases}$$

Remark 2. In the sequel, for brevity we write $q_{t\dots tx\dots xy\dots y}$, with i copies of t , j copies of x , and k copies of y , instead of $q_{(i,j,k)}$. Under this convention, the tangent-covering coordinates over Equation (1) are denoted by the corresponding derivative symbols of q .

3. Recursion Operators for the Modified Veronese Web Equation

We now construct recursion operators for Equation (1) from its spectral-parameter covering. The construction proceeds in two steps. First, the compatibility of a scalar covering fixes the relation between the auxiliary parameters and produces the parameter λ . Second, the tangent covering is used to convert this compatible covering into an auto-Bäcklund system for the linearized equation. This system maps one symmetry characteristic to another and hence defines the direct recursion operator. The inverse operator is then obtained by solving the same relations in the opposite direction.

We start from the following two-parameter covering ansatz

$$w_t = a u_t w_x, \quad w_y = b u_y w_x,$$

where a and b are auxiliary constants in the covering. They measure the relative weights of the t - and y -directions in the lifted total derivatives. Such parameters are introduced in order to obtain a one-parameter family of compatible dispersionless coverings, which is the usual source of a spectral parameter in covering-based constructions of recursion operators [5,8]. The compatibility condition $D_y(w_t) = D_t(w_y)$ gives

$$a u_{ty} w_x + a u_t w_{xy} = b u_{ty} w_x + b u_y w_{xt}.$$

Using

$$w_{xy} = b(u_{xy} w_x + u_y w_{xx}), \quad w_{xt} = a(u_{tx} w_x + u_t w_{xx}),$$

and cancelling the common term $abu_tu_yw_{xx}$, we obtain

$$(a - b)u_{ty} = -ab(u_tu_{xy} - u_yu_{tx}).$$

Since Equation (1) implies $u_{ty} = u_tu_{xy} - u_yu_{tx}$, we have

$$(a - b + ab)(u_tu_{xy} - u_yu_{tx}) = 0.$$

Thus, in the nontrivial case, the auxiliary parameters must satisfy

$$a - b + ab = 0.$$

This condition reduces the preliminary two-parameter ansatz to a one-parameter compatible covering. Solving it gives

$$a = \frac{b}{1 + b}.$$

By introducing the spectral parameter $b = \lambda^{-1}$, we obtain

$$a = (1 + \lambda)^{-1}.$$

The corresponding one-parameter covering is thus given by

$$\begin{cases} w_t = (1 + \lambda)^{-1}u_tw_x, \\ w_y = \lambda^{-1}u_yw_x, \end{cases} \tag{9}$$

where $\lambda \in \mathbb{R} \setminus \{0, -1\}$ is the spectral parameter. The covering Equation (9) is the starting point for the construction below. We use it to compare the commutator relations associated with the covering and the evolutionary vector field generated by a symmetry characteristic φ . This leads to an overdetermined system for a new characteristic ψ .

We now apply this construction to the covering Equation (9). From Equation (9), we have

$$X_1 = D_t - (1 + \lambda)^{-1}u_tD_x, \quad X_2 = D_y - \lambda^{-1}u_yD_x.$$

In the notation $X_j = \sum_{i=1}^3 Q_j^i D_{x^i}$, with $(x^1, x^2, x^3) = (t, x, y)$, the nonzero coefficients are

$$Q_1^1 = 1, \quad Q_1^2 = -(1 + \lambda)^{-1}u_t, \quad Q_2^2 = -\lambda^{-1}u_y, \quad Q_2^3 = 1.$$

All the remaining coefficients Q_j^i are zero.

Let $Z = \zeta_1 D_t + \zeta_2 D_x + \zeta_3 D_y$. Substituting X_1, X_2 , and Z into

$$[X_j, Z] = \sum_{i=1}^3 E_\varphi(Q_j^i) D_{x^i}, \quad j = 1, 2,$$

and comparing the coefficients of D_t, D_x , and D_y , we obtain

$$X_1(\zeta_1) = X_1(\zeta_3) = 0, \quad X_2(\zeta_1) = X_2(\zeta_3) = 0,$$

and

$$\begin{aligned} X_1(\zeta_2) + (1 + \lambda)^{-1}Z(u_t) &= -(1 + \lambda)^{-1}D_t(\varphi), \\ X_2(\zeta_2) + \lambda^{-1}Z(u_y) &= -\lambda^{-1}D_y(\varphi). \end{aligned}$$

Therefore, for $\zeta = \zeta_1$ or $\zeta = \zeta_3$, we get

$$\begin{cases} D_t(\zeta) = (1 + \lambda)^{-1}u_t D_x(\zeta), \\ D_y(\zeta) = \lambda^{-1}u_y D_x(\zeta). \end{cases} \tag{10}$$

For ζ_2 , using

$$Z(u_t) = \zeta_1 u_{tt} + \zeta_2 u_{tx} + \zeta_3 u_{ty}, \quad Z(u_y) = \zeta_1 u_{ty} + \zeta_2 u_{xy} + \zeta_3 u_{yy},$$

the two remaining equations give

$$\begin{cases} D_t(\zeta_2) = (1 + \lambda)^{-1}u_t D_x(\zeta_2) - (1 + \lambda)^{-1}u_{tt}\zeta_1 - (1 + \lambda)^{-1}u_{tx}\zeta_2 \\ \quad - (1 + \lambda)^{-1}u_{ty}\zeta_3 - (1 + \lambda)^{-1}D_t(\varphi), \\ D_y(\zeta_2) = \lambda^{-1}u_y D_x(\zeta_2) - \lambda^{-1}u_{ty}\zeta_1 - \lambda^{-1}u_{xy}\zeta_2 \\ \quad - \lambda^{-1}u_{yy}\zeta_3 - \lambda^{-1}D_y(\varphi). \end{cases} \tag{11}$$

Thus systems (Equations (10) and (11)) are obtained from the coefficient comparison in the commutator relation. The compatibility of (11) follows from the symmetry condition for φ on Equation (1).

Theorem 1. Let $a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3$ be associated with the covering Equation (9). This expression gives a shadow of a nonlocal symmetry of Equation (1) whenever it can be represented as an \mathbb{R} -linear combination of the following two functions:

$$\psi = (1 + \lambda)^{-1}u_t\zeta_1 + \zeta_2 + \lambda^{-1}u_y\zeta_3,$$

and

$$\eta = \frac{1}{q_x}\zeta, \tag{12}$$

where q is a solution of the covering Equation (9), while ζ satisfies Equation (10).

Remark 3. The factor multiplying ζ in the right-hand side of Equation (12) obeys the identity

$$\tilde{D}_t\tilde{D}_y(v) = u_t\tilde{D}_x\tilde{D}_y(v) + u_{xy}\tilde{D}_t(v) - u_{tx}\tilde{D}_y(v) - u_y\tilde{D}_t\tilde{D}_x(v). \tag{13}$$

It follows from this relation that $\frac{1}{q_x}$ determines a shadow of a nonlocal symmetry of Equation (1) with respect to the covering Equation (9).

Since ζ_1 and ζ_3 solve Equation (10), and ζ_2 is governed by Equation (11), the function ψ introduced above satisfies the following overdetermined system.

Indeed, from

$$\psi = (1 + \lambda)^{-1}u_t\zeta_1 + \zeta_2 + \lambda^{-1}u_y\zeta_3,$$

we first compute its total derivative with respect to t :

$$\begin{aligned} D_t(\psi) &= (1 + \lambda)^{-1}u_{tt}\zeta_1 + (1 + \lambda)^{-1}u_t D_t(\zeta_1) + D_t(\zeta_2) \\ &\quad + \lambda^{-1}u_{ty}\zeta_3 + \lambda^{-1}u_y D_t(\zeta_3). \end{aligned}$$

Using Equations (10) and (11), this becomes

$$D_t(\psi) = (1 + \lambda)^{-2}u_t^2D_x(\zeta_1) + (1 + \lambda)^{-1}u_tD_x(\zeta_2) + \lambda^{-1}(1 + \lambda)^{-1}u_tu_yD_x(\zeta_3) - (1 + \lambda)^{-1}u_{tx}\zeta_2 + (\lambda^{-1} - (1 + \lambda)^{-1})u_{ty}\zeta_3 - (1 + \lambda)^{-1}D_t(\varphi).$$

Since

$$\lambda^{-1} - (1 + \lambda)^{-1} = \lambda^{-1}(1 + \lambda)^{-1},$$

and, on Equation (1),

$$u_{ty} = u_tu_{xy} - u_yu_{tx},$$

the preceding expression can be rewritten as

$$D_t(\psi) = (1 + \lambda)^{-1}u_tD_x(\psi) - (1 + \lambda)^{-1}u_{tx}\psi - (1 + \lambda)^{-1}D_t(\varphi).$$

Similarly, differentiating ψ with respect to y gives

$$D_y(\psi) = \lambda^{-1}u_yD_x(\psi) - \lambda^{-1}u_{xy}\psi - \lambda^{-1}D_y(\varphi).$$

Therefore, the required system for ψ is obtained as follows:

$$\begin{cases} D_t(\psi) = (1 + \lambda)^{-1}u_tD_x(\psi) - (1 + \lambda)^{-1}u_{tx}\psi - (1 + \lambda)^{-1}D_t(\varphi), \\ D_y(\psi) = \lambda^{-1}u_yD_x(\psi) - \lambda^{-1}u_{xy}\psi - \lambda^{-1}D_y(\varphi). \end{cases} \tag{14}$$

System Equation (14) is compatible whenever φ is a symmetry characteristic of Equation (1). Moreover, any solution ψ of Equation (14) is again a symmetry characteristic. Hence, Equation (14) defines the direct recursion operator $\psi = \mathcal{R}(\varphi)$ for Equation (1).

Solving the two equations in Equation (14) for $D_t(\varphi)$ and $D_y(\varphi)$, respectively, gives the inverse transformation. More precisely, multiplying the first equation in Equation (14) by $1 + \lambda$ and rearranging the terms yields

$$D_t(\varphi) = -(1 + \lambda)D_t(\psi) + u_tD_x(\psi) - u_{tx}\psi.$$

Similarly, multiplying the second equation in Equation (14) by λ gives

$$D_y(\varphi) = u_yD_x(\psi) - \lambda D_y(\psi) - u_{xy}\psi.$$

Thus we obtain the following system:

$$\begin{cases} D_t(\varphi) = -(1 + \lambda)D_t(\psi) + u_tD_x(\psi) - u_{tx}\psi, \\ D_y(\varphi) = u_yD_x(\psi) - \lambda D_y(\psi) - u_{xy}\psi. \end{cases} \tag{15}$$

System Equation (15) is compatible whenever ψ is a symmetry characteristic of Equation (1). Thus, Equation (15) defines the inverse recursion operator $\varphi = \mathcal{R}^{-1}(\psi)$. Together, systems Equations (14) and (15) can be viewed as auto-Bäcklund transformations for the tangent covering of Equation (1).

4. Actions of Recursion Operators on Contact Symmetry Characteristics and Nonlocal Shadows

We next compute the action of the two recursion operators on the contact symmetry characteristics of Equation (1), in the standard Lie-symmetry framework [35]. For a given characteristic φ , the direct operator \mathcal{R} is obtained by solving Equation (14) for ψ .

Conversely, for a prescribed characteristic ψ , the inverse operator \mathcal{R}^{-1} is obtained from Equation (15). The resulting images are not always local: some have local representatives, while others appear as shadows of nonlocal symmetries. This distinction is important for interpreting the table and examples below.

Theorem 2. *Let*

$$\mathbf{v} = \sum_{i=1}^p \zeta^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha},$$

be a vector field on an open subset $M \subset X \times U$. Its n -th prolongation is

$$\text{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^q \sum_{\mathbf{I}} \phi_\alpha^{\mathbf{I}}(x, u^{(n)}) \frac{\partial}{\partial u_{\mathbf{I}}^\alpha},$$

where the second summation is taken over all multi-indices $\mathbf{I} = (i_1, \dots, i_k)$ of length k , with $1 \leq k \leq n$, and each component of \mathbf{I} belongs to $\{1, \dots, p\}$. The prolongation coefficients are given by

$$\phi_\alpha^{\mathbf{I}} = D_{\mathbf{I}} \left(\phi_\alpha - \sum_{i=1}^p \zeta^i u_i^\alpha \right) + \sum_{i=1}^p \zeta^i u_{\mathbf{I},i}^\alpha,$$

where

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \quad u_{\mathbf{I},i}^\alpha = \frac{\partial u_{\mathbf{I}}^\alpha}{\partial x^i},$$

and $D_{\mathbf{I}}$ denotes the corresponding total derivative operator.

For Equation (1), we consider an infinitesimal vector field of the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}. \tag{16}$$

Here ξ, ζ, τ , and ϕ are smooth functions of (x, y, t, u) . Then the second prolongation of \mathbf{v} , restricted to the derivatives that occur in Equation (1), is

$$\text{pr}^{(2)}\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{ty} \frac{\partial}{\partial u_{ty}}.$$

The required prolongation coefficients are explicitly given as follows:

$$\phi^x = \phi_x + \phi_u u_x - (\xi_x + \xi_u u_x) u_x - (\zeta_x + \zeta_u u_x) u_y - (\tau_x + \tau_u u_x) u_t, \tag{17a}$$

$$\phi^y = \phi_y + \phi_u u_y - (\xi_y + \xi_u u_y) u_x - (\zeta_y + \zeta_u u_y) u_y - (\tau_y + \tau_u u_y) u_t, \tag{17b}$$

$$\phi^t = \phi_t + \phi_u u_t - (\xi_t + \xi_u u_t) u_x - (\zeta_t + \zeta_u u_t) u_y - (\tau_t + \tau_u u_t) u_t, \tag{17c}$$

$$\begin{aligned} \phi^{xy} &= \phi_{xy} + \phi_{xu} u_y + \phi_{yu} u_x + \phi_{uu} u_x u_y + \phi_u u_{xy} \\ &\quad - \xi_{xy} u_x - \xi_x u_{xy} - \xi_y u_{xx} - \xi_{xu} u_x u_y - \xi_{yu} u_x^2 - \xi_u (2u_x u_{xy} + u_{xx} u_y) - \xi_{uu} u_x^2 u_y \\ &\quad - \zeta_{xy} u_y - \zeta_x u_{yy} - \zeta_y u_{xy} - \zeta_{xu} u_y^2 - \zeta_{yu} u_x u_y - \zeta_u (u_x u_{yy} + 2u_y u_{xy}) - \zeta_{uu} u_x u_y^2 \\ &\quad - \tau_{xy} u_t - \tau_x u_{ty} - \tau_y u_{xt} - \tau_{xu} u_t u_y - \tau_{yu} u_t u_x - \tau_u (u_t u_{xy} + u_x u_{ty} + u_y u_{xt}) - \tau_{uu} u_t u_x u_y, \end{aligned} \tag{17d}$$

$$\begin{aligned} \phi^{xt} &= \phi_{xt} + \phi_{xu} u_t + \phi_{tu} u_x + \phi_{uu} u_x u_t + \phi_u u_{xt} \\ &\quad - \xi_{xt} u_x - \xi_x u_{xt} - \xi_t u_{xx} - \xi_{xu} u_x u_t - \xi_{tu} u_x^2 - \xi_u (u_t u_{xx} + 2u_x u_{xt}) - \xi_{uu} u_t u_x^2 \\ &\quad - \zeta_{xt} u_y - \zeta_x u_{ty} - \zeta_t u_{xy} - \zeta_{xu} u_t u_y - \zeta_{tu} u_x u_y - \zeta_u (u_t u_{xy} + u_x u_{ty} + u_y u_{xt}) - \zeta_{uu} u_t u_x u_y \\ &\quad - \tau_{xt} u_t - \tau_x u_{tt} - \tau_t u_{xt} - \tau_{xu} u_t^2 - \tau_{tu} u_t u_x - \tau_u (2u_t u_{xt} + u_{tt} u_x) - \tau_{uu} u_t^2 u_y, \end{aligned} \tag{17e}$$

$$\begin{aligned} \phi^{ty} &= \phi_{ty} + \phi_{tu} u_y + \phi_{yu} u_t + \phi_{uu} u_t u_y + \phi_u u_{ty} \\ &\quad - \xi_{ty} u_x - \xi_t u_{xy} - \xi_y u_{xt} - \xi_{tu} u_x u_y - \xi_{yu} u_t u_x - \xi_u (u_t u_{xy} + u_x u_{ty} + u_y u_{xt}) - \xi_{uu} u_t u_x u_y \\ &\quad - \zeta_{ty} u_y - \zeta_t u_{yy} - \zeta_y u_{ty} - \zeta_{tu} u_y^2 - \zeta_{yu} u_t u_y - \zeta_u (u_t u_{yy} + 2u_y u_{ty}) - \zeta_{uu} u_t u_y^2 \\ &\quad - \tau_{ty} u_t - \tau_t u_{ty} - \tau_y u_{tt} - \tau_{tu} u_t u_y - \tau_{yu} u_t^2 - \tau_u (2u_t u_{ty} + u_{tt} u_y) - \tau_{uu} u_t^2 u_y. \end{aligned} \tag{17f}$$

The infinitesimal invariance condition is obtained by applying $\text{pr}^{(2)}\mathbf{v}$ to Equation (1). This gives

$$\phi^{ty} - \phi^t u_{xy} - \phi^{xy} u_t + \phi^y u_{tx} + \phi^{xt} u_y = 0.$$

Substituting the coefficients in Equation (17) into this relation and using Equation (1), we obtain the determining equations for ξ, ζ, τ , and ϕ . By separating the coefficients of the independent monomials formed by the derivatives of u , the determining system reduces to

$$\begin{aligned} \tau_x &= 0, & \tau_y &= 0, & \tau_u &= 0, \\ \xi_x &= \phi_u, & \xi_y &= 0, & \xi_t &= 0, & \xi_u &= 0, \\ \zeta_x &= 0, & \zeta_t &= 0, & \zeta_u &= 0, \\ \phi_y &= 0, & \phi_t &= 0. \end{aligned}$$

Solving this system yields

$$\begin{aligned} \xi &= A_0(x), \\ \zeta &= C_1(y), \\ \tau &= B_1(t), \\ \phi &= -A'_0(x)u + A_1(x). \end{aligned}$$

Therefore, the vector field Equation (16) gives the following infinitesimal generators:

$$\begin{aligned} \mathbf{v}_1 &= A_0 \frac{\partial}{\partial x} - A'_0(x)u \frac{\partial}{\partial u}, \\ \mathbf{v}_2 &= A_1 \frac{\partial}{\partial u}, \\ \mathbf{v}_3 &= B_1 \frac{\partial}{\partial t}, \\ \mathbf{v}_4 &= C_1 \frac{\partial}{\partial y}. \end{aligned}$$

Using the characteristic form of these generators, the contact symmetry algebra of Equation (1) is generated by

$$\begin{aligned} \varphi_0(A_0) &= A_0 u_x - A'_0 u, \\ \varphi_1(A_1) &= A_1, \\ \varphi_2(B_1) &= B_1 u_t, \\ \varphi_3(C_1) &= C_1 u_y. \end{aligned}$$

Their nonzero commutation relations are as follows. Here $j, k \in \{0, 1\}$, and A_j and \tilde{A}_k denote two arbitrary smooth functions of x . The prime denotes differentiation with respect to x . Then

$$\{\varphi_j(A_j), \varphi_k(\tilde{A}_k)\} = \begin{cases} \varphi_{j+k}(\tilde{A}_k A'_j - \tilde{A}'_k A_j), & 0 \leq j+k \leq 1, \\ 0, & j+k > 1. \end{cases}$$

Let $\mathfrak{g}_j = \{\varphi_j(A_j) \mid A_j \in C^\infty(\mathbb{R})\}$, $0 \leq j \leq 1$, and define $\mathfrak{h} = \{\varphi_2(B_1) \mid B_1 \in C^\infty(\mathbb{R})\}$, $\mathfrak{k} = \{\varphi_3(C_1) \mid C_1 \in C^\infty(\mathbb{R})\}$. Then the contact symmetry algebra decomposes as $\text{sym}_0(\mathcal{E}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{h} \oplus \mathfrak{k}$. Moreover, the subspaces \mathfrak{g}_j , $0 \leq j \leq 1$ satisfy the grading relation

$$\{\mathfrak{g}_j, \mathfrak{g}_k\} = \begin{cases} \mathfrak{g}_{j+k}, & 0 \leq j+k \leq 1, \\ 0, & j+k > 1. \end{cases}$$

We next evaluate the inverse recursion operator on the above generators. For $\psi = \varphi_j$, $j = 1, 2, 3$, system Equation (15) admits the following local representatives:

$$\begin{aligned} \mathcal{R}^{-1}(\varphi_1(A_1)) &= -\varphi_0(A_1), \\ \mathcal{R}^{-1}(\varphi_2(B_1)) &= -(1 + \lambda)\varphi_2(B_1), \\ \mathcal{R}^{-1}(\varphi_3(C_1)) &= -\lambda\varphi_3(C_1). \end{aligned}$$

The generator $\varphi_0(A_0)$ gives a different type of image. In this case, \mathcal{R}^{-1} does not give a purely local representative in the contact symmetry algebra but produces a nonlocal symmetry shadow:

$$\mathcal{R}^{-1}(\varphi_0(A_0)) = m_0 - \lambda\varphi_0(A_0), \tag{18}$$

where the potential m_0 is determined by the compatible system

$$\begin{cases} m_{0,t} = A_0(u_t u_{xx} - u_{tx} u_x - u_x) + A'_0(u_{tx} u + u) - A''_0 u_t u, \\ m_{0,y} = A_0(u_y u_{xx} - u_{xy} u_x) + A'_0 u u_{xy} - A''_0 u u_y. \end{cases} \tag{19}$$

Equivalently, m_0 is the potential associated with the conservation law

$$\begin{aligned} dm_0 \wedge dx &= (A_0(u_t u_{xx} - u_{tx} u_x - u_x) + A'_0(u_{tx} u + u) - A''_0 u_t u) dt \wedge dx \\ &\quad + (A_0(u_y u_{xx} - u_{xy} u_x) + A'_0 u u_{xy} - A''_0 u u_y) dy \wedge dx, \end{aligned}$$

which is a conservation law of Equation (1). Therefore, for the generator $\varphi_0(A_0)$, the inverse recursion operator does not preserve the local contact symmetry space. Instead, it produces a family of shadows of nonlocal symmetries represented by Equation (18) and associated with the compatible system Equation (19).

We now consider the direct recursion operator. Since system Equation (14) determines ψ up to solutions of the homogeneous system obtained by setting $\varphi = 0$, we fix representatives modulo this homogeneous part. With this convention, the following local representatives are obtained:

$$\begin{aligned} \mathcal{R}(\varphi_0(A_0)) &= -\varphi_1(A_0), \\ \mathcal{R}(\varphi_2(B_1)) &= -\frac{1}{1 + \lambda}\varphi_2(B_1), \\ \mathcal{R}(\varphi_3(C_1)) &= -\frac{1}{\lambda}\varphi_3(C_1). \end{aligned}$$

The remaining case $\varphi = \varphi_1(A_1)$ does not reduce to a local contact symmetry characteristic in this representative choice. Substituting $\varphi = A_1$ into Equation (14) gives

$$\begin{cases} D_t(\psi) = (1 + \lambda)^{-1}(u_t D_x(\psi) - u_{tx}\psi), \\ D_y(\psi) = \lambda^{-1}(u_y D_x(\psi) - u_{xy}\psi). \end{cases}$$

Thus, $\mathcal{R}(\varphi_1(A_1))$ is represented by a nonlocal symmetry shadow determined by the above system.

For clarity, the recursion operators obtained above and their actions on the infinitesimal symmetry characteristics are summarized in Table 1. The table distinguishes local representatives from nonlocal symmetry shadows, so that the roles of \mathcal{R} and \mathcal{R}^{-1} are stated without ambiguity.

Table 1. Summary of the recursion operators and infinitesimal symmetry characteristics for Equation (1).

Object	Definition	\mathcal{R} -Image	\mathcal{R}^{-1} -Image
\mathcal{R}	$\begin{cases} D_t(\psi) = (1 + \lambda)^{-1}(u_t D_x(\psi) - u_{tx}\psi - D_t(\varphi)), \\ D_y(\psi) = \lambda^{-1}(u_y D_x(\psi) - u_{xy}\psi - D_y(\varphi)) \end{cases}$	$\psi = \mathcal{R}(\varphi)$	–
\mathcal{R}^{-1}	$\begin{cases} D_t(\varphi) = -(1 + \lambda)D_t(\psi) + u_t D_x(\psi) - u_{tx}\psi, \\ D_y(\varphi) = u_y D_x(\psi) - \lambda D_y(\psi) - u_{xy}\psi \end{cases}$	–	$\varphi = \mathcal{R}^{-1}(\psi)$
$\varphi_0(A_0)$	$A_0 u_x - A_0' u$	$-\varphi_1(A_0)$	$m_0 - \lambda \varphi_0(A_0)$
$\varphi_1(A_1)$	A_1	nonlocal shadow determined by $\begin{cases} D_t(\psi) = (1 + \lambda)^{-1}(u_t D_x(\psi) - u_{tx}\psi), \\ D_y(\psi) = \lambda^{-1}(u_y D_x(\psi) - u_{xy}\psi) \end{cases}$	$-\varphi_0(A_1)$
$\varphi_2(B_1)$	$B_1 u_t$	$-\frac{1}{1 + \lambda} \varphi_2(B_1)$	$-(1 + \lambda) \varphi_2(B_1)$
$\varphi_3(C_1)$	$C_1 u_y$	$-\frac{1}{\lambda} \varphi_3(C_1)$	$-\lambda \varphi_3(C_1)$

Here $A_0, A_1 \in C^\infty(x)$, $B_1 \in C^\infty(t)$, and $C_1 \in C^\infty(y)$. The prime denotes differentiation with respect to x , and the potential m_0 is defined by Equation (19). The entries involving m_0 and the defining system in the row of $\varphi_1(A_1)$ correspond to nonlocal symmetry shadows; the remaining displayed images are local representatives.

Illustrative Examples

Based on Table 1, we give three examples to verify the actions of \mathcal{R} and \mathcal{R}^{-1} by direct substitution into Equations (14) and (15).

Example 2. Let $B_1(t) = t$. Then

$$\varphi = \varphi_2(t) = tu_t.$$

According to the action of \mathcal{R} on $\varphi_2(B_1)$, we set

$$\psi = -\frac{1}{1 + \lambda} \varphi_2(t) = -\frac{t}{1 + \lambda} u_t.$$

A direct substitution into Equation (14) gives

$$D_t(\psi) = -\frac{1}{1 + \lambda} (u_t + tu_{tt}) = (1 + \lambda)^{-1} (u_t D_x(\psi) - u_{tx}\psi - D_t(\varphi)),$$

and

$$D_y(\psi) = -\frac{t}{1 + \lambda} u_{ty} = \lambda^{-1} (u_y D_x(\psi) - u_{xy}\psi - D_y(\varphi)).$$

Hence

$$\mathcal{R}(\varphi_2(t)) = -\frac{1}{1 + \lambda} \varphi_2(t).$$

Repeated application gives

$$\mathcal{R}^n(\varphi_2(t)) = \left(-\frac{1}{1 + \lambda}\right)^n tu_t, \quad n = 0, 1, 2, \dots$$

Thus $\varphi_2(t)$ generates a local symmetry sequence under \mathcal{R} .

Example 3. Let $A_0(x) = x$. Then

$$\varphi = \varphi_0(x) = xu_x - u.$$

According to Table 1, the direct recursion operator should give

$$\psi = \mathcal{R}(\varphi_0(x)) = -\varphi_1(x) = -x.$$

Indeed,

$$D_x(\psi) = -1, \quad D_t(\psi) = D_y(\psi) = 0,$$

and

$$D_t(\varphi) = xu_{tx} - u_t, \quad D_y(\varphi) = xu_{xy} - u_y.$$

Substituting these expressions into Equation (14), we obtain

$$(1 + \lambda)^{-1}(u_t D_x(\psi) - u_{tx}\psi - D_t(\varphi)) = 0 = D_t(\psi),$$

and

$$\lambda^{-1}(u_y D_x(\psi) - u_{xy}\psi - D_y(\varphi)) = 0 = D_y(\psi).$$

Therefore,

$$\mathcal{R}(\varphi_0(x)) = -\varphi_1(x).$$

This verifies the local action of \mathcal{R} from φ_0 to φ_1 .

Example 4. Let $A_0(x) = x^2$. Then

$$\psi = \varphi_0(x^2) = x^2 u_x - 2xu.$$

Motivated by Equation (18), set

$$\varphi = m_0 - \lambda(x^2 u_x - 2xu).$$

Here m_0 is the potential defined by Equation (19). For $A_0 = x^2$, system Equation (19) becomes

$$\begin{cases} (m_0)_t = x^2(u_t u_{xx} - u_{tx} u_x - u_x) + 2x(u_{tx} u + u) - 2u_t u, \\ (m_0)_y = x^2(u_y u_{xx} - u_{xy} u_x) + 2xu u_{xy} - 2u u_y. \end{cases}$$

Using these two relations, we have

$$D_t(\varphi) = -(1 + \lambda)D_t(\psi) + u_t D_x(\psi) - u_{tx}\psi,$$

and

$$D_y(\varphi) = u_y D_x(\psi) - \lambda D_y(\psi) - u_{xy}\psi.$$

Thus $\varphi = m_0 - \lambda\varphi_0(x^2)$ satisfies Equation (15). Hence

$$\mathcal{R}^{-1}(\varphi_0(x^2)) = m_0 - \lambda\varphi_0(x^2),$$

which verifies the nonlocal shadow produced by the inverse recursion operator.

5. Conclusions

In this paper, we studied the modified Veronese web equation by using the spectral-parameter covering and tangent-covering approach. Starting from a compatible linear covering, we derived a direct recursion operator and an inverse recursion operator, given by Equation (14) and Equation (15), respectively. These two compatible systems act on symmetry characteristics and can be interpreted as auto-Bäcklund transformations for the tangent covering of Equation (1).

The main mathematical significance of the results is that they reveal a recursive structure in the symmetry space of the modified Veronese web equation. The obtained operators relate symmetry characteristics, describe their action on the contact symmetry algebra,

and produce both local representatives and shadows of nonlocal symmetries associated with conservation-law coverings. Hence, the results provide additional information on the covering-based nonlocal geometric structure of Equation (1) and are consistent with recent studies on recursion operators and nonlocal symmetries for multidimensional integrable equations [8,10–12,20].

There are also some limitations. The present paper focuses on the construction of recursion operators and on verifiable examples of their actions, but it does not give a complete classification of all local and nonlocal symmetry hierarchies generated by \mathcal{R} and \mathcal{R}^{-1} . Future work may extend this construction to other higher-dimensional Veronese-web-type equations and develop algorithmic tools for generating symmetry sequences and nonlocal shadows in broader classes of integrable models [36]. Another possible direction, beyond the scope of the present paper, is to investigate whether the recursion operators obtained here can be related to Hamiltonian or bi-Hamiltonian structures; this issue is not addressed in the present paper.

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