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Generalized B-Curvature Tensor in Lorentzian Para-Kenmotsu Manifold with Semi-Symmetric Metric Connection

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Abstract

The main object of this work is to study the generalized B-curvature tensor in an n-dimensional Lorentzian para-Kenmotsu (briefly, $(LPK)_n$) manifold along a semi-symmetric metric connection $\bar{\nabla}$. First, in an $(LPK)_n$-manifold, we explore certain flatness conditions, namely, $\bar{B}(Y,Z)X = 0$, $\bar{B}(Y,Z)\zeta = 0$, $g(\bar{B}(\varphi Y, \varphi Z)\varphi X, \varphi W) = 0$, and $\bar{B}(Y,Z) \cdot \varphi = 0$ conditions, which all result in an η-Einstein manifold. Furthermore, in an $(LPK)_n$-manifold, we study the curvature conditions $\bar{B}.Q = 0$ and $\bar{B}.\bar{Q} = 0$, which provide the scalar curvature. The generalized B-curvature tensor blends the features of different curvature tensors, allowing researchers to study conditions like semi-symmetry, pseudo-symmetry in a unified framework. Conditions like B-semi-symmetry correspond to conservation laws or stability properties in physical systems.

Keywords: generalized B-curvature tensor; Lorentzian para-Kenmotsu manifold; Ricci tensor; curvature tensor; linear connection

MSC: 53C15; 53C25

1. Introduction

In [1], Shaikh and Kundu introduced the equivalency of various geometric structures obtained by same restriction imposed on different curvature tensors. For this aim, they defined a (0, 4)-type tensor, which is linear combination of the Riemann-Christoffel curvature tensor, Ricci tensor, metric tensor, and scalar curvature, and describes various curvature tensors as its particular cases. The set of all B-tensors are denoted by $\mathscr{B}$.

The generalized B-curvature tensor is a refinement of classical curvature tensors that allows for deeper exploration of geometric structures. Standard tensors (like Riemann, Ricci, and Weyl) often fail to distinguish subtle geometric properties in $(LPK)_n$-manifold. The generalized B-tensor introduces additional flexibility to study vanishing, semi-symmetric, and more. Existing studies on $(LPK)_n$-manifold emphasize Ricci solitons or Weyl curvature conditions.

The generalized B-curvature tensor introduces new geometric invariants and conditions that were not previously studied, making it a novel tool for both classification and physical interpretation. Think of the manifold like a fabric stretched evenly in all directions (Einstein case). In the η-Einstein case, one thread (the ζ-direction) is woven tighter or looser, giving the fabric a special anisotropy. That "thread" controls how the geometry bends

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differently along that axis. The stated results in Sections 4.1–4.3 and 5 that "η-Einstein manifold" means the manifold belongs to a special family where curvature is uniform, except in one distinguished direction. That makes the geometry easier to study and gives it potential physical interpretations.

The idea of almost para-contact manifolds was introduced by Sato [2]. According to Kaneyuki and his collaborator [3], the main variation among an almost para-contact manifold is the signature of metric. In 1989, Matsumoto [4] used a structure vector field 'ζ' instead of 'ζ' in an almost para contact manifold and associated a Lorentzian metric with this resulting structure, and called it a Lorentzian almost para contact manifold. The "para-Kenmotsu" condition encodes a specific interaction between the metric and a para-contact structure, giving a controlled way to study curvature and hypersurfaces in a pseudo-Riemannian setting. Afterwards, para-Kenmotsu manifolds were a great focus of geometers and brought forward the significant characteristics of such manifolds [5–9].

In 1924, Friedmann and Schouten proposed the concept of a semi-symmetric linear connection on differentiable manifold. A semi-symmetric metric connection preserves the metric but allows torsion, specifically tied to a 1-form. In [10], the authors studied a semi-symmetric metric connection on submanifolds of a Riemannian manifold. Certain properties of a semi-symmetric metric connection on a Riemannian manifold have been studied by De and De [11]. Recently, the authors Haseeb and Prasad investigated Kenmotsu and Lorentzian para-Sasakian manifolds with a semi-symmetric metric connection by satisfying certain curvature conditions [12,13]. The semi-symmetric metric connection on indefinite Kenmotsu manifold have been studied by the authors Kumar et al. [14]. In 2009, the authors Shukla et. al. [15] studied φ-Ricci symmetric Kenmotsu and illustrated some of the results through an example.

This paper is organized as follows: Section 1 covers the introduction, corresponding concepts, and brief histories. Section 2 contains preliminaries, where some fundamental results are given, which are used in subsequent sections. In Section 3, the generalized B-curvature tensor of $(LPK)_n$-manifold with connection $\bar{\nabla}$ is described. In Section 4, we explore certain curvature conditions on $(LPK)_n$-manifolds, namely, $\bar{B}(Y,Z)X = 0$, $\bar{B}(Y,Z)\zeta = 0$, and $g(\bar{B}(\varphi Y, \varphi Z)\varphi X, \varphi W) = 0$. In Section 5, we study φ-$\bar{B}$ semi-symmetric $(LPK)_n$. In a $(LPK)_n$, the curvature conditions $\bar{B}.Q = 0$ and $\bar{B}.\bar{Q} = 0$ have been studied in Sections 6 and 7, respectively.

2. Preliminaries

Let M be a n-dimensional Lorentzian metric manifold. If it is endowed with a structure $(\varphi, \zeta, \eta, g)$, where φ is a (1,1) tensor field, ζ is a vector field, η is a 1-form on M, and g is a Lorentz metric,fulfilling [16]

$$\varphi^2 Y = Y + \eta(Y)\zeta, \qquad g(\varphi Y, \varphi Z) = g(Y,Z) + \eta(Y)\eta(Z), \tag{1}$$

$$\eta(\zeta) = -1, \qquad g(Y,\zeta) = \eta(Y), \tag{2}$$

for any vector fields Y, Z on M, then it is called a Lorentzian almost para-contact manifold. In a Lorentzian almost para-contact manifold, the coming relations are valid:

$$\varphi\zeta = 0, \quad \eta(\varphi Y) = 0, \tag{3}$$

$$\Phi(Y,Z) = \Phi(Z,Y) \tag{4}$$

here $\Phi(Y,Z) = g(Y, \varphi Z)$.

Definition 1. *A Lorentzian almost para-contact manifold M is called a Lorentzian para-Kenmotsu (briefly, $(LPK)_n$) manifold if [17,18]*

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$$(\nabla_Y\varphi)Z = -g(\varphi Y, Z)\zeta - \eta(Z)\varphi Y, \tag{5}$$

for any vector fields Y, Z on $(LPK)_n$-manifold.
In an $(LPK)_n$-manifold, we have

$$\nabla_Y\zeta = -Y - \eta(Y)\zeta, \tag{6}$$

$$(\nabla_Y\eta)Z = -g(Y, Z) - \eta(Y)\eta(Z), \tag{7}$$

where ∇ indicates the operator of covariant differentiation respecting to the Lorentzian metric g. Further, in an $(LPK)_n$-manifold, the following relations are valid [17–19]:

$$g(R(Y, Z)X, \zeta) = \eta(R(Y, Z)X) = g(Z, X)\eta(Y) - g(Y, X)\eta(Z), \tag{8}$$

$$R(\zeta, Y)Z = g(Y, Z)\zeta - \eta(Z)Y, \tag{9}$$

$$R(Y, Z)\zeta = \eta(Z)Y - \eta(Y)Z, \tag{10}$$

$$R(\zeta, Y)\zeta = Y + \eta(Y)\zeta, \tag{11}$$

$$S(Y, \zeta) = (n-1)\eta(Y), S(\zeta, \zeta) = -(n-1), \tag{12}$$

$$r = n(n-1), Q\zeta = (n-1)\zeta, \tag{13}$$

$$S(\varphi Y, \varphi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z), \tag{14}$$

for any vector fields Y, Z and X on M, where S, R and Q denotes the Ricci tensor, the curvature tensor, and the Ricci operator on $(LPK)_n$.

Definition 2. *The generalized B-curvature tensor on a Riemannian (or semi-Riemannian) manifold is given by [1]*

$$\begin{aligned} B(Y, Z)X &= a_0R(Y, Z)X + a_1[S(Z, X)Y - S(Y, X)Z \\ &+ g(Z, X)QY - g(Y, X)QZ] + 2a_2r[g(Z, X)Y - g(Y, X)Z], \end{aligned} \tag{15}$$

where a_0, a_1, a_2 are scalars. Also, see [20–22].

From (15), we have

$$\begin{aligned} B(\zeta, Y)Z &= (a_0 + a_1(n-1) + 2a_2r)[g(Y, X)\zeta - \eta(X)Y] \\ &+ a_1(S(Y, X)\zeta - \eta(X)QY) \end{aligned} \tag{16}$$

$$\begin{aligned} B(Y, \zeta)X &= (a_0 + a_1(n-1) + 2a_2r)[\eta(X)Y - g(Y, X)\zeta] \\ &+ a_1(\eta(X)QY - S(Y, X)\zeta), \end{aligned} \tag{17}$$

$$\begin{aligned} B(Y, Z)\zeta &= (a_0 + a_1(n-1) + 2a_2r)[\eta(Z)Y - \eta(Y)Z] \\ &+ a_1(\eta(Z)QY - \eta(Y)QZ). \end{aligned} \tag{18}$$

These equations will be applied in subsequent sections.

Definition 3. *A linear connection $\bar{\nabla}$ on M is called a semi-symmetric metric connection if its torsion tensor $\bar{T}$ satisfies*

$$\bar{T}(Y, Z) = \bar{\nabla}_Y Z - \bar{\nabla}_Z Y - [Y, Z],$$

satisfies

$$\bar{T}(Y,Z) = \eta(Z)Y - \eta(Y)Z,$$

for all $Y, Z \in \chi(M)$*, the torsion tesnor is skew-symmetric and expressible as the tensor product of 1-form and a vector-field. Here,* $\chi(M)$ *is the set of differentiable vector fields on M.*

The connection $\bar{\nabla}$ is called a semi-symmetric metric connection [23], if $\bar{\nabla}_Y g = 0$. A relation between the connections $\bar{\nabla}$ and ∇ is given by

$$\bar{\nabla}_Y Z = \nabla_Y Z + \eta(Z)Y - g(Y,Z)\zeta, \tag{19}$$

here, ∇ represents the Levi-Civita connection.

In an $(LPK)_n$-manifold, we have

$$\bar{\nabla}_X \xi = -2X - 2\eta(X)\xi, \quad (\bar{\nabla}_X \eta)Y = -2g(X,Y) - 2\eta(X)\eta(Y). \tag{20}$$

The Riemannian Christoffel curvature tensor with a connection $\bar{\nabla}$ is given by

$$\bar{R}(Y,Z)X = \bar{\nabla}_Y \bar{\nabla}_Z X - \bar{\nabla}_Z \bar{\nabla}_Y X - \bar{\nabla}_{[Y,Z]} X. \tag{21}$$

Using relation (19) in (21), we have

$$\begin{aligned} \bar{R}(Y,Z)X &= R(Y,Z)X + 3g(Z,X)Y - 3g(Y,X)Z + 2\eta(Z)\eta(X)Y \\ &- 2\eta(Y)\eta(X)Z + 2g(Z,X)\eta(Y)\zeta - 2g(Y,X)\eta(Z)\zeta, \end{aligned} \tag{22}$$

$$\bar{R}(\zeta,Z)X = 2g(Z,X)\zeta - 2\eta(X)Z, \tag{23}$$

$$\bar{R}(Y,\zeta)X = 2\eta(X)Y - 2g(Y,X)\zeta, \tag{24}$$

$$\bar{R}(Y,Z)\zeta = 2\eta(Z)Y - 2\eta(Y)Z. \tag{25}$$

Let $\{e_i\}, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $Y = W = e_i$ and taking summation over i, we have

$$\begin{aligned} \sum_{i=1}^{n} g(\bar{R}(e_i,Z)X,e_i) &= \sum_{i=1}^{n} [g(R(e_i,Z)X,e_i) + 3g(Z,X)g(e_i,e_i) - 3g(e_i,X)g(Z,e_i) \\ &+ 2\eta(Z)\eta(X)g(e_i,e_i) - 2g(e_i,\zeta)\eta(X)g(Z,e_i) \\ &+ 2g(Z,X)g(e_i,\zeta)g(\zeta,e_{i,}) - 2g(e_i,X)\eta(Z)g(\zeta,e_i)]. \end{aligned}$$

On simplification, the above relation gives

$$\bar{S}(Z,X) = S(Z,X) + (3n-5)g(Z,X) + 2(n-2)\eta(Z)\eta(X). \tag{26}$$

By contracting (26) over X and Z, we have

$$\sum_{i=1}^{n} \bar{S}(e_i,e_i) = \sum_{i=1}^{n} [S(e_i,e_i) + (3n-5)g(e_i,e_i) + 2(n-2)g(e_i,\zeta)g(e_i,\zeta)],$$

which gives

$$\bar{r} = r + 3n^2 - 7n + 4. \tag{27}$$

Putting $X = \zeta$ in (26) and using (12), we have

$$\bar{S}(Z,\zeta) = 2(n-1)\eta(Z). \tag{28}$$

Also, from (26) it follows that

$$\bar{Q}Z = QZ + (3n-5)Z + 2(n-2)\eta(Z)\zeta. \tag{29}$$

Here, $\bar{S}$, $\bar{R}$, $\bar{r}$, and $\bar{Q}$ denote the Ricci tensor, the curvature tensor, the scalar curvature, and the Ricci operator with respect to the connection $\bar{\nabla}$ on the $(LPK)_n$-manifold.

Definition 4. *An $(LPK)_n$-manifold is called an η-Einstein manifold, if its $S(\neq 0)$ is of the form*

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

here, a and b are scalar functions on M. If $b = 0$, then the manifold becomes an Einstein manifold [24].

3. Generalized *B*-Curvature Tensor in (LPK)$_n$-Manifold with Semi-Symmetric Metric Connection

In this section, we deal with the generalized B-curvature tensor with semi-symmetric metric connection in the $(LPK)_n$-manifold.
For this purpose, we derive the generalized B-curvature tensor [25,26] with the connection $\bar{\nabla}$ as

$$\begin{aligned}\bar{B}(Y,Z)X &= a_0\bar{R}(Y,Z)X + a_1[\bar{S}(Z,X)Y - \bar{S}(Y,X)Z + g(Z,X)\bar{Q}Y \\ &\quad - g(Y,X)\bar{Q}Z] + 2a_2\bar{r}[g(Z,X)Y - g(Y,X)Z],\end{aligned}$$

Now, using (22), (26), (27), and (29) in the above relation, we obtain

$$\begin{aligned}\bar{B}(Y,Z)X &= a_0R(Y,Z)X + a_1[S(Z,X)Y - S(Y,X)Z + g(Z,X)QY \\ &- g(Y,X)QZ] + (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4)) \\ &\quad \times[g(Z,X)Y - g(Y,X)Z] + 2(a_0 + (n-2)a_1)[\eta(Z)\eta(X)Y \\ &- \eta(Y)\eta(X)Z + g(Z,X)\eta(Y)\zeta - g(Y,X)\eta(Z)\zeta].\end{aligned} \tag{30}$$

From (30), we can easily find

$$\begin{aligned}\bar{B}(\zeta,Z)X &= (2a_0 + (5n-7)a_1 + 2a_2(r + 3n^2 - 7n + 4))[g(Z,X)\zeta - \eta(X)Z] \\ &+ a_1[S(Z,X)\zeta - \eta(X)QZ],\end{aligned} \tag{31}$$

$$\begin{aligned}\bar{B}(Y,\zeta)X &= (2a_0 + (5n-7)a_1 + 2a_2(r + 3n^2 - 7n + 4))[\eta(X)Y - g(Y,X)\zeta] \\ &+ a_1[\eta(X)QY - S(Y,X)\zeta],\end{aligned} \tag{32}$$

$$\begin{aligned}\bar{B}(Y,Z)\zeta &= (2a_0 + (5n-7)a_1 + 2a_2(r + 3n^2 - 7n + 4))[\eta(Z)Y - \eta(Y)Z] \\ &+ a_1[\eta(Z)QY - \eta(Y)QZ].\end{aligned} \tag{33}$$

4. Certain Flatness Conditions on (LPK)$_n$-Manifold

This section deals with the study of certain flatness conditions in the framework of the $(LPK)_n$-manifold.

4.1. Generalized $\bar{B}$-Flat (LPK)$_n$-Manifold

Definition 5. *An $(LPK)_n$-manifold is said to be generalized $\bar{B}$-flat if*

$$\bar{B}(Y,Z)X = 0, \tag{34}$$

for any Y, Z, X on $(LPK)_n$.

Let an $(LPK)_n$-manifold be generalized $\bar{B}$-flat, i.e., $\bar{B}(Y,Z)X = 0$. Then, (30) takes the form

$$\begin{aligned} a_0 R(Y,Z)X + a_1[S(Z,X)Y - S(Y,X)Z + g(Z,X)QY \\ - g(Y,X)QZ] + (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4)) \\ [g(Z,X)Y - g(Y,X)Z] + 2(a_0 + (n-2)a_1)[\eta(Z)\eta(X)Y \\ - \eta(Y)\eta(X)Z + g(Z,X)\eta(Y)\zeta - g(Y,X)\eta(Z)\zeta] = 0. \end{aligned} \tag{35}$$

Applying the inner product on (35) with W, we have

$$\begin{aligned} a_0 g(R(Y,Z)X,W) + a_1[S(Z,X)g(Y,W) - S(Y,X)g(Z,W) + g(Z,X)S(Y,W) \\ - g(Y,X)S(Z,W)] + (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4))[g(Z,X)g(Y,W) \\ - g(Y,X)g(Z,W)] + 2(a_0 + (n-2)a_1)[\eta(Z)\eta(X)g(Y,W) - \eta(Y)\eta(X)g(Z,W) \\ + \eta(Y)\eta(W)g(Z,X) - \eta(Z)\eta(W)g(Y,X)] = 0. \end{aligned}$$

Let $e_i, i = 1,2,...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $Y = W = e_i$ and taking summation over i, we have

$$\begin{aligned} a_0 \sum_{i=1}^{n} g(R(e_i,Z)X,e_i) + a_1 \sum_{i=1}^{n} [S(Z,X)g(e_i,e_i) - S(e_i,X)g(Z,e_i) + g(Z,X)S(e_i,e_i) \\ - g(e_i,X)S(Z,e_i)] + (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4)) \sum_{i=1}^{n} [g(Z,X)g(e_i,e_i) \\ - g(e_i,X)g(Z,e_i)] + 2(a_0 + (n-2)a_1) \sum_{i=1}^{n} [\eta(Z)\eta(X)g(e_i,e_i) - g(e_i,\zeta)\eta(X)g(Z,e_i) \\ + g(e_i,\zeta)g(e_i,\zeta)g(Z,X) - \eta(Z)g(e_i,\zeta)g(e_i,X)] = 0, \end{aligned}$$

which after some steps calculations gives

$$S(Z,X) = \lambda_1 g(Z,X) + \lambda_2 \eta(Z)\eta(X), \tag{36}$$

where $\lambda_1 = -\frac{[(3n-5)a_0 + 2(3n^2-9n+7)a_1 + (a_1 + 2(n-1)a_2)(r+3n^2-7n+4)]}{a_0 + a_1(n-2)}$ and $\lambda_2 = -2(n-2)$. Hence, we state the following theorem:

Theorem 1. *An $(LPK)_n$-manifold along with semi-symmetric metric connection satisfying condition generalized $\bar{B}$-flat, then the manifold is an η-Einstein manifold of the form* (36)*, provided $a_0 + a_1(n-2) \neq 0$ (i.e., $\bar{B}$ defines a non-degenerate curvature structure).*

The η-Einstein condition essentially decomposes the Ricci tensor into two geometrically meaningful components: one proportional to the metric (isotropic curvature) and another proportional to the tensor product of the contact 1-form η (anisotropic curvature) with itself.

4.2. Generalized ζ-$\bar{B}$-Flat $(LPK)_n$-Manifold

In this subsection, we study generalized ζ-$\bar{B}$-flat $(LPK)_n$-manifold, i.e., $\bar{B}(Y,Z)\zeta = 0$. Thus, it follows from (30) that

$$\begin{aligned} a_0 R(Y,Z)\zeta + a_1[S(Z,\zeta)Y - S(Y,\zeta)Z + g(Z,\zeta)QY - g(Y,\zeta)QZ] \\ +(3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4))[g(Z,\zeta)Y - g(Y,\zeta)Z] \\ +2(a_0 + (n-2)a_1)[\eta(Z)\eta(\zeta)Y - \eta(Y)\eta(\zeta)Z] = 0. \end{aligned} \tag{37}$$

Taking the inner product of (37) with W and using (2), (10) and (12), we have

$$\begin{aligned}(2a_0+(5n-7)a_1+2a_2(r+3n^2-7n+4))[\eta(Z)g(Y,W)-\eta(Y)g(Z,W)]\\ +a_1[\eta(Z)S(Y,W)-\eta(Y)S(Z,W)]=0.\end{aligned} \tag{38}$$

Substitute $Z=\zeta$ in (38), we lead to

$$\begin{aligned}S(Y,W)=&-\frac{[2a_0+(5n-7)a_1+2a_2(r+3n^2-7n+4)]}{a_1}g(Y,W)\\ &-\frac{2[a_0+(3n-4)a_1+a_2(r+3n^2-7n+4)]}{a_1}\eta(Y)\eta(W),\ a_1\neq 0.\end{aligned} \tag{39}$$

Hence, we state the following theorem:

Theorem 2. *An $(LPK)_n$-manifold along with semi-symmetric connection satisfying condtion generelized ζ-$\bar{B}$-flat, then manifold is an η-Einstein manifold of the form (39), provided $a_1\neq 0$.*

Next, from (30), we have

$$\begin{aligned}\bar{B}(Y,Z)\zeta\ =\ &a_0R(Y,Z)\zeta+a_1[S(Z,\zeta)Y-S(Y,\zeta)Z+g(Z,\zeta)QY-g(Y,\zeta)QZ]\\ &+[3a_0+2(3n-5)a_1+2a_2(r+3n^2-7n+4)-2(a_0+(n-2)a_1)]\\ &\times[\eta(Z)Y-\eta(Y)Z],\end{aligned}$$

which can be written as

$$\bar{B}(Y,Z)\zeta=B(Y,Z)\zeta+[a_0+(4n-6)a_1+2(3n^2-7n+4)a_2]\eta(Y)\eta(Z), \tag{40}$$

where

$$\begin{aligned}B(Y,Z)\zeta\ =\ &a_0R(Y,Z)\zeta+a_1[S(Z,\zeta)Y-S(Y,\zeta)Z+g(Z,\zeta)QY-g(Y,\zeta)QZ]\\ +\ &2a_2r(\eta(Z)Y-\eta(Y)Z).\end{aligned}$$

If the scalars a_0,a_1,a_2 are related by $a_0+2(2n-3)a_1+2(n-1)(3n-4)a_2=0$, then (40) reduces to $\bar{B}(Y,Z)\zeta=B(Y,Z)\zeta$.

Thus, we have the following corollary:

Corollary 1. *An $(LPK)_n$-manifold is generalized ζ-B-flat with respect to the semi-symmetric connection if and only if the manifold is also generelized ζ-B-flat with respect to the Levi-Civita connection, provided $a_0+2(2n-3)a_1+2(n-1)(3n-4)a_2=0$.*

4.3. φ-Generalized $\bar{B}$-Flat $(LPK)_n$-Manifold

In this subsection, we study φ-generalized $\bar{B}$-flat $(LPK)_n$-manifold, i.e., $g(\bar{B}(\varphi Y,\varphi Z)\varphi X,\varphi W)=0$. Thus, in account of (30), we have

$$\begin{aligned}&a_0g(R(\varphi Y,\varphi Z)\varphi X,\varphi W)+a_1[S(\varphi Z,\varphi X)g(\varphi Y,\varphi W)-S(\varphi Y,\varphi X)g(\varphi Z,\varphi W)\\ &+g(\varphi Z,\varphi X)S(\varphi Y,\varphi W)-g(\varphi Y,\varphi X)S(\varphi Z,\varphi W)]\\ &+(3a_0+2(3n-5)a_1+2a_2(r+3n^2-7n+4))[g(\varphi Z,\varphi X)g(\varphi Y,\varphi W)\\ &-g(\varphi Y,\varphi X)g(\varphi Z,\varphi W)]\\ &+2(a_0+(n-2)a_1)[\eta(\varphi Z)\eta(\varphi X)g(\varphi Y,\varphi W)-\eta(\varphi Y)\eta(\varphi X)g(\varphi Z,\varphi W)\\ &+\eta(\varphi Y)\eta(\varphi W)g(\varphi Z,\varphi X)-\eta(\varphi Z)\eta(\varphi W)g(\varphi Y,\varphi X)]=0.\end{aligned}$$

In view of (1) and (14), the above expression takes the form

$$
\begin{aligned}
&a_0 g(R(\varphi Y, \varphi Z)\varphi X, \varphi W) + a_1[S(Z,X)g(Y,W) - S(Y,X)g(Z,W) \\
&+g(Z,X)S(Y,W) - g(Y,X)S(Z,W) + S(Z,X)\eta(Y)\eta(W) - S(Y,X)\eta(Z)\eta(W) \\
&+S(Y,W)\eta(Z)\eta(X) - S(Z,W)\eta(Y)\eta(X) + (n-1)g(Y,W)\eta(Z)\eta(X) \\
&-(n-1)g(Z,W)\eta(Y)\eta(X) + (n-1)g(Z,X)\eta(Y)\eta(W) - (n-1)g(Y,X)\eta(Z)\eta(W)] \\
&+(3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4))[g(Z,X)g(Y,W) - g(Y,X)g(Z,W) \\
&+g(Y,W)\eta(Z)\eta(X) + g(Z,X)\eta(Y)\eta(W) - g(Z,W)\eta(Y)\eta(X) - g(Y,X)\eta(Z)\eta(W)] = 0.
\end{aligned}
$$

Let $e_i, i = 1, 2, ...n$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $Y = W = e_i$ and taking summation over i, we have

$$
\begin{aligned}
a_0 \sum_{i=1}^{n} g(R(\varphi e_i, \varphi Z)\varphi X, \varphi e_i) + a_1 \sum_{i=1}^{n} [S(Z,X)g(e_i,e_i) - S(e_i,X)g(Z,e_i) \\
+g(Z,X)S(e_i,e_i) - g(e_i,X)S(Z,e_i) + S(Z,X)g(e_i,\zeta)g(e_i,\zeta) \\
+S(e_i,e_i)\eta(Z)\eta(X) - S(Z,e_i)g(e_i,\zeta)\eta(X) + (n-1)g(e_i,e_i)\eta(Z)\eta(X) \\
-(n-1)g(Z,e_i)g(e_i,\zeta)\eta(X) + (n-1)g(Z,X)g(e_i,\zeta)g(e_i,\zeta) \\
-(n-1)g(e_i,X)\eta(Z)g(e_i,\zeta)] + (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4)) \\
\sum_{i=1}^{n}[g(Z,X)g(e_i,e_i) - g(e_i,X)g(Z,e_i) + g(e_i,e_i)\eta(Z)\eta(X) \\
+g(Z,X)g(e_i,\zeta)g(e_i,\zeta) - g(Z,e_i)g(e_i,\zeta)\eta(X) - g(e_i,X)\eta(Z)g(e_i,\zeta)] = 0.
\end{aligned}
$$

This gives

$$
\begin{aligned}
(a_0 + (n-3)a_1)S(Z,X) &= -[(3n-7)(a_0 + (2n-3)a_1) + 2(n-2)(3n^2 - 7n + 4)a_2 \\
&+ (a_1 + 2(n-2)a_2)r]g(Z,X) - [4(n-2)a_0 + (7n^2 - 27n + 24)a_1 \\
&+ 2(n-2)(3n^2 - 7n + 4)a_2 + (a_1 + 2(n-2)a_2)r]\eta(Z)\eta(X),
\end{aligned}
$$

which is of the form

$$S(Z,X) = \lambda_1 g(Z,X) + \lambda_2 \eta(Z)\eta(X), \tag{41}$$

where

$$\lambda_1 = -\frac{(3n-7)(a_0 + (2n-3)a_1) + 2(n-2)(3n^2 - 7n + 4)a_2 + (a_1 + 2(n-2)a_2)r}{a_0 + (n-3)a_1},$$

and

$$\lambda_2 = -\frac{4(n-2)a_0 + (7n^2 - 27n + 24)a_1 + 2(n-2)(3n^2 - 7n + 4)a_2 + (a_1 + 2(n-2)a_2)r}{a_0 + (n-3)a_1}.$$

Thus, we state the following result:

Theorem 3. *An $(LPK)_n$-manifold along with a semi-symmetric metric connection is φ-generalized $\bar{B}$-flat , then the manifold is an η-Einstein manifold of the form (41), provided $a_0 + a_1(n-3) \neq 0$.*

Definition 6. *An (LP-K)$_n$-manifold is called φ-Ricci-symmetric with a semi-symmetric metric connection $\bar{\nabla}$ if*

$$\varphi^2((\bar{\nabla}_X Q)(Y)) = 0, \tag{42}$$

for any X, Y on (LP-K)$_n$-manifold [27]. In case, X, Y are orthogonal to ζ, then (LP-K)$_n$ is named locally φ-Ricci-symmetric.

From (41), it follows that

$$QZ = \lambda_1 Z + \lambda_2 \eta(Z)\xi. \tag{43}$$

Differentiating covariantly (43) along Y w. r. t. $\bar{\nabla}$, we have

$$\begin{aligned}(\bar{\nabla}_Y Q)Z + Q(\bar{\nabla}_Y Z) &= \lambda_1 \bar{\nabla}_Y Z - \frac{(a_1 + 2(n-2)a_2)dr(Y)}{a_0 + (n-1)a_1}(Z + \eta(Z)\xi) \\ &+ \lambda_2[(\bar{\nabla}_Y \eta)(Z)\xi + \eta(\bar{\nabla}_Y Z)\xi + \eta(Z)\bar{\nabla}_Y \xi]. \end{aligned} \tag{44}$$

By virtue of (43), (44) takes the form

$$\begin{aligned}(\bar{\nabla}_Y Q)Z &= -\frac{(a_1 + 2(n-2)a_2)dr(Y)}{a_0 + (n-1)a_1}(Z + \eta(Z)\xi) \\ &+\lambda_2[(\bar{\nabla}_Y \eta)(Z)\xi + \eta(Z)\bar{\nabla}_Y \xi]. \end{aligned} \tag{45}$$

By using (20) in (45), we have

$$\begin{aligned}(\bar{\nabla}_Y Q)Z &= -\frac{(a_1 + 2(n-2)a_2)dr(Y)}{a_0 + (n-1)a_1}(Z + \eta(Z)\xi) \\ &-2\lambda_2[g(Y,Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi]. \end{aligned} \tag{46}$$

By operating φ^2 on both sides of (46) and using (1), (3), we arrive at

$$\varphi^2((\bar{\nabla}_Y Q)Z) = -\frac{(a_1 + 2(n-2)a_2)dr(Y)}{a_0 + (n-3)a_1}[Z + \eta(Z)\xi] - 2\lambda_2 \eta(Z)[Y + \eta(Y)\xi]. \tag{47}$$

If Z is orthogonal to ξ, then (47) provides

$$\varphi^2((\bar{\nabla}_Y Q)Z) = -\frac{(a_1 + 2(n-2)a_2)dr(Y)}{a_0 + (n-3)a_1} Z. \tag{48}$$

If r is constant, then from (48), it follows that

$$\varphi^2((\bar{\nabla}_Y Q)Z) = 0. \tag{49}$$

Thus, we have the following corollary:

Corollary 2. *A φ-generalized $\bar{B}$-flat $(LPK)_n$-manifold of a constant scalar curvature with a semi-symmetric metric connection $\bar{\nabla}$ is locally φ-Ricci-symmetric.*

5. φ-Generalized $\bar{B}$-Semi-Symmetric (LPK)$_n$-Manifold

Definition 7. *An $(LPK)_n$-manifold is said to be φ-generalized $\bar{B}$-semi-symmetric if*

$$\bar{B}(Y,Z) \cdot \varphi = 0, \tag{50}$$

for any Y, Z on M. See [28].

In this section, we study a φ-generalized $\bar{B}$-semi-symmetric (LPK)$_n$-manifold, i.e., $\bar{B}(Y,Z) \cdot \varphi = 0$. This implies that

$$(\bar{B}(Y,Z) \cdot \varphi)X = \bar{B}(Y,Z)\varphi X - \varphi \bar{B}(Y,Z)X = 0. \tag{51}$$

Making use of (30) in (51), we have

$$\begin{aligned} &a_0 R(Y,Z)\varphi X + a_1[S(Z,\varphi X)Y - S(Y,\varphi X)Z + g(Z,\varphi X)QY - g(Y,\varphi X)QZ] \\ &+ (3a_0 + 2(3n-5)a_1 + 2a_2(r + 3n^2 - 7n + 4))[g(Z,\varphi X)Y - g(Y,\varphi X)Z] + 2(a_0 + (n-2)a_1) \end{aligned}$$

$$[g(Z,\varphi X)\eta(Y)\zeta - g(Y,\varphi X)\eta(Z)\zeta] - a_0\varphi(R(Y,Z)X) - a_1 S(Z,X)\varphi Y + a_1 S(Y,X)\varphi Z$$
$$- a_1 g(Z,X)\varphi(QY) + a_1 g(Y,X)\varphi(QZ) - (3a_0 + 2(3n-5)a_1 + 2a_2(r+3n^2-7n+4))$$
$$[g(Z,X)\varphi Y - g(Y,X)\varphi Z] - 2(a_0 + (n-2)a_1)[\eta(Z)\eta(X)\varphi Y - \eta(Y)\eta(X)\varphi Z] = 0. \tag{52}$$

Putting $Y = \zeta$ in (52) and making use of (2), (3), (9), and (12), we have

$$a_1 S(Z,\varphi X)\zeta + [2a_0 + (5n-7)a_1 + 2a_2(r+3n^2-7n+4)] \times$$
$$[g(Z,\varphi X)\zeta + \eta(X)\varphi Z] + a_1\eta(\mathcal{X})\varphi(QZ) = 0. \tag{53}$$

Interchanging X by φX in (53) and using (1), (3), we obtain

$$S(Z,X) = -\frac{[2a_0 + (5n-7)a_1 + 2a_2(r+3n^2-7n+4)]}{a_1} g(Z,X) \tag{54}$$
$$-\frac{2[a_0 + (3n-4)a_1 + a_2(r+3n^2-7n+4)]}{a_1}\eta(Z)\eta(X), \quad a_1 \neq 0.$$

Thus, we have the following result:

Theorem 4. *An $(LPK)_n$-manifold along with a semi-symmetric metric connection satisfying condition φ-generalized $\bar{B}$-semi-symmetric, then the manifold is an η-Einstein manifold of the form (54).*

6. (LPK)$_n$-Manifold Satisfying the Curvature Condition $\bar{B}.Q = 0$

In this section, we study the $(LPK)_n$-manifold satisfying $(\bar{B}(Y,Z).Q)X = 0$. This implies

$$\bar{B}(Y,Z)QX - Q(\bar{B}(Y,Z)X) = 0. \tag{55}$$

Put $Z = \zeta$ in (55), we have

$$\bar{B}(Y,\zeta)QX - Q(\bar{B}(Y,\zeta)X) = 0,$$

which in view of (32) turns to

$$(2a_0 + (5n-7)a_1 + 2a_2(r+3n^2-7n+4))(n-1)[\eta(X)Y - g(Y,X)\zeta]$$
$$- 2(a_0 + (2n-3)a_1 + a_2(r+3n^2-7n+4))[\eta(X)QY + S(Y,X)\zeta]$$
$$- a_1 S(Y,QX)\zeta - a_1\eta(X)Q(QY) = 0. \tag{56}$$

Performing the inner product of (56) with ζ, we find

$$a_1 S(Y,QX) + 2(a_0 + (2n-3)a_1 + a_2(r+3n^2-7n+4))S(Y,X)$$
$$- (2a_0 + (5n-7)a_1 + 2a_2(r+3n^2-7n+4))(n-1)g(Y,X) = 0. \tag{57}$$

Contracting (57) over Y and X, we obtain

$$a_1\sum_{i=1}^{n} S(e_i,Qe_i) + 2(a_0 + (2n-3)a_1 + a_2(r+3n^2-7n+4))\sum_{i=1}^{n} S(e_i,e_i)$$
$$- (2a_0 + (5n-7)a_1 + 2a_2(r+3n^2-7n+4))(n-1)\sum_{i=1}^{n} g(e_i,e_i) = 0,$$

$$2a_2r^2 + 2(a_0 + (2n-3)a_1 + 2(n-1)^2a_2)r$$
$$- n(n-1)(2a_0 + (5n-7)a_1 + 2(3n^2-7n+4)a_2) + a_1 trQ^2 = 0. \tag{58}$$

The relation (58) is quadratic in r. Let $a = 2a_2$, $b = 2(a_0 + (2n-3)a_1 + 2(n-1)^2a_2)$ and $c = -n(n-1)(2a_0 + (5n-7)a_1 + 2(3n^2-7n+4)a_2) + a_1 trQ^2$. After applying the well-known formula $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it yields the two values of a scalar curvature. Thus, we have the following result:

Theorem 5. *An $(LPK)_n$-manifold along with a semi-symmetric metric connection satisfying $\bar{B}.Q = 0$ gives the scalar curvature in the quadratic form described in (58).*

7. (LPK)$_n$-Manifold Satisfying the Curvature Condition $\bar{B}.\bar{Q} = 0$

In this section, we study the $(LPK)_n$-manifold satisfying $(\bar{B}(Y,Z).\bar{Q})X = 0$. This implies

$$\bar{B}(Y,Z)\bar{Q}X - \bar{Q}(\bar{B}(Y,Z)X) = 0. \tag{59}$$

Put $Z = \zeta$ in (59), we have

$$\bar{B}(Y,\zeta)\bar{Q}X - \bar{Q}(\bar{B}(Y,\zeta)X) = 0. \tag{60}$$

In view of (32), (60) takes the form

$$\begin{aligned}
&-2(a_0 + (3n-5)a_1 + a_2(r + 3n^2 - 7n + 4))[S(Y,X)\zeta + \eta(X)QY] \\
&\quad - (n-3)(2a_0 + (5n-7)a_1 + 2a_2(r + 3n^2 - 7n + 4))[\eta(X)Y + g(Y,X)\zeta] \\
&\quad - 8(n-2)(a_0 + (3n-4)a_1 + a_2(r + 3n^2 - 7n + 4))\eta(X)\eta(Y)\zeta \\
&\quad - a_1 S(Y,QX)\zeta - a_1\eta(\mathcal{X}_1)Q(QY) = 0,
\end{aligned}$$

which by taking the inner product with ζ becomes

$$\begin{aligned}
&a_1 S(Y,QX) + 2(a_0 + (3n-5)a_1 + (r + 3n^2 - 7n + 4)a_2)S(Y,X) \\
&\quad + (n-3)(2a_0 + (5n-7)a_1 + 2(r + 3n^2 - 7n + 4)a_2)g(Y,X) \\
&\quad + 4(n-2)(a_0 + (3n-4)a_1 + (r + 3n^2 - 7n + 4)a_2)\eta(X)\eta(Y) = 0.
\end{aligned} \tag{61}$$

Contracting (61) over Y and X, we obtain

$$\begin{aligned}
&a_1 \sum_{i=1}^{n} S(e_i, Qe_i) + 2(a_0 + (3n-5)a_1 + (r + 3n^2 - 7n + 4)a_2) \sum_{i=1}^{n} S(e_i, e_i) \\
&\quad + (n-3)(2a_0 + (5n-7)a_1 + 2(r + 3n^2 - 7n + 4)a_2) \sum_{i=1}^{n} g(e_i, e_i) \\
&\quad + 4(n-2)(a_0 + (3n-4)a_1 + (r + 3n^2 - 7n + 4)a_2) \sum_{i=1}^{n} g(e_i,\zeta)g(e_i,\zeta) = 0,
\end{aligned}$$

$$\begin{aligned}
&a_1 trQ^2 + 2(a_0 + (3n-5)a_1 + (r + 3n^2 - 7n + 4)a_2)r \\
&\quad + n(n-3)(2a_0 + (5n-7)a_1 + 2(r + 3n^2 - 7n + 4)a_2) \\
&\quad - 4(n-2)(a_0 + (3n-4)a_1 + (r + 3n^2 - 7n + 4)a_2) = 0.
\end{aligned}$$

We arrange the above relation as follows:

$$\begin{aligned}
&2a_2 r^2 + 2(a_0 + (3n-5)a_1 + 4(n-1)(n-2)a_2)r + (5n^3 - 34n^2 + 61n - 32)a_1 \\
&\quad + 2(n-1)(n-4)(a_0 + (3n^2 - 7n + 4))a_2 + a_1 trQ^2 = 0.
\end{aligned} \tag{62}$$

The relation (62) is quadratic in r. Let $a = 2a_2$, $b = 2(a_0 + (2n-3)a_1 + 2(n-1)^2 a_2)$, and $c = -n(n-1)(2a_0 + (5n-7)a_1 + 2(3n^2 - 7n + 4)a_2) + a_1 trQ^2$. After applying the well-known formula $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it yields the two values of the scalar curvature. Thus, we have the following result:

Theorem 6. *An $(LPK)_n$-manifold along with semi-symmetric metric connection satisfying $\bar{B}.\bar{Q} = 0$ gives the scalar curvature in quadratic form as described in Equation (62).*

8. Example

We consider the 3-dimensional manifold $M = \{(x_1, x_2, z) \in \mathbb{R}^3 : z > 0\}$, where (x_1, x_2, z) are the standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, and e_3 be the vector fields on M defined by

$$e_1 = z\frac{\partial}{\partial x_1}, \quad e_2 = z\frac{\partial}{\partial x_2}, \quad e_3 = z\frac{\partial}{\partial z} = \zeta,$$

which are linearly independent at each point p of M. Let g be the Lorentzian metric defined by

$$g(e_i, e_i) = 1, \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad g(e_3, e_3) = -1,$$
$$g(e_i, e_j) = 0, \quad \text{for} \quad i \neq j, \quad 1 \leq i, j \leq 3.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3) = g(X, \zeta)$ for all $X \in \chi(M)$, and let φ be the (1, 1)-tensor field defined by

$$\varphi e_1 = -e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0. \tag{63}$$

By applying linearity of φ and g, we have

$$\eta(\zeta) = g(\zeta, \zeta) = -1, \qquad \varphi^2 X = X + \eta(X)\zeta,$$

$$\text{and,} \qquad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \chi(M)$. Thus, for $e_3 = \zeta$, the structure $(\varphi, \zeta, \eta, g)$ defines a Lorentzian almost para-contact metric structure on M. Then, we have

$$[e_i, e_j] = 0, \quad \text{if} \ \ i \neq j, \quad \text{and} \ \ i, j = 1, 2, \tag{64}$$

$$[e_i, e_3] = -e_i, \quad \text{for} \ \ i = 1, 2. \tag{65}$$

By using the well-known Koszul's formula, we find

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Now, let

$$X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3,$$

$$Y = \sum_{j=1}^{3} Y^j e_j = Y^1 e_1 + Y^2 e_2 + Y^3 e_3,$$

$$Z = \sum_{k=1}^{3} Z^k e_k = Z^1 e_1 + Z^2 e_2 + Z^3 e_3,$$

for all $X, Y, Z \in \chi(M)$. Also, one can easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \varphi)Y = -g(\varphi X, Y)\zeta - \eta(Y)\varphi X.$$

Therefore, the manifold is a Lorentzian para-Kenmotsu manifold.
From the above results, we can easily obtain the non-vanishing components of the curvature tensor as follows:

$$R(e_1,e_2)e_1 = -e_2, R(e_1,e_2)e_2 = e_1, R(e_1,e_3)e_1 = -e_3,$$

$$R(e_1,e_3)e_3 = e_1, R(e_2,e_3)e_2 = -e_3, R(e_2,e_3)e_3 = -e_2,$$

from which it is clear that

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{66}$$

Thus, the manifold is of a constant curvature. Also, we calculate the Ricci tensors as follows:

$$S(e_1,e_1) = S(e_2,e_2) = 2, \quad S(e_3,e_3) = -2.$$

Hence, we find

$$r = S(e_1,e_1) + S(e_2,e_2) - S(e_3,e_3) = 6.$$

By contracting (22), it follows that

$$S(Y,Z) = 2g(Y,Z), \qquad r = 6, \tag{67}$$

which are same as the values of Ricci tensor and scalar curvature.
The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ has been given by:

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\zeta. \tag{68}$$

Using this relation, we find:

$$\bar{\nabla}_{e_1} e_1 = -2e_3, \bar{\nabla}_{e_1} e_2 = 0, \bar{\nabla}_{e_1} e_3 = 0,$$

$$\bar{\nabla}_{e_2} e_1 = 0, \bar{\nabla}_{e_2} e_2 = -2e_3, \bar{\nabla}_{e_2} e_3 = 0,$$

$$\bar{\nabla}_{e_3} e_1 = 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \quad \bar{\nabla}_{e_3} e_3 = -2e_3,$$

Also, one can easily verify that

$$\bar{\nabla}_X \xi = -2X - 2\eta(X)\xi \quad \text{and} \quad (\bar{\nabla}_X \varphi)Y = -2g(\varphi X, Y)\zeta - 2\eta(Y)\phi X.$$

Therefore, this shows that the manifold is a Lorentzian para-Kenmotsu manifold along a semi-symmetric metric connection.
From the above results, we obtain the non-vanishing components of the curvature tensor as:

$$\bar{R}(e_1,e_2)e_1 = -4e_2, \bar{R}(e_1,e_2)e_2 = 4e_1, \bar{R}(e_1,e_3)e_1 = -2e_3,$$

$$\bar{R}(e_1,e_3)e_3 = -2e_1, \bar{R}(e_2,e_3)e_2 = -2e_3, \bar{R}(e_2,e_3)e_3 = -2e_2,$$

from which it is clear that

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + 3g(Y,Z)X - 3g(X,Z)Y + 2\eta(Y)\eta(Z)X \\ &\quad -2\eta(X)\eta(Z)Y + 2g(Y,Z)\eta(X)\zeta - 2g(X,Z)\eta(Y)\zeta. \end{aligned} \tag{69}$$

Also, we calculate the Ricci tensors along the semi-symmetric metric connection, as follows:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = 6, \quad \bar{S}(e_3,e_3) = -4.$$

Hence, we find

$$\bar{r} = \bar{S}(e_1,e_1) + \bar{S}(e_2,e_2) - \bar{S}(e_3,e_3) = 16.$$

By contracting (69), it follows that

$$\bar{S}(X,Y) = 6g(X,Y) + 2\eta(X)\eta(Y). \tag{70}$$

By using (26) in (70), we find $S(X,Y) = 2g(X,Y)$. Hence, $QX = 2X$, which implies that $\varphi^2((\bar{\nabla}_W Q)X) = 0$. Thus, we observe that the manifold under consideration is φ-Ricci symmetric with respect to $\bar{\nabla}$.

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