

Article

Spectral Models for Subsidy Allocation in Industrial Systems

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Abstract

This paper studies subsidy allocation in interconnected industrial systems using the spectral theory of positive matrices. The allocation is characterized by the Perron eigenvector of a cost matrix describing inter-factory interactions. We show that convergence to equilibrium is exponential and governed by the spectral ratio. A systemic resilience index based on spectral separation is introduced to quantify both stability and robustness under perturbations. The results demonstrate that stability and fairness arise from the spectral structure of the system.

Keywords: positive matrices; spectral radius; spectral gap; convergence rate; stability analysis; allocation models; Perron–Frobenius theory

1. Introduction

Linear allocation mechanisms arise naturally in interconnected economic and industrial systems where agents exchange intermediate goods and financial resources. When interactions between components are described by nonnegative coefficients, the resulting structure can be represented by a positive matrix governing the dynamics of the system. In such settings, long-term behavior is determined by spectral properties of the underlying matrix [1–4].

The Perron–Frobenius theory provides a fundamental analytical framework for studying positive linear systems. For primitive positive matrices, the spectral radius is a simple dominant eigenvalue associated with a strictly positive eigenvector [1,2]. This eigenvector determines the unique stable distribution toward which iterated linear processes converge. Such spectral structures play a central role in input–output economics [5], Markov processes [6], network centrality measures [7], and growth models in interconnected systems [8].

Such models are particularly relevant in contexts where production systems are highly interconnected, such as energy networks, manufacturing clusters, and regional supply chains. In these settings, understanding how local interactions aggregate into global equilibrium is essential for effective policy design.

In this paper, we formulate a subsidy allocation problem in interconnected industrial systems as a spectral problem for a positive matrix. The matrix represents cost or transfer coefficients between production units, while the equilibrium allocation is identified with the positive eigenvector corresponding to the spectral radius. Under primitivity assumptions, Perron–Frobenius theory guarantees existence and uniqueness of the allocation up to scaling [1–3].

Beyond this classical result, we focus on stability and convergence speed. The rate at which the allocation mechanism stabilizes depends on the spectral gap, that



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is, the ratio between the dominant eigenvalue and the modulus of the second largest eigenvalue [3,4,6]. Convergence estimates for matrix powers are closely related to spectral separation properties and are fundamental in the analysis of linear dynamical systems [2,4]. We interpret these estimates in the context of economic stabilization and production balance mechanisms.

Furthermore, we examine robustness of the allocation with respect to perturbations of the cost matrix. Sensitivity of the Perron eigenvector to parameter variations has been studied in matrix perturbation theory and plays an important role in applications where coefficients are subject to uncertainty [3,9]. Structural changes in cost interactions may therefore influence equilibrium outcomes, and understanding this dependence is crucial for assessing stability of allocation mechanisms.

Recent studies have emphasized spectral and numerical aspects of nonnegative matrices in networked and large-scale systems [10–14]. Iterative methods for computing Perron vectors and stability estimates in positive dynamical systems have attracted renewed attention in recent years.

Recent developments in network economics and spectral graph theory have further emphasized the role of eigenvector-based measures in describing centrality, resilience, and systemic importance in complex interconnected systems. Eigenvector-based rankings and spectral indicators have been used to quantify the propagation of shocks, the robustness of networks, and the structural importance of nodes in production and financial systems. These perspectives provide additional motivation for interpreting Perron eigenvectors as equilibrium allocation mechanisms in interconnected industrial structures [15–17].

Numerical examples complement the theoretical analysis. They demonstrate convergence toward the equilibrium allocation, quantify convergence speed, and illustrate sensitivity to parameter variations. These results highlight that fairness and stability are not externally imposed conditions but emerge as intrinsic consequences of the spectral structure of positive matrices.

The main contribution of this paper is threefold. First, we derive explicit quantitative bounds linking the spectral gap to the rate of convergence of allocation dynamics. Second, we introduce a systemic resilience index that captures robustness under perturbations. Third, we provide an economic interpretation of spectral properties in terms of stability and fairness of subsidy allocation.

From an applied perspective, such allocation mechanisms are relevant for industrial policy design, subsidy distribution schemes, and the stabilization of supply chains. In particular, the model provides a theoretical foundation for understanding how structural interdependencies between production units influence both efficiency and robustness of resource allocation.

The paper is organized as follows. Section 2 introduces the model framework. Section 3 presents illustrative examples. Section 4 develops the main theoretical results. Section 5 reviews relevant spectral theory. Section 6 provides numerical illustrations, and Section 7 discusses limitations and possible extensions.

2. Model Formulation

In this paper, we employ methods from linear algebra and the spectral theory of positive matrices. The models under consideration are based on finite real matrices with nonnegative or positive entries, which describe transfer or cost relationships among the individual components of the system. The analysis is purely theoretical in nature and does not involve empirical or statistical data.

The central mathematical tool is the Perron–Frobenius theory. We use Perron’s theorem for positive matrices to establish the existence of a dominant eigenvalue and an

associated positive eigenvector, and Wielandt's lemma to analyze spectral properties and the convergence of matrix powers. Attention is devoted to conditions of primitivity, which ensure the uniqueness of the positive eigenvector and the aperiodic convergence of linear dynamical processes.

The models are formulated as linear systems or recursive relations describing the temporal evolution of the system states. Long-term behavior is analyzed through sequences of matrix powers and their spectral properties. The proofs rely on standard methods of linear algebra, including matrix norms, eigenvalues, eigenvectors, and properties of invariant subspaces.

To illustrate the theory, we consider deterministic example models of industrial and regional systems in which economic interactions are formalized using positive matrices. These examples serve exclusively as mathematical motivation and do not constitute empirical validation of the model. All results are established analytically, without the use of numerical simulations or experimental methods.

Although the present work is purely theoretical, it is worth noting that matrix multiplication and related operations admit highly efficient implementations. Modern fast matrix multiplication algorithms significantly reduce computational complexity and allow large-scale spectral problems to be handled in practice. Moreover, many matrix operations can be reduced to matrix multiplication, implying that advances in multiplication algorithms directly improve the performance of a broad class of linear algebraic computations. A comprehensive overview of fast matrix multiplication and its applications can be found in Respondek [18]. This observation highlights the practical relevance of the presented spectral framework for large interconnected systems.

In practical terms, the matrix coefficients may be interpreted as technological or cost dependencies between production units, while the Perron eigenvector represents a stable proportional allocation of resources across the system. Such structures arise naturally in input–output models, regional production systems, and network-based economic models.

3. Spectral Characterization of the Subsidy Allocation Problem

In this section, we present the main results of the paper through two model examples that illustrate the application of the theory of nonnegative and positive matrices in the analysis of linear systems. The purpose of both examples is to demonstrate how the spectral properties of matrices are reflected in the long-term behavior of the system, as well as in the existence and structure of stable distributions.

In the first example, we consider a dynamic model of goods redistribution among regions, which is described by a column-stochastic matrix. We analyze the convergence of the sequence of matrix powers and show that the long-term distribution of goods corresponds to the limiting state of the system determined by the spectral properties of the matrix.

In the second example, we examine a subsidy allocation problem in an industrial system consisting of three factories. The problem is formulated as a linear system with a positive matrix of cost coefficients. We show that a fair and optimal subsidy allocation is equivalent to a positive eigenvector associated with the spectral radius of the matrix, and that Perron's theorem guarantees the existence and uniqueness of such a solution.

Taken together, these two examples illustrate that the theory of positive matrices provides a unified and mathematically rigorous approach to the analysis of stability, equilibrium, and optimal allocations in linear dynamical systems.

Theorem 1. *Subsidy Allocation Problem.*

We consider a stylized linear allocation model inspired by industrial input–output relations.

Suppose that we have three factories T_1, T_2, T_3 that produce different products and trade intermediate goods among themselves. The state provides support to all three factories in the form of subsidies.

The subsidies are distributed fairly: each factory uses only its own share of the subsidy for its production, while at the same time purchasing subsidized intermediate goods from the other factories. In return, the state requires the total production to be as large as possible, which is achieved under a fair allocation of subsidies.

Let us now examine the following matrix:

$$c_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (1)$$

Each coefficient c_{ij} represents the unit cost contribution of inputs from factory j to the production of factory i . The variables c_{ii} denote production levels, which correspond to proportional shares of the total subsidy allocation in the equilibrium state.

The entry c_{ij} represents the cost incurred by the j -th factory in producing one unit of output due to the use of products from the i -th factory. Naturally, all coefficients satisfy $c_{ij} > 0$.

The diagonal entries c_{ii} correspond to the costs of using a factory's own products, while the off-diagonal entries represent the costs of using products from other factories.

We now ask whether the subsidy is distributed fairly. Each factory receives a subsidy directly from the state. However, its production is also subsidized indirectly, since it uses both its own products and the products of other factories in its production process, and these products have already been subsidized.

If the i -th factory produces u_i units per year, then its indirect benefit from subsidies is given by

$$(c_{1i} + c_{2i} + c_{3i}) u_i = \gamma_i u_i. \quad (2)$$

At the same time, the other factories also derive an indirect benefit from subsidies through their use of the products of the i -th factory. In this way, factories T_1, T_2 , and T_3 obtain a total indirect benefit

$$c_{i1}u_1 + c_{i2}u_2 + c_{i3}u_3 \quad (3)$$

from the production of the i -th factory.

For the allocation of subsidies to be fair, the indirect benefit received by each factory must equal the indirect benefit provided to the other factories through the use of its products. Since this condition must hold for every factory, the indirectly received benefit must be equal to the indirectly provided benefit for each factory. Therefore, the following condition must be satisfied:

$$\begin{aligned} \gamma_1 \cdot u_1 - c_{11} \cdot u_1 - c_{12} \cdot u_2 - c_{13} \cdot u_3 &= 0, \\ \gamma_2 \cdot u_1 - c_{21} \cdot u_1 - c_{22} \cdot u_2 - c_{23} \cdot u_3 &= 0, \\ \gamma_3 \cdot u_1 - c_{31} \cdot u_1 - c_{32} \cdot u_2 - c_{33} \cdot u_3 &= 0, \end{aligned} \quad (4)$$

where

$$\begin{aligned} \gamma_1 &= c_{11} + c_{21} + c_{31}, \\ \gamma_2 &= c_{12} + c_{22} + c_{32} \\ \gamma_3 &= c_{13} + c_{23} + c_{33}. \end{aligned} \quad (5)$$

We seek solutions of the linear system

$$\begin{aligned} u_1 &= \frac{c_{11} \cdot u_1 + c_{12} \cdot u_2 + c_{13} \cdot u_3}{\gamma_1}, \\ u_2 &= \frac{c_{21} \cdot u_1 + c_{22} \cdot u_2 + c_{23} \cdot u_3}{\gamma_2}, \\ u_3 &= \frac{c_{31} \cdot u_1 + c_{32} \cdot u_2 + c_{33} \cdot u_3}{\gamma_3}. \end{aligned} \quad (6)$$

Moreover, we impose the constraints $u_1 > 0$, $u_2 > 0$, and $u_3 > 0$, since the number of units produced by each factory must be positive.

At the same time, we wish to satisfy the state's requirement that the total production of all factories be as large as possible. Hence, we seek a positive triple (u_1, u_2, u_3) solving the above linear system such that, for every ordered positive triple (x_1, x_2, x_3) that also solves the system, namely

$$\begin{aligned} 0 < x_1 &= \frac{c_{11} \cdot x_1 + c_{12} \cdot x_2 + c_{13} \cdot x_3}{\gamma_1}, \\ 0 < x_2 &= \frac{c_{21} \cdot x_1 + c_{22} \cdot x_2 + c_{23} \cdot x_3}{\gamma_2}, \\ 0 < x_3 &= \frac{c_{31} \cdot x_1 + c_{32} \cdot x_2 + c_{33} \cdot x_3}{\gamma_3}, \end{aligned} \quad (7)$$

the inequality $u_1 + u_2 + u_3 \geq x_1 + x_2 + x_3$ holds.

Does this system have a solution?

In 1905, the German mathematician O. Perron proved a fundamental theorem for real matrices A whose entries are all positive. Among other statements, Perron's theorem guarantees that for such matrices A , the eigenvalue equation

$$Ax = \rho(A)x \quad (8)$$

has a solution x with strictly positive components, i.e., $x_i > 0$ for all i .

Here $\rho(A)$ denotes the spectral radius of the matrix A . Perron's theorem also states that the eigenspace corresponding to the eigenvalue $\rho(A)$ is one-dimensional.

The matrix

$$A = \begin{bmatrix} \frac{c_{11}}{\gamma_1} & \frac{c_{12}}{\gamma_1} & \frac{c_{13}}{\gamma_1} \\ \frac{c_{21}}{\gamma_2} & \frac{c_{22}}{\gamma_2} & \frac{c_{23}}{\gamma_2} \\ \frac{c_{31}}{\gamma_3} & \frac{c_{32}}{\gamma_3} & \frac{c_{33}}{\gamma_3} \end{bmatrix} \quad (9)$$

has all entries positive. By Perron's theorem, there exists a vector

$$x = [x_1, x_2, x_3]^{tr} \quad (10)$$

with strictly positive components such that

$$\begin{aligned} \frac{c_{11} \cdot x_1 + c_{12} \cdot x_2 + c_{13} \cdot x_3}{\gamma_1} &= \rho(A) \cdot x_1, \\ \frac{c_{21} \cdot x_1 + c_{22} \cdot x_2 + c_{23} \cdot x_3}{\gamma_2} &= \rho(A) \cdot x_2, \\ \frac{c_{31} \cdot x_1 + c_{32} \cdot x_2 + c_{33} \cdot x_3}{\gamma_3} &= \rho(A) \cdot x_3. \end{aligned} \quad (11)$$

Let us denote

$$g = [\gamma_1, \gamma_2, \gamma_3]^{tr}. \quad (12)$$

It is clear that

$$g^{tr} A = g^{tr}. \quad (13)$$

Therefore,

$$g^{tr} x = g^{tr} Ax = g^{tr} \rho(A)x = \rho(A) g^{tr} x. \quad (14)$$

Since the scalar $g^{tr} x$ is nonzero, it follows that $\rho(A) = 1$.

Every vector of the form $u = \tau x$, where $\tau > 0$, is a solution of the linear system (6) and, of course, satisfies the inequalities $u_1 > 0, u_2 > 0, u_3 > 0$.

Perron's theorem further states that the eigenspace corresponding to the eigenvalue

$$Ay = \rho(A)y \quad (15)$$

is one-dimensional. Consequently, every solution of the linear system (6) under the constraints $u_1 > 0, u_2 > 0, u_3 > 0$ is a positive scalar multiple of the vector x .

The sum

$$\tau \cdot x_1 + \tau \cdot x_2 + \tau \cdot x_3 \quad (16)$$

represents the total production of all three factories combined.

The cost of this production is given by

$$\gamma_1 \cdot (\tau \cdot x_1) + \gamma_2 \cdot (\tau \cdot x_2) + \gamma_3 \cdot (\tau \cdot x_3). \quad (17)$$

This cost cannot exceed the total state subsidy σ . Therefore, we choose

$$\begin{aligned} u_1 &= \frac{\sigma}{\gamma_1 \cdot x_1 + \gamma_2 \cdot x_2 + \gamma_3 \cdot x_3} \cdot x_1, \\ u_2 &= \frac{\sigma}{\gamma_1 \cdot x_1 + \gamma_2 \cdot x_2 + \gamma_3 \cdot x_3} \cdot x_2, \\ u_3 &= \frac{\sigma}{\gamma_1 \cdot x_1 + \gamma_2 \cdot x_2 + \gamma_3 \cdot x_3} \cdot x_3. \end{aligned} \quad (18)$$

We obtain the solution (6) under the constraints $u_1 > 0, u_2 > 0, u_3 > 0$, which is optimal. In this case, the i -th factory receives $\gamma_i u_i$ as its share of the total subsidy.

Perron's theorem thus provides an answer to the question posed earlier: the subsidy allocation problem is always solvable, regardless of the choice of the parameters c_{ij} .

Perron's theorem will be stated precisely and discussed in detail in the following section.

For example, if one factory has stronger technological connections with others, the model assigns it a larger share of the subsidy. This reflects its higher systemic importance within the production network.

Relation to Input–Output Models

The present framework is closely related to classical input–output analysis initiated by Wassily Leontief [5]. In Leontief's model, production equilibrium is determined by solving the linear system

$$x = Ax + d, \quad (19)$$

which yields

$$x = (I - A)^{-1}d, \quad (20)$$

provided that the spectral radius satisfies $\rho(A) < 1$; see [5,19]. The Leontief inverse $(I - A)^{-1}$ determines the production levels required to meet exogenous demand d . Convergence of the associated Neumann series follows directly from the spectral radius condition [2,3].

In contrast, the present study adopts a fundamentally different perspective. Rather than prescribing an external demand vector, we analyze an endogenous equilibrium structure determined by the dominant eigenvector of a primitive positive matrix. The Perron eigenvector characterizes a proportional allocation regime governed entirely by internal structural interactions [2,3].

The key distinction lies in the role of the spectral radius. In the Leontief framework, stability requires $\rho(A) < 1$, ensuring boundedness and convergence of the inverse operator [5]. In the present model, the spectral radius is not constrained to be less than one; instead, it governs normalization and stabilization speed through spectral separation. The

spectral gap, which plays no central role in the classical input–output setting, becomes decisive for convergence rate and robustness analysis in the Perron-based formulation.

Thus, while both approaches rely on linear operators describing inter-industry relations, the Leontief model focuses on demand-driven equilibrium levels, whereas the present framework characterizes structurally determined proportional allocation patterns arising from the internal spectral geometry of the system matrix.

4. Quantitative Stability and Robustness

While the existence and uniqueness of the Perron equilibrium follow from classical spectral theory, the dynamic and structural properties of the allocation mechanism require a quantitative analysis. It is necessary to determine (i) the rate at which iterative redistribution converges to equilibrium and (ii) the dependence of the equilibrium allocation on structural perturbations of the cost coefficients. The following theorem provides explicit bounds governed by the spectral gap of the system matrix.

Theorem 2. *Quantitative Stability and Robustness of Subsidy Allocation.*

Let $A \in \mathbb{R}^{n \times n}$ be a primitive positive matrix representing inter-factory cost interactions. Let:

- $\rho = \rho(A)$ be the spectral radius,
- $v \gg 0$ be the normalized Perron eigenvector,
- λ_2 be the eigenvalue of second largest modulus,
- $\gamma = \frac{|\lambda_2|}{\rho} < 1$ be the spectral ratio.

Define the iterative subsidy mechanism:

$$x_{k+1} = \frac{Ax_k}{\|Ax_k\|_1} \quad (21)$$

Then the following statements hold:

The proof relies on standard spectral decomposition and perturbation theory (see Horn and Johnson [3], Stewart and Sun [9]).

Proof of Theorem 2. Exponential Convergence. $\square$

Since A is primitive and positive, the Perron–Frobenius theorem applies. The spectral radius is a simple eigenvalue. All other eigenvalues satisfy

$$|\lambda| < \rho. \quad (22)$$

Let v and w denote the right and left Perron eigenvectors normalized so that

$$w^T v = 1. \quad (23)$$

By spectral decomposition, we may write

$$A^k = \rho^k v w^T + R_k, \quad (24)$$

where the remainder term satisfies

$$\|R_k\| \leq C |\lambda_2|^k \quad (25)$$

for some constant $C > 0$, and where $|\lambda_2|$ denotes the modulus of the second largest eigenvalue.

Let the iteration be (21).

Using the decomposition above, we obtain

$$A^k x_0 = \rho^k v w^T x_0 + R_k x_0. \quad (26)$$

Since $w^T x_0 > 0$, dividing by $\|A^k x_0\|_1 \sim \rho^k$ yields

$$x_k = v + O\left(\left(\frac{|\lambda_2|}{\rho}\right)^k\right). \quad (27)$$

Hence,

$$\|x_k - v\| \leq C \left(\frac{|\lambda_2|}{\rho}\right)^k, \quad (28)$$

which proves exponential convergence governed by the spectral ratio

$$\gamma = \frac{|\lambda_2|}{\rho}. \quad (29)$$

Sensitivity Estimate.

We consider the restriction of the operator to the invariant subspace complementary to the Perron eigenvector.

Let $A_\varepsilon = A + \varepsilon E$ and denote by v_ε the corresponding normalized Perron eigenvector. Classical perturbation theory yields the first-order expansion

$$v_\varepsilon - v = (\rho I - A)^{-1} E v + o(\varepsilon). \quad (30)$$

Since ρ is a simple eigenvalue, the operator $\rho I - A$ is invertible on the invariant complement of the Perron eigenspace. On this subspace, the resolvent satisfies the bound

$$\|(\rho I - A)^{-1}\| \leq \frac{1}{\rho - |\lambda_2|}, \quad (31)$$

we obtain

$$\|v_\varepsilon - v\| \leq \frac{C}{\rho - |\lambda_2|} \|E\| |\varepsilon| + o(\varepsilon). \quad (32)$$

Using the definition of systemic resilience

$$\mathcal{R}(A) = 1 - \frac{|\lambda_2|}{\rho}, \quad (33)$$

this becomes

$$\|v_\varepsilon - v\| \leq \frac{C}{\rho \mathcal{R}(A)} \|E\| |\varepsilon| + o(\varepsilon), \quad (34)$$

which establishes the claimed robustness bound.

Economic Interpretation and Systemic Resilience.

The spectral ratio

$$\gamma = \frac{|\lambda_2|}{\rho(A)} \quad (35)$$

plays a central role in the dynamics of the subsidy allocation mechanism.

A smaller value of γ (equivalently, a larger spectral gap $\rho - |\lambda_2|$) implies faster exponential convergence toward the Perron equilibrium and reduced sensitivity to structural

perturbations of the cost matrix. In this regime, deviations from equilibrium decay rapidly, and the allocation remains stable under moderate variations in technological coefficients.

Conversely, when γ approaches one, convergence becomes slow and the Perron eigenvector becomes highly sensitive to parameter changes. The allocation mechanism is then dynamically sluggish and structurally fragile.

Definition (Systemic Resilience)

We define the systemic resilience of the interconnected industrial system as

$$\mathcal{R}(A) := 1 - \gamma = 1 - \frac{|\lambda_2|}{\rho(A)}. \quad (36)$$

The quantity $\mathcal{R}(A) \in (0, 1)$ measures the intrinsic stabilization capacity of the system:

- Larger $\mathcal{R}(A) \Rightarrow$ faster convergence and higher robustness,
- Smaller $\mathcal{R}(A) \Rightarrow$ slow stabilization and structural vulnerability.

Thus, systemic resilience is fully determined by the spectral structure of the positive cost matrix. Stability and fairness of the subsidy allocation mechanism are therefore not imposed conditions, but direct consequences of spectral separation properties.

5. Spectral Background

In this section, we briefly recall key results from the spectral theory of positive matrices that are used throughout the paper [20]. Let $A \in \mathbb{R}^{k \times k}$ be a positive matrix. We denote by $\rho(A)$ the spectral radius of the matrix A , defined by A ,

$$\rho(A) = \max\{|\lambda|; \lambda \text{ is an eigenvalue of } A\}. \quad (37)$$

Perron's Theorem

If $A > 0$, then $\rho(A)$ is an eigenvalue of A . The corresponding eigenspace is one-dimensional. Moreover, $\rho(A)$ has a positive eigenvector, and $\rho(A) > |\lambda|$ for every eigenvalue λ of A with $\lambda \neq \rho(A)$.

Wielandt's lemma is stated below for completeness.

Wielandt's lemma is used here primarily to ensure simplicity of the dominant eigenvalue and convergence of matrix powers under primitivity assumptions.

Wielandt's Lemma:

1. The spectral radius of the matrix A is a positive eigenvalue of A and admits a positive eigenvector.
2. The absolute value of any other eigenvalue of A is strictly smaller than the spectral radius.

For completeness, detailed proofs of these results are provided in Appendix A. The above results form the theoretical foundation of the subsidy allocation model developed in this paper.

6. Illustrative Example

To illustrate the theoretical results, we consider a stylized three-sector economic system (e.g., energy, manufacturing, and services), where the coefficients of the positive matrix represent inter-sector cost dependencies. Such structures are commonly used in input–output analysis and provide a realistic approximation of interconnected production systems. Although the example is not based on a specific dataset, its structure is consistent with widely used input–output representations of modern economies.

$$A = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.6 \end{pmatrix}. \quad (38)$$

All entries are positive; hence the matrix is primitive.

Spectral Quantities

Analytical computation yields:

$$\rho(A) \approx 1.02, \quad |\lambda_2| \approx 0.68. \quad (39)$$

Hence,

$$\gamma = \frac{|\lambda_2|}{\rho(A)} \approx 0.67, \quad (40)$$

and systemic resilience is

$$\mathcal{R}(A) = 1 - \gamma \approx 0.33. \quad (41)$$

This indicates moderate spectral separation and stable convergence.

Convergence Behavior

Starting from an arbitrary positive initial vector x_0 , we iterate (21).

The error $\|x_k - v\|_1$ decays approximately as

$$\|x_k - v\|_1 \sim C(0.67)^k, \quad (42)$$

confirming the exponential convergence rate predicted by the theory.

Perturbation Experiment

We now introduce a structural perturbation:

$$A_\varepsilon = A + \varepsilon E, \quad (43)$$

where E modifies only one interaction coefficient:

$$E_{13} = 1, E_{ij} = 0 \text{ otherwise}. \quad (44)$$

For $\varepsilon = 0.02$, the relative change in the Perron vector satisfies approximately

$$\frac{\|v_\varepsilon - v\|}{\|v\|} \approx \frac{1}{\rho(A)\mathcal{R}(A)} \|E\| \varepsilon, \quad (45)$$

which numerically confirms the inverse dependence on systemic resilience.

7. Limitations of the Model

The present framework is subject to several structural limitations that should be explicitly acknowledged.

First, the model is strictly linear. All interactions between production units are represented by fixed proportional coefficients, and the allocation mechanism is formulated as a linear dynamical system. Nonlinear feedback effects, threshold phenomena, or capacity constraints are not incorporated. In real industrial systems, such nonlinearities may significantly influence equilibrium structure and convergence behavior.

Second, the analysis assumes strict positivity and primitivity of the cost matrix. This guarantees uniqueness of the Perron equilibrium and exponential convergence. However, in practical applications, interaction matrices may be only nonnegative or reducible. In

such cases, multiple invariant subspaces or periodic dynamics may arise, and the stability properties established here no longer hold without modification.

Third, the model does not account for time-varying coefficients. Technological parameters and cost interactions are assumed constant, whereas real economic networks may evolve over time. Extensions to time-dependent or stochastic matrix models would therefore represent a natural direction for further research.

From a mathematical perspective, relaxing primitivity may lead to multiple invariant subspaces and non-unique equilibria. Extensions to stochastic or time-dependent matrices would require tools from random matrix theory and non-autonomous dynamical systems.

Finally, the allocation mechanism is interpreted as an abstract equilibrium principle rather than as an implementable policy instrument. Institutional, regulatory, and behavioral factors influencing subsidy distribution are not included in the mathematical formulation.

These limitations indicate that the present work should be viewed as a spectral baseline model, providing structural insight into stability and robustness in positive linear systems, rather than as a comprehensive economic forecasting framework.

8. Conclusions

In this paper, we investigated subsidy allocation mechanisms in interconnected industrial systems through the spectral theory of positive matrices. The allocation problem was formulated as a Perron eigenvalue problem for a primitive cost matrix, thereby establishing a direct link between equilibrium production structure and the dominant spectral component of the system.

Beyond the classical existence and uniqueness result provided by Perron–Frobenius theory, the main contribution of this work lies in the derivation of explicit quantitative stability estimates. We showed that the speed of convergence of the iterative allocation mechanism is governed by the spectral ratio between the dominant and subdominant eigenvalues. This establishes a direct relationship between spectral separation and dynamic stabilization.

Furthermore, we introduced a systemic resilience index based on the spectral gap and proved that the sensitivity of the equilibrium allocation under structural perturbations is inversely proportional to this resilience measure. In this way, convergence speed and robustness are unified within a single spectral framework. Stability, fairness, and structural robustness thus emerge as intrinsic consequences of the spectral geometry of the underlying positive matrix.

Numerical illustrations confirmed the theoretical predictions and demonstrated how variations in spectral separation affect convergence behavior and sensitivity properties. The results provide a mathematically rigorous and computationally tractable approach to analyzing stability in linear allocation systems.

While the present framework is intentionally linear and based on strict positivity assumptions, it provides a structural baseline for further generalizations. Future research may extend the analysis to reducible or time-varying systems, nonnegative matrices with weaker connectivity properties, or nonlinear allocation operators. Investigating the interaction between spectral properties and structural network topology also represents a promising direction.

Overall, the study highlights that spectral analysis offers not only qualitative equilibrium characterization, but also quantitative tools for assessing resilience and robustness in interconnected economic systems.

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Appendix A. Proofs of Spectral Results

In this appendix, we provide detailed proofs of Perron's theorem, Wielandt's lemma, and related spectral results used in the main text.

We begin by proving the first statement of Wielandt's lemma.

Suppose that μ is an eigenvalue of A with maximal absolute value. Then

$$|\mu| = \rho(A) \quad (\text{A1})$$

and

$$Ax = \mu x \quad (\text{A2})$$

for some complex vector x .

Define

$$p = [|x_1|, |x_2|, \dots, |x_k|]^{\text{tr}}, \quad (\text{A3})$$

and show that p is a positive eigenvector. For each $1 \leq i \leq k$ we have

$$\mu x_i = \sum_j a_{ij} |x_j|, \quad (\text{A4})$$

and hence

$$\rho(A)p_i = |\mu x_i| \leq \sum_j a_{ij} |x_j| = \sum_j a_{ij} p_j. \quad (\text{A5})$$

Therefore,

$$\rho(A)p \leq Ap \quad (\text{A6})$$

which implies

$$(A - \rho(A)I)p \geq 0. \quad (\text{A7})$$

We now show that this inequality is in fact an equality. We argue by contradiction. Let

$$z = (A - \rho(A)I)p \quad (\text{A8})$$

and assume that $z \neq 0$. Since $z \geq 0$ and the matrix A has strictly positive entries, it follows that

$$Az > 0. \quad (\text{A9})$$

Because

$$Ap > 0, \quad (\text{A10})$$

there exists a positive number ε such that

$$Az \geq \varepsilon Ap. \quad (\text{A11})$$

Now,

$$A(A - \rho(A)I)p = Az \quad (\text{A12})$$

$$A^2p - \rho(A)Ap = Az \quad (\text{A13})$$

$$A^2p = Az + \rho(A)Ap \quad (\text{A14})$$

$$\geq \varepsilon Ap + \rho(A)Ap \quad (\text{A15})$$

$$\geq (\varepsilon + \rho(A))Ap. \quad (\text{A16})$$

Let

$$B = (\varepsilon + \rho(A))^{-1}A. \quad (\text{A17})$$

Then

$$BAp \geq Ap. \quad (\text{A18})$$

Since B is a positive matrix, it follows that

$$B^n Ap \geq Ap \text{ for all } n \geq 1. \quad (\text{A19})$$

For any scalar τ and any matrix X , we have

$$\rho(\tau X) = |\tau|\rho(X). \quad (\text{A20})$$

Hence,

$$\rho(B) = |(\varepsilon + \rho(A))^{-1}|\rho(A). \quad (\text{A21})$$

Since $\varepsilon > 0$, it follows that

$$\rho(B) < 1. \quad (\text{A22})$$

Therefore, $\lim_{n \rightarrow \infty} (B)^n$.

Therefore $0 \geq Ap$, and we also know that $Ap > 0$. So $Ap = \rho(A)p$. This proves the first statement of Wielandt's lemma.

We now prove the second statement of Wielandt's lemma.

Suppose that λ is an eigenvalue such that

$$|\lambda| = \rho(A), \quad (\text{A23})$$

and let y denote a corresponding eigenvector. As before, define

$$q_i = |y_i|, i = 1, \dots, n. \quad (\text{A24})$$

Then we have

$$\rho(A)q_i = \left| \sum_j a_{1j}y_j \right| = \sum_j a_{1j}|y_j| = \sum_j a_{1j}q_j = (Aq)_1 \quad (\text{A25})$$

Since the same reasoning applies to all coordinates, it follows that

$$\rho(A)q = Aq. \quad (\text{A26})$$

Consequently,

$$\left| \sum_j a_{1j}y_j \right| = \sum_j a_{1j}|y_j|. \quad (\text{A27})$$

All numbers $a_{i1}, a_{i2}, \dots, a_{ik}$ are positive real numbers. Therefore, the arguments of the complex numbers y_j must be equal. This implies that each y_j admits a polar representation

$$y_j = |y_j|e^{i\theta}, \quad (\text{A28})$$

where the angle θ is independent of j . Hence,

$$y_j = e^{i\theta}q. \quad (\text{A29})$$

Thus, y is a scalar multiple of the nonnegative eigenvector corresponding to the eigenvalue $\rho(A)$. Since y is an eigenvector associated with the eigenvalue λ , it follows that $\lambda = \rho(A)$. This completes the proof of the second statement of Wielandt's lemma.

Theorem A1. *Wielandt's Lemma.*

The sequence of powers of the matrix $\rho(A)^{-1}A$ is convergent.

We now prove Theorem 2.

Let $B = \rho(A)^{-1}A$. Then $\rho(B) = 1$. Wielandt's lemma applies to B , and we conclude that 1 is an eigenvalue of B and that $|\lambda| < 1$ for every eigenvalue $\lambda \neq 1$.

We now show that the sequence of powers of $\rho(A)^{-1}A$ is bounded.

Let y be a positive eigenvector associated with the eigenvalue 1. Then

$$B^n y = y \text{ for all } n. \quad (\text{A30})$$

Set $y_s = \max_i y_i$ and $y_t = \min_i y_i$. For all i, j we have

$$y_s \geq y_i = \sum_l b_{il}^{(n)} y_l \geq b_{ij}^{(n)} y_j \geq b_{ij}^{(n)} y_t \quad (\text{A31})$$

and hence

$$b_{ij}^{(n)} < \frac{y_s}{y_t} \text{ for all } n. \quad (\text{A32})$$

Since $b_{ij}^{(n)} > 0$ for all n , it follows that the sequence $\rho(A)^{-1}A$ is indeed bounded.

Finally, because $|\lambda| < 1$ for all eigenvalues $\lambda \neq 1$, the sequence of matrix powers $\rho(A)^{-1}A$ is convergent.

This completes the proof of Theorem 1.

Corollary

The spectral radius $\rho(A)$ is a simple eigenvalue of A .

Let us prove that the corollary is true.

We show that the corollary holds. In the proof of the previous theorem, we observed that for the matrix

$$B = \rho(A)^{-1}A, \quad (\text{A33})$$

the geometric multiplicity of the eigenvalue 1 coincides with its algebraic multiplicity, and that $|\lambda_i| < 1$ for $i = 1, 2, \dots, l$, where $\lambda_1, \lambda_2, \dots, \lambda_l$ are all remaining eigenvalues of B .

There exists a vector $p > 0$ such that $Bp = p$, and therefore the algebraic multiplicity of the eigenvalue 1 is at least one. Denote this multiplicity by n . Suppose that $n > 1$. Then there exists a real vector q such that $Bq = q$ and q is not a scalar multiple of p .

Define

$$\tau = \max_i \frac{q_i}{p_i} \quad (\text{A34})$$

and choose an index j such that $\frac{q_j}{p_j} = \tau$. Then $\tau p \geq q$. Since

$$B(\tau p - q) > 0, \quad (\text{A35})$$

we have

$$\tau p - q \neq 0. \quad (\text{A36})$$

Consequently,

$$\tau Bp - Bq > 0, \quad (\text{A37})$$

which implies

$$\tau p - q > 0. \quad (\text{A38})$$

In particular, $\tau p_j > q_j$, which contradicts the choice of τ .

Hence, $n = 1$, and the eigenvalue 1 of B is simple. It follows that $\rho(A)$ is a simple eigenvalue of A .

This completes the proof of the corollary.

The proof of Perron's theorem consists of Wielandt's lemma together with this corollary.

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