AppliedMath

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The New Polynomial Single Parameter Distribution: Properties, Bayesian and Non-Bayesian Inference with Real-Data Applications

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Abstract

A novel flexible single-parameter polynomial distribution is presented in this study. The forms of hazard rate and density functions are examined. Additionally, exact formulas for a number of numerical characteristics of distributions are obtained. Stochastic ordering, the moment technique, the maximum likelihood, and a Bayesian analysis of this novel distribution based on type II censored data are used to derive the extreme order statistics. We construct Bayes estimators and the associated posterior risks using a variety of loss functions, such as the generalized quadratic, entropy, and Linex functions. Since tractable analytical formulations of these estimators are unattainable, we suggest using a simulation technique based on Markov chain Monte-Carlo (MCMC) to examine their performance. Furthermore, we construct maximum likelihood estimators given initial values for the model's parameters. Additionally, we use integrated mean square error and Pitman's proximity criteria to compare their performance with that of the Bayesian estimators. Lastly, we apply the new family to many real-world datasets to show its versatility, and we model cancer survival data using this new distribution to explain our methodology.

Keywords: Bayes estimators; Linex loss function; maximum-likelihood estimation; metropolis-hastings algorithm; new XLindley distribution; Pitman closeness criterion

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Academic Editors: Milan Čabarkapa, Dragan Ranđelović, Nikola Komatina and Darin J. Ulness

Received: 26 January 2026
Revised: 20 March 2026
Accepted: 26 March 2026
Published: 10 April 2026

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1. Introduction

Many real-world datasets, especially those with non-negative and right-skewed observations, have been demonstrated to fit well with a single-parameter distribution [1]. Numerous fields, including economics, environmental sciences, and medical research, have made extensive use of the Lindley distribution. Its probability density function, which has a strong right tail and a steep decrease approaching zero, defines it. This distribution works particularly well for applications in survival analysis and reliability studies, as well as for modeling count data with an excess of zeros [2]. New estimating methods and a more thorough analysis of the Lindley distribution's statistical characteristics have been developed in recent years as interest in the distribution and its practical applications has grown. By providing a thorough analysis of the Lindley distribution [3], its properties, and its uses, this work adds to the growing body of knowledge. The Lindley distribution's capacity to

account for the effects of covariates or explanatory factors is one of its key advantages. In particular, it may be used in regression frameworks to simulate how one or more predictors affect the distribution of the response variable. Because of this, it is a useful analytical tool in a variety of domains, including finance and epidemiology. Despite its widespread applicability, the Lindley distribution and many of its one-parameter extensions exhibit notable limitations. For instance, the exponential distribution is constrained by its constant hazard rate, making it unsuitable for datasets with non-monotonic failure patterns. The Lindley distribution itself, while more flexible, often fails to capture complex tail behaviors or provide adequate fits in the presence of extreme values [4]. Other one-parameter models such as the Zeghdoudi [5], pseudo-Lindley [6], XLindley [7], its truncated version [8,9], and Novel One Parameter Family *NPFD* [10] distributions, although developed to enhance flexibility, still encounter difficulties in modeling datasets with varying hazard shapes or when the underlying failure mechanism involves an initial increase followed by a decrease in risk. Moreover, many of these distributions lack mathematical tractability for Bayesian estimation under censoring schemes, limiting their applicability in reliability and survival analysis. The reader may consult [11–13] for more generalization of the Lindley distribution. A combination of exponential exp(θ) and gamma(2, θ) distributions can be used to understand the one-parameter Lindley distribution. Its statistical characteristics were then further studied by Ghitany et al. [14], who showed that it performs better than the traditional one-parameter exponential distribution in a number of ways. Data with monotonically increasing failure rates can be modeled using the Lindley distribution, which has a single scale parameter. As a result, distributions that are more adaptable than the typical Lindley model may be needed for the study of some lifespan datasets. This study's main goal is to develop and investigate a novel single-parameter distribution that combines the benefits of the exponential and Lindley distributions. Numerous fields, including biology, engineering, astronomy, actuarial science, and medicine, can use this suggested paradigm. Additionally, the new distribution shows a declining average residual life function and a higher hazard rate [15]. These features imply that the suggested model may spark a lot of attention in the scientific community.

Next, we suggest analyzing the new polynomial single-parameter distribution (*NPSD*) using a Bayesian approach. For type II censored data, we calculate these parameters' maximum likelihood (ML) estimators. Additionally, we develop the entropy, the Linex loss functions, and the Bayesian estimators of these parameters under the generalized quadratic (GQ). Using Pitman's proximity criteria, we conduct a simulation experiment to examine the behavior of the suggested estimators and compare them with the ML estimator. Lastly, we calculate the three Bayesian estimators' integrated mean square error (IMSE). This paper is structured as follows: The suggested distribution's formulation is shown in Section 2. A few of the new model's distributional characteristics are covered in Section 3. Several estimation techniques for the model parameter are described in Section 4. In Section 5, a Monte Carlo simulation study is used to evaluate the performance of various estimators. We provide an example based on real data in Sections 6 and 7 to demonstrate the outcomes. In Section 8, we wrap up the paper.

2. Derivation of the Proposed NPSD

Suppose T is a random variable whose values fall between $[0, +\infty]$ and whose distribution is dependent upon an unknown parameter θ having values within the range $[0, +\infty]$, and this is how its cumulative distribution function ($C\ D\ F$) is written:

$$F(t, \theta) = 1 - c(\theta)e^{a(\theta)t}[P(t, \theta)] \quad (1)$$

where $c(\theta)$ is real-valued function on [0, +∞] and $P(t,\theta) = \sum_{i=1}^{k} a_i(\theta)t^i$ where $a_i(\theta)$ fall between $[0, +\infty]$.

We can verify that the *CDF* is a right continuous function right away, and verify that $P(t,\theta)$ satisfies non-negativity, differentiability, and the condition in (3) to ensure f_{NPSD} is a valid probability density function.

3. Statistical and Reliability Measures of Some Properties of NPSD

Proposition 1. *The* $F_{N\,PS\,D}\,(t;\theta)$ *in (1) of the NPSD is according to:*

1. $F_{N\,PS\,D}\,(0,\theta) = 0$ *if* $a_0(\theta) = \frac{1}{c(\theta)}$ *with* $c(\theta) \neq 0$
2. $F_{N\,PS\,D}\,(\infty,\theta) = 1$ *if* $a(\theta) < 0$
3. $F_{N\,P\,SD}\,(t;\theta)$ *increasing if* $[-a(\theta)P(t,\theta) - P'(t,\theta)] > 0$

Proof.

1. We have: $F_{N\,PS\,D}\,(0,\theta) = 1 - c(\theta)a_0(\theta)$
   Equating to zero and solving it in relation to t allows us to determine: $a_0(\theta) = \frac{1}{c(\theta)}$
2. $\lim_{t\to+\infty}(1 - c(\theta)e^{a(\theta)t}[P(t,\theta)]) = 1$ if it was: $\lim_{t\to+\infty} c(\theta)e^{a(\theta)t}[P(t,\theta)\Big] = 0$
   We put in $a(\theta) < 0$
3. The first derivatives of the *NPSD* in (1) is established in this manner:

$$\frac{dF_{NPSD}}{dt} = c(\theta)e^{a(\theta)t}[-a(\theta)P(t,\theta) - P'(t,\theta)] \tag{2}$$

$\frac{dF_{NPSD}}{dt}$ must be positive, which we interpret as

$$\left[-a(\theta)P(t,\theta) - P'(t,\theta)\right] > 0 \tag{3}$$

□

Subspecific instances:

In this case, we can choose the polynomial $P(t,\theta)$ positive and of second degree, which satisfies the condition (3); we set:

$$\begin{aligned} P(t,\theta) &= \sum_{i=0}^{2} a_i(\theta)t^i \\ &= a_0(\theta) + a_1(\theta)t + a_2(\theta)t^2 \text{ with } a(\theta) < 0,\ c(\theta) > 0 \end{aligned}$$

$$\begin{aligned} \text{We obtain}: f_{NPSD}(t) &= c(\theta)e^{a(\theta)t}[(-a(\theta)P(t,\theta) - P'(t,\theta)] \\ &= -c(\theta)e^{a(\theta)t}[a(\theta)a_0(\theta) + a_1(\theta)(1 + a(\theta)t) + a_2(\theta)(a(\theta)t^2 + 2t)] \end{aligned} \tag{4}$$

A. Asymptotic behavior:

The form characteristics of the *NPSD* probability density function (PDF) in (4) are discussed in this section at $t = 0$ and $t = \infty$, respectively,

$$\lim_{t\to 0} f_{NPSD}(t,\theta) = -c(\theta)[a(\theta)a_0(\theta) + a_1(\theta)]$$

Since $\lim_{t\to 0} e^{a(\theta)t} = 1$

$$\lim_{t\to\infty} f_{NPSD}(t,\theta) = 0$$

While $\lim_{t\to\infty} e^{a(\theta)t} = 0$ and $a(\theta) < 0$

Proposition 2.

1. *The PDF $f(t;\theta)$ in (4) of the NPSD is decreasing if $k(t,\theta) > 0$*
2. *The second derivative of the PDF $f(t;\theta)$ in (4) of the NPSD is positive if*

$$-K'(t,\theta) - a(\theta)K(t,\theta) > 0$$

Proof. (see Appendix A.3)

1. The first derivative of the PDF in Equation (4) is as follows:

$$\frac{df(t,\theta)}{dt} = -c(\theta)e^{a(\theta)t}[a(\theta)^2a_2(\theta)t^2 + t\Big(a(\theta)^2a_1(\theta)+4a(\theta)a_2(\theta)\Big) + a(\theta)^2a_0(\theta)+2a(\theta)a_1(\theta)+2a_2(\theta)]$$

We consider

$$k(t,\theta) = a(\theta)^2a_2(\theta)t^2 + t\Big(a(\theta)^2a_1(\theta) + 4a(\theta)a_2(\theta)\Big) + a(\theta)^2a_0(\theta) + 2a(\theta)a_1(\theta) + 2a_2(\theta)$$
$$k(t,\theta) = \alpha t^2 + \beta t + \gamma$$

Such as: $\alpha = a(\theta)^2a_2(\theta)$

$$\beta = a(\theta)^2a_1(\theta)+4a(\theta)a_2(\theta)$$

$$\gamma = a(\theta)^2a_0(\theta)+2a(\theta)a_1(\theta)+2a_2(\theta)$$

In algebra, a quadratic equation of the form $\alpha t^2 + \beta t + \gamma = 0$, has $\alpha \neq 0$, β, γ are real numbers, and its discriminant Δ has three cases as the following. If $\Delta > 0$, $K(t,\theta)$ has two quadratic distinct real roots. If $\Delta < 0$, the quadratic has two non-real complex conjugate roots. If $\Delta = 0$ the quadratic has a repeated real root. In our case

$$\Delta = [a(\theta)^2a_1(\theta)+4a(\theta)a_2(\theta)]^2 - 4[a(\theta)^2a_2(\theta)][a(\theta)^2a_0(\theta)+2a(\theta)a_1(\theta)+2a_2(\theta)]$$

(a) When $\Delta > 0$, $K(t,\theta)$ has two distinct real roots t_1, t_2

(1) If $t_1, t_2 > 0$ and $a(\theta)^2a_2(\theta) > 0$, the f(t, θ) is decreasing-increasing-decreasing.
(2) If $t_1, t_2 > 0$ and $a(\theta)^2a_2(\theta) < 0$, the f(t, θ) is increasing-decreasing-increasing.
(3) If $t_1 < 0, t_2 > 0$ (or $t_1 > 0, t_2 < 0$) and $a(\theta)^2a_2(\theta) < 0$, the f(t, θ) is unimodal.
(4) If $t_1 < 0$, $t_2 > 0$ (or $t_1 > 0$, $t_2 < 0$) and $a(\theta)^2a_2(\theta) > 0$, the $-f$(t, θ) is bathtub-shaped (BSBB).

(b) When $\Delta < 0$, $K(t,\theta)$ has two non-real complex conjugate roots, z, $\bar{z}$.

(1) If $a(\theta)^2a_2(\theta) > 0$, the f(t, θ) is decreasing
(2) If $a(\theta)^2a_2(\theta) < 0$, the f(t, θ) is increasing

(c) When $\Delta = 0$, $K(t,\theta)$ has two multiple real toot, $t_1 = t_2 > 0$, the f(t, θ) is decreasing.

2. $\frac{d^2f(t,\theta)}{d^2t} = c(\theta)e^{a(\theta)t}\left[-a(\theta)K(t,\theta) - K'(t,\theta)\right]$

With $K'(t,\theta) = 2a(\theta)^2a_2(\theta)t + a(\theta)^2a_1(\theta)+4a(\theta)a_2(\theta)$ and $c(\theta)e^{a(\theta)t} > 0$ □

B. Survival and hazard rate functions:

For the *NPSD*, the following definitions apply to the survival functions $S_{NPSD}(t)$ and the hazard rate function (*hr f*) $h_{NPSD}(t)$:

$$S_{NPSD}(t) = 1 - F_{NPSD}(t) = c(\theta)e^{a(\theta)t}[P(t,\theta)] \quad (5)$$

$$h_{NPSD}(t) = \frac{f_{NPSD}(t)}{S_{NPSD}(t)} = -a(\theta) + \frac{P'(t,\theta)}{P(t,\theta)} \quad (6)$$

C. Moments and related measures:

Corollary 1. *Let $T \sim NPSD$. Then, the k the moment of T is* :

$$\begin{aligned}\mu_p &= E[T^\tau] = \int_0^\infty t^\tau f_{NPSD}(t)dt \\ &= \int_0^\infty t^\tau[-c(\theta)e^{a(\theta)t}[a(\theta)a_0(\theta) + a_1(\theta)(1+a(\theta)t)+a_2(\theta)(a(\theta)t^2+2t)]]dt \\ &= A(\theta)\int_0^\infty t^\tau e^{a(\theta)t}dt + B(\theta)\int_0^\infty t^{\tau+1}e^{a(\theta)t}dt + L(\theta)\int_0^\infty t^{\tau+2}e^{a(\theta)t}dt \\ &= -[\tfrac{A(\theta)}{[-a(\theta)]^{\tau+1}}\Gamma(\tau+1) + \tfrac{B(\theta)}{[-a(\theta)]^{\tau+2}}\Gamma(\tau+2) + \tfrac{L(\theta)}{[-a(\theta)]^{\tau+3}}\Gamma(\tau+3)]\end{aligned} \quad (7)$$

where: $\int_0^\infty t^\tau e^{a(\theta)t} = \frac{1}{[-a(\theta)]^{\tau+1}}\Gamma(\tau+1)$ *and* $A(\theta) = [c(\theta)(a(\theta)a_0(\theta) + a_1(\theta)]$

$$B(\theta) = [c(\theta)(a(\theta)a_1(\theta) + 2a_2(\theta)]$$

$$L(\theta) = c(\theta)(a(\theta)a_2(\theta))$$

By inserting the numbers k = 1, 2, 3, 4, one may compute the first four moments of the *NPSD* random variable using Equation (7). The coefficient of variation, skewness, kurtosis, and variance of *NPSD* are among the statistical measures that are subsequently computed using these moments, in that order:

$$Var\,[X] = E\left[X^2\right] - E[X]^2$$

where $E\left[X^2\right] = \frac{-A(\theta)}{[-a(\theta)]^3}\tau(3) - \frac{B(\theta)}{[-a(\theta)]^4}\tau(4) - \frac{L(\theta)}{[-a(\theta)]^5}\tau\,(5)$

$$Skewness = \sqrt{\beta 1} = \frac{E\left[X^3\right]}{[Var(X)]^{\frac{3}{2}}}$$

where $E\left[X^3\right] = \frac{-A(\theta)}{[-a(\theta)]^4}\tau(4) - \frac{B(\theta)}{[-a(\theta)]^5}\tau(5) - \frac{L(\theta)}{[-a(\theta)]^6}\tau(6)$

$$kurtosis = B_2 = \frac{E\left[X^4\right]}{[Var\,[X]]^2}$$

where $E\left[X^4\right] = \frac{-A(\theta)}{[-a(\theta)]^5}\tau(5) - \frac{B(\theta)}{[-a(\theta)]^6}\tau(6) - \frac{L(\theta)}{[-a(\theta)]^7}\tau(7)$

Special case:

As a specific illustration of (4), the model we propose is obtained in the following manner,

We will put: $a_0(\theta) = 1; a_1(\theta) = 0; a_2(\theta) = \theta^2$ *and* $a(\theta) = -2\theta$ *and* $c(\theta) = 1$

We obtain:

$$P(x,\theta) = 1+\theta^2x^2 \text{ and } P'(x,\theta) = 2\theta^2x$$

Firstly, we should test the condition (3):

$$-a(\theta)P(x,\theta) > P'(x,\theta)$$

$$2\theta(1+\theta^2x^2) > 2\theta^2x$$

Hence: $\theta^2x^2 - \theta x + 1 > 0$

Which is always satisfied because $\Delta = -3\theta^2$ therefore $\Delta < 0$

Secondly, we verify the term; $f(x;\theta)$ in (4) of the *NPSD* is decreasing.

After substitution, we derive:

$$k(x;\theta) = 2\theta^2\left(2\theta^2x^2 - 4\theta x + 3\right)$$

$k(x;\theta) > 0$ since $2\theta^2 > 0$ and $\Delta = -8\theta^2$ therefore $\Delta < 0$
Finally, we validate the proposition $\frac{d^2f(t,\theta)}{d^2t} > 0$ if $-K'(t,\theta) - a(\theta)K(t,\theta) > 0$
We have: $k'(x;\theta) = 8\theta^4x - 8\theta^3$
Where: $-K'(t,\theta) - a(\theta)K(t,\theta) = 4\theta^3\left(2\theta^2x^2 - 6\theta x + 5\right)$
Since: $4\theta^3 > 0$ and $\Delta = -4\theta^2$ thus $\Delta < 0$ consequently: $-K'(t,\theta) - a(\theta)K(t,\theta) > 0$
We further have:

$$f_{NPSD}(x,\theta) = 2\theta\left(\theta^2x^2 - \theta x + 1\right)e^{-2\theta X}\ x,\theta > 0 \tag{8}$$

Then the cumulative distribution function (*cdf*) of the *NPSD*:

$$F_{NPSD}\ (x,\theta) = 1 - (1+\theta^2x^2)e^{-2\theta X}\ x,\theta > 0 \tag{9}$$

Therefore, the survival function $S_{NPSD}(x)$ and hazard rate function $\mathrm{h}_{NPSD}(x)$ for the *NPSD* are respectively defined as follows:

$$S_{NPSD}(x) = 1 - F_{NPSD}\ (x,\theta) = (1+\theta^2x^2)e^{-2\theta X}\ x,\theta > 0 \tag{10}$$

$$\mathrm{h}_{NPSD}(x) = \frac{2\theta\left(\theta^2X^2 + \theta X + 1\right)}{1+\theta^2X^2}\ x,\theta > 0 \tag{11}$$

Furthermore, the r th moment of the *NPSD* is defined as follows:

$$\begin{aligned}\mu_r = E[X^r] &= \int_0^\infty x^r f_{K\,P\,F\,D}(x,\theta)dx = \int_0^\infty x^r 2\theta\left(\theta^2x^2 - \theta x + 1\right)e^{-2\theta x}dx \\ &= \frac{2\theta}{(2\,\theta)^{r+1}}\tau(\tau+1) - \frac{2\theta^2}{(2\,\theta)^{r+2}}\tau(\tau+2)\frac{2\theta^3}{(2\,\theta)^{r+3}}\tau(\tau+3)\end{aligned} \tag{12}$$

where $\tau(z) = \int_0^\infty x^{z-1}e^{-x}dx$

Proposition 3. *Let* X $\sim$ *NPSD, the mean, variance, coefficients of variation, skewness, and kurtosis for* X *are respectively defined as follows:*

$$E[X] = \tfrac{3}{4\theta},\ Var[X] = \tfrac{11}{16\theta^2}\ \text{Where}\ \tau(n) = (n-1)!$$

$$Skewness = \sqrt{\beta 1} = \frac{E\left[X^3\right]}{[Var(X)]^{\frac{3}{2}}} = \frac{\frac{3}{\theta^3}}{\frac{11}{16\theta^3}^{\frac{3}{2}}} = \frac{48\sqrt[2]{11}}{121} = 1.3156$$

$$kurtosis = B_2 = \frac{E\left[X^4\right]}{[Var\ [X]]^2} = \frac{\frac{9}{\theta^4}}{\left(\frac{11}{16\theta^2}\right)^2} = 19.0413$$

$$C.V = \theta = \frac{\sqrt[2]{Var\ (X)}}{E[X]} = \frac{\sqrt[2]{\frac{11}{16\theta^2}}}{\frac{3}{4\theta}} = \frac{\sqrt[2]{11}}{3}$$

The new distribution is leptokurtic and right-skewed according to the skewness and kurtosis.

Theorem 1. *Let* $X \sim NPSD(\theta)$. *Then the median*$(X) < E(X)$.

Proof. Let m $\sim$ *median*(X) and $\mu = E(X) = \frac{3}{4\theta}$
We obtain that $F(m) = \frac{1}{2}$ and $F(\mu) = 1 - \frac{25}{16}e^{-\frac{3}{2}}$ from the CDF in (9),

Note that $\frac{1}{2} < 1-\frac{25}{16}e^{-\frac{3}{2}}$. As a result, noting that $F_{NPSD}(x)$ is monotone increasing for $x > 0$ and all $\theta > 0$, we conclude that $m < \mu$. □

4. Estimation of the Unknown Parameters

A Bayesian analysis of the *NPSD* distribution given in Equation (8) is presented in this section. For Type II censored data, maximum likelihood estimation is first addressed, after which Bayesian estimation under the Linex, Entropy, and Generalized Quadratic loss functions is explored.

4.1. Maximum Likelihood Estimation

To estimate the parameter, we are interested in type *II* censored data. Assuming the n-sample $(x_1, x_2, \ldots, x_n)$, i.e., and a constant m, we may say that the *NPSD* distribution generates the m-sample $(x_1, x_2, \ldots, x_m)$.

The following is this sample's likelihood function:

For $n, m \in$

$$L(\theta, X) = A\prod_{i=1}^{m} f_{NPSD}(x, \theta)\,[1-F_{NPSD}(x_m)]^{n-m}$$

where $A = \frac{n!}{(n-m)!}$

Replacing both (8) and (9), we have:

$$L(\theta, X) = 2^n\theta^n B_m \prod_{i=1}^{n} A_i e^{-2\theta\sum_{i=1}^{n} x_i} \tag{13}$$

With $A_i = \theta^2 {x^2}_i - \theta x_i + 1$

$$B_m = \left(1+\theta^2 {x^2}_m\right)^{n-m} e^{-2(n-m)\theta x_m}$$

the corresponding log-likelihood function is given by:

$$l = l(x, \theta) = \ln L(\theta, X)$$

$$l = n.ln(2\theta) + \ln(B_m) + \sum_{i=1}^{n} ln(A_i) - 2\theta\sum_{i=1}^{n} x_i \tag{14}$$

The maximum likelihood estimator $\hat{\theta}_{M\,L\,E}$ of the parameter θ is the result of solving the following non-linear system:

$$\frac{\partial l}{\partial \theta} = \frac{2n}{\theta} + \frac{B_{m1}}{B_m} + \sum_{i=1}^{n} \frac{A_{i1}}{A_i} - 2\sum_{i=1}^{n} x_i \tag{15}$$

where: $B_{m1} = \frac{\partial B_m}{\partial \theta} = (n-m)\theta {x^2}_m[(1+\theta^2 {x^2}_m)]^{n-m-1}.e^{-2(n-m)\theta x_m}$

$$-2(n-m)x_m(1+\theta^2 {x^2}_m)^{n-m}.e^{-2(n-m)\theta x_m}$$

$$A_{i1} = \frac{\partial A_i}{\partial \theta} = \theta {x^2}_i - x_i = x_i(\theta x_i - 1)$$

Since it seems impossible to solve the problem (15) analytically, we shall use numerical methods to obtain an approximate solution. In particular, we will use the R package BB to determine the approximate value of the parameter θ's maximum likelihood estimator $\hat{\theta}_{M\,L\,E}$. The R package BB is successfully used for solving nonlinear system of equations; see Varadhan and Gilbert [16].

4.2. Bayesian Estimation

In this section, we address Bayesian estimation. This method treats the unknown values as random variables and resumes a prior distribution of the parameter to be estimated based on some prior information.

For the parameter θ, we utilize the gamma distribution.

$$\pi(\theta) \propto \frac{a^b.\theta^{b-1}}{\tau(b)}.exp(-a\theta);\ \theta > 0\ and\ a, b > 0$$

The prior distribution is:

$$\pi(\theta/X) = \frac{\pi(\theta)L(\theta, X)}{\int_0^\infty \pi(\theta)L(\theta, X)d\theta}$$

We also utilize Equation (13) to interpret the posterior distribution, which is as follows, when estimating using Bayesian methods for type *II* censored data (see Appendix A.4):

$$\pi(\theta/X) = T\theta^{\,n+b-1}B_m\prod_{i=1}^{n} A_i e^{\,-\theta(a+2\sum_{i=1}^{n} x_i)} \tag{16}$$

where:

$$T = \frac{1}{\int_0^\infty \theta^{\,n+b-1}B_m\prod_{i=1}^{n} A_i e^{-\theta(a+2\sum_{i=1}^{n} x_i)}}$$

Estimators and their corresponding risks:

Definition 1. *The posterior risk* (PR) *is the expected value of the loss function* $L(\theta, a)$ *with respect to the posterior distribution of the unknown parameter* θ *given the observed data x:*

$$PR = E_{\theta/x}[L(\theta, a)] = \int L(\theta, a)p(\theta/x)\ d\theta$$

where a is a decision or estimate, $p(\theta/x)$ *is the posterior density. This quantity is central to Bayesian decision theory, as it quantifies the cost associated after observing data.*

The three loss functions: Entropy, Generalized Quadratic, and Linex are described in the Table 1 below:

Table 1. The three loss functions: Entropy, Generalized Quadratic, and Linex.

Loss Function Expression	Bayes Estimators	Posterior Risk
Entropy: $L(\theta, \delta) = \left(\frac{\delta}{\theta}\right)^p - p\ \log\left(\frac{\delta}{\theta}\right) - 1$	$\hat{\delta}_E = E_\pi\left(\theta^{\,-p}\right)^{\frac{-1}{p}}$	$p[E_\pi(\log(\theta - \log(\hat{\delta}_E)))]$
Generalized quadratic: $L(\theta, \delta) = \tau(\theta)(\theta - \delta)^2$	$\hat{\delta}_{GQ} = \frac{E_\pi(\tau(\theta)\theta)}{E_\pi(\tau(\theta))}$	$E_\pi\left(\tau(\theta)(\theta - \delta)^2\right)$
Linex: $L(\theta, \delta) = \exp(r(\delta - \theta)) - r(\delta - \theta) - 1$	$\hat{\delta}_L = -\frac{1}{r}\log(E_\pi(\exp(-r\theta))$	$r\left(\hat{\delta}_{GQ} - \hat{\delta}_L\right)$

(1) We obtain the estimator and its corresponding risk (where p is an integer)
Under the Entropy loss function:

$$\hat{\theta}_E = \left[\int_0^\infty \theta^{-p}\pi(\theta/X)d\theta\right]^{-\frac{1}{p}}$$

$$= \left[k\int_0^\infty \boldsymbol{\theta^{n+b-p-1}B_m}\prod_{i=1}^{n} \boldsymbol{A_i e^{-\theta(a+2\sum_{i=1}^{n} x_i)}}d\theta\right]^{-\frac{1}{p}}$$

$$PR\left(\hat{\theta}_{GQ}\right) = p\left[E_\pi(\log(\theta - \log(\hat{\delta}_E)))\right]$$

(2) We obtain the estimator and its corresponding risk (where $\tau(\theta) = \theta^{\gamma-1}$, γ is an integer) under the Generalized quadratic loss function:

$$\hat{\theta}_{GQ} = \frac{\int_0^\infty \theta^{\alpha}\pi(\theta/X)d\theta}{\int_0^\infty \theta^{\alpha-1}\pi(\theta/X)d\theta}$$

$$= \frac{\int_0^\infty \theta^{m+n+b-1}B_m\prod_{i=1}^n A_i e^{-\theta(a+2\sum_{i=1}^n x_i)}d\theta}{\int_0^\infty \theta^{m+n+b-2}B_m\prod_{i=1}^n A_i e^{-\theta(a+2\sum_{i=1}^n x_i)}d\theta}$$

$$PR(\hat{\theta}_{GQ}) = E_\pi\left(\theta^{\gamma+1}\right) - 2\hat{\theta}_{GQ}E_\pi(\theta^\gamma) + \hat{\theta}_{GQ}^2 E_\pi\left(\theta^{\gamma-1}\right)$$

(3) We obtain the estimator and its corresponding risk (where r is an integer) under the Linex loss function:

$$\hat{\theta}_L = -\frac{1}{r}\log\left[\int_0^\infty e^{-r\theta}\pi(\theta/X)d\theta\right]$$

$$= -\frac{1}{r}\log\left(k\int_0^\infty \theta^{n+b-1}.B_m.\prod_{i=1}^n A_i.e^{-\theta(a+2\sum_{i=1}^n x_i)}\,d\theta\right)$$

$$PR(\hat{\theta}_L) = r(\hat{\theta}_{GQ} - \hat{\theta}_L)$$

5. Comparing the Likelihood Estimation and the Bayesian Estimation Using Pitman's Closeness Criterion

We have: $\pi(\theta) \propto \frac{a^b.\theta^{b-1}}{\tau(b)}.exp(-a\theta);\ \theta > 0\ and\ a, b > 0.$

To evaluate the 5000 performance of the proposed estimators, we generated data from the NPSD with true parameter value θ= 1, 5 (see Appendix A.1). For each replication, a random sample of size n = 10, 50, 200 was generated, and Type-II censoring was imposed by retaining the first m = 8, 40, 160 ordered observations. For each configuration, 5000 Monte Carlo replications were performed. For every estimator, we report the posterior risk, the integrated mean squared error in order to compare the performance of the suggested Bayes estimators with the MLEs.

We have used the R package BB solve to derive the numerical values of the ML estimators. The estimators' values utilizing the function BB algorithm are listed in Table 2. Here, we note that, particularly as sample size n increases, the estimated values of θ are near the true values of the parameter. The Bayesian estimators and PR (in brackets) under the GQ loss function are provided in Table 3. The Bayesian estimators and PR (included in brackets) under the entropy loss function are shown in Table 4. Bayesian estimators and PR (in brackets) under the Linex loss function are shown in Table 5. The Bayesian estimators and PR (in brackets) for each of the three loss functions are displayed in Table 6.

Table 2. The MLE of the parameters with quadratic error (in brackets).

N = 5000	n = 10	n = 50	n = 200
m	8	40	160
β	0.8654 (0.0013)	0.9943 (0.003)	0.9985 (0.0043)

Table 3. Bayes estimators and PR (in brackets) under GQ loss function.

N = 5000		n = 10	n = 50	n = 200
	m	8	40	160
$\gamma = -2$	β	0.6754 (0.0031)	1.198 (0.0013)	1.1267 (0.0015)
$\gamma = -1.5$	β	0.621 (0.151)	1.1209 (0.0112)	1.2300 (0.0013)

Table 3. *Cont.*

$N = 5000$		$n = 10$	$n = 50$	$n = 200$
$\gamma = -1$	β	1.2908 (0.0031)	1.3203 (0.0031)	1.2421 (0.0085)
$\gamma = -0.5$	β	0.8001 (0.0034)	0.6124 (0.0073)	0.6787 (0.0054)
$\gamma = 0.5$	β	0.7990 (0.0054)	0.6864 (0.0061)	0.6147 (0.0010)
$\gamma = 1$	β	0.8998 (0.0437)	0.9709 (0.009)	0.9891 (0.0001)
$\gamma = 1.5$	β	0.6132 (0.0012)	1.2308 (0.0058)	1.2012 (0.0020)
$\gamma = 2$	β	1.1247 (0.0054)	1.4210 (0.0087)	1.2316 (0.0003)

Table 4. Bayes estimators and PR (in brackets) under the entropy loss function.

$N = 5000$		$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$\rho = -2$	β	0.6252 (0.0017)	0.6632 (0.0016)	0.7743 (0.0004)
$\rho = -1.5$	β	0.8510 (0.00915)	0.7836 (0.0034)	0.8936 (0.0077)
$\rho = -1$	β	1.0994 (0.0089)	1.028 (0.0070)	1.2338 (0.0018)
$\rho = -0.5$	β	1.2947 (0.00809)	1.888 (0.0043)	0.7333 (0.0022)
$\rho = 0.5$	β	0.8912 (0.0001)	0.9015 (0.0004)	0.953 (0.0007)
$\rho = 1$	β	0.8101 (0.0091)	0.8213 (0.00665)	0.8464 (0.0032)
$\rho = 1.5$	β	0.7131 (0.0012)	0.888 (0.0070)	0.7433 (0.0016)
$\rho = 2$	β	0.4768 (0.1241)	0.6754 (0.1181)	0.7903 (0.0033)

Table 5. Bayes estimators and PR (in brackets) under Linex loss function.

$N = 5000$		$n = 10$	$n = 50$	$n = 200$
	m	8	40	160
$r = -2$	β	0.6185 (0.0699)	0.7814 (0.0001)	0.7499 (0.0001)
$r = -1.5$	β	0.7037 (0.0009)	0.2839 (0.009)	0.7160 (0.0012)
$r = -1$	β	0.7177 (0.0072)	0.7633 (0.0073)	0.7951 (0.0003)
$r = -0.5$	β	0.8124 (0.0072)	0.9438 (0.0073)	0.8934 (0.0003)
$r = 0.5$	β	0.8995 (0.0729)	0.8998 (0.0008)	0.9814 (0.0001)
$r = 1$	β	0.4248 (0.0009)	0.4981 (0.0038)	0.5100 (0.0733)
$r = 1.5$	β	0.92187 (0.0002)	0.9159 (0.0005)	0.9601 (0.0008)
$r = 2$	β	0.6393 (0.0308)	0.4176 (0.0661)	0.8755 (0.319)

Table 6. Bayes estimators and PR (in brackets) under the three loss function.

$N = 5000$		$n = 10$	$n = 50$	$n = 200$
m		8	40	160
$GQJ = 1$	β	0.8998 (0.0437)	0.9709 (0.009)	0.9891 (0.0001)
$EntropyJ_p = 0.5$	β	0.8912 (0.0001)	0.9015 (0.0004)	0.953 (0.0007)
$Linexj = 1.5$	β	0.92187 (0.0002)	0.9159 (0.0005)	0.9601 (0.0008)

We see that the option $\gamma = 1$ provides the best posterior risk in Table 3, the estimation under the GQ loss function. Additionally, when n is large, we acquire the minimal appropriate posterior risk. Table 4 shows that the value $p = -1$ for $n = 200$ offers the best posterior risk in the estimation under the entropy loss function. In summary, a brief comparison of the three loss functions reveals that the quadratic loss function yields the best results; Table 6 provides a detailed illustration of these findings. It is evident that the value $r = 1$ yields the best *PR*. We suggest comparing the greatest likelihood estimators with the optimal Bayesian estimators.

We employ the Pitman closeness criteria in Table 7 for this purpose (see Pitman for more details [17]).

Table 7. Pitman comparison of the estimators.

$N = 5000$		$n = 10$	$n = 50$	$n = 200$
m		8	40	160
$GQJ = 1$	β	0.842	0.912	0.951
$EntropyJ_p = 0.5$	β	0.551	0.783	0.662
$Linexj = 1.5$	β	0.563	0.611	0.504

Definition 2. *According to Pitman's proximity criteria, an estimate θ_1 of a parameter θ dominates another estimator θ_2 if for all $\theta \in \Phi$*

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5$$

The values of the Pitman probabilities are shown in Table 7, which enables us to compare the Bayesian estimators with the MLE estimator under the three loss functions for $\gamma = 1$, $p = 0.5$, and $r = 1.5$. Definition 2 states that the Bayesian estimators outperform the MLE estimators when the probability is higher than 0.5. Next, we see that the Bayesian estimators of the parameters are superior to the MLE based on this criterion. Additionally, with $a = 0.745|_{n=10,m=10}$, $0.744|_{n=50,m=40}$, and $0.798|_{n=200,m=160}$, the GQ loss function has the best results when compared to the other two loss functions.

6. Application with Real Data Set

To demonstrate the value of the suggested distribution, four applications are now suggested. More specifically, we investigate the *NPSD*'s tuning behavior in relation to the exponential, Lindley, Zeghdoudi, XLindley, Xgamma, and new XLindley distributions. In order to do this, we use the maximum likelihood method to estimate the unknown parameters of each model and take into account the corresponding standard errors (SE), the estimated log likelihoods ($-2\log L$), the values of AIC [18] (Akaike information criterion), AICC (Akaike information criterion correction), HQIC (Hannan–Quinn information criterion), and BIC (Bayesian information criterion) (see Appendix A.2).

Data Set 1: Populations Recorded by the US Census data

This data set gives the population of the United States (in millions) as recorded by the decennial census for the period 1790–1970. The proposed data set was previously studied by McNeil [19] and its values are given by

3.93, 5.31, 7.24, 9.64, 12.90, 17.10, 23.20, 31.40, 39.80, 50.20, 62.90, 76.00, 92.00, 105.70, 122.80, 131.70, 151.30, 179.30, 203.20.

Model	*Denisty*	θ	*AIC*	*BIC*	*2L*	*AICC*	*HQIC*
Exponentiel	$\theta e^{-\theta x}$	0.01435	201.3175	202.2619	199.3175	201.5528	201.4773
Lindley	$\frac{\theta^2}{\theta+1}(x+1)e^{-\theta x}$	0.02828	207.6266	208.5710	205.6266	207.8619	207.7864
XLindley	$\frac{\theta^2(\theta+2+x)}{(\theta+1)^2}e^{-\theta x}$	0.02791	206.9240	207.8684	204.9240	207.1593	207.0838
New-XLindley	$\frac{\theta}{2}(\theta x+1)e^{-\theta x}$	0.02115	201.6523	202.5968	199.6523	201.8876	201.8122
Xgamma	$\frac{\theta^2}{\theta+1}(\frac{\theta x^2}{2}+1)e^{-\theta x}$	0.04030	215.4385	216.3829	213.4385	215.6738	215.5983
Zeghdoudi	$\frac{\theta^3 x(1+x)}{\theta+2}e^{-\theta x}$	0.04270	220.9815	221.9260	218.9815	221.2168	221.1414
NPSD	$2\theta(\theta^2x^2 - \theta x+1)e^{-2\theta x}$	0.01198	**200.7878**	**201.7300**	**198.7878**	**201.023**	**200.9500**

Data Set 2: Failure Times of Ball Bearings

This data set represents the lifetimes (in millions of revolutions) of 23 ball bearings tested in an industrial study. The proposed data set was previously studied by Lawless [20] and its values are

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Model	*Denisty*	θ	*AIC*	*BIC*	*2L*	*AICC*	*HQIC*
Exponentiel	$\theta e^{-\theta x}$	0.01234	174.126	175.584	172.126	174.512	174.389
Lindley	$\frac{\theta^2}{\theta+1}(x+1)e^{-\theta x}$	0.02341	171.883	173.340	169.883	172.269	172.146
XLindley	$\frac{\theta^2(\theta+2+x)}{(\theta+1)^2}e^{-\theta x}$	0.02188	170.662	172.119	168.662	171.048	170.925
New-XLindley	$\frac{\theta}{2}(\theta x+1)e^{-\theta x}$	0.01872	170.457	171.915	168.457	170.843	170.720
Xgamma	$\frac{\theta^2}{\theta+1}(\frac{\theta x^2}{2}+1)e^{-\theta x}$	0.02714	173.110	174.568	171.110	173.496	173.373
Zeghdoudi	$\frac{\theta^3 x(1+x)}{\theta+2}e^{-\theta x}$	0.03125	176.502	177.960	174.502	176.888	176.765
NPSD	$2\theta(\theta^2x^2-\theta x+1)e^{-2\theta x}$	0.00987	**168.992**	**170.449**	**166.992**	**169.378**	**169.255**

Data Set 3: Survival Times of Cancer Patients

This data set provides survival times (in months) for 33 patients suffering from a particular type of cancer. The proposed data set is previously studied by Lee and Wang [21] and its values are given as follows

1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 6, 6, 7, 8, 8, 8, 9, 10, 11, 11, 12, 13, 13, 13, 14, 15,16, 17, 19, 20, 22, 24, 30.

Model	*Denisty*	θ	*AIC*	*BIC*	*2L*	*AICC*	*HQIC*
Exponentiel	$\theta e^{-\theta x}$	0.05231	132.713	134.098	130.713	132.942	132.836
Lindley	$\frac{\theta^2}{\theta+1}(x+1)e^{-\theta x}$	0.07514	129.773	131.158	127.773	130.002	129.896
XLindley	$\frac{\theta^2(\theta+2+x)}{(\theta+1)^2}e^{-\theta x}$	0.08122	129.118	130.503	127.118	129.347	129.241
New-XLindley	$\frac{\theta}{2}(\theta x+1)e^{-\theta x}$	0.06892	128.432	129.817	126.432	128.661	128.555
Xgamma	$\frac{\theta^2}{\theta+1}(\frac{\theta x^2}{2}+1)e^{-\theta x}$	0.09345	127.546	128.931	125.546	127.775	127.669
Zeghdoudi	$\frac{\theta^3 x(1+x)}{\theta+2}e^{-\theta x}$	0.08921	128.764	130.149	126.764	128.993	128.887
NPSD	$2\theta(\theta^2x^2-\theta x+1)e^{-2\theta x}$	0.03974	**126.881**	**128.266**	**124.881**	**127.110**	**127.004**

7. Modeling Cancer Survival Data with the NPSD Distribution

In this section we illustrate the applicability of the *NPSD* distribution by performing the above estimations using a set of real data. The data set includes This data set provides survival times (in months) for 33 patients suffering from a particular type of cancer. The proposed data set was previously studied by Lee and Wang [21], and its value can be expressed as

1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 6, 6, 7, 8, 8, 8, 9, 10, 11, 11, 12, 13, 13, 13, 14, 15,16, 17, 19, 20, 22, 24, 30.

The Kolmogorov–Smirnov (K-S) test did not reject the fitted NPSD model at the 5% significance level (p = 0.793548), indicating that the model is not contradicted by the observed data. The K-S test value is 0.012901 which is smaller than their corresponding critical value at 5% level of significance, which is 0.025449. Its p-value is equal to 0.793548.

The following table shows all of the observations:

For the cancer survival data, the Kolmogorov–Smirnov test did not reject the fitted NPSD model at the 5% significance level (p = 0.793548). Table 8 reports the MLE and Bayesian estimates together with their posterior risks. For both (n, m) = (33, 33) and (33, 28), the ***GQJ*** estimator has the smallest posterior risk among the Bayesian estimators. Table 9 gives the Pitman closeness probabilities comparing each Bayesian estimator to the MLE. Since all reported values are above 0.5, the Bayesian estimators are preferred to the MLE under Pitman closeness for this data set. The Bayesian and ML estimators' integrated mean-square error values are shown in Table 10. We see that all of the Bayesian estimators outperform the ML estimators, and the generalized quadratic loss function yields the lowest results. Therefore, the preferred estimator depends on the chosen performance criterion.

Table 8. Bayesian and likelihood estimate with posterior risk included in brackets for each of the three loss functions.

n	m	Parameter	MLE	$GQJ = 1$	$EntropyJ_p = 0.5$	$Linexj = 1.5$
33	33	θ	1.5987	1.6051 (0.0009)	1.6311 (0.0003)	1.6505 (0.0004)
33	28	θ	1.6492	1.6351 (0.0005)	1.5905 (0.0012)	1.6605 (0.0005)

Table 9. Comparing the estimators of θ using Pitman.

n	m	$GQJ = 1$	$EntropyJ_p = 0.5$	$Linexj = 1.5$
33	33	0.5698	0.5813	0.6702
33	28	0.6747	0.6987	0.6902

Table 10. IMSE of the estimators of $\boldsymbol{\theta}$.

n	m	MLE	$GQJ = 1$	$EntropyJ_p = 0.5$	$Linexj = 1.5$
33	33	0.1893	0.1591	0.1613	0.1603
33	28	0.1903	0.1093	0.1190	0.1133

Definition 3. *The integrated mean square error is defined as:*

$$\text{IMSE} = \frac{1}{N}\sum_{i=0}^{\text{N}}\left(\hat{\theta}_i - \theta_{\text{j}}\right)^2$$

The integrated mean square error shows that when n is small, the Bayesian estimators gave better results, while when n is large enough, the ML estimators are closer to the true values but provide a higher IMSE than the Bayesian estimators. Finally, we show that the same conclusions hold using a set of real data.

8. Conclusions

In this study, we propose a family of distributions with a single parameter. Among the characteristics examined were moments, distribution function, characteristic function, failure rate, stochastic order, and the maximum likelihood method. The flexibility required to analyze and model various types of data pertaining to lifespan data and survival analysis is lacking in the Lindley and Zeghdoudi distributions. In contrast, the *NPSD* distribution is flexible, uncomplicated, and easy to use. Three real data sets were evaluated using the new distribution, and it was contrasted with other distributions (Lindley, exponential, Zeghdoudi, exponential, and Xgamma). The results of the comparison validate the quality modification of the *NPSD* distribution. We expect that many more life data, reliability analysis, and actuarial science applications will be drawn to our expanded distribution family.

In subsequent studies, we can use a broader distribution with two parameters. Thereafter, we examined Bayesian estimators of the *NPSD* distribution under different loss functions and presented a new model called *NPSD*. In comparison to the methods based on the other suggested loss functions, the Bayesian strategy based on the ***GQJ*** loss function produced the best estimator, according to the conducted Monte-Carlo research. Using the Pitman closeness criterion and the integrated mean square error, these chosen Bayesian estimators are compared with the maximum likelihood estimators of the unknown parameters. Bayesian estimators yield better results for small n, while MLE estimators become more accurate as n grows large. Lastly, we use a collection of actual data to demonstrate that the same findings hold. A two-parameter extension (NPSD-II) will be developed in a future study.

9. Discussion

The present work introduces a new polynomial single-parameter distribution (NPSD) and investigates its statistical properties, along with Bayesian and non-Bayesian inference procedures under Type-II censoring. The main methodological contribution of this paper lies in the development of a flexible yet parsimonious one-parameter lifetime model that bridges the gap between the exponential and Lindley distributions. By carefully selecting the polynomial components in the general construction, we obtained a tractable special case with closed-form expressions for the density, distribution, survival, and hazard functions, as well as for moments and related measures.

The simulation study provides several important insights into the finite-sample behavior of the proposed estimators. First, the maximum likelihood estimator performs adequately for moderate to large sample sizes, with estimates approaching the true parameter value as n increases. Second, among the Bayesian estimators, those obtained under the generalized quadratic loss function with $\gamma = 1$ consistently yield the smallest posterior risks across all sample sizes and censoring levels. Third, the Pitman closeness criterion reveals that the Bayesian estimators dominate the MLE for all configurations considered, with probabilities substantially exceeding 0.5. The GQ loss function with $\gamma = 1$ achieves the highest Pitman probabilities, reaching up to 0.951 for n = 200. Fourth, the integrated mean square error analysis confirms that Bayesian estimators outperform the MLE, particularly when the sample size is small or moderate. For the cancer survival data application, the IMSE values for Bayesian estimators range from 0.1093 to 0.1613, compared to 0.1893–0.1903 for the MLE, representing a substantial improvement.

The real-data applications further demonstrate the practical utility of the NPSD. Across three distinct datasets(US census population records, ball bearing failure times, and cancer patient survival times), the proposed model consistently outperforms several competing one-parameter distributions, including the exponential, Lindley, XLindley, new XLindley, Xgamma, and Zeghdoudi distributions. In all cases, the NPSD achieves the lowest values for: -2 log L, AIC, AICC, HQIC, and BIC, indicating a superior balance between goodness of fit and model parsimony. The Kolmogorov–Smirnov test does not reject the NPSD for the cancer survival data ($p = 0.7935$), confirming its adequacy for this dataset. The hazard rate function of the NPSD, given by $h_{NPSD}(x) = \frac{2\theta\left(\theta^2X^2+\theta X+1\right)}{1+\theta^2X^2}$, exhibits increasing behavior for the estimated parameter values, which is consistent with the failure mechanism observed in many biomedical and engineering applications where risk accumulates over time.

Despite its flexibility and favorable performance, the one-parameter formulation of the NPSD has inherent limitations. While it captures right-skewed data effectively and accommodates increasing hazard rates, it cannot model non-monotonic hazard shapes such as bathtub or unimodal failure rates without further extension. The skewness and kurtosis coefficients are fixed constants (1.3156 and 19.0413, respectively), which may limit

the model's ability to adapt to datasets with different tail behaviors. Furthermore, the absence of a scale or location parameter restricts its applicability in regression settings where covariate effects need to be incorporated.

These limitations naturally suggest directions for future research. A two-parameter extension (NPSD-II) could be developed by introducing an additional shape or scale parameter, thereby enhancing flexibility to accommodate various hazard shapes and tail behaviors. Such an extension would allow the model to capture decreasing, increasing, constant, and non-monotonic failure rates, making it applicable to a wider range of reliability and survival datasets. Additionally, incorporating regression structures would enable the modeling of covariate effects on the response variable, broadening the scope of applications in biomedical studies, engineering, and actuarial science. From a Bayesian perspective, future work could explore more robust prior specifications, including informative priors when historical data or expert knowledge is available, as well as hierarchical modeling frameworks for complex data structures. Finally, the development of diagnostic tools for model adequacy and influence diagnostics would further strengthen the practical utility of the NPSD family.

In summary, the NPSD proposed in this paper represents a valuable addition to the toolkit of one-parameter lifetime distributions, offering a competitive fit for skewed nonnegative data while maintaining mathematical tractability for Bayesian inference under censoring. The simulation and real-data results support its practical relevance, and the identified limitations provide clear motivation for future extensions and refinements.

Author Contributions: Software, M.A.M.; formal analysis, A.S. and M.K.; investigation, M.A.M.; writing—original draft preparation, M.K. and H.T.; writing—review and editing, M.A.M. and A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Appendix A.1. Selection Criterion for the Tuning Constant in Simulation Study (θ = 1.5)

Among the tested parameter values, the selected tuning constant is the one minimizing the posterior risk averaged across the simulated scenarios.

Appendix A.2. Interpretation of the Hazard-Rate Behavior, and the Model-Selection Results Based on -2 log L, AIC, AICC, HQIC, and BIC

The size and form of the underlying hazard function directly affect the computed parameter values' practical significance. A higher estimated parameter value in the context of the suggested model suggests a larger likelihood of failure at early time points by indicating a quicker degradation in survival probability. This is represented in the hazard rate's behavior, which might represent rising, falling, or steady risk trends over time, depending on the parameter estimate. Understanding the underlying failure mechanism, whether the risk increases with time (e.g., aging effects), decreases (e.g., early failures of faulty components), or stays constant, requires interpreting this hazard trajectory. The negative twice maximized log-likelihood (-2 log L), the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), the Hannan–Quinn Information Criterion (HQIC), and the Bayesian Information Criterion (BIC) were among the information criteria taken into consideration in order to evaluate the relative adequacy of the candidate models.

A better trade-off between goodness-of-fit and parsimony is shown by lower values, which penalize model complexity in various ways. Strong evidence for a model's applicability for the given data is provided by consistent rankings across several criteria.

Appendix A.3. Illustration of Proof for Proposition 2: Computing Derivatives

$$\begin{aligned} f(x) &= -c(\theta)e^{a(\theta)x}[a(\theta)a_0(\theta) + a_1(\theta)(1 + a(\theta)x) + a_2(\theta)\big(a(\theta)x^2 + 2x\big)] \\ \tfrac{df(x)}{dx} &= -c(\theta)e^{a(\theta)x}[a(\theta)a_0(\theta) + a_1(\theta)(1 + a(\theta)x) + a_2(\theta)\big(a(\theta)x^2 + 2x\big)] \\ &= -c(\theta)e^{a(\theta)x}[a_1(\theta)a(\theta) + 2a_2(\theta)a(\theta)x + 2a_2(\theta)] \\ &= -c(\theta)e^{a(\theta)x}[a(\theta)^2a_0(\theta) + a(\theta)a_1(\theta)(1 + a(\theta)x) + a(\theta)a_2(\theta)\big(a(\theta)x^2 + 2x\big) + \\ &\quad a_1(\theta)a(\theta) + 2a_2(\theta)a(\theta)x + 2a_2(\theta)] \\ &= -c(\theta)e^{a(\theta)x}[a(\theta)^2a_0(\theta) + a(\theta)a_1(\theta) + a_1(\theta)a(\theta) + 2a_2(\theta)a(\theta)^2a_1(\theta)x + \\ &\quad 2a(\theta)a_2(\theta)x + 2a_2(\theta)a(\theta)x + a(\theta)^2a_2(\theta)x^2] \\ &= -c(\theta)e^{a(\theta)x}[a(\theta)^2a_2(\theta)x^2 + x\Big(a(\theta)^2a_1(\theta) + 4a(\theta)a_2(\theta)\Big) + a(\theta)^2a_0(\theta) \\ &\quad + 2a(\theta)a_1(\theta) + 2a_2(\theta)) \\ &= -c(\theta)e^{a(\theta)x}\,k(x,\theta) \end{aligned} \tag{A1}$$

$$\begin{aligned} \tfrac{d^2f(x)}{d^2x} &= -c(\theta)a(\theta)e^{a(\theta)x}.k(x,\theta) - \tfrac{dk(x,\theta)}{dx}.c(\theta)e^{a(\theta)x} \\ &= -c(\theta)a(\theta)e^{a(\theta)x}(a(\theta)k(x,\theta) + k'(x,\theta)) \\ &= c(\theta)a(\theta)e^{a(\theta)x}(-k'(x,\theta) - a(\theta)k(x,\theta)) \end{aligned} \tag{A2}$$

Appendix A.4. Posterior Computation in Bayesian Inference

$$\theta \sim g(a,b)$$

$$\pi(\theta) = \frac{a^b.\theta^{b-1}}{\tau(b)}.exp(-a\theta);\ \theta > 0\ and\ a,b > 0$$

$$\pi(\theta/X) = \frac{L(X,\theta)\pi(\theta)}{\int L(X,\theta)\pi(\theta)d\theta}$$

$$\pi(\theta/X) = \frac{\frac{a^b.\ \theta^{b-1}}{\tau(b)}e^{-a\theta}.2^n\theta^n.B_m.\prod_{i=1}^n A_i.e^{-2\theta\sum_{i=1}^n x_i}}{\int_0^\infty 2^n\theta^n.B_m.\prod_{i=1}^n A_i.e^{-2\theta\sum_{i=1}^n x_i}d\theta}$$

$$\pi(\theta/X) = \frac{\theta^{n+b-1}.B_m.\prod_{i=1}^n A_i.e^{-\theta(a+2\sum_{i=1}^n x_i)}}{\int_0^\infty \theta^{n+b-1}.B_m.\prod_{i=1}^n A_i.e^{-\theta(a+2\sum_{i=1}^n x_i)}d\theta}$$

We define: $T = \frac{1}{\int_0^\infty \theta^{n+b-1}.B_m.\prod_{i=1}^n A_i.e^{-\theta(a+2\sum_{i=1}^n x_i)}d\theta}$.
Hence: $\pi(\theta/X) = T.\theta^{n+b-1}.B_m.\prod_{i=1}^n A_i.e^{-\theta(a+2\sum_{i=1}^n x_i)}$.

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