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Tripled Fixed Points and Tripled Best Proximity Points in Modular Function Spaces

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Abstract

We establish a modular-space framework for the study of tripled fixed points and tripled best proximity points. Under suitable assumptions on the underlying modular (convexity, the Δ_2 property, uniform continuity, and uniform convexity-type properties), we prove that Banach theorems guarantee the existence, uniqueness, and convergence of modular iterative schemes. In particular, we develop results for cyclic ρ -Kannan contraction maps and pairs, showing that both tripled fixed points and tripled best proximity points arise uniquely and attract all iterative trajectories. An illustrative example in the space $L^2[0, 1]$ with integral operators demonstrates the applicability of the theory and the predicted rate of convergence. These results extend classical fixed point methods to a broader modular setting and open the way for applications in nonlinear functional equations.

Keywords: tripled fixed points; tripled best proximity points; modular function spaces; cyclic ρ -Kannan contractions; Banach-type fixed-point theorems; Δ_2 property; integral operators



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1. Introduction

The Banach contraction principle [1] is a cornerstone of fixed point theory and has inspired a vast body of literature devoted to extensions and generalizations in various directions. In particular, the introduction of coupled fixed points by Bhaskar and Lakshmikantham [2] opened a new line of research, which was further developed by Berinde [3,4], Lakshmikantham, Ćirić [5], and many others. Coupled fixed-point results have found applications in matrix and integral equations [6].

The best proximity point theory, initiated in the research of Eldred and Veeramani [7] and Sehgal and Singh [8], addresses the problem of approximating fixed points when self-mappings are replaced by non-self mappings between two sets with nontrivial intersections. It is worth mentioning that the first ideas with different notions were suggested by the authors of [8], and later, a simplified model was proposed by the authors of [7], which is currently exploited in a substantial number of generalizations and applications.

The rapid development in the aforementioned field stimulated further results for generalized contractions, monotone operators, and ordered metric spaces [9–12]. The

study of multi-point fixed points has since become an active direction, connecting with approximation theory, optimization, and nonlinear functional analysis.

At the same time, modular function spaces provide a natural setting for extending classical fixed-point theory beyond normed spaces initiated by Kozłowski [13,14] and Musielak [15,16], unifying and generalizing classical Banach space structures. The theory of fixed points in a modular function space was initiated by Khamsi, Kozłowski, and Reich [17]. Important results on fixed points in modular function spaces and the related geometric properties of these spaces are reported by the authors of [18]. The theory of best proximity points was proposed by the authors of [19], where new geometric properties were obtained and existence and uniqueness results were proven.

Rapid development has occurred in the fixed-point theory of Musielak–Orlicz spaces [20–22] and variable exponent spaces [23–26].

Tripled fixed points and tripled best proximity points were introduced by the authors of [27,28]. Some recent applications of the tripled fixed-point notion were presented by the authors of [29] for solving systems of nonlinear matrix equations, and the authors of [30] also reported an investigating equilibrium in oligopoly markets with three dominating players.

According to the introduction of the Kannan contraction condition [31], it can be used in the search of fixed points for discontinuous maps, such as step functions. In this paper, we attempt to develop a theory of tripled fixed points and tripled best proximity points in modular function spaces and apply it in solving impulse integral equations. Let us recall a map $T : L_2[0, 1] \rightarrow L_2[0, 1]$, where we denote all functions with a Lebesgue integrable square by $L_2[0, 1]$, and we search for a contractive condition of Kannan's type:

$$\begin{aligned}\|Tf - Tg\|_2 &= \sqrt{\int_0^1 (Tf(x) - Tg(t))^2 dt} \\ &\leq \gamma \left(\sqrt{\int_0^1 (f(x) - Tf(x))^2 dt} + \sqrt{\int_0^1 (g(x) - Tg(x))^2 dt} \right) \\ &= \gamma (\|f - Tf\|_2 + \|g - Tg\|_2),\end{aligned}$$

where $\|\cdot\|_2$ is the Hilbert's norm in $L_2[0, 1]$, and $\gamma \in (0, 1/2)$. A few inequalities can be used. In comparison, we have

$$\int_0^1 (Tf(x) - Tg(t))^2 dt \leq \gamma \left(\int_0^1 (f(x) - Tf(x))^2 dt + \int_0^1 (g(x) - Tg(x))^2 dt \right),$$

where $\|\cdot\|_2^2$ and the square of the Hilbert norm is a modular function that can be used to calculate the modular distance between two points.

2. Preliminaries

For the sake of brevity, we only recall fundamental notions and results from the theory of modular function spaces that are directly relevant to the subsequent results. For a more complete background on the theory of modular function spaces, we refer the reader to the research of the authors of [13,14,17] and the monographs published by the authors of [15,18].

We will follow the concepts and notations used by the authors of [18], which are utilized in the literature.

Let Ω be a nonempty set and Σ be a non-trivial σ algebra of subsets of Ω . Let $\mathcal{P}$ be a δ ring of subsets of Ω such that for any $E \in \mathcal{P}$ and $A \in \Sigma$, $E \cap A \in \mathcal{P}$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup_n K_n$.

By $\mathcal{E}$, we denote the linear space of all functions with supports that belong to $\mathcal{P}$. By $\mathcal{M}_\infty$, we denote the space of all extended measurable functions $f : \Omega \rightarrow [-\infty, \infty]$ such that there exists a function sequence $g_n \in \mathcal{E}$, with $|g_n| \leq |f|$ and for all $\omega \in \Omega$, $g_n(\omega) \rightarrow f(\omega)$.

For $A \subset \Omega$, we write $\mathbf{1}_A$ for the characteristic function of A .

Let us mention that, in view of the fact that we are considering measurable functions, we will always understand the equality of two functions and the limit convergence a.e. (almost everywhere).

Definition 1 ([18]). Let $\rho : \mathcal{M}_\infty \rightarrow [0, \infty)$ be a non-trivial even and convex function. We say that ρ is a convex regular function pseudomodular if the following is the case:

1. $\rho(0) = 0$;
2. ρ is monotone in the sense that $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
3. ρ is orthogonally subadditive, $\rho(f\mathbf{1}_{A \cup B}) \leq \rho(f\mathbf{1}_A) + \rho(f\mathbf{1}_B)$, where $A, B \in \Sigma$ with $A \cap B = \emptyset$ and $f \in \mathcal{M}_\infty$;
4. ρ possesses the Fatou property whenever $f_n(\omega) \uparrow |f(\omega)|$ for every $\omega \in \Omega$ entails $\rho(f_n) \uparrow \rho(f)$, for $f_n \in \mathcal{M}_\infty$;
5. ρ is order-continuous on $\mathcal{E}$, as $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ imply $\rho(g_n) \downarrow 0$;

Definition 2 ([18], p. 116). A modular function ρ is called uniformly continuous if, for any $L > 0$ and any $\varepsilon > 0$, there exists $\delta(L, \varepsilon) > 0$ such that, if $\rho(x) \leq L$ and $\rho(y) < \delta$, the inequality $|\rho(y + x) - \rho(x)| < \varepsilon$ holds and is denoted by $\rho \in \mathfrak{R}$.

If $\lim_{n \rightarrow \infty} \rho(x_n - x) = 0$ and $\{x_n\}_{n=1}^\infty$ is a bounded sequence, then, for any $y \in L_\rho$, there holds $\rho(x - y) = \lim_{n \rightarrow \infty} \rho(x_n - y)$.

Analogously to the measure-space setting, a set $A \in \Sigma$ is ρ -null if $\rho(g\mathbf{1}_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ -almost everywhere if the exceptional set is ρ -null. As usual, we identify measurable sets for which their symmetric difference is ρ -null and measurable functions that differ only on a ρ -null set. With this convention, we set where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is regarded as an equivalence class of functions with equal ρ almost everywhere. When no ambiguity arises, we simply write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 3 ([18]). We say that ρ is a regular convex modular function if $\rho(f) = 0$ implies $f = 0, \rho$.

The class of all non-zero regular convex modular functions defined on Ω will be denoted by $\mathfrak{R}$.

Definition 4 ([18]). Let $\rho : \mathcal{M} \rightarrow [0, \infty)$ be a convex modular function. The associated modular function space is as follows:

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \rightarrow 0 \text{ as } \lambda \rightarrow 0\},$$

and it is equipped with the following Luxemburg norm:

$$\|f\|_\rho = \inf \left\{ \alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq 1 \right\}.$$

Definition 5 ([18]). Let L_ρ be a modular function space. Then, we use the following notations:

1. $f_n \rightarrow f$ (ρ) if $\rho(f_n - f) \rightarrow 0$;
2. $\{f_n\}$ is ρ -Cauchy if $\rho(f_m - f_n) \rightarrow 0$ as $m, n \rightarrow \infty$;
3. L_ρ is said to be ρ -complete whenever every ρ -Cauchy sequence is ρ -convergent;

4. $A \subset L_\rho$ is ρ -closed if the limit f of every ρ -convergent sequence $\{f_n\}_{n=1}^\infty \subset A$ belongs to A ;
5. $A \subset L_\rho$ is ρ -bounded if there is $f \in L_\rho$ and a constant M such that $\rho(f, g) < M$ for every $g \in A$;
6. ρ has the Δ_2 property if $\rho(f_n) \rightarrow 0$ implies $\rho(2f_n) \rightarrow 0$.

Theorem 1 ([18]). Let ρ be a convex modular. Then, we have the following:

1. $(L_\rho, \|\cdot\|_\rho)$ is complete;
2. If $\rho(\alpha f_n) \rightarrow 0$ for some $\alpha > 0$, then a subsequence converges ρ -a.e. to 0;
3. (Fatou property) If $f_n \rightarrow f$ ρ -a.e., then $\rho(f) \leq \liminf \rho(f_n)$.

Theorem 2 ([18]). The space (L_ρ, ρ) is ρ -complete.

Theorem 3 ([18]). If ρ satisfies the Δ_2 property, then ρ -convergence is equivalent to norm convergence in L_ρ .

Different generalizations of the notion for uniform convexity in normed spaces have been introduced and investigated by the authors of [16,18] in their study of the geometry of modular function spaces. We will recall only the notion UC1, which is relevant in our investigation.

Definition 6 ([18]). A modular ρ satisfies condition UC1 if there exists a function $\delta_1 : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\rho(f), \rho(g) \leq r, \rho(f - g) \geq \varepsilon \Rightarrow \rho\left(\frac{f+g}{2}\right) \leq r - \delta_1(r, \varepsilon).$$

Definition 7 ([19]). Let (X, ρ) be a modular function space, and let $A, B \subseteq X$ be nonempty subsets. The modular distance between A and B is defined as follows:

$$d_\rho(A, B) := \inf\{\rho(a - b) \mid a \in A, b \in B\}.$$

For brevity, we denote $d_\rho = d_\rho(A, B)$.

The next lemmas generalize the results from [7] in the context of modular function spaces instead of uniformly convex Banach spaces.

Lemma 1 ([19]). Let $\rho \in \mathfrak{R}$. Let ρ be UC1, and it has the Δ_2 property. Let $A \subset L_\rho$ be a ρ -closed and convex subset, $B \subset L_\rho$ be a ρ -closed subset, and $A \cup B$ be ρ -bounded. If the sequences $\{u_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \subset A$ and $\{v_n\}_{n=1}^\infty \subset B$ are such that

1. $\lim_{n \rightarrow \infty} \rho(w_n - v_n) = d_\rho$;
2. For every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for every $m > n \geq N_0$, the inequality $\rho(u_m - v_n) \leq d_\rho + \varepsilon$ holds;

then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $m > n \geq N_1$, there holds the inequality $\rho(u_m - w_n) < \varepsilon$.

Lemma 2 ([19]). Let $\rho \in \mathfrak{R}$. Let ρ be UC1, and it has the Δ_2 property; let A be a ρ -closed and convex subset of L_ρ , B be a ρ -closed subset of L_ρ , and $A \cup B$ be ρ -bounded. If the sequences $\{u_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \subset A$ and $\{v_n\}_{n=1}^\infty \subset B$ are such that

1. $\lim_{n \rightarrow \infty} \rho(w_n - v_n) = d_\rho$;
2. $\lim_{n \rightarrow \infty} \rho(u_n - v_n) = d_\rho$;

then $\lim_{n \rightarrow \infty} \rho(u_n - w_n) = 0$.

Lemma 3 ([19]). Let $\rho \in \mathfrak{R}$. ρ has the Δ_2 property, and it is be uniformly continuous; let $A, B \subset L_\rho$ denote subsets, and $A \cup B$ is ρ -bounded. If the sequences $\{u_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \subset A$ and $\{v_n\}_{n=1}^\infty \subset B$ are such that

1. $\lim_{n \rightarrow \infty} \rho(w_n - u_n) = 0$;
2. $\lim_{n \rightarrow \infty} \rho(w_n - v_n) = d_\rho$;

then $\lim_{n \rightarrow \infty} \rho(u_n - v_n) = d_\rho$.

3. Results

In this section, we work within a modular function space L_ρ and study two complementary notions: tripled fixed points and tripled best proximity points generated by ρ -Kannan-type contractions. We begin by introducing the modular distance between sets and formal definitions of the two concepts. Next, we present existence and uniqueness theorems together with constructive iteration schemes that converge (in the modular sense) to the desired triples. The results are established under standard assumptions on ρ (Fatou property, Δ_2 , UC1, and uniform continuity) and mild geometric conditions on the underlying subsets (non-emptiness, ρ -closedness, ρ -boundedness, and convexity).

3.1. Tripled Fixed Points for Kannan-Type Mappings in Modular Function Spaces

The notion of coupled fixed points was introduced by the authors of [32]. More recently, a more concise version was proposed by the authors of [2], which can be considered as the starting point for investigations of the theory of coupled, tripled, and n -tupled fixed points. The introduction of n -tupled fixed points is carried out by the authors of [12]. Following the authors of [12], we recall the n -tupled fixed-point definition in the particular case of tripled fixed points.

Definition 8 ([12]). Let A be a nonempty subset of a modular function space X and $F: A \times A \times A \rightarrow A$. An ordered triple $(x, y, z) \in A \times A \times A$ is a tripled fixed point of F in A if

$$x = F(x, y, z), \quad y = F(y, z, x), \quad z = F(z, x, y).$$

It is worth mentioning that the in-depth investigation carried out by the authors of [33] presents a connection between coupled fixed points and fixed points. We rewrite this connection in the context of tripled fixed points. We take F from Definition 8 and define $T: A^3 \rightarrow A^3$ as $T(x, y, z) = (F(x, y, z), F(y, z, x), F(z, x, y))$; then, the ordered triple (x, y, z) is a tripled fixed point for (F) if and only if it is a fixed point for T . For the rest of this study, we use the notation T for the functions defined above.

As the geometry of modular function spaces can be very unusual, there are no usable results concerning Cartesian products of such spaces, in contrast to what is known for metric and normed spaces. Therefore, we provide proofs following the classical technique from the authors of [2,32] instead of that proposed by the authors of [33].

The following notation will be useful for fitting certain formulas into the text field and simplifying them.

Definition 9. Let $A \subset X$ be nonempty and $F: A^3 \rightarrow A$. For any initially chosen ordered triple of points $(x_0, y_0, z_0) \in A^3$, we consider the iteratively constructed sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty \subset A$, defined by $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, z_n, x_n)$, and $z_{n+1} = F(z_n, x_n, y_n)$ or equivalently $(x_{n+1}, y_{n+1}, z_{n+1}) = T(x_n, y_n, z_n)$ for $n \geq 0$.

In what follows, we will always assume that the iterated sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty, \{z_n\}_{n=0}^\infty \subset A$ are those defined in Definition 9.

Definition 10. Let $A \subset X$ be nonempty and $F: A^3 \rightarrow A$. We call F a ρ -Kannan contraction map if there exists $\alpha \in [0, \frac{1}{2})$ such that, for all $x, y, z, u, v, w \in A$,

$$\rho(F(x, y, z) - F(u, v, w)) \leq \alpha(\rho(x - F(x, y, z)) + \rho(u - F(u, v, w))).$$

Theorem 4. Let $\mathfrak{R}$ and let $A \subset L_\rho$ be nonempty, ρ -bounded, and ρ -closed. If $F: A^3 \rightarrow A$ is a ρ -Kannan contraction map, then the following is the case:

1. F possesses a unique tripled fixed point $(x, y, z) \in A^3$;
2. For any initially chosen triple $(x_0, y_0, z_0) \in A^3$, the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty$ converge (in the modular sense) to the unique tripled fixed point (x, y, z) .

Proof. Let $(x_0, y_0, z_0) \in A^3$ be an arbitrary initial triple, and we consider the sequences given by the following iteration scheme: $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$.

Using the ρ -Kannan contraction property, for each $n \geq 1$, we have the following:

$$\begin{aligned} \rho(x_{n+1} - x_n) &= \rho(F(x_n, y_n, z_n) - F(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leq \alpha(\rho(x_n - F(x_n, y_n, z_n)) + \rho(x_{n-1} - F(x_{n-1}, y_{n-1}, z_{n-1}))) \\ &= \alpha(\rho(x_{n+1} - x_n) + \rho(x_n - x_{n-1})). \end{aligned}$$

Analogous inequalities hold cyclically for the following:

$$\begin{aligned} \rho(y_{n+1} - y_n) &= \rho(F(y_n, z_n, x_n) - F(y_{n-1}, z_{n-1}, x_{n-1})) \\ &\leq \alpha(\rho(y_{n+1} - y_n) + \rho(y_n - y_{n-1})) \end{aligned}$$

and

$$\begin{aligned} \rho(z_{n+1} - z_n) &= \rho(F(z_n, x_n, y_n) - F(z_{n-1}, x_{n-1}, y_{n-1})) \\ &\leq \alpha(\rho(z_{n+1} - z_n) + \rho(z_n - z_{n-1})). \end{aligned}$$

We define the following:

$$S_n(x_0, y_0, z_0) = \rho(x_{n+1} - x_n) + \rho(y_{n+1} - y_n) + \rho(z_{n+1} - z_n),$$

and we will use this notation for the rest of this study.

Summing the last three inequalities, we obtain

$$S_n(x_0, y_0, z_0) \leq \alpha(S_n(x_0, y_0, z_0) + S_{n-1}(x_0, y_0, z_0)),$$

and obtain the following key inequality:

$$S_n(x_0, y_0, z_0) \leq \frac{\alpha}{1 - \alpha} S_{n-1}(x_0, y_0, z_0),$$

with the crucial assumption $0 < \alpha < \frac{1}{2}$.

Setting $\beta = \frac{\alpha}{1 - \alpha} \in (0, 1)$, iterating yields the limit when $n \rightarrow \infty$:

$$S_n(x_0, y_0, z_0) \leq \beta^n S_0(x_0, y_0, z_0) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We define

$$W_{n,m}(x_0, y_0, z_0) = \rho(x_{n+p} - x_n) + \rho(y_{n+p} - y_n) + \rho(z_{n+p} - z_n), \quad (1)$$

and this notation we will use for the rest of this study.

Next, we show that the sequences are Cauchy. Indeed, for arbitrary $n, p \in \mathbb{N}$:

$$W_{n+p,n}(x_0, y_0, z_0) \leq \beta^n W_{p,0}(x_0, y_0, z_0).$$

Since A is ρ -bounded, there exists $M > 0$ with $\rho(u - v) \leq M$ for all $u, v \in A$. Thus,

$$W_{n+p,n}(x_0, y_0, z_0) \leq 3M\beta^n.$$

Given that $\beta^n \rightarrow 0$, for any $\varepsilon > 0$, $N \in \mathbb{N}$ is chosen such that, for all $n \geq N$, $3M\beta^n < \varepsilon$. Hence, the sequences are Cauchy in nature.

By the ρ -completeness of L_ρ and ρ -closedness of A , there exist limits $(x, y, z) \in A^3$ satisfying the following:

$$\lim_{n \rightarrow \infty} \rho(x_n - x) = 0, \quad \lim_{n \rightarrow \infty} \rho(y_n - y) = 0, \quad \lim_{n \rightarrow \infty} \rho(z_n - z) = 0.$$

We now prove that (x, y, z) is a tripled fixed point. Using the uniform continuity of ρ and the Kannan contraction condition, we observe the following:

$$\begin{aligned} \rho(x - F(x, y, z)) &= \lim_{n \rightarrow \infty} \rho(x_{n+1} - F(x, y, z)) \\ &= \lim_{n \rightarrow \infty} \rho(F(x_n, y_n, z_n) - F(x, y, z)) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(x_n - F(x_n, y_n, z_n)) + \rho(x - F(x, y, z))) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(x_n - x_{n+1}) + \rho(x - F(x, y, z))) \\ &= \alpha\rho(x - F(x, y, z)). \end{aligned}$$

From the assumption that $\alpha \in [0, 1/2)$, it follows that $\rho(x - F(x, y, z)) = 0$, i.e., $x = F(x, y, z)$. In a similar fashion, we obtain the next two chains of inequalities:

$$\begin{aligned} \rho(y - F(y, z, x)) &= \lim_{n \rightarrow \infty} \rho(y_{n+1} - F(y, z, x)) \\ &= \lim_{n \rightarrow \infty} \rho(F(y_n, z_n, x_n) - F(y, z, x)) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(y_n - F(y_n, z_n, x_n)) + \rho(y - F(y, z, x))) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(y_n - y_{n+1}) + \rho(y - F(y, z, x))) \\ &= \alpha\rho(y - F(y, z, x)). \end{aligned}$$

and

$$\begin{aligned} \rho(z - F(z, x, y)) &= \lim_{n \rightarrow \infty} \rho(z_{n+1} - F(z, x, y)) \\ &= \lim_{n \rightarrow \infty} \rho(F(z_n, x_n, y_n) - F(z, x, y)) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(z_n - F(z_n, x_n, y_n)) + \rho(z - F(z, x, y))) \\ &\leq \lim_{n \rightarrow \infty} \alpha(\rho(z_n - z_{n+1}) + \rho(z - F(z, x, y))) \\ &= \alpha\rho(z - F(z, x, y)). \end{aligned}$$

It follows that $(x, y, z) = T(x, y, z)$, i.e., (x, y, z) is a tripled fixed point for F .

To establish uniqueness, suppose that (u, v, w) is another tripled fixed point. Then,

$$\begin{aligned} \rho(x - u) &= \rho(F(x, y, z) - F(u, v, w)) \\ &\leq \alpha(\rho(x - F(x, y, z)) + \rho(u - F(u, v, w))) = 0. \end{aligned}$$

Similarly, $\rho(y - v) = \rho(z - w) = 0$. Indeed, using the symmetry in the definition of the tripled fixed points $x = F(x, y, z)$, $y = F(y, z, x)$, and $z = F(z, x, y)$ and $u = F(u, v, w)$, $v = F(v, w, u)$, and $w = F(w, u, v)$, we obtain the following inequalities:

$$\begin{aligned}\rho(y - v) &= \rho(F(y, z, x) - F(v, w, u)) \\ &\leq \alpha(\rho(y - F(y, z, x)) + \rho(v - F(v, w, u))) = 0\end{aligned}$$

and

$$\begin{aligned}\rho(z - w) &= \rho(F(z, x, y) - F(w, u, v)) \\ &\leq \alpha(\rho(z - F(z, x, y)) + \rho(w - F(w, u, v))) = 0.\end{aligned}$$

Therefore, $\rho(y - v) = \rho(z - w) = 0$, and thus, $(x, y, z) = (u, v, w)$.

Finally, convergence to the unique tripled fixed point (x, y, z) for an arbitrary initial triple $(u_0, v_0, w_0) \in A^3$, $(u_0, v_0, w_0) \neq (x_0, y_0, z_0)$, follows from the proof: the iterated sequences converge to a tripled fixed point, and by uniqueness, the fixed point must be (x, y, z) .

Suppose that the tripled fixed point (x, y, z) does not comprise equal elements, i.e.,

$$\rho(x - y) + \rho(y - z) + \rho(z - x) > 0.$$

From

$$\begin{aligned}\rho(x - y) &= \rho(F(x, y, z) - F(y, z, x)) \\ &\leq \alpha(\rho(x - F(x, y, z)) + \rho(y - F(y, z, x))) = 0,\end{aligned}$$

$$\begin{aligned}\rho(y - z) &= \rho(F(y, z, x) - F(z, x, y)) \\ &\leq \alpha(\rho(y - F(y, z, x)) + \rho(z - F(z, x, y))) = 0,\end{aligned}$$

and

$$\begin{aligned}\rho(z - x) &= \rho(F(z, x, y) - F(x, y, z)) \\ &\leq \alpha(\rho(z - F(z, x, y)) + \rho(x - F(x, y, z))) = 0\end{aligned}$$

it follows that $\rho(x - y) = \rho(y - z) = \rho(z - x) = 0$, and consequently, the tripled fixed point of F is (x, x, x) . $\square$

3.2. Tripled Best Proximity Points in Modular Function Spaces

Definition 11. Let A and B be nonempty subsets of a modular function space L_p , and let $F: A \times A \times A \rightarrow B$. An ordered triple $(x, y, z) \in A \times A \times A$ is called a tripled best proximity point of F if

$$\rho(x - F(x, y, z)) = \rho(y - F(y, z, x)) = \rho(z - F(z, x, y)) = d_\rho(A, B).$$

For $F: A^3 \rightarrow B$ and $G: B^3 \rightarrow A$, the following notations are used:

$$T(x, y, z) = (F(x, y, z), F(y, z, x), F(z, x, y)): A^3 \rightarrow B^3$$

for $(x, y, z) \in A^3$ and

$$H(x, y, z) = (G(x, y, z), G(y, z, x), G(z, x, y)): B^3 \rightarrow A^3$$

for $(x, y, z) \in B^3$.

Definition 12. Let $A, B \subset L_p$ be nonempty. Let $F: A^3 \rightarrow B$ and $G: B^3 \rightarrow A$. For any triple $(x_0, y_0, z_0) \in A^3$, define the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty$ by

$$(x_{2n+1}, y_{2n+1}, z_{2n+1}) = T(x_{2n}, y_{2n}, z_{2n})$$

and

$$(x_{2n+2}, y_{2n+2}, z_{2n+2}) = H(x_{2n+1}, y_{2n+1}, z_{2n+1})$$

for $n \geq 0$.

Definition 13. Let $A, B \subset X$ be nonempty, $F: A^3 \rightarrow B$, and $G: B^3 \rightarrow A$. We call (F, G) a cyclic ρ -Kannan contraction pair provided that $\alpha \in [0, \frac{1}{2})$ exists such that, for all $(x, y, z) \in A^3$ and $(u, v, w) \in B^3$,

$$\rho(F(x, y, z) - G(u, v, w)) \leq \alpha(\rho(x - F(x, y, z)) + \rho(u - G(u, v, w))) + (1 - 2\alpha)d_\rho(A, B).$$

Lemma 4. We assume the hypotheses of Theorem 3.8. If $(x_0, y_0, z_0) \in A^3$ and the iterated sequences are given by (4), then

$$\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+1}) = d_\rho, \quad \lim_{n \rightarrow \infty} \rho(y_{2n} - y_{2n+1}) = d_\rho, \quad \lim_{n \rightarrow \infty} \rho(z_{2n} - z_{2n+1}) = d_\rho.$$

Proof. Let $(x_0, y_0, z_0) \in A^3$ be an arbitrary initial triple and consider the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty$.

Since (F, G) is a cyclic ρ -Kannan contraction pair with $\alpha \in [0, \frac{1}{2})$, for each $n \geq 1$, we have

$$\begin{aligned} \rho(x_{2n+1} - x_{2n}) &= \rho(F(x_{2n}, y_{2n}, z_{2n}) - G(x_{2n-1}, y_{2n-1}, z_{2n-1})) \\ &\leq \alpha(\rho(x_{2n} - x_{2n+1}) + \rho(x_{2n-1} - x_{2n})) + (1 - 2\alpha)d_\rho. \end{aligned}$$

The corresponding inequalities for y_n and z_n are obtained through cyclic permutation:

$$\begin{aligned} \rho(y_{2n+1} - y_{2n}) &\leq \alpha(\rho(y_{2n} - y_{2n+1}) + \rho(y_{2n-1} - y_{2n})) + (1 - 2\alpha)d_\rho, \\ \rho(z_{2n+1} - z_{2n}) &\leq \alpha(\rho(z_{2n} - z_{2n+1}) + \rho(z_{2n-1} - z_{2n})) + (1 - 2\alpha)d_\rho. \end{aligned}$$

Due to the symmetry, the inequality types hold for

$$\rho(x_{2n+2} - x_{2n+1}), \rho(y_{2n+2} - y_{2n+1}), \text{ and } \rho(z_{2n+2} - z_{2n+1}).$$

We use the notation defined in the previous paragraph

$$S_{n+1}(x_0, y_0, z_0) = \rho(x_{n+1} - x_n) + \rho(y_{n+1} - y_n) + \rho(z_{n+1} - z_n)$$

and we sum the three inequalities, yielding

$$S_{n+1}(x_0, y_0, z_0) \leq \frac{\alpha}{1 - \alpha} S_{n-1}(x_0, y_0, z_0) + 3 \left(1 - \frac{2\alpha}{1 - \alpha} \right) d_\rho.$$

The condition $\alpha \in [0, \frac{1}{2})$ ensures that $\beta = \frac{\alpha}{1 - \alpha} \in [0, 1)$. Through algebraic manipulation, we obtain the following

$$S_{n+1}(x_0, y_0, z_0) - 3d_\rho \leq \beta(S_n(x_0, y_0, z_0) - 3d_\rho).$$

By iteration, this results in

$$S_n(x_0, y_0, z_0) - 3d_\rho \leq \beta^n(S_1(x_0, y_0, z_0) - 3d_\rho).$$

Since $\beta \in [0, 1)$, we conclude the following:

$$\lim_{n \rightarrow \infty} (S_n(x_0, y_0, z_0) - 3d_\rho) = 0 \quad \text{and thus} \quad \lim_{n \rightarrow \infty} S_n(x_0, y_0, z_0) = 3d_\rho,$$

which implies

$$\lim_{n \rightarrow \infty} \rho(x_{n+1} - x_n) = \lim_{n \rightarrow \infty} \rho(y_{n+1} - y_n) = \lim_{n \rightarrow \infty} \rho(z_{n+1} - z_n) = d_\rho.$$

In particular, for subsequences with even indices,

$$\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+1}) = \lim_{n \rightarrow \infty} \rho(y_{2n} - y_{2n+1}) = \lim_{n \rightarrow \infty} \rho(z_{2n} - z_{2n+1}) = d_\rho.$$

The case where $\alpha = 0$ is automatically included, and the contraction condition reduces to $\rho(F(x, y, z) - G(u, v, w)) \leq d_\rho$, rendering all distances $\rho(x_{n+1} - x_n) = d_\rho$ trivially.

This completes the proof of the lemma. $\square$

Lemma 5. Let $\rho \in \mathfrak{R}$ satisfy UC1, and it possesses the Δ_2 property and is uniformly continuous. Let $A, B \subset L_\rho$ be convex, ρ -closed, and ρ -bounded subsets, and let $F: A^3 \rightarrow B$ and $G: B^3 \rightarrow A$ form a cyclic ρ -Kannan contraction pair. Given any initial triple $(x_0, y_0, z_0) \in A^3$ and iterative sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty$, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $m > n \geq N_0$ with odd $m + n$, we have

$$\rho(x_m - x_n) + \rho(y_m - x_n) + \rho(z_m - x_n) < 3d_\rho + \varepsilon.$$

Proof. For the modular distance $d_\rho = d_\rho(A, B)$, Lemma 4 establishes that $\lim_{n \rightarrow \infty} S_n = 3d_\rho$, where we use the following notation from the previous section:

$$S_n(x_0, y_0, z_0) = \rho(x_n - x_{n+1}) + \rho(y_n - y_{n+1}) + \rho(z_n - z_{n+1}).$$

For indices $m > n$, $m + n$ is odd. We consider two cases: m is an even number, and m is an odd one.

When $m = 2k$ is even and $n = 2l + 1$ is odd, the contraction property yields the following:

$$\begin{aligned} \rho(x_{2k+1} - x_{2l+2}) &= \rho(F(x_{2k}, y_{2k}, y_{2k}) - G(x_{2l+1}, y_{2l+1}, z_{2l+1})) \\ &\leq \alpha(\rho(x_{2k} - F(x_{2k}, y_{2k}, y_{2k}) + \rho(x_{2l+1} - G(x_{2l+1}, y_{2l+1}, y_{2l+1}))) \\ &\quad + (1 - 2\alpha)d_\rho \\ &= \alpha(\rho(x_{2k} - x_{2k+1}) + \rho(x_{2l+1} - x_{2l+2})) + (1 - 2\alpha)d_\rho. \end{aligned}$$

We finally obtain the following:

$$\rho(x_{2k+1} - x_{2l+2}) - d_\rho \leq \alpha(\rho(x_{2k} - x_{2k+1}) - d_\rho) + \alpha(\rho(x_{2l+1} - x_{2l+2}) - d_\rho).$$

In the following, we have analogous inequalities for y_n and z_n . Using the definition of the iterated sequences $y_{2n+1} = F(y_{2n}, z_{2n}, x_{2n})$ and $y_{2n+2} = G(y_{2n+1}, z_{2n+1}, x_{2n+1})$, we obtain the following:

$$\rho(y_{2k+1} - y_{2l+2}) - d_\rho \leq \alpha(\rho(y_{2k} - y_{2k+1}) - d_\rho) + \alpha(\rho(y_{2l} - y_{2l+1}) - d_\rho)$$

and

$$\rho(z_{2k+1} - z_{2l+2}) - d_\rho \leq \alpha(\rho(z_{2k} - z_{2k+1}) - d_\rho) + \alpha(\rho(z_{2l} - z_{2l+1}) - d_\rho).$$

Summing these last three inequalities results in the following:

$$\begin{aligned} W_{2k+1,2l+2}(x_0, y_0, z_0) &\leq \alpha(S_{2k+1}(x_0, y_0, z_0) - d_\rho) + \alpha(S_{2l+2}(x_0, y_0, z_0) - d_\rho) \\ &\leq \alpha^{2k+1}(S_0(x_0, y_0, z_0) - d_\rho) + \alpha^{2l+2}(S_0(x_0, y_0, z_0) - d_\rho), \end{aligned}$$

where $W_{2k+1,2l+2}(x_0, y_0, z_0)$ is defined by the authors of (1).

When $m = 2k + 1$ is odd and $n = 2l$ is even, the results follows through identical arguments by permuting the roles of the even and odd indices. The symmetric nature of the contraction conditions yields parallel inequalities:

$$\begin{aligned} W_{2k+2,2l+1}(x_0, y_0, z_0) &\leq \alpha(S_{2k+2}(x_0, y_0, z_0) - d_\rho) + \alpha(S_{2l}(x_0, y_0, z_0) - d_\rho) \\ &\leq \alpha^{2k+2}(S_0(x_0, y_0, z_0) - d_\rho) + \alpha^{2l}(S_0(x_0, y_0, z_0) - d_\rho). \end{aligned}$$

By combining the previous odd and even cases, the following inequality is obtained:

$$\begin{aligned} W_{m,n}(x_0, y_0, z_0) &\leq \alpha(S_m(x_0, y_0, z_0) - d_\rho) + \alpha(S_n(x_0, y_0, z_0) - d_\rho) \\ &\leq \alpha^m(S_0(x_0, y_0, z_0) - d_\rho) + \alpha^n(S_0(x_0, y_0, z_0) - d_\rho). \end{aligned}$$

Consequently, for each $\varepsilon > 0$, some $N_0 \in \mathbb{N}$ exists such that, whenever $m > n \geq N_0$ and $n + m$ is odd, we obtain

$$\rho(x_m - x_n) + \rho(y_m - y_n) + \rho(z_m - z_n) < 3d_\rho + \varepsilon.$$

□

Lemma 6. Under the hypotheses of Lemma 5, if for the iterated sequences $\{x_n\}_{n \geq 0} \subset A$, $\{y_n\}_{n \geq 0} \subset B$, and $\{z_n\}_{n \geq 0} \subset A$, it holds that $\{x_{2n}\}_{n \geq 0}$, $\{y_{2n}\}_{n \geq 0}$, and $\{z_{2n}\}_{n \geq 0}$ are ρ -convergent to x , y , and z , respectively, then

$$\rho(x - F(x, y, z)) = d_\rho, \quad \rho(y - G(y, z, x)) = d_\rho, \quad \rho(z - F(z, x, y)) = d_\rho.$$

Proof. From Lemma 4, we have equalities $\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+1}) = d_\rho$, $\lim_{n \rightarrow \infty} \rho(x_{2n+2} - x_{2n+1}) = d_\rho$, and $\lim_{n \rightarrow \infty} \rho(z_{2n} - z_{2n+1}) = d_\rho$. Thus, from Lemma 2, we obtain $\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+2}) = 0$. From $\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n-1}) = d_\rho$ and $\lim_{n \rightarrow \infty} \rho(x_{2n} - x) = 0$ and applying Lemma 3, we obtain $\lim_{n \rightarrow \infty} \rho(x_{2n-1} - x) = d_\rho$.

By similar arguments, we obtain $\lim_{n \rightarrow \infty} \rho(y_{2n-1} - y) = d$ and $\lim_{n \rightarrow \infty} \rho(z_{2n-1} - z) = d$. Using the uniform continuity of the modular function ρ , we have

$$\begin{aligned} \rho(F(x, y, z) - x) &\leq \lim_{n \rightarrow \infty} \rho(F(x, y, z) - x_{2n}) \\ &= \lim_{n \rightarrow \infty} \rho(F(x, y, z) - G(x_{2n-1}, y_{2n-1}, z_{2n-1})) \\ &\leq \lim_{n \rightarrow \infty} [\alpha \rho(x - F(x, y, z)) + \alpha \rho(x_{2n-1} - x_{2n}) + (1 - 2\alpha)d] \\ &= \alpha \rho(x - F(x, y, z)) + (1 - \alpha)d_\rho. \end{aligned}$$

Thus, $\rho(F(x, y, z) - x) \leq d_\rho$. By similar arguments, we obtain the inequalities $\rho(F(y, z, x) - y) \leq d_\rho$ and $\rho(F(z, x, y) - z) \leq d_\rho$. Consequently, $\rho(F(x, y, z) - x) = \rho(F(y, z, x) - y) = \rho(F(z, x, y) - z) = d_\rho$. □

Theorem 5. Let $\rho \in \mathfrak{R}$ satisfy UC1, and it possesses the Δ_2 property and is uniformly continuous. Let $A, B \subset L_\rho$ be convex ρ -closed and ρ -bounded subsets, and $F: A^3 \rightarrow B$ and $G: B^3 \rightarrow A$ are a cyclic ρ -Kannan contraction pair. Then, there exists a unique ordered triple $(x, y, z) \in A$ such that

$$\rho(x - F(x, y, z)) + \rho(y - F(y, z, x)) + \rho(z - F(z, x, y)) = 3d_\rho(A, B).$$

The ordered tripled (x, y, z) is the unique fixed points for the map $H(T(x, y, z))$, i.e.,

$$\begin{aligned}x &= G(F(x, y, z), F(y, z, x), F(z, x, y)) \\y &= G(F(y, z, x), F(z, x, y), F(x, y, z)) \\z &= G(F(z, x, y), F(x, y, z), F(y, z, x))\end{aligned}$$

and $(F(x, y, z), F(y, z, x), F(z, x, y))$ is a unique best proximity point of G in B .

Moreover, for every initial triple $(x_0, y_0, z_0) \in A^3$, the iterated sequences satisfy

$$\lim_{n \rightarrow \infty} \rho(x_{2n} - x) = \lim_{n \rightarrow \infty} \rho(y_{2n} - y) = \lim_{n \rightarrow \infty} \rho(z_{2n} - z) = 0,$$

and

$$\begin{aligned}\lim_{n \rightarrow \infty} \rho(x_{2n+1} - F(x, y, z)) &= d_\rho, \\ \lim_{n \rightarrow \infty} \rho(y_{2n+1} - F(y, z, x)) &= d_\rho, \\ \lim_{n \rightarrow \infty} \rho(z_{2n+1} - F(z, x, y)) &= d_\rho.\end{aligned}$$

Proof. Let $(x_0, y_0, z_0) \in A \times A \times A$ be arbitrary initial points. We define iterative sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$. By the assumptions and definition of maps F and G , it follows that $(x_{2n}, y_{2n}, z_{2n}) \in A^3$ and $(x_{2n+1}, y_{2n+1}, z_{2n+1}) \in B^3$ for $n = 0, 1, 2, \dots$.

From Lemma 4, we have the fundamental limit $\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+1}) = d_\rho$. Applying Lemma 5, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $m, n \geq N_0$, the following holds: $\rho(x_{2m} - x_{2n+1}) < d_\rho + \varepsilon$. From Lemma 1, by setting $w_n = x_{2n}$, $u_m = x_{2m}$, and $v_n = x_{2n+1}$, the existence of $N_1 \in \mathbb{N}$ follows such that the inequality $\rho(u_m - w_{2n}) = \rho(x_{2m} - x_{2n}) < \varepsilon$ holds for all $m, n \geq N_1$. Therefore, $\{x_{2n}\}_{n=0}^\infty$ is a ρ -Cauchy sequence.

The proof for sequences $\{y_{2n}\}_{n=0}^\infty$ and $\{z_{2n}\}_{n=0}^\infty$ is carried out in a similar fashion.

By the ρ -completeness of L_ρ and ρ -closedness of A , $x, y, z \in A$ exists such that $\lim_{n \rightarrow \infty} \rho(x_{2n} - x) = 0$, $\lim_{n \rightarrow \infty} \rho(y_{2n} - y) = 0$, and $\lim_{n \rightarrow \infty} \rho(z_{2n} - z) = 0$. By Lemma 6, (x, y, z) is a tripled best proximity point for the map F .

For the proof that (x, y, z) is a fixed point for the map $H(T(x, y, z))$, we first consider the equation for x . We will $F(x, y, z) \in B$ and $G(F(x, y, z), F(y, z, x), F(z, x, y)) \in A$ use in the next chain of inequalities, and therefore, we have the following:

$$\rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) \geq d_\rho.$$

Thus, the following holds:

$$\begin{aligned}d_\rho &\leq \rho(x - G(F(x, y, z), F(y, z, x), F(z, x, y))) \\ &= \lim_{n \rightarrow \infty} \rho(x_{2n+2} - G(F(x, y, z), F(y, z, x), F(z, x, y))) \\ &= \lim_{n \rightarrow \infty} \rho(G(x_{2n+1}, y_{2n+1}, z_{2n+1}) - G(F(x, y, z), F(y, z, x), F(z, x, y))) \\ &\leq \alpha \lim_{n \rightarrow \infty} \rho(x_{2n+1} - x_{2n}) + \alpha \rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) + (1 - 2\alpha)d_\rho \\ &\leq \alpha d_\rho + \alpha d_\rho + (1 - 2\alpha)d_\rho = d_\rho.\end{aligned}$$

Consequently, $x = G(F(x, y, z), F(y, z, x), F(z, x, y))$. The proof for

$$y = G(F(y, z, x), F(z, x, y), F(x, y, z)) \text{ and } z = G(F(z, x, y), F(x, y, z), F(y, z, x))$$

is carried out in a similar fashion.

Assume that there exists another tripled best proximity point $(u, v, w) \in A^3$ of F in A . Then, it is a fixed point for the map $H(T(u, v, w))$.

Applying the contraction property between (x, y, z) and (u, v, w) and taking $x = G(F(x, y, z), F(y, z, x), F(z, x, y))$ and $u = G(F(u, v, w), F(v, w, u), F(w, u, v))$, we obtain

$$\begin{aligned} & \rho(x - F(u, v, w)) \\ &= \rho(G(F(x, y, z), F(y, z, x), F(z, x, y)) - F(u, v, w)) \\ &\leq \alpha \rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) + \rho(u - F(u, v, w)) + (1 - 2\alpha)d_\rho \\ &= \alpha[\rho(F(x, y, z) - x) + \rho(F(u, v, w) - u)] + (1 - 2\alpha)d_\rho = d_\rho. \end{aligned}$$

By equality $\rho(F(u, v, w) - x) - \rho(F(u, v, w) - u) = d_\rho$ and Lemma 2, it follows that $x = u$.

The proof for $y = v$ and $z = w$ is carried out in a similar manner.

From the inequality

$$\begin{aligned} P_1 &= \rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) \\ &\leq \alpha \rho(x - F(x, y, z)) + \alpha \rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) + (1 - 2\alpha)d_\rho \end{aligned}$$

we obtain

$$\begin{aligned} (1 - \alpha)d_\rho &\leq (1 - \alpha)\rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) \\ &\leq \alpha \rho(x - F(x, y, z)) + (1 - 2\alpha)d_\rho = (1 - \alpha)d_\rho \end{aligned}$$

and thus, it follows that $\rho(F(x, y, z) - G(F(x, y, z), F(y, z, x), F(z, x, y))) = d_\rho$.

By similar arguments, we obtain

$$\rho(F(y, z, x) - G(F(y, z, x), F(z, x, y), F(x, y, z))) = d_\rho$$

and

$$\rho(F(z, x, y) - G(F(z, x, y), F(x, y, z), F(y, z, x))) = d_\rho$$

and therefore, $(F(x, y, z), F(y, z, x), F(z, x, y))$ is a best proximity point of G in B . The uniqueness is proven in a similar fashion as that for the best proximity point of F in A .

From Lemma 5, it follows that, for every $\varepsilon > 0$, there exists N such that, for any $n, m \geq N$, $\rho(x_{2n} - x_{2m+1}) \leq d_\rho + \varepsilon$ holds. Therefore, applying Lemma 1 and $\lim_{n \rightarrow \infty} \rho(x_{2n} - x_{2n+1}) = d_\rho$, it follows that $\{x_{2n+1}\}_{n=1}^\infty$ is a Cauchy sequence. From the completeness of L_ρ and the closeness of B , it follows that $\bar{x} \in B$ such that $\lim_{n \rightarrow \infty} \rho(x_{2n+1} - \bar{x}) = 0$.

By similar arguments, we obtain $\lim_{n \rightarrow \infty} \rho(y_{2n+1} - \bar{y}) = 0$ and $\lim_{n \rightarrow \infty} \rho(z_{2n+1} - \bar{z}) = 0$ for some $\bar{y}, \bar{z} \in B$.

We will show that $\bar{x} = F(x, y, z)$, $\bar{y} = F(y, z, x)$, and $\bar{z} = F(z, x, y)$.

From $\lim_{n \rightarrow \infty} \rho(x_{2n+1} - \bar{x}) = 0$, $\lim_{n \rightarrow \infty} \rho(x_{2n+1} - x_{2n}) = d_\rho$, and Lemma 3, it follows that $\lim_{n \rightarrow \infty} \rho(x_{2n} - \bar{x}) = d$. Using the uniform continuity of ρ , we obtain $\rho(x - \bar{x}) = \lim_{n \rightarrow \infty} \rho(x_{2n} - \bar{x}) = d_\rho$. By $\rho(x - F(x, y, z)) = d_\rho$ and Lemma 1, it follows that $\bar{x} = F(x, y, z)$.

The same arguments are used to prove $\bar{y} = F(y, z, x)$ and $\bar{z} = F(z, x, y)$.

Verification is still required for the following: the sequences x_n, y_n, z_n generated by iterations are ρ -convergent to the unique tripled best proximity point (x, y, z) of F in A .

Indeed, starting from an arbitrary initial point $(u_0, v_0, w_0) \in A \times A \times A$, one obtains that the corresponding iterated sequences u_n, v_n, w_n are ρ -convergent to some tripled best proximity point (u, v, w) of F in A . Since the tripled best proximity point (x, y, z) of F in A is unique, it follows that the sequences u_n, v_n , and w_n are, in fact, ρ -convergent to this unique point (x, y, z) . $\square$

4. Examples

The example used illustration is based on the classic example of Kannan's maps [31]. For the function

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ \frac{1}{8}, & x \in (\frac{1}{2}, 1] \end{cases},$$

it is known that it satisfies the mentioned condition and Kannan's theorem, i.e.,

$$|f(x) - f(y)| \leq \frac{1}{9}(|x - f(x)| + |y - f(y)|).$$

Thus, there exists a unique fixed point of f .

Let the following be the case:

$$f(x) = \begin{cases} \frac{x}{50}, & x \in [0, \frac{1}{2}], \\ \frac{x}{50} + \frac{1}{20}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

We will prove that there exists $\gamma \in [0, \frac{1}{2})$ such that the inequality

$$\begin{aligned} P_2 &= \int_0^1 \int_0^t [f(x(s)y(s)z(s)) - f(u(s)v(s)w(s))]^2 ds dt \\ &\leq \gamma \left(\int_0^1 \left[x(t) - \int_0^t f(x(s)y(s)z(s)) ds \right]^2 dt + \int_0^1 \left[u(s) - \int_0^t f(u(s)v(s)w(s)) ds \right]^2 dt \right) \end{aligned}$$

holds for every $s \in [0, 1]$ and for all functions $0 \leq x(s), y(s), z(s), u(s), v(s), w(s) \leq 1$ with squares that are Lebesgue integrable on $[0, 1]$.

The inequality proof:

For brevity, we set $A(s) = x(s)y(s)z(s)$, $B(s) = u(s)v(s)w(s)$, and

$$h(s) = f(A(s)) - f(B(s)).$$

We prove first that that for any $s \in [0, 1]$,

$$h(s)^2 \leq \frac{1}{9}((x(s) - f(A(s)))^2 + (u(s) - f(B(s)))^2). \quad (2)$$

We will consider three cases for $A(s)$ and $B(s)$, depending on whether they are greater or less than $\frac{1}{2}$.

(Case I) If $A(s), B(s) \leq \frac{1}{2}$, then $f(A(s)) = \frac{A(s)}{50}$ and $f(B(s)) = \frac{B(s)}{50}$. Thus, $h(s) = \frac{A(s) - B(s)}{50}$. Because $A(s) \leq x(s)$ and $B(s) \leq u(s)$, the inequalities

$$x(s) - f(A(s)) \geq x(s) - \frac{A(s)}{50} \geq \frac{49}{50}A(s), \quad u(s) - f(B(s)) \geq u(s) - \frac{B(s)}{50} \geq \frac{49}{50}B(s),$$

hold, and thus, we have the following:

$$\begin{aligned} h^2(s) &= \frac{(A(s) - B(s))^2}{2500} \\ &\leq \frac{(A(s))^2 + (B(s))^2}{2500} \\ &\leq \left(\frac{50}{49}\right)^2 \frac{(x(s) - f(A(s)))^2 + (u(s) - f(B(s)))^2}{2500} \\ &\leq \frac{1}{9}((x(s) - f(A(s)))^2 + (u(s) - f(B(s)))^2). \end{aligned}$$

(Case II) In $A(s), B(s) > \frac{1}{2}$, $f(A(s)) = \frac{A(s)}{50} + \frac{1}{20}$ and $f(B(s)) = \frac{B(s)}{50} + \frac{1}{20}$ hold, and again, $h(s) = \frac{A(s)-B(s)}{50}$. Since $x(s) \geq A(s) > \frac{1}{2}$, we obtain

$$x(s) - f(A(s)) \geq \frac{49}{50}A(s) - \frac{1}{20} \geq \frac{11}{25},$$

and the same inequality $u(s) - f(B(s)) \geq \frac{11}{25}$ holds as well, resulting in the inequality

$$\begin{aligned} h^2(s) &= \frac{(A(s)-B(s))^2}{2500} \\ &\leq \frac{(A(s))^2 + (B(s))^2}{2500} \leq \frac{2}{2500} \leq \frac{1}{9} \left(\frac{11}{25} + \frac{11}{25} \right) \\ &\leq \frac{1}{9} \left((x(s) - f(A(s)))^2 + (u(s) - f(B(s)))^2 \right). \end{aligned}$$

(Case III) If, without loss of generality, we can assume $A(s) \leq \frac{1}{2} < B(s) \leq 1$, then

$$|h(s)| \leq \frac{1}{20} + \frac{1}{50} = \frac{7}{100}.$$

From the assumption in Case III, we have $u(s) \geq B(s) > \frac{1}{2}$. Consequently,

$$u(s) - f(B(s)) \geq \frac{1}{2} - \left(\frac{1}{50} + \frac{1}{20} \right) = \frac{43}{100},$$

and hence

$$h^2(s) \leq \frac{49}{10000} \leq \frac{1}{9} \left((x(s) - f(A(s)))^2 + (u(s) - f(B(s)))^2 \right).$$

Integrating (2) over s yields the following:

$$\int_0^1 h(s)^2 ds \leq \frac{1}{9} \left(\int_0^1 (x(s) - f(x(s)y(s)z(s)))^2 ds + \int_0^1 (u(s) - f(u(s)v(s)w(s)))^2 ds \right). \quad (3)$$

Let us recall the Hardy inequality for $h \in L_2[0, 1]$; the following holds:

$$\int_0^1 \left(\int_0^t h(s) ds \right)^2 dt \leq \frac{1}{2} \int_0^1 h(s)^2 ds. \quad (4)$$

Using (4) and (3), we obtain

$$\begin{aligned} \int_0^1 \left(\int_0^t h(s) ds \right)^2 dt &\leq \frac{1}{2} \int_0^1 h(s)^2 ds \\ &\leq \frac{1}{18} \int_0^1 (x(s) - f(x(s)y(s)z(s)))^2 ds \\ &\quad + \frac{1}{18} \int_0^1 (u(s) - f(u(s)v(s)w(s)))^2 ds. \end{aligned}$$

Example: We consider the integral equation

$$x(s) = \int_0^t f(x^3(s)) ds \quad (5)$$

in the Lebesgue space $L_2[0, 1]$.

Let us endow the Lebesgue space $L_2[0, 1]$ with the modular function $\rho(x) = \int_0^1 x^2(s) ds$. It is easy to check that ρ is a uniformly convex modular function, and it satisfies the Δ_2 property and UC1. The map $T(x, y, z) = \int_0^t f(x(s), y(s), z(s))$ satisfies the assumptions of

Theorem 4. Therefore, there exists a unique $(x, y, z) \in L_\rho \times L_\rho \times L_\rho$, which is a solution of the following system of equations:

$$\begin{cases} x(s) = \int_0^t f(x(s), y(s), z(s)) ds \\ y(s) = \int_0^t f(y(s), z(s), x(s)) ds \\ z(s) = \int_0^t f(z(s), x(s), y(s)) ds \end{cases} \quad (6)$$

Moreover, $x(s) = y(s) = z(s)$; therefore, it is a solution of (5). From the fact that ρ is a uniformly convex modular function, it satisfies the Δ_2 property and UC1. Furthermore, it follows that it is not only a solution of (5) in $L_2[0, 1]$ but also a sequence of successive iterations that converges to the solution in $L_2[0, 1]$.

Let us choose an initial estimate $x_0(s) = s$. We introduce $H(s) = 0$ if $s \notin [0, 1]$ and $H(s) = 1$ if $s \in [0, 1]$. We obtain the following:

$$\begin{aligned} x_1(s) = & 0.005 s^4 - 0.005 s^4 H(s-1) - 0.05 s H(s-1) + 0.05 s H(s-0.7937005260) \\ & - 0.0396850263 H(s-0.7937005260) + 0.055 H(s-1) + 0.05 s \end{aligned}$$

$$\begin{aligned} x_2(s) = & 0.1250000000 \times 10^{-5} s \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 1.923076923 \times 10^{-10} s^{13} \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 1.500000000 \times 10^{-8} s^{10} \left(H(s-1) - H(s-0.7937005260) \right) \\ & + 6.614171050 \times 10^{-9} s^9 \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 4.285714286 \times 10^{-7} s^7 \left(H(s-1) - H(s-0.7937005260) \right) \\ & + 3.968502630 \times 10^{-7} s^6 \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 9.449407875 \times 10^{-8} s^5 \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 0.5000000000 \times 10^{-5} s^4 \left(H(s-1) - H(s-0.7937005260) \right) \\ & + 0.7937005260 \times 10^{-5} s^3 \left(H(s-1) - H(s-0.7937005260) \right) \\ & - 0.4724703937 \times 10^{-5} s^2 \left(H(s-1) - H(s-0.7937005260) \right) \\ & + 6.724920582 \times 10^{-7} H(s-1) \\ & - 1.626150282 \times 10^{-8} H(s-0.7937005260) + 0.05 s \end{aligned}$$

The next iterations become more complicated. One can verify that neither $x(s) = 0$ nor $x(s) = \alpha s$ for some $\alpha \in \mathbb{R}$ is a solution of (5).

5. Discussion

The results presented in this study extend the classical contraction framework to the setting of modular function spaces. By establishing theorems for both tripled fixed points and tripled best proximity points, we demonstrate that modular convexity and uniform continuity properties are sufficient for guaranteeing the existence, uniqueness, and modular convergence of the constructed iterative schemes. In particular, our results show that the structure of modular spaces, although more general than normed or metric spaces, still supports high-complexity fixed-point phenomena, such as the tripled and cyclic cases.

The use of the modular function—rather than a norm—allows one to capture a wider class of function spaces, including Orlicz and Musielak–Orlicz spaces, which are central in applications where growth conditions vary locally. The Δ_2 property and UC1 convexity assumptions play a crucial role in transferring geometric properties known from uniformly convex Banach spaces to the modular context. The proposed approach is, therefore, suitable for treating discontinuous or impulsive operators, which often fail to

satisfy classical Lipschitz-type or Banach contractive inequalities but comply with Kannan-type inequalities.

The illustrative example—based on integral operators in $L_2[0, 1]$ —confirms that the theoretical framework can be implemented constructively. The results predict convergence behavior and provide analytical tools for studying nonlinear functional or impulse integral equations and systems arising in applications. Moreover, the modular distance concept provides a natural method for extending best proximity point results to non-self mappings between subsets of modular function spaces.

6. Conclusions

In this study, we developed a unified modular framework for studying the tripled fixed points and tripled best proximity points generated by ρ -Kannan type contractions. The main theoretical contributions can be summarized as follows. With respect to existence and uniqueness, the convexity, Δ_2 property, and uniform continuity of the modular function $\rho(\cdot)$ are supported under mild assumptions. Modular convergence is proposed through iterative schemes that converge to the unique tripled fixed (or best proximity) point in the modular distance function, establishing a constructive procedure that generalizes Banach's and Kannan's classical results to modular settings. With respect to cyclic ρ -Kannan contractions for non-self mappings, we have shown that cyclic contraction maps generate unique tripled best proximity points, thereby extending proximity theory to modular function spaces. An illustrative example is used to present applications involving integral-type operators in $L_2[0, 1]$, demonstrating the effectiveness and broad applicability of the obtained results.

These findings highlight the versatility of modular function spaces as a natural environment for the extension of fixed points and the best proximity point theory. They provide new perspectives for solving nonlinear matrix, integral, and differential equations under generalized modular conditions.

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