

Article

Feedback Control Design for Time-Delay Systems Based on the Manabe Polynomial Concept Under Unmodeled Input Delay

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Abstract

Time delays are inherent in modern motion-control and electric-drive loops due to sensing, filtering, sampling and computation, communication, and actuation scheduling. When such delays are only partially known, they can markedly reduce stability margins and narrow the admissible range of state-feedback gains, especially in high-bandwidth servo applications. This paper develops a design-oriented state-feedback framework for delay-affected plants based on the Manabe polynomial concept and the Coefficient Diagram Method (CDM). The plant is represented as a chain of integrators of order two to four with an effective input gain, and the feedback gain is synthesized for the nominal delay-free model by matching a standard Manabe/CDM characteristic polynomial using the classical CDM stability-index pattern. When an unmodeled input delay is present, the closed loop is governed by a delay-dependent characteristic equation. By introducing a normalized representation, the analysis yields explicit delay-stability limits that directly translate into a lower bound on the equivalent time constant used for tuning. The degradation of the phase margin and gain margin with increasing normalized delay is quantified as design charts, and a simple phase-margin-based inequality is proposed for selecting the tuning time constant, with gain-margin checks recommended as a verification step.

Keywords: time-delay systems; unmodeled input delay; Manabe polynomial; coefficient diagram method (CDM); state-feedback; pole placement; chain of integrators; servo drives; robustness margins; phase margin; gain margin

1. Introduction

1.1. Motivation

Time delays are an inherent feature of modern feedback loops in motion-control and drive applications. They arise from sensing chains (signal conditioning and filtering), sampling and computation, communication buses, and actuation scheduling (e.g., PWM update and zero-order hold). In high-bandwidth servo drives even a small delay may markedly reduce the phase margin, excite oscillatory modes, and impose a strict upper limit on achievable closed-loop speed. The practical difficulty is that the delay is rarely a single, fixed constant: it is distributed across hardware and software layers, may vary with computational load and operating conditions, and is therefore often only partially known or effectively unmodeled. Recent studies on time-delay systems confirm that uncertain or time-varying delays can strongly restrict reachable performance and may threaten stability when neglected in synthesis [1,2].

This work targets a design-oriented treatment of such unmodeled input delay in state-feedback loops. Rather than relying on heavyweight optimization, we seek explicit,



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lightweight rules that link an engineer-chosen transient time scale to delay margins. Although real drives also exhibit parameter uncertainty (e.g., inertia and electrical-parameter drift), the present paper deliberately focuses on the delay mechanism and uses a benchmark plant structure to isolate and quantify how delay alone constrains feasible bandwidth and classical robustness margins.

1.2. Related Work

State-feedback is a cornerstone of modern control, enabling systematic pole placement and shaping in the state space. Ackermann's formula established a direct polynomial link between desired closed-loop dynamics and the feedback vector [3]. However, selecting poles that remain satisfactory under implementation effects is nontrivial; for structured uncertainty, parameter-space viewpoints provide an interpretable robustness analysis framework [4]. In parallel, polynomial design traditions have long exploited coefficient patterns to shape responses; the Butterworth polynomial is a classic example where the coefficient structure encodes a characteristic trade-off between flatness and roll-off [5]. These ideas motivate controller synthesis procedures that remain transparent at the characteristic-polynomial level.

Within this lineage, Manabe's Coefficient Diagram Method (CDM) prescribes a desired characteristic polynomial via stability indices and an equivalent time constant, tuned and visualized using a coefficient diagram [6,7]. CDM provides explicit coefficient-level rules that connect time response with robustness-related coefficient ratios, and its methodological foundations and practical guidelines are summarized in [8,9]. CDM-inspired concepts have also been embedded into state-feedback structures for practical plants [10,11] and in drive-oriented studies [12].

For delay-affected systems, contemporary robust synthesis frequently relies on Lyapunov–Krasovskii functionals and LMI-based optimization, or on predictor-type compensation when a delay model is available [1,13]. In drive applications, delay compensation is often treated in predictive control and digital current control schemes; examples include Smith-structure compensation in model predictive current control [14] and explicit input-delay compensation in deadbeat predictive current control [15]. More model-lean approaches, such as time-delay estimation/control, have also been explored as practical mechanisms to mitigate unmodeled dynamics and delays [16]. Recent servo-oriented work also addresses uncertain time-delay effects from a broader application perspective; for example, Ma et al. review representative control approaches for servo systems exhibiting uncertain time delays and discuss compensation/estimation strategies, providing complementary context to delay uncertainty in high-performance servos [17]. While these approaches are powerful, they typically do not provide a simple coefficient-level pathway that links a chosen closed-loop time scale directly to admissible delay ratios and classical margins in a form convenient for fast engineering tuning.

Beyond the LMI- and predictor-based literature on delay-affected systems, there is also representative work that specifically addresses stabilization/regulation of chains of integrators under input delays using delay-compensation and/or adaptive mechanisms. Druzhinina and Sedova construct a stabilizing delay-feedback structure for an even-order integrator chain [18]. Koo, Choi, and Lim study output-feedback regulation under an unbounded time-varying input delay and employ a time-varying gain-scaling mechanism [19]. More recently, Oh and Choi propose an adaptive zero-order-hold triggered output-feedback scheme that copes with unknown input delay and unknown interexecution time [20].

The objective and setting of the present paper are complementary to these delay-compensation/adaptive-delay designs. We intentionally keep the controller synthesis delay-free (fixed state-feedback tuned via a Manabe/CDM polynomial template) and

instead quantify the robustness of the resulting closed loop against an unmodeled constant input delay in a design-oriented manner. The key outcome is a set of explicit admissible normalized delay ratios, $\delta = T_d/\tau$ (stability limits and PM/GM-based charts), which can be used as lightweight tuning and verification guidance when the delay is only partially known or treated as uncertainty rather than explicitly compensated.

Finally, we note that a direct one-to-one numerical benchmarking is outside the scope of this paper—both against integrator-chain delay-compensation/adaptive-delay designs [18–20] and against established predictor-/Smith-type compensation or optimization/LMI-based robust designs. Such approaches typically assume an explicit delay model and implement delay-dependent compensation (predictors/Smith structures), or they adopt different information patterns (e.g., observers/output feedback, time-varying gains) and optimization objectives than the present contribution. In contrast, our setting intentionally keeps synthesis delay-free (fixed state-feedback gains tuned via a Manabe/CDM template) and treats the delay as uncertainty; hence, the proposed $\delta = T_d/\tau$ limits and the PM/GM charts should be interpreted as a lightweight tuning/verification baseline for the intrinsic delay tolerance of delay-free fixed-gain designs (before any explicit delay compensation is introduced).

1.3. Contribution and Paper Organization

This paper develops a Manabe/CDM-based, coefficient-oriented state-feedback framework to quantify and tune the effect of an unmodeled input delay. The choice of a pure integrator-chain plant is deliberate: it is a transparent and design-useful benchmark in servo and electric-drive loop synthesis, capturing the dominant kinematic relations while keeping the characteristic-polynomial shaping fully interpretable. The objective is not to provide a complete high-fidelity model of real plants (flexible modes, nonlinearities, or detailed delay distributions), but to isolate and quantify how an effective delay alone constrains classical robustness margins and the admissible range of the Manabe time constant τ . The resulting δ -based limits and charts should therefore be read as lightweight baseline guidance for tuning and verifying delay tolerance when the delay is only partially known. The main contributions are:

- A unified normalization for Manabe/CDM pole-shaping under input delay, showing that delay robustness is governed primarily by the dimensionless ratio $\delta = T_d/\tau$, where τ is the equivalent time constant of the target Manabe polynomial.
- Explicit delay-stability limits $\delta_{\max,N}$ for chain-of-integrator benchmarks with $N \in \{2,3,4\}$ under the standard CDM indices, yielding direct lower bounds on τ for a given delay estimate.
- Phase-margin (PM) and gain-margin (GM) degradation laws versus δ , summarized as design charts; a simple PM-based inequality is provided for selecting τ to meet a required robustness level, with GM treated as a verification step supported by a compact rational approximation.
- A comparative illustration against common pole-selection patterns (repeated real poles and repeated oscillatory prototypes) under matched dominant time constants, highlighting transient-response and control-effort trade-offs.

This paper is organized as follows: Section 2 introduces the chain-of-integrators benchmark and the nominal state-feedback synthesis by polynomial matching. Section 3 summarizes the Manabe polynomial construction and its scaling properties. Section 4 analyzes the impact of an unmodeled input delay, derives normalized stability limits and PM/GM degradation characteristics, and formulates practical tuning rules in terms of $\delta = T_d/\tau$. Section 5 concludes this paper and outlines future work.

2. Preliminaries: Plant and Controller Synthesis

2.1. Chain-of-Integrators Model and Meaning in Electric Drives

A recurring approximation in motion-control and servo-drive analysis is the chain-of-integrators model, which captures the dominant kinematic relations between position and its successive time derivatives. In this paper, the nominal plant is represented by an N -fold integrator with an input gain

$$G(s) = \frac{b}{s^N}, \quad N \in \{2, 3, 4\}, \quad b > 0, \quad (1)$$

where b aggregates the effective gain of the actuation channel as seen from the outer-loop controller. Although (1) is an idealized model, it serves as a transparent benchmark for studying fundamental robustness phenomena (in particular, sensitivity to unmodeled delays and gain mismatch) while remaining directly interpretable in servo applications.

2.1.1. State Vector and Indexing Convention

We define the state vector as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^\top \in \mathbb{R}^N, \quad (2)$$

and adopt the 1-based indexing convention (MATLAB R2025a). With the output defined as $y(t) = x_1(t)$, the remaining states represent successive time derivatives of the output:

$$x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad \dots, \quad x_N = y^{(N-1)}. \quad (3)$$

2.1.2. State-Space Realization

A standard controllable canonical realization of (1) is

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} b u(t), \quad y(t) = \mathbf{C} \mathbf{x}(t), \quad (4)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{N \times 1}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times N}. \quad (5)$$

In this convention, the input acts on the highest derivative state via $\dot{x}_N(t) = b u(t)$, while the output is simply $y(t) = x_1(t)$.

2.1.3. Practical Meaning for Electric Drives

For a single-mass electric drive under an outer-loop position or speed controller, the effective closed-loop dynamics often admit a low-order description dominated by integral relations:

- $N = 1$ (single integrator) corresponds to position as the integral of velocity. This abstraction is appropriate when the inner velocity loop is much faster than the outer position loop and can be approximated by a static gain.
- $N = 2$ (double integrator) is the classical model for position control with acceleration as a direct input, i.e., when the commanded torque/current channel can be viewed as producing acceleration after lumping the inertia and torque constant into b .
- $N = 3$ becomes relevant when an additional dynamic stage is explicitly represented (e.g., inner-loop actuator dynamics, a filtered torque reference, or a deliberately in-

troduced auxiliary state), leading to additional integrator-like behavior in the outer-loop model.

- $N = 4$ is a convenient benchmark for position regulation with an additional integral action (or auxiliary state) introduced to remove the steady-state offset under disturbances or parameter mismatch. In this case, the fourth state may represent an integral-of-error state or a higher-order kinematic relation, depending on the chosen state augmentation.

In practice, the precise physical meaning of states beyond $(y, \dot{y})$ depends on the modeling level and on the structure of inner loops; nevertheless, the chain-of-integrators model (4) provides a unified mathematical setting for systematic pole-shaping and robustness studies across these cases.

2.1.4. Scope and Limitations

The model (1) deliberately neglects non-ideal effects such as actuator saturation, discrete-time implementation, friction nonlinearities, structural resonances, and unmodeled transport delays. These effects are addressed in this work only through two aggregated uncertainty mechanisms: (i) an unmodeled input delay introduced later in this paper, and (ii) bounded uncertainty of the effective gain b . Despite its simplicity, the chain-of-integrators benchmark remains a useful and widely adopted starting point for deriving design-oriented guidelines.

Although the normalization and the delay-robustness analysis developed later can, in principle, be carried out for any order N , we deliberately focus on $N = 2, 3, 4$ for three engineering reasons. First, these orders admit a direct and widely used interpretation in drive and motion-control architectures: $N = 2$ captures the dominant position–acceleration relation under fast inner loops; $N = 3$ reflects an additional integrator-like stage (e.g., actuator/command dynamics or an auxiliary state); and $N = 4$ is a convenient benchmark when an extra integral action/state augmentation is introduced for offset rejection. For $N > 4$, a pure integrator chain typically no longer corresponds to the physical plant dynamics of servo drives; higher-order behavior is instead dominated by non-integrator effects (e.g., elastic modes/resonances, sampling/hold dynamics, and nonlinearities), so the “ N -fold integrator” benchmark becomes less representative.

Second, for the adopted Manabe/CDM template, the admissible normalized delay ratio $\delta_{\max, N}$ shrinks very rapidly with increasing N (the computed values for $N = 2, 3, 4$ already exhibit an approximately geometric, near-exponential decay). This trend is consistent with the increasing phase lag and phase slope of an N -integrator chain and with the higher normalized crossover frequencies implied by the fixed polynomial template. As a result, already at $N = 4$, the stability requirement $\tau > T_d / \delta_{\max, 4}$ imposes a very conservative bandwidth (in our case, $\tau \gtrsim 9.4 T_d$), and even more restrictive bounds are expected for $N > 4$.

Third, extending the same charts to $N > 4$ would therefore mainly reinforce the same qualitative message—severely reduced delay tolerance and strongly limited bandwidth—without providing additional actionable insight beyond the $N = 4$ case. If needed, the numerical procedure presented in Section 4.2 applies directly to higher orders.

2.2. State-Feedback Pole Placement for the Nominal Plant

This subsection summarizes the classical full state-feedback design by pole placement for the nominal, delay-free plant introduced in Section 2.1. The design is performed for a single-input controllable pair and yields a feedback gain that assigns a prescribed closed-loop characteristic polynomial. A constructive coordinate-free formula due to Ackermann is commonly used as a theoretical reference for this procedure [3]; see, e.g., [4].

2.2.1. Nominal Plant Model for Synthesis

In this paper, the input gain $b > 0$ is treated as known, and the controller is designed using the delay-free model

$$\dot{x}(t) = A x(t) + B b u(t), \quad y(t) = C x(t). \quad (6)$$

The subsequent robustness analysis focuses on an unmodeled input delay introduced in the actuation channel, while other modeling imperfections are outside the present scope.

2.2.2. Full State-Feedback Law (Two Degrees of Freedom)

We adopt the standard two-degree-of-freedom structure

$$u(t) = -K x(t) + k_r r(t), \quad (7)$$

where $K \in \mathbb{R}^{1 \times N}$ is the state-feedback gain and $k_r \in \mathbb{R}$ is a static reference prefilter. For $r(t) \equiv 0$, the closed-loop dynamics are governed by

$$\dot{x}(t) = A_{cl} x(t), \quad A_{cl} := A - B b K. \quad (8)$$

2.2.3. Pole Placement Objective

Let the desired closed-loop characteristic polynomial be specified as

$$\Delta_d(s) = \prod_{i=1}^N (s - p_i) = s^N + \alpha_{N-1} s^{N-1} + \cdots + \alpha_1 s + \alpha_0, \quad (9)$$

where p_i are the target poles (e.g., obtained from a Manabe polynomial in later sections). If the pair $(A, B b)$ is controllable, then there exists a gain K , such that

$$\det(sI - A_{cl}) = \Delta_d(s). \quad (10)$$

2.2.4. Ackermann Formula (Single-Input Case)

For a controllable single-input system, a compact closed-form expression for K is given by Ackermann's formula [3]. Define the controllability matrix

$$C = [B b \quad A B b \quad \cdots \quad A^{N-1} B b], \quad e_N = [0 \quad \cdots \quad 0 \quad 1]^\top. \quad (11)$$

Then

$$K = e_N^\top C^{-1} \Delta_d(A), \quad K = [k_1 \quad \cdots \quad k_{N-1} \quad k_N]^\top, \quad (12)$$

where the matrix polynomial is evaluated as

$$\Delta_d(A) = A^N + \alpha_{N-1} A^{N-1} + \cdots + \alpha_1 A + \alpha_0 I. \quad (13)$$

In numerical studies, K is typically computed using robust algorithms (e.g., MATLAB place), while (12) remains a useful theoretical reference.

2.2.5. Reference Tracking via a Static Prefilter

To compare step responses across designs, we choose k_r , such that the nominal (delay-free) closed loop has unity steady-state gain from a constant reference, i.e., $y_{ss} = r_{ss}$ for constant $r(t) = r_{ss}$. For the stable matrix A_{cl} , the steady state satisfies $0 = A_{cl} x_{ss} + B b k_r r_{ss}$; hence,

$$k_r = -\frac{1}{C A_{cl}^{-1} B b}. \quad (14)$$

For the plant defined by (4) and (5), the prefilter expression (14) simplifies to

$$k_r = k_1. \quad (15)$$

This choice yields a convenient baseline for assessing how an unmodeled input delay distorts the nominal step response. (Later, when an auxiliary integral state is introduced, the reference injection can alternatively be formulated through an error-integrator channel; the present form (14) is kept here for clarity and direct comparability.)

2.2.6. Remark on State Availability

The results above assume that the full state vector $x(t)$ is available for feedback. In practical implementations, unmeasured components can be reconstructed by an observer; however, state estimation is not the focus of the present study.

3. Manabe Polynomials

3.1. Historical Notes and Context

The Manabe polynomial concept originated as an algebraic pole-shaping idea in the Japanese control community and later became widely associated with the *Coefficient Diagram Method* (CDM) [6–9]. The central motivation is to provide a practically convenient way to prescribe closed-loop dynamics directly through the coefficients of a desired characteristic polynomial rather than by specifying pole locations one-by-one. In this framework, the polynomial coefficients are generated from a small set of shaping parameters that are interpretable from an engineering perspective; notably, a dominant time constant (setting the overall speed of the response) and a set of stability indices (shaping relative damping and the distribution of modes).

From the standpoint of robust control design, Manabe-style polynomials are appealing because (i) they impose structured, repeatable coefficient patterns across different orders, and (ii) they allow one to express key time-scale relations explicitly (e.g., how closed-loop bandwidth scales with the chosen time constant). These properties make the approach suitable as a backbone for systematic studies of robustness with respect to unmodeled delays and plant-parameter mismatch, which is the main focus of this paper.

3.2. Construction of a Manabe Polynomial

Consider a real-coefficient polynomial of order N written in ascending powers,

$$P(s) = a_0 + a_1s + a_2s^2 + \cdots + a_Ns^N, \quad a_i > 0. \quad (16)$$

Following the CDM terminology, define the stability indices

$$\gamma_i := \frac{a_i^2}{a_{i-1}a_{i+1}}, \quad i = 1, 2, \dots, N-1, \quad (17)$$

and interpret the dominant time constant as a scaling parameter that sets the overall speed of response. A convenient normalization is to set

$$a_0 = 1, \quad a_1 = \tau, \quad (18)$$

where $\tau > 0$ plays the role of the dominant time constant.

Given τ and chosen indices γ_i , the remaining coefficients can be generated recursively from (17):

$$a_2 = \frac{a_1^2}{\gamma_1 a_0}, \quad a_{i+1} = \frac{a_i^2}{\gamma_i a_{i-1}}, \quad i = 2, \dots, N-1. \quad (19)$$

The classical Manabe choice uses a fixed pattern of indices, typically

$$\gamma_1 = 2.5, \quad \gamma_i = 2, \quad i \geq 2, \quad (20)$$

which yields a family of polynomials whose pole patterns scale with $1/\tau$.

For control design and pole placement, it is often convenient to work with the monic form in descending powers,

$$\tilde{P}(s) = s^N + \alpha_{N-1}s^{N-1} + \cdots + \alpha_1s + \alpha_0, \quad (21)$$

obtained by reversing the coefficient order and normalizing by the leading coefficient a_N .

Explicit Manabe Polynomials for $N = 1, 2, 3, 4$

Using the normalization $a_0 = 1$, $a_1 = \tau$ and the stability-index definition

$$\gamma_1 = \frac{a_1^2}{a_0 a_2}, \quad (22)$$

the Manabe coefficients follow directly.

Order $N = 1$. For $N = 1$ the polynomial is simply

$$P_{M,1}(s; \tau) = a_0 + a_1s = 1 + \tau s. \quad (23)$$

Order $N = 2$. With the classical Manabe choice $\gamma_1 = 2.5$, the coefficient a_2 becomes

$$a_2 = \frac{a_1^2}{\gamma_1 a_0} = \frac{\tau^2}{2.5}; \quad (24)$$

hence,

$$P_{M,2}(s; \tau) = 1 + \tau s + \frac{\tau^2}{2.5}s^2. \quad (25)$$

In monic, descending-power form this reads

$$\tilde{P}_{M,2}(s; \tau) = s^2 + \frac{2.5}{\tau}s + \frac{2.5}{\tau^2}. \quad (26)$$

Order $N = 3$. Using $\gamma_2 = 2$ and the recursion,

$$a_3 = \frac{a_2^2}{\gamma_2 a_1} = \frac{(\tau^2/2.5)^2}{2\tau} = \frac{\tau^3}{12.5}, \quad (27)$$

yields

$$P_{M,3}(s; \tau) = 1 + \tau s + \frac{\tau^2}{2.5}s^2 + \frac{\tau^3}{12.5}s^3. \quad (28)$$

The corresponding monic, descending-power form is

$$\tilde{P}_{M,3}(s; \tau) = s^3 + \frac{5}{\tau}s^2 + \frac{12.5}{\tau^2}s + \frac{12.5}{\tau^3}. \quad (29)$$

Order $N = 4$. We use $\gamma_3 = 2$ and the recursion again,

$$a_4 = \frac{a_3^2}{\gamma_3 a_2} = \frac{(\tau^3/12.5)^2}{2(\tau^2/2.5)} = \frac{\tau^4}{125}; \quad (30)$$

hence,

$$P_{M,4}(s; \tau) = 1 + \tau s + \frac{\tau^2}{2.5}s^2 + \frac{\tau^3}{12.5}s^3 + \frac{\tau^4}{125}s^4. \quad (31)$$

The corresponding monic, descending-power form is

$$\tilde{P}_{M,4}(s; \tau) = s^4 + \frac{10}{\tau}s^3 + \frac{50}{\tau^2}s^2 + \frac{125}{\tau^3}s + \frac{125}{\tau^4}. \quad (32)$$

To illustrate the properties of Manabe polynomials, we simulated the unit-step response of a fourth-order chain-of-integrators plant controlled by full state-feedback. Two tests were performed for the dominant time constant values $\tau = 1.5$ s and $\tau = 0.75$ s. The results are shown in Figure 1. As can be seen, reducing the time constant by a factor of two yields a noticeably faster transient; however, it also leads to significantly larger excursions of the internal states (x_2 – x_4) and a substantially higher required control effort $u(t)$.

Figure 1 is intended as a direct illustration of the τ –control–effort trade-off inherent to Manabe/CDM tuning. When τ is reduced (right column), the output x_1 settles faster, but the internal-state excursions (x_2 – x_4) become larger and the required actuation $u(t)$ increases markedly (higher peak and stronger transient activity). In the workflow used later in Section 4, this observation is essential: choosing a smaller τ increases bandwidth, but it also makes the design more sensitive to unmodeled delay through $\delta = T_d/\tau$.

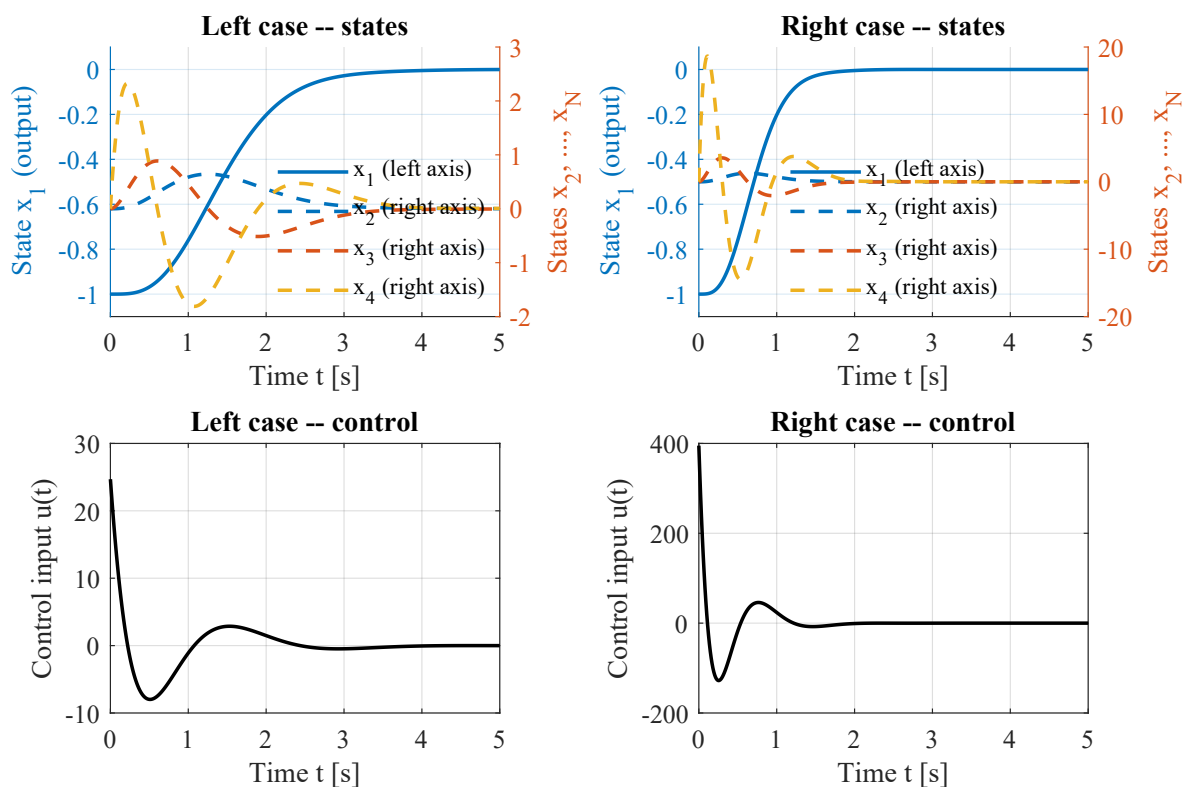


Figure 1. Step responses of the fourth-order chain-of-integrators closed loop with Manabe-based pole placement and full state-feedback. Top row: state trajectories (output state x_1 on the left axis; internal states x_2 – x_4 on the right axis). Bottom row: control input $u(t)$. (**Left Column**): $\tau = 1.5$ s. (**Right Column**): $\tau = 0.75$ s.

Remark 1. Introducing the dimensionless variable $\sigma = \tau s$, the normalized polynomials,

$$\tilde{P}_{M,N}(s; \tau) = \tau^{-N} \tilde{Q}_{M,N}(\sigma), \quad \sigma = \tau s, \quad (33)$$

have coefficients independent of τ , and their roots scale as $p_i = \sigma_i/\tau$. For $N = 4$, (33) specializes to

$$\tilde{P}_{M,4}(s; \tau) = \frac{1}{\tau^4} (\sigma^4 + 10\sigma^3 + 50\sigma^2 + 125\sigma + 125), \quad \sigma = \tau s, \quad (34)$$

which highlights the scaling $p_i = \sigma_i/\tau$.

3.3. Pole Patterns and Step-Response Comparison (Fourth-Order Illustration)

To illustrate the qualitative properties of Manabe-style pole shaping, we compare the fourth-order Manabe polynomial (32) with two commonly used reference designs of the same order: (i) a quadruple real pole and (ii) a repeated second-order oscillatory pattern.

3.3.1. Three Fourth-Order Reference Polynomials

(A) Manabe Polynomial (Order 4)

We use $\tilde{P}_{M,4}(s; \tau)$ from (32). Its poles satisfy the normalized equation

$$\sigma^4 + 10\sigma^3 + 50\sigma^2 + 125\sigma + 125 = 0, \quad \sigma = \tau s,$$

which yields two complex-conjugate pairs (numerically, in normalized coordinates)

$$\sigma_{1,2} \approx -2.5 \pm j 3.44095, \quad \sigma_{3,4} \approx -2.5 \pm j 0.81230, \quad (35)$$

and therefore $p_k = \sigma_k/\tau$.

(B) Quadruple Real Pole (Order 4)

A standard “aperiodic” benchmark is a repeated real pole at $-1/\tau$:

$$\tilde{P}_{(R,4)}(s; \tau) = (s + \frac{1}{\tau})^4 = s^4 + \frac{4}{\tau}s^3 + \frac{6}{\tau^2}s^2 + \frac{4}{\tau^3}s + \frac{1}{\tau^4}. \quad (36)$$

All poles coincide at $p = -1/\tau$ with multiplicity four.

(C) Squared Second-Order Oscillatory Pattern (Order 4)

Let the second-order prototype be

$$s^2 + 2\zeta\omega_n s + \omega_n^2, \quad \zeta \in (0, 1), \quad \omega_n > 0.$$

Then the fourth-order benchmark is its square:

$$\tilde{P}_{O,4}(s; \zeta, \omega_n) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^2 = s^4 + 4\zeta\omega_n s^3 + (2 + 4\zeta^2)\omega_n^2 s^2 + 4\zeta\omega_n^3 s + \omega_n^4. \quad (37)$$

This corresponds to a double complex-conjugate pair

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2},$$

each of multiplicity two. For a direct time-scale comparison with Manabe polynomials, a convenient choice is $\omega_n = 1/\tau$ (and a user-chosen ζ), while alternative matching criteria (e.g., equal 2% settling time) can also be used.

Matching the Dominant Time Constant Across Pole-Selection Methods

To compare different pole-selection rules in a consistent manner, we match the dominant time constant of the resulting closed-loop dynamics. The Manabe design is used as the reference and is parameterized by $\tau_M > 0$. For a fixed polynomial family, Manabe

poles scale as $p_i = \sigma_i / \tau_M$, where σ_i are the roots of the normalized polynomial obtained for $\tau_M = 1$. Hence, the dominant real part can be written as

$$\Re(p_{\text{dom},M}) = -\frac{c_N}{\tau_M}, \quad c_N := -\max_i \Re(\sigma_i) > 0, \quad (38)$$

where c_N depends only on the order N and on the selected Manabe stability indices.

For the repeated-real-pole design, the desired characteristic polynomial is $\Delta_R(s) = (s + 1/\tau_R)^N$, i.e., all closed-loop poles are located at $p = -1/\tau_R$. Therefore,

$$\Re(p_{\text{dom},R}) = -\frac{1}{\tau_R}. \quad (39)$$

Matching (39) with (38) yields the time-constant mapping

$$\tau_R = \frac{\tau_M}{c_N}. \quad (40)$$

For the oscillatory-based design, the second-order prototype is defined as $s^2 + 2\zeta\omega_n s + \omega_n^2$ with $\omega_n = 1/\tau_O$ and, in this study, $\zeta = \sqrt{2}/2$. For $0 < \zeta \leq 1$, the prototype poles satisfy

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \Rightarrow \Re(p_{\text{dom},O}) = -\frac{\zeta}{\tau_O}, \quad (41)$$

and raising the prototype to a power (to obtain higher-order polynomials) changes only pole multiplicities, not their locations. Equating (41) with (38) gives

$$\tau_O = \frac{\zeta}{c_n} \tau_M, \quad \zeta = \frac{\sqrt{2}}{2}. \quad (42)$$

In summary, for a given Manabe parameter τ_M , the comparison is performed with τ_R and τ_O selected according to (40)–(42), ensuring that the dominant exponential decay rate (i.e., the dominant real part of the closed-loop poles) is identical across the considered pole-selection methods.

3.3.2. Qualitative Comparison

The three patterns exhibit distinct qualitative behavior, as shown in Figure 2:

- The repeated real pole (36) typically yields a monotonic response (no oscillations) but with a comparatively “heavy” transient tail due to pole multiplicity.
- The squared oscillatory pattern (37) produces pronounced oscillatory transients whose overshoot and ringing depend strongly on (ζ, ω_n) ; pole multiplicity tends to increase peaking for a given damping ratio.
- The Manabe polynomial (32) yields a structured pair-of-pairs pole distribution (35) that often provides a favorable compromise between fast initial convergence and limited oscillation, while retaining transparent time-scale control through τ .

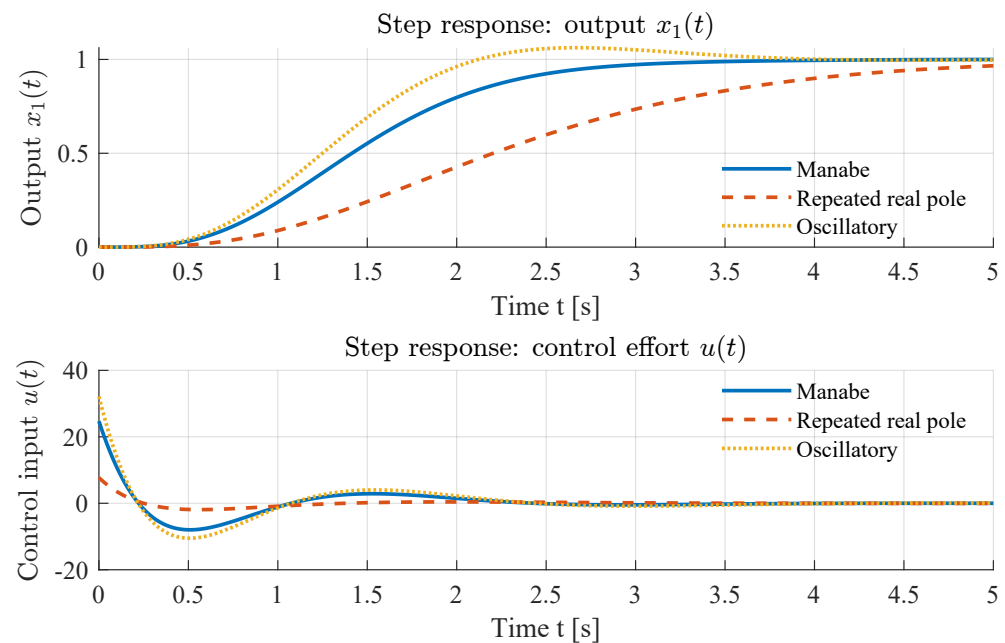


Figure 2. Comparison of three pole-selection methods for the $N = 4$ chain-of-integrators plant with full state-feedback and a step reference. Top: output trajectory $x_1(t)$ (plant output). Bottom: control input $u(t)$. The Manabe-based design uses $\tau_M = 1.5$ s; the repeated real-pole design uses $\tau_R = 0.6$ s; the oscillatory-based design uses $\tau_O = 0.42$ s with $\zeta = \sqrt{2}/2$. The time constants τ_R and τ_O are selected to match the dominant exponential decay rate (dominant real part of the closed-loop poles) to the Manabe reference.

4. Effect of Unmodeled Input Delay

4.1. Sources of Input Delay in Drive Control Loops

In practical motion-control and electric-drive systems, input delay is rarely a single, well-defined constant. Instead, it is an aggregate effect of several mechanisms distributed across sensing, computation, and actuation. In many implementations, the controller operates in discrete time, which introduces an inherent latency between sampling the measurements and updating the actuator command. This latency is often close to one sampling period and may increase due to task scheduling, multi-rate execution, or interrupt-driven architectures.

A major contribution comes from measurement conditioning and filtering. Anti-aliasing filters placed before analog-to-digital conversion add phase lag that is commonly approximated as an equivalent delay within the control bandwidth. Additional digital filtering—used to suppress noise, ripple, or quantization effects—introduces further group delay. This is especially relevant in current and speed estimation loops, where filtering is applied to measured currents, voltages, or encoder signals.

Input delay also appears due to the discrete nature of sensing and estimation. For example, position is measured at sampling instants, while speed is frequently computed from the incremental position over the elapsed sampling interval. As a result, the speed feedback effectively represents a past average value rather than an instantaneous quantity, which behaves as an additional delay in the feedback path. Similar effects occur when acceleration or disturbance estimates are obtained by discrete differentiation or observers with filtering.

Finally, modern drive systems increasingly rely on distributed control architectures. Communication over serial links and fieldbuses (e.g., when sensors, drives, and supervisory controllers are physically separated) introduces transmission and buffering delays. In addition, computational pipelines in embedded processors and power-electronic modulation (e.g., PWM update and zero-order hold behavior) can shift the effective application

time of the control input. Since these contributions may vary with operating conditions and computational load, the resulting delay is often only partially known and should be treated as unmodeled or uncertain in robust controller design. Within the control bandwidth of a linear loop, the individual latency contributions from sensing, computation, communication, and actuation can often be approximated by their sum and represented as a single effective lumped delay, which motivates the scalar input delay T_d used in this study. More detailed delay structures (e.g., distributed effects, multiple delays, or output/measurement delays) are important in practice and constitute natural extensions of the present design-oriented robustness analysis.

4.2. Effect of Unmodeled Input Delay on Stability

We study a chain of N integrators, $N \in \{2, 3, 4\}$, with input gain $b > 0$ and an input delay.

The restriction to $N \in \{2, 3, 4\}$ is deliberate and follows the drive-architecture interpretation discussed in Section 2.1, together with the rapidly shrinking admissible normalized delay ratios $\delta_{\max, N}$ as N increases (which makes $N = 4$ already a strongly bandwidth-limited benchmark).

We denote the (possibly unmodeled) delay by T_d :

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ &\vdots \\ \dot{x}_{N-1}(t) &= x_N(t), \\ \dot{x}_N(t) &= b u(t - T_d). \end{aligned} \quad (43)$$

The controller is full state-feedback,

$$u(t) = -Kx(t), \quad K = \begin{bmatrix} k_1 & k_2 & \cdots & k_N \end{bmatrix}, \quad (44)$$

designed as if $T_d = 0$ by matching the closed-loop characteristic polynomial to a Manabe/CDM template [6,7,9]; see also [18–20] for complementary designs.

4.2.1. Nominal Manabe Polynomials for $\gamma_1 = 2.5$ and $\gamma_i = 2.0$

In this paper we use the standard CDM stability indices $\gamma_1 = 2.5$ and $\gamma_i = 2.0$ for $i \geq 2$. Using the same dimensionless Laplace variable $\sigma = \tau s$ as in Section 3, the corresponding dimensionless Manabe polynomials become:

$$N = 2: \quad P_2(\sigma) = \sigma^2 + 2.5\sigma + 2.5, \quad (45)$$

$$N = 3: \quad P_3(\sigma) = \sigma^3 + 5\sigma^2 + 12.5\sigma + 12.5, \quad (46)$$

$$N = 4: \quad P_4(\sigma) = \sigma^4 + 10\sigma^3 + 50\sigma^2 + 125\sigma + 125. \quad (47)$$

Therefore, the delay-free target characteristic polynomial in s has the form

$$P_M(s; \tau) = \frac{1}{\tau^N} P_N(\sigma) = s^N + \frac{\alpha_{N-1}}{\tau} s^{N-1} + \frac{\alpha_{N-2}}{\tau^2} s^{N-2} + \cdots + \frac{\alpha_0}{\tau^N}, \quad (48)$$

where $\sigma = \tau s$ and the coefficients α_i are read directly from (45) to (47). Matching $P_M(s; \tau)$ yields feedback gains that scale approximately as $k_i \sim 1/\tau^{N-i+1}$ (for fixed b), so decreasing τ increases the closed-loop bandwidth.

4.2.2. Characteristic Equation with Input Delay

When the input delay T_d is present, the closed-loop characteristic equation becomes a quasi-polynomial:

$$s^N + \left(\frac{\alpha_{N-1}}{\tau} s^{N-1} + \frac{\alpha_{N-2}}{\tau^2} s^{N-2} + \dots + \frac{\alpha_0}{\tau^N} \right) e^{-sT_d} = 0. \quad (49)$$

The term e^{-sT_d} introduces an additional phase lag $-\omega T_d$, which reduces stability margins and may destabilize a controller that is stable in the nominal, delay-free design [1,2].

4.2.3. Stability Depends Primarily on the Ratio T_d/τ

Using the normalized variable $\sigma = \tau s$ and the normalized delay

$$\delta = \frac{T_d}{\tau}, \quad (50)$$

Equation (49) can be rewritten in the compact form

$$\sigma^N + (\alpha_{N-1}\sigma^{N-1} + \dots + \alpha_1\sigma + \alpha_0) e^{-\sigma\delta} = 0. \quad (51)$$

Hence, for a fixed Manabe template (fixed α_i), stability is governed by the single dimensionless ratio $\delta = T_d/\tau$. There exists a critical value $\delta_{\max,N}$, such that the closed loop is stable for $0 \leq \delta < \delta_{\max,N}$ and loses stability at $\delta = \delta_{\max,N}$.

4.2.4. Delay Margin for the Adopted Manabe Template

To determine the delay margin for the adopted Manabe/CDM template, we analyze the first stability boundary of the normalized quasi-polynomial

$$\sigma^N + Q_N(\sigma) e^{-\sigma\delta} = 0, \quad \sigma = \tau s, \quad \delta = \frac{T_d}{\tau}, \quad (52)$$

where $Q_N(\sigma) = \alpha_{N-1}\sigma^{N-1} + \dots + \alpha_1\sigma + \alpha_0$ is obtained from the standard stability indices $\gamma_1 = 2.5$ and $\gamma_i = 2.0$ ($i \geq 2$); cf. (45)–(47). The critical ratio $\delta_{\max,N}$ is the smallest positive δ for which (52) has a purely imaginary root $\sigma = j\Omega$ ($\Omega > 0$). Substituting $\sigma = j\Omega$ yields the standard magnitude–phase conditions

$$|Q_N(j\Omega)| = \Omega^N, \quad (53)$$

$$\Omega\delta = \arg(Q_N(j\Omega)) - \arg(-(j\Omega)^N) + 2\pi k, \quad k \in \mathbb{Z}. \quad (54)$$

The delay margin corresponds to the first crossing; hence, we take the smallest positive δ generated by (53)–(54) (typically with $k = 0$).

For each N , $\delta_{\max,N}$ is obtained from the first imaginary-axis root $\sigma = j\Omega$ of (52). Substituting $\sigma = j\Omega$ yields the standard magnitude–phase conditions (53)–(54). For $N = 2$ the boundary can be computed in closed form, while for $N \in \{3, 4\}$, it is found by a one-dimensional numerical search in Ω . For the adopted Manabe/CDM template (standard indices) the resulting critical ratios are:

$$\delta_{\max,2} \approx 0.454, \quad \delta_{\max,3} \approx 0.210, \quad \delta_{\max,4} \approx 0.106. \quad (55)$$

These values immediately give a practical stability constraint on the selectable equivalent time constant:

$$\tau > \frac{T_d}{\delta_{\max,N}}. \quad (56)$$

Equivalently,

$$\tau_{\min} \approx \begin{cases} 2.20 T_d, & N = 2, \\ 4.77 T_d, & N = 3, \\ 9.42 T_d, & N = 4. \end{cases} \quad (57)$$

This result formalizes an important design trade-off: as N increases, the admissible ratio T_d/τ decreases for this standard Manabe template, so maintaining stability requires a more conservative (larger) τ .

4.3. Phase Margin and Gain Margin: Use as Delay-Robustness Indicators

Beyond stability, it is useful to quantify how much additional unmodeled phase lag or gain mismatch the loop can tolerate. We use the classical phase margin (PM) and gain margin (GM) of an associated open-loop transfer $L(s)$.

4.3.1. Phase Margin (PM)

Let ω_c satisfy $|L(j\omega_c)| = 1$. The phase margin is

$$PM = 180^\circ + \angle L(j\omega_c).$$

An unmodeled input delay contributes an extra phase lag $-\omega T_d$; therefore, PM is the most direct indicator of robustness against delay.

4.3.2. Gain Margin (GM)

Let ω_π satisfy $\angle L(j\omega_\pi) = -180^\circ$. The gain margin is

$$GM = \frac{1}{|L(j\omega_\pi)|}, \quad GM_{\text{dB}} = 20 \log_{10}(GM).$$

In this paper, GM is treated mainly as a verification metric once τ has been selected from a PM-driven constraint.

4.4. Computing the Phase Margin as a Function of the Normalized Delay δ

We evaluate the phase margin of the Manabe-template loop as a function of the normalized delay

$$\delta = \frac{T_d}{\tau} \in [0, \delta_{\max, N}),$$

using the normalized open-loop model

$$L_N(\sigma; \delta) = \frac{Q_N(\sigma)}{\sigma^N} e^{-\sigma\delta}, \quad Q_N(\sigma) = \alpha_{N-1}\sigma^{N-1} + \cdots + \alpha_1\sigma + \alpha_0, \quad (58)$$

where $\{\alpha_i\}$ are fixed by the adopted Manabe/CDM template (Section 4.2).

4.4.1. Gain Crossover Is Independent of δ

Since $|e^{-j\Omega\delta}| = 1$, the gain crossover frequency $\Omega_{c,N}$ is determined by the delay-free rational factor:

$$\left| \frac{Q_N(j\Omega_{c,N})}{(j\Omega_{c,N})^N} \right| = 1, \quad (59)$$

and therefore does not depend on δ .

4.4.2. Exact Affine Law for $PM_N(\delta)$

Define the delay-free phase margin (at $\delta = 0$) as

$$PM_N(0) = 180^\circ + \angle\left(\frac{Q_N(j\Omega_{c,N})}{(j\Omega_{c,N})^N}\right). \quad (60)$$

The input delay contributes an additional phase lag $-\Omega_{c,N}\delta$ (radians); hence,

$$PM_N(\delta) = PM_N(0) - \Omega_{c,N}\delta \cdot \frac{180^\circ}{\pi}, \quad 0 \leq \delta < \delta_{\max,N}. \quad (61)$$

4.4.3. PM-Driven Tuning Rule for τ

Given a required phase margin PM_{req} , the corresponding admissible normalized delay is

$$\delta_{\text{PM},N} = \frac{PM_N(0) - PM_{\text{req}}}{\Omega_{c,N}} \cdot \frac{\pi}{180^\circ}, \quad 0 < PM_{\text{req}} < PM_N(0). \quad (62)$$

Together with the stability constraint $\delta < \delta_{\max,N}$, a conservative design choice is

$$\delta \leq \min\{\delta_{\max,N}, \delta_{\text{PM},N}\}.$$

Using $\delta = T_d/\tau$ gives the explicit time-constant bound

$$\tau \geq \frac{T_d}{\min\{\delta_{\max,N}, \delta_{\text{PM},N}\}}. \quad (63)$$

Figure 3 illustrates the resulting linear degradation of $PM_N(\delta)$ over the stable range, and visualizes the exact affine law in (61): because the delay term has unit magnitude, the gain crossover $\Omega_{c,N}$ is fixed by (59), and the delay contributes only an additional phase lag $-\Omega_{c,N}\delta$. Hence the slope of each line is fully determined by $\Omega_{c,N}$, and the intercept is the nominal margin $PM_N(0)$. This makes Figure 3 a design chart: for a required PM_{req} , one reads the admissible δ (or uses (62)) and converts it to a direct lower bound on τ via (63).

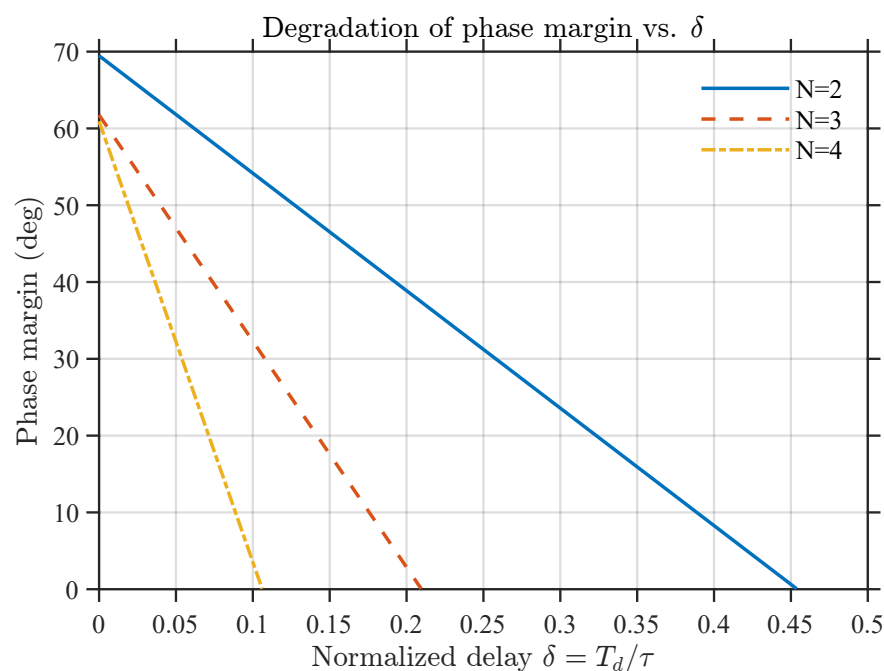


Figure 3. Phase-margin degradation $PM_N(\delta)$ versus normalized delay $\delta = T_d/\tau$ for Manabe/CDM state-feedback designs ($N = 2, 3, 4$). Curves are shown in the stable range $0 \leq \delta < \delta_{\max,N}$; increasing N reduces delay robustness.

Given an estimated effective input delay T_d and a target margin PM_{req} , Table 1 provides (i) the stability limit $\delta_{\max,N}$ and (ii) the pair $(\Omega_{c,N}, PM_N(0))$ needed to compute $\delta_{PM,N}$ from (62). A conservative selection is then $\delta \leq \min\{\delta_{\max,N}, \delta_{PM,N}\}$, which translates to $\tau \geq T_d/\delta$. For completeness, the physical gain-crossover frequency is $\omega_{c,N} = \Omega_{c,N}/\tau$.

Table 1. Characteristic values for the adopted Manabe/CDM template ($\gamma_1 = 2.5$, $\gamma_i = 2$ for $i \geq 2$): delay-stability limit $\delta_{\max,N}$, gain crossover $\Omega_{c,N}$ (from (59)), and nominal phase margin $PM_N(0)$.

N	$\delta_{\max,N}$	$\Omega_{c,N}$	$PM_N(0)$ [°]
2	0.4541	2.6696	69.46
3	0.2097	5.1388	61.75
4	0.1062	10.0163	60.92

4.5. Computing the Gain Margin as a Function of the Normalized Delay δ

This subsection outlines how to compute the gain margin (GM) of the delayed loop as a function of the normalized delay

$$\delta = \frac{T_d}{\tau} \in [0, \delta_{\max,N}),$$

for the Manabe-template designs introduced earlier. For a fixed polynomial template (fixed α_i), GM depends only on the dimensionless parameter δ .

4.5.1. Open-Loop Model Used for Margin Evaluation

As in Section 4.4, the delayed implementation is represented by an input delay e^{-sT_d} in the actuation channel. Using the normalized variable $\sigma = s\tau$, the open-loop transfer function is

$$L_N(\sigma; \delta) = \frac{Q_N(\sigma)}{\sigma^N} e^{-\sigma\delta}, \quad Q_N(\sigma) = \alpha_{N-1}\sigma^{N-1} + \dots + \alpha_1\sigma + \alpha_0, \quad (64)$$

with α_i fixed by the adopted Manabe/CDM template (Section 4.2).

The gain margin is defined at the phase-crossover frequency $\Omega_{\pi,N}(\delta)$, i.e., the smallest $\Omega > 0$ for which the open-loop phase equals -180° :

$$\angle L_N(j\Omega_{\pi}; \delta) = -180^\circ. \quad (65)$$

Then the (linear) gain margin is

$$GM_N(\delta) = \frac{1}{|L_N(j\Omega_{\pi,N}(\delta); \delta)|}, \quad (66)$$

and in decibels

$$GM_{\text{dB},N}(\delta) = 20 \log_{10} GM_N(\delta). \quad (67)$$

4.5.2. Dependence on δ and Numerical Evaluation

The gain margin is defined at the phase-crossover frequency $\Omega_{\pi,N}(\delta)$, i.e., the smallest $\Omega > 0$ such that (65) holds. Because the delay contributes an additional phase lag $-\Omega\delta$ (radians), $\Omega_{\pi,N}$ depends on δ , and consequently $GM_N(\delta)$ is generally a nonlinear function of δ . Figure 4 summarizes $GM_N(\delta)$ for $N = 2, 3, 4$ over the stable range $0 \leq \delta < \delta_{\max,N}$.

In contrast to PM, the gain margin is evaluated at the phase crossover, and the phase condition (65) contains the delay term explicitly. Therefore, the crossover frequency $\Omega_{\pi,N}(\delta)$ shifts with δ , which makes $GM_N(\delta)$ intrinsically nonlinear. Figure 4 should be interpreted

as a verification chart: after selecting τ primarily from the PM constraint, one checks whether the corresponding $\delta = T_d/\tau$ yields an acceptable GM level for the expected gain uncertainty.

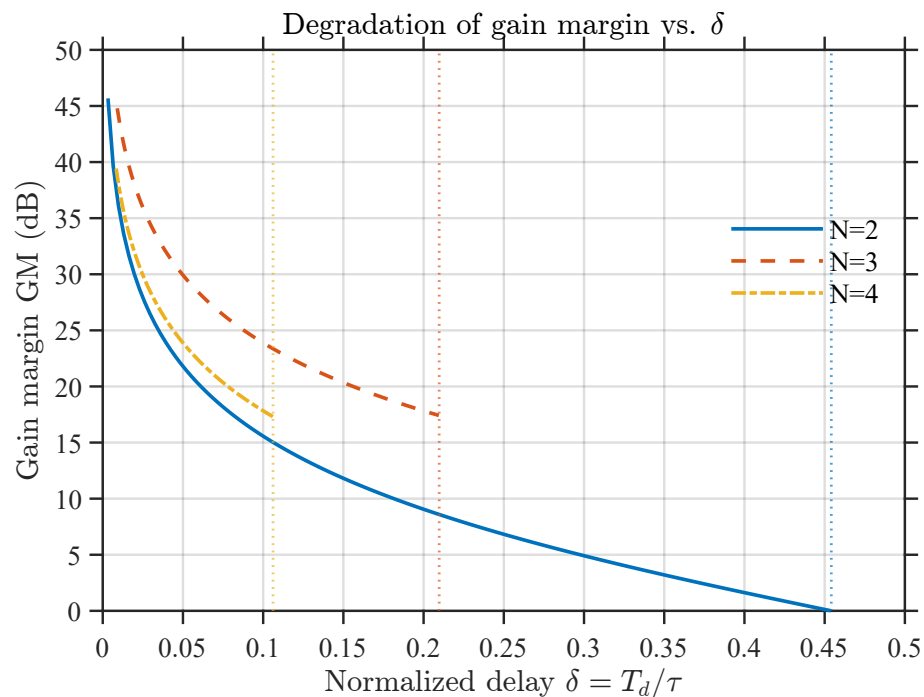


Figure 4. Degradation of the gain margin GM (in dB) as a function of the normalized input delay $\delta = T_d/\tau$ for Manabe/CDM-based state-feedback designs of orders $N = 2, 3, 4$. For each N , the curve is evaluated only in the stable range $\delta \in [0, \delta_{\max,N})$, and the corresponding vertical dashed line marks the delay-stability limit $\delta_{\max,N}$.

4.5.3. Closed-Form Reduction for $N = 2$

For $N = 2$, $Q_2(\sigma) = 2.5(1 + \sigma)$ and

$$L_2(\sigma; \delta) = \frac{2.5(1 + \sigma)}{\sigma^2} e^{-\sigma\delta}. \quad (68)$$

The phase-crossover condition $\angle L_2(j\Omega_\pi; \delta) = -180^\circ$ reduces to

$$\arctan(\Omega_\pi) = \Omega_\pi \delta, \quad (69)$$

(with both sides in radians), solved for the smallest $\Omega_\pi > 0$ for a given $\delta > 0$. Since $|e^{-j\Omega\delta}| = 1$, the magnitude at the crossover is

$$|L_2(j\Omega_\pi; \delta)| = \frac{2.5\sqrt{1 + \Omega_\pi^2}}{\Omega_\pi^2}; \quad (70)$$

hence,

$$GM_2(\delta) = \frac{\Omega_\pi^2}{2.5\sqrt{1 + \Omega_\pi^2}}, \quad GM_{2,\text{dB}}(\delta) = 20 \log_{10} \left(\frac{\Omega_\pi^2}{2.5\sqrt{1 + \Omega_\pi^2}} \right). \quad (71)$$

4.5.4. Extension to $N \in \{3, 4\}$

For higher orders the same procedure applies: for each δ solve the one-dimensional equation

$$\angle \left(\frac{Q_N(j\Omega_\pi)}{(j\Omega_\pi)^N} \right) - \Omega_\pi \delta \cdot \frac{180^\circ}{\pi} = -180^\circ \quad (72)$$

for the smallest $\Omega_\pi > 0$, and then evaluate

$$GM_N(\delta) = \left| \frac{Q_N(j\Omega_\pi)}{(j\Omega_\pi)^N} \right|^{-1}, \quad GM_{dB,N}(\delta) = 20 \log_{10} GM_N(\delta). \quad (73)$$

4.5.5. Role in the Tuning Workflow

In this paper, τ is selected primarily from the PM-driven constraint (Section 4.4). The gain margin is then checked a posteriori using $GM_N(\delta)$ (Figure 4) to ensure adequate tolerance to loop-gain variations.

4.5.6. Rat22 Approximation for Quick Checks

For fast engineering evaluation, the numerically obtained curves can be approximated by

$$GM_N(\rho) \approx \frac{p_{0,N} + p_{1,N}\rho + p_{2,N}\rho^2}{1 + q_{1,N}\rho + q_{2,N}\rho^2}, \quad \rho := \frac{\delta}{\delta_{\max,N}} \in [0, 1), \quad (74)$$

with

$$GM_{N,dB}(\rho) = 20 \log_{10}(GM_N(\rho)). \quad (75)$$

The parameters are listed in Table 2.

Table 2. Rat22 approximation parameters for the gain-margin curve $GM_N(\rho)$, where $\rho = \delta/\delta_{\max,N} \in [0, 1)$. The approximation is $GM_N(\rho) \approx \frac{p_{0,N} + p_{1,N}\rho + p_{2,N}\rho^2}{1 + q_{1,N}\rho + q_{2,N}\rho^2}$.

N	$p_{0,N}$	$p_{1,N}$	$p_{2,N}$	$q_{1,N}$	$q_{2,N}$
2	7.11477×10^4	-1.16339×10^5	2.37407×10^4	5.12713×10^4	-7.15903×10^4
3	2.28809×10^6	-2.98547×10^6	2.65952×10^4	3.05452×10^5	-3.95813×10^5
4	4.72113×10^5	-4.45326×10^5	3.88957×10^3	6.38031×10^4	-5.96211×10^4

The main advantage of the proposed framework is its design-oriented nature: for the adopted Manabe/CDM template, the closed-loop delay sensitivity is governed primarily by the dimensionless ratio $\delta = T_d/\tau$, which enables (i) explicit stability limits $\delta_{\max,N}$ and the corresponding lower bound on τ , and (ii) a particularly simple PM-based inequality to size τ for a required robustness level, supported by compact design charts. In addition, all results are expressed in normalized form and therefore transfer across plants of the same integrator-chain order and template indices. The main limitation is the intentionally narrow benchmark scope: the plant is restricted to pure chains of integrators ($N \in \{2, 3, 4\}$), and the delay is modeled as a lumped input delay in continuous time. Consequently, the presented rules should be interpreted as lightweight screening and tuning guidelines that isolate the effect of unmodeled input delay, rather than as a universal delay-robust synthesis method for general servo plants with flexible modes, nonlinearities, sampling/computation effects, or distributed/output delays. Experimental validation and extensions to more realistic models are therefore left to future work.

5. Conclusions and Future Work

This paper investigated the Manabe/CDM-based state-feedback design for plants approximated by a chain of N integrators in the presence of an unmodeled input delay. The Manabe polynomial concept provides a fast and convenient way to synthesize closed-loop characteristic polynomials through a small number of interpretable parameters, most notably the equivalent time constant τ and the stability-index pattern. In the comparative study, the resulting closed-loop behavior was generally superior to that obtained with common reference patterns, such as multiple repeated real poles or repeated oscillatory

sections, especially when comparable dominant time constants were enforced. This makes the Manabe/CDM template attractive as a practical pole-shaping rule for high-order servo-type loops.

The contribution of this paper is primarily a transparent, normalized set of design rules linking τ to an estimated unmodeled input delay via $\delta = T_d/\tau$ (including explicit $\delta_{\max,N}$ values and a simple PM-driven tuning inequality). At the same time, the results are derived for an idealized integrator-chain benchmark with a lumped input delay, and should be used as a fast engineering guideline; extensions to higher-fidelity servo models, discrete-time effects, and experimental validation are part of planned future work.

A key practical motivation is that input delay in drive loops is distributed across measurement, computation, and actuation and is therefore often only partially known. As detailed in Section 4, this motivates a design workflow that explicitly constrains $\delta = T_d/\tau$ to preserve stability and robustness margins. The presented analysis confirms that neglecting such delays degrades closed-loop properties and, for sufficiently small τ (high bandwidth), may lead to loss of stability. For the adopted Manabe/CDM template and orders $N \in \{2, 3, 4\}$, the work provided explicit stability limits in terms of the normalized delay $\delta = T_d/\tau$, summarized by the critical values $\delta_{\max,N}$ and supported by numerical verification.

To enable efficient tuning of Manabe polynomial coefficients in the presence of delays, this paper proposed margin-based design rules formulated in terms of phase margin (PM) and gain margin (GM). The PM-oriented approach leads to a particularly simple and design-friendly inequality linking the admissible τ to the estimated delay T_d , the plant order N , and the required phase-margin level. The GM condition is less amenable to a similarly compact synthesis inequality; therefore, it is recommended to use the PM-based constraint as the primary design rule and subsequently verify the GM requirement using the provided $GM(\delta)$ plots. For engineering convenience, the gain-margin curves can additionally be approximated using the Rat22 rational model in the normalized variable $\rho = \delta/\delta_{\max,N}$, with the fitted parameters listed in Table 2.

Overall, the comparison of PM- and GM-based constraints indicates that the phase-margin requirement is typically the decisive factor when selecting τ under unmodeled input delay, while the gain-margin requirement should be treated as a verification step after τ has been fixed. This establishes a clear and lightweight design workflow: select the order-dependent Manabe template, choose τ from the PM inequality for the estimated T_d and the desired robustness level, and finally confirm that the resulting GM remains acceptable over the expected delay range.

Future work will focus on the robustness of the proposed procedure with respect to variations in plant parameters beyond the input gain, including uncertainty in inertial and damping terms and deviations from the ideal chain-of-integrators structure. Further directions include extending the analysis to discrete-time implementations (sampling and computation delay treated explicitly), incorporating actuator constraints (saturation and rate limits), and validating the design charts in high-fidelity simulation case studies and experimentally on representative drive and motion-control platforms.

In particular, extensions to multiple/distributed delay structures (including explicit sampling/computation delays) and to higher-fidelity servo models with flexible modes and nonlinearities are planned; the present δ -based charts are intended as a baseline reference for tuning and for verifying the delay tolerance of delay-free fixed-gain designs.

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