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Analytic Solutions and Conservation Laws of a 2D Generalized Fifth-Order KdV Equation with Power Law Nonlinearity Describing Motions in Shallow Water Under a Gravity Field of Long Waves

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Abstract

The Korteweg–de Vries (KdV) equation is a nonlinear evolution equation that reflects a wide variety of dispersive wave occurrences with limited amplitude. It has also been used to describe a range of major physical phenomena, such as shallow water waves that interact weakly and nonlinearly, acoustic waves on a crystal lattice, lengthy internal waves in density-graded oceans, and ion acoustic waves in plasma. The KdV equation is one of the most well-known soliton models, and it provides a good platform for further research into other equations. The KdV equation has several forms. The aim of this study is to introduce and investigate a (2+1)-dimensional generalized fifth-order KdV equation with power law nonlinearity (gFKdVp). The research methodology employed is the Lie group analysis. Using the point symmetries of the gFKdVp equation, we transform this equation into several nonlinear ordinary differential equations (ODEs), which we solve by employing different strategies that include Kudryashov's method, the (G'/G)expansion method, and the power series expansion method. To demonstrate the physical behavior of the equation, 3D, density, and 2D graphs of the obtained solutions are presented. Finally, utilizing the multiplier technique and Ibragimov's method, we derive conserved vectors of the gFKdVp equation. These include the conservation of energy and momentum. Thus, the major conclusion of the study is that analytic solutions and conservation laws of the gFKdVp equation are determined.

Keywords: 2D generalized fifth-order KdV equation with power law nonlinearity; Lie group analysis; analytic solutions; conserved vectors

MSC: 35B06; 35L65; 37J15; 37K05

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1. Introduction

Over the years, the construction of mathematical models has aided in the effective description of a variety of physical processes in engineering, finance, and natural sciences. Nonlinear partial differential equations (PDEs) typically represent the important physical processes of the model, including genetic engineering, the spread of infectious diseases, the behavior of tidal waves, the pricing of derivatives, and many others. For instance, the

Shigesada–Kawasaki–Teramoto cross-diffusion model was investigated in [1] to prove the existence of global weak solutions for an arbitrary number of species. Also, the authors of [2] studied the shallow water wave equation, which is mostly used in fluid mechanics as the model that describes the flow of fluids below a pressured surface. In [3], a nonlinear (1+1)-dimensional fifth-order equation was examined and multiple soliton solutions were obtained. In addition, the authors of [4] studied the new extended (3+1)-dimensional Boussinesq equation and obtained multiple soliton solutions, soliton molecules, and travelling wave solutions of the equation.

To comprehend these physical processes, we must seek the exact solutions of these nonlinear PDEs. Thus, one of the major topics in mathematical physics is finding exact solutions to nonlinear PDEs that appear in many physical applications. Most of the time, finding exact solutions to these equations is challenging.

Currently, there is no efficient method for finding the general solution of nonlinear PDEs, although there are methodologies and mathematical tools that have been developed by researchers that may help in finding a specific type of solution. In this paper, our attention is focused on employing Lie group analysis [5,6] to find analytic solutions. A point symmetry of a differential equation (DE) is a transformation that maps solutions of the DE to other solutions of the same DE. Once symmetry of the DE is found, one may use it to lower the order of the DE. This process is called symmetry reduction. Several researchers have successfully introduced methods for finding specific analytic solutions to nonlinear PDEs. These include the Hirota method [7], Cole–Hopf transformation approach [8], extended simplest equation technique [9], symmetry method [10], neural network method [11], Mittag–Leffler kernel technique [12], modified simple equation method [13], Fokas method [14], hybrid technique for improved accuracy [15], extended homoclinic test approach [16], sine-Gordon equation expansion technique [17], $(G'/G)-$ expansion method [18], power series solution technique [19], Painlevé expansion technique [20], tanh–coth technique [21], the mapping approach [22], homotopy perturbation technique [23], generalized extended rational expansion method [24], Kudryashov method [25], and Bäcklund transformation technique [26].

Lu et al. [27] considered the $(2+1)$-dimensional generalized fifth-order Korteweg–de Vries (KdV) equation

$$\alpha u_{xxxxx} + \beta u u_{xxx} + \gamma u_x u_{xx} + \delta u^2 u_x + u_t + u_y = 0, \tag{1}$$

where α, β, γ, and δ are the coefficients of the fifth-order dispersion term and high-order nonlinear term, which are arbitrary nonzero real numbers. Equation (1) describes motions of long waves in shallow water under a gravity field and in a two-dimensional nonlinear lattice. The variables t and x represent time and space, respectively. In Equation (1), the authors took $\alpha = 1$, $\beta = \gamma = 15$, and $\delta = 45$ and obtained the equation

$$u_{xxxxx} + 15 u u_{xxx} + 15 u_x u_{xx} + 45 u^2 u_x + u_t + u_y = 0. \tag{2}$$

Furthermore,upon solving Equation (2) using the Hirota bilinear method, a kind of lump solution and two classes of interaction were discussed [27].

Moreover, new solutions were found in [28] by employing the Hirota bilinear form and the authors investigated the interactions between the lump-type solutions and the double exponential functions.

Lastly, Guo et al. [29] used the Hirota bilinear forms with the Hirota direct method and, as a result, the single soliton and the single periodic wave solutions were obtained.

In this paper, we further generalize Equation (2) and obtain the (2+1)-dimensional generalized fifth-order Korteweg–de Vries equation with power law nonlinearity (gFKdVp) equation

$$u_{xxxxx} + auu_{xxx} + au_xu_{xx} + 3au^nu_x + u_t + u_y = 0, \tag{3}$$

where a is an arbitrary nonzero real number and n is a nonzero constant.

With a motivation to expand the area of applications of Equation (3), we employ Lie group analysis to obtain new analytic solutions. To the best of our knowledge, the results obtained in this paper are new and have not been reported.

The rest of the paper is organized as follows. Lie point symmetries are obtained in Section 2. In Section 3, we construct conserved vectors of Equation (3) using the general multiplier methodology and Ibragimov's method. Symmetry reductions of Equation (3) are performed and, thereafter, analytic solutions are constructed in Section 4. Also, dynamical behavior of certain solutions are depicted. Finally, in Section 5, concluding remarks are given.

2. Point Symmetries of (3)

First and foremost, Lie theory is used to compute point symmetries of the form

$$\mathcal{S} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u},$$

with coefficient functions τ, ξ, ϕ, η depending on t, x, y, and u. Using the Lie invariance condition [30], the generator $\mathcal{S}$ is a symmetry of (3), provided

$$\mathcal{S}^{[5]}\Delta|_{\Delta=0} = 0, \tag{4}$$

holds with $\Delta = u_{xxxxx} + auu_{xxx} + au_xu_{xx} + 3au^nu_x + u_t + u_y$. Here, $\mathcal{S}^{[5]}$ stands for the fifth extension of $\mathcal{S}$, given by

$$\mathcal{S}^{[5]} = \mathcal{S} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_3\frac{\partial}{\partial u_y} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}} + \zeta_{22222}\frac{\partial}{\partial u_{xxxxx}}, \tag{5}$$

and $\zeta_1, \zeta_2, \zeta_3, \zeta_{22}, \zeta_{222}$, and ζ_{22222} are determined by

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi) - u_yD_t(\phi),\\
\zeta_2 &= D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) - u_yD_x(\phi),\\
\zeta_3 &= D_y(\eta) - u_tD_y(\tau) - u_xD_y(\xi) - u_yD_y(\phi),\\
\zeta_{22} &= D_x(\zeta_2) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\phi),\\
\zeta_{222} &= D_x(\zeta_{22}) - u_{txx}D_x(\tau) - u_{xxx}D_x(\xi) - u_{xxy}D_x(\phi),\\
\zeta_{22222} &= D_x(\zeta_{2222}) - u_{txxxx}D_x(\tau) - u_{xxxxx}D_x(\xi) - u_{xxxxy}D_x(\phi),
\end{aligned}$$

with total derivatives D_t, D_x, and D_y, defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{ty}\frac{\partial}{\partial u_t} + u_{xy}\frac{\partial}{\partial u_x} + \cdots.
\end{aligned}$$

With the help of the scientific program Mathematica, expanding (4) and separating the derivatives of u, we obtain a system of linear homogeneous PDEs. Solving these equations yields

$$\tau = F^1(y-t), \;\; \xi = F^2(y-t), \;\; \phi = F^3(y-t), \;\; \eta = 0, \tag{6}$$

where F^i, $i = 1, \ldots, 3$ are arbitrary functions of their arguments. Thus, the point symmetries of the gFKdVp Equation (3) are given by:

$$\begin{aligned} \mathcal{S}_1 &= F^1(y-t)\frac{\partial}{\partial t}, \\ \mathcal{S}_2 &= F^2(y-t)\frac{\partial}{\partial x}, \\ \mathcal{S}_3 &= F^3(y-t)\frac{\partial}{\partial y}. \end{aligned}$$

The first symmetry represents the time translation, while the second and third symmetries are space translations. We obtain an extra fourth symmetry of Equation (3) when $n = 2$, and it is given by

$$\mathcal{S}_4 = 5F^4(y-t)t\frac{\partial}{\partial t} + F^4(y-t)x\frac{\partial}{\partial x} + 5F^4(y-t)t\frac{\partial}{\partial y} - 2F^4(y-t)u\frac{\partial}{\partial y}.$$

3. Conservation Laws for (3)

The conserved vectors for the gFKdVp Equation (3) are computed in this section. To derive the conserved vectors, we apply two methods, namely the general multiplier technique and Ibragimov's theorem.

3.1. Conserved Vectors via the Multiplier Approach

3.1.1. Conserved Vectors for (3)

We first compute the fourth-order multipliers $\mathcal{M}$, that is $\mathcal{M}$ depends on t, x, y, u, u_x, u_{xx}, u_{xxx} and u_{xxxx} [11] by using

$$\frac{\delta}{\delta u}\left\{\mathcal{M}\left[u_{xxxxx} + auu_{xxx} + au_xu_{xx} + 3au^nu_x + u_t + u_y\right]\right\} = 0. \tag{7}$$

Here, $\delta/\delta u$ is a Euler operator that has the form

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_tD_x\frac{\partial}{\partial u_{tx}} - D_x^4\frac{\partial}{\partial u_{xxxx}} + D_tD_x^3\frac{\partial}{\partial u_{txxx}} + \cdots, \tag{8}$$

and D_t, D_x and D_y represent the total derivatives. The following seven linear determining equations are obtained by expanding (7):

$$\mathcal{M}_t + \mathcal{M}_t = 0, \;\; \mathcal{M}_x = 0, \;\; \mathcal{M}_u = 0, \;\; \mathcal{M}_{u_x} = 0, \;\; \mathcal{M}_{u_{xx}} = 0, \;\; \mathcal{M}_{u_{xxx}} = 0, \;\; \mathcal{M}_{u_{xxxx}} = 0,$$

which, upon solving for $\mathcal{M}$, yields

$$\mathcal{M} = F(y-t), \tag{9}$$

where $F(y-t)$ is an arbitrary function of its argument. Thus, we obtain the following conserved vector (T^t, T^x, T^y), corresponding to the multiplier $F(y-t)$ [11], whose components are

$$
\begin{aligned}
T^t &= F(y-t)u,\\
T^x &= F(y-t)\left(\frac{3a}{n+1}u^{n+1} + auu_{xx} + u_{xxxx}\right),\\
T^y &= F(y-t)u.
\end{aligned}
$$

It should be noted that by using the zeroth, first, second, and third-order multipliers, we obtain the same conserved vector as given above.

3.1.2. Conserved Vectors of (3) When $a = 15$ and $n = 2$

We now consider a special case of (3) when $a = 15$, $n = 2$ and compute the fourth-order multipliers Y, viz., $\mathrm{Y} = \mathrm{Y}(t, x, y, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$. Thus, expanding

$$
\frac{\delta}{\delta u}\left\{\mathrm{Y}\left[u_{xxxxx} + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x + u_t + u_y\right]\right\} = 0, \tag{10}
$$

as before, yields 12 linear determining equations

$$
\begin{aligned}
&\mathrm{Y}_{xx} = 0,\ \mathrm{Y}_{u_t} = 0,\ \mathrm{Y}_{u_y} = 0,\ \mathrm{Y}_{u_{xxx}} = 0,\ \mathrm{Y}_{xu_{xx}} = 0,\ \mathrm{Y}_{xu_{xxxx}} = 0,\\
&\mathrm{Y}_{u_{xx}u_{xx}} = 0,\ \mathrm{Y}_{u_{xx}u_{xxxx}} = 0,\ \mathrm{Y}_{u_{xxxx}u_{xxxx}} = 0,\ \mathrm{Y}_{u_x} - 9u_x\mathrm{Y}_{u_{xxxx}} = 0,\\
&\mathrm{Y}_t + 45u^2\mathrm{Y}_x + 30u_{xx}\mathrm{Y}_x + \mathrm{Y}_y = 0,\ \mathrm{Y}_u + 9u^2\mathrm{Y}_{u_{xxxx}} - 9u_{xx}\mathrm{Y}_{u_{xxxx}} - 3u\mathrm{Y}_{u_{xx}} = 0.
\end{aligned}
$$

Solving the preceding equations for Y, we obtain

$$
\begin{aligned}
\mathrm{Y} = {} & (x - 45tu^2 - 30tu_{xx})G(y-t) + \frac{1}{18}(12u^3 + 18uu_{xx} + 9u_x^2 + 2u_{xxxx})H(y-t)\\
& + \frac{1}{2}(3u^2 + 2u_{xx})I(y-t) + J(y-t),
\end{aligned} \tag{11}
$$

where G, H, I and J are arbitrary functions of their argument. Consequently, we acquire the following four multipliers:

$$
\begin{aligned}
\mathrm{Y}_1 &= (x - 45tu^2 - 30tu_{xx})G(y-t),\\
\mathrm{Y}_2 &= \frac{1}{18}(12u^3 + 18uu_{xx} + 9u_x^2 + 2u_{xxxx})H(y-t),\\
\mathrm{Y}_3 &= \left(\frac{3}{2}u^2 + u_{xx}\right)I(y-t),\\
\mathrm{Y}_4 &= J(y-t).
\end{aligned}
$$

Thus, we obtain the following four conserved vectors:

Case 1. Considering first the multiplier $\mathrm{Y}_1 = (x - 45tu^2 - 30tu_{xx})G(y-t)$, the associated conserved vector is (T_1^t, T_1^x, T_1^y), whose components are

$$
\begin{aligned}
T_1^t = {} & -15\,u^3tG - 15\,u_tu_{xx}\,G + u_xG,\\
T_1^x = {} & 90\,uu_{xxx}\,u_x\,tG - Gu_{xxx} + 15\,uGtu_{xy} - 675\,u^3u_{xx}\,tG - 45\,u^2u_{xxxx}\,tG\\
& - 90\,u_{xx}\,Gtu_x{}^2 + 15\,uu_{tx}\,Gt + 15\,uu_{xx}\,xG - 15\,u_x\,Gtu_t - 15\,u_x\,Gtu_y\\
& + 15\,u_{xxx}{}^2Gt - 30\,u_{xx}\,Gtu_{xxxx} - 405\,u^5Gt + u_{xxxx}\,xG - 270\,uu_{xx}{}^2tG\\
& + 15\,u^3Gx,\\
T_1^y = {} & -15\,u^3tG - 15\,u_tu_{xx}\,G + u_xG.
\end{aligned}
$$

Case 2. The multiplier $Y_2 = \frac{1}{18}(12u^3 + 18uu_{xx} + 9u_x^2 + 2u_{xxxx})H(y-t)$ yields the conserved vector (T_2^t, T_2^x, T_2^y) with components

$$\begin{aligned}
T_2^t &= \frac{1}{18}\,u\Big(3\,u^3 + 6\,uu_{xx} + 3\,{u_x}^2 + u_{xxxx}\Big)H,\\
T_2^x &= 6\,u^2{u_{xx}}^2H - \frac{1}{18}\,Hu_{xx}\,u_{tx} + \frac{1}{3}\,uu_{xxx}{}^2H - \frac{1}{3}\,u^2u_{xy}\,H + 10\,u^4u_{xx}\,H + \frac{2}{3}\,u^3u_{xxxx}\,H\\
&\quad + \frac{1}{2}\,{u_x}^2Hu_{xxxx} + \frac{15}{2}\,u^3H{u_x}^2 + \frac{1}{18}\,Hu_x\,u_{txx} - \frac{1}{3}\,u^2u_{tx}\,H + 5\,u^6H + \frac{3}{2}\,{u_x}^4H\\
&\quad + \frac{1}{18}\,{u_{xxxx}}^2H + \frac{3}{2}\,uu_{xx}\,H{u_x}^2 + 3\,u^2u_{xxx}\,Hu_x + \frac{1}{9}\,{u_{xx}}^3H + \frac{1}{3}\,uHu_t\,u_x\\
&\quad + \frac{1}{3}\,uHu_x\,u_y - \frac{1}{3}\,u_x\,u_{xx}\,Hu_{xxx} + uu_{xx}\,u_{xxxx}\,H + \frac{1}{18}\,u_{xxx}\,Hu_t + \frac{1}{18}\,u_{xxx}\,Hu_y\\
&\quad - \frac{1}{18}\,Hu_{xx}\,u_{xy} + \frac{1}{18}\,Hu_x\,u_{xxy} - \frac{1}{18}\,Huu_{txxx} - \frac{1}{18}\,Huu_{xxxy},\\
T_2^y &= \frac{1}{18}\,u\Big(3\,u^3 + 6\,uu_{xx} + 3\,{u_x}^2 + u_{xxxx}\Big)H.
\end{aligned}$$

Case 3. The associated conserved vector (T_3^t, T_3^x, T_3^y) for the third multiplier, $Y_3 = \left(\frac{3}{2}u^2 + u_{xx}\right)I(y-t)$, is given by

$$\begin{aligned}
T_3^t &= \frac{1}{2}\,u^3I + \frac{1}{2}\,uu_{xx}\,I,\\
T_3^x &= 3\,{u_x}^2u_{xx}\,I + 9\,u{u_{xx}}^2I + \frac{1}{2}\,u_x\,Iu_t + \frac{1}{2}\,u_x\,Iu_y + u_{xx}\,Iu_{xxxx} + \frac{27}{2}u^5I\\
&\quad - \frac{1}{2}\,{u_{xxx}}^2I - 3\,uu_{xxx}\,u_x\,I + \frac{45}{2}u^3u_{xx}\,I - \frac{1}{2}\,uu_{xy}\,I - \frac{1}{2}\,uu_{tx}\,I\\
&\quad + \frac{3}{2}\,u^2u_{xxxx}\,I,\\
T_3^y &= \frac{1}{2}\,u^3I + \frac{1}{2}\,uu_{xx}\,I.
\end{aligned}$$

Case 4. Lastly, the fourth multiplier $Y_4 = J(y-t)$ gives the corresponding conserved vector (T_4^t, T_4^x, T_4^y), whose components are

$$\begin{aligned}
T_4^t &= uJ,\\
T_4^x &= 15\,u^3J + u_{xxxx}\,J + 15\,uu_{xx}\,J,\\
T_4^y &= uJ.
\end{aligned}$$

It should be noted that using the zeroth and first-order multipliers, we obtain only one conserved vector, which is the conserved vector in Case 4. However, the second and third-order multipliers yield three conserved vectors that are given in Cases 1, 3, and 4 above.

3.2. Conserved Vectors via Ibragimov's Method

3.2.1. Conserved Vectors of (3)

Ibragimov's method [31] is employed to construct the conserved vectors of the gFKdVp Equation (3). According to this method, each infinitesimal generator provides a conserved quantity.

To obtain the adjoint equation of (3), we use

$$F^* \equiv \frac{\delta}{\delta u}\left[v\left\{u_{xxxxx} + auu_{xxx} + au_xu_{xx} + 3au^nu_x + u_t + u_y\right\}\right] = 0, \tag{12}$$

where v is the new variable depending on t, x and y. Here,

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} - D_x^5\frac{\partial}{\partial u_{xxxxx}} + \cdots$$

is a Euler operator and D_t, D_x, and D_y are total derivatives given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + u_{ty}\frac{\partial}{\partial u_t} + v_{ty}\frac{\partial}{\partial v_t} + \cdots.
\end{aligned} \tag{13}$$

Expanding Equation (12), we obtain the adjoint equation as

$$v_y + 3au^n v_x + 2av_x u_{xx} + 2au_x v_{xx} + auv_{xxx} + v_{xxxxx} + v_t = 0. \tag{14}$$

The Lagrangian of Equations (3) and (14) is [31]

$$\mathcal{L} = v_{xx}u_{xxx} + v(auu_{xxx} + au_x u_{xx} + 3au^n u_x + u_t + u_y). \tag{15}$$

Extending the symmetries of (3) to the new variable v gives the generator

$$\mathcal{K} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} - \eta_1\frac{\partial}{\partial v},$$

where $\eta_1 = v[\Phi + D_t(\tau) + D_x(\xi) + D_y(\phi)]$, and the value of Φ is determined from $\mathcal{S}^{[3]}(F) = \Phi(F)$. The formulae [31]

$$H^i = \xi^i\mathcal{L} + W^\alpha\left(\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_k\frac{\partial\mathcal{L}}{\partial u_{ik}^\alpha}\right) + D_k(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_{ik}^\alpha}, \tag{16}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, is used to calculate the conserved vectors associated with the infinitesimal generators based on the Lagrangian (15). Consequently, we can now write down the conserved vectors corresponding to the three Lie point symmetries of (3), where F^i is equal to one:

Case 1. For the time translation symmetry $\mathcal{S}_1 = \partial/\partial t$, the associated conserved vector is (T_1^t, T_1^x, T_1^y), whose components are

$$\begin{aligned}
T_1^t &= vu_y + 3au^n vu_x + avu_x u_{xx} + auvu_{xxx} + v_{xx}u_{xxx},\\
T_1^x &= -3au^n vu_t - au_x v_x u_t - avu_{xx}u_t - auv_{xx}u_t - v_{xxxx}u_t\\
&\quad + u_{xxxx}v_t + auv_x u_{tx} + v_{xxx}u_{tx} - u_{xxx}v_{tx} - auvu_{txx} - v_{xx}u_{txx},\\
T_1^y &= -vu_t.
\end{aligned}$$

Case 2. The space translation symmetry $\mathcal{S}_2 = \partial/\partial x$ produces the conserved vector (T_2^t, T_2^x, T_2^y), whose components are

$$
\begin{aligned}
T_2^t &= -vu_x,\\
T_2^x &= vu_y - au_x{}^n v_x + auv_x u_{xx} - auu_x v_{xx} - v_{xx}u_{xxx} + u_{xx}v_{xxx} + v_x u_{xxxx}\\
&\quad + auv_x u_{xx} - auu_x v_{xx} - v_{xx}u_{xxx} + u_{xx}v_{xxx} + v_x u_{xxxx} - u_x v_{xxxx} + vu_t\\
&\quad - u_x v_{xxxx} + vu_t vu_y - au_x^2 v_x,\\
T_2^y &= -vu_x.
\end{aligned}
$$

Case 3. Lastly, the space translation symmetry $\mathcal{S}_3 = \partial/\partial y$ yields the conserved vector (T_3^t, T_3^x, T_3^y), whose components are

$$
\begin{aligned}
T_3^t &= -vu_y,\\
T_3^x &= auv_x u_{xy} - 3au^2 vu_y - au_y u_x v_x - avu_y u_{xx} - auu_y v_{xx} - auvu_{xxy}\\
&\quad - v_{xx}u_{xxy} - v_{xy}u_{xxx} + u_{xy}v_{xxx} + v_y u_{xxxx} - u_y v_{xxxx} - 3au^n vu_y - au_y u_x v_x\\
&\quad + auv_x u_{xy} - avu_y u_{xx} - auu_y v_{xx} - auvu_{xxy} - v_{xx}u_{xxy} - v_{xy}u_{xxx}\\
&\quad + u_{xy}v_{xxx} + v_y u_{xxxx} - u_y v_{xxxx},\\
T_3^y &= 3au^n vu_x + avu_x u_{xx} + auvu_{xxx} + v_{xx}u_{xxx} + vu_t.
\end{aligned}
$$

3.2.2. Conserved Vectors of (3) When $n = 2$

When $n = 2$ in Equation (3), we have an additional symmetry

$$\mathcal{S}_4 = 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial y},$$

with $F^4(y - t) = 1$. Thus, the associated conserved vector for the above symmetry is (T_4^t, T_4^x, T_4^y) with

$$
\begin{aligned}
T_4^t &= 15atu_x u^2 v + 5atu_{xxx}uv + 5atu_x u_{xx}v - xu_x v - 2uv + 5tu_{xxx}v_{xx},\\
T_4^x &= au_x v_x u - 15atu_y u^2 v - 2av_{xx}u^2 - 15atu_t u^2 v + 5atv_x u_{xy}u - 6au_{xx}uv + axu_{xx}v_x u\\
&\quad - 5atu_y v_{xx}u - axu_x v_{xx}u - 5atu_{xxy}uv - 5atu_t v_{xx}u + 5atv_x + u_{tx}u - 5atu_{txx}uv\\
&\quad - 5atu_t u_{xx}v - 2v_{xxxx}u + xu_y v + xu_t v - 6au^3 v - 5atu_x u_y v_x - 5atu_t u_x v_x - axu_x{}^2 v_x\\
&\quad - 5tv_{xx}u_{xxy} - 5tu_{xxx}v_{xy} + 5tv_{xxx}u_{xy} + 5tu_{xxxx}v_y - 5tu_y v_{xxxx} - 5tu_t v_{xxxx} + 5tv_t u_{xxxx}\\
&\quad + 5tv_{xxx}u_{tx} - 5tu_{xxx}v_{tx} - 5tv_{xx}u_{txx} - 4u_{xx}v_{xx} - u_{xxx}v_x - xu_{xxx}v_{xx}\\
&\quad + 3u_x v_{xxx} + xu_{xx}v_{xxx} + xu_{xxxx}v_x - xu_x v_{xxxx} - 5atu_{xx}u_y v,\\
T_4^y &= 15atu_x u^2 v + 5atu_{xxx}uv + 5atu_x u_{xx}v - xu_x v - 2uvtu_{xxx}v_{xx}.
\end{aligned}
$$

4. Analytic Solutions of Equation (3)

We utilize symmetry analysis in conjunction with a number of other techniques, such as Kudryashov's, (G'/G)-expansion and power series expansion methods to create analytic solutions for Equation (3). Symmetry reductions are first performed with the help of point symmetries of (3), which leads to nonlinear ODEs, which are then solved.

4.1. Symmetry Reductions and Solutions Using $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$

We first take into consideration the three translation symmetries $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, and perform symmetry reductions. Taking linear combinations of these translation symmetries and letting F^is be equal to one, we obtain symmetry $\mathcal{S} = \mathcal{S}_1 + \alpha\mathcal{S}_2 + \mu\mathcal{S}_3$, where α and μ

are constants. This symmetry $\mathcal{S}$ will provide the travelling wave solutions of Equation (3). The associated Lagrangian system for $\mathcal{S}$ yields the three invariants

$$f = x - \alpha t, \;\; g = y - \mu t, \;\; \theta(f,g) = u(x,y,t). \tag{17}$$

Treating f, g as new independent variables and θ as the new dependent variable, Equation (3) transforms to nonlinear PDE

$$\theta_{fffff} + a\theta\theta_{fff} + a\theta_f\theta_{ff} + 3a\theta^n\theta_f - \alpha\theta_f - \mu\theta_g + \theta_g = 0. \tag{18}$$

The nonlinear PDE (18) has two translation symmetries, namely,

$$\Gamma_1 = \frac{\partial}{\partial f}, \;\; \Gamma_2 = \frac{\partial}{\partial g}. \tag{19}$$

Consider the symmetry $\Gamma = \Gamma_1 + \nu\Gamma_2$ (ν is a constant). The two invariants associated with the symmetry Γ are $z = g - \nu f$, $\psi = \theta$ and using these invariants Equation (18) reduces to the nonlinear ODE

$$\nu^5\psi'''''' + a\nu^3\psi\psi''' + a\nu^3\psi'\psi'' + 3a\nu\psi^n\psi' - (1 - \mu + \alpha\nu)\psi' = 0. \tag{20}$$

We consider Equation (20) with $a = 15$ and $n = 2$, namely

$$\nu^5\psi'''''' + 15\nu^3\psi\psi''' + 15\nu^3\psi'\psi'' + 45\nu\psi^2\psi' - (1 - \mu + \alpha\nu)\psi' = 0. \tag{21}$$

Equation (2) was studied in [27–29] and analytic solutions were found. Here, we employ two different techniques and obtain new and different solutions. The first technique we employ is Kudryashov's method.

Solutions for (21) using Kudryashov's method

Kudryashov's method [25] assumes that the solution to Equation (21) is of the form

$$\psi(z) = \sum_{i=0}^{\mathcal{V}} C_i\mathcal{K}^i(z), \tag{22}$$

where C_i, $i = 0, 1, \ldots, \mathcal{V}$ are constants and $\mathcal{K}(z)$ solves the Riccati equation

$$\mathcal{K}'(z) = \mathcal{K}^2(z) - \mathcal{K}(z). \tag{23}$$

It is known that the solution of the above equation is

$$\mathcal{K}(z) = \frac{1}{1 + e^z}. \tag{24}$$

Firstly, balancing the linear term of the highest order with the nonlinear term, we find $\mathcal{V} = 2$ [32]. Thus, the solution (22) can be written as

$$\psi(z) = C_0 + C_1\mathcal{K}(z) + C_2\mathcal{K}^2(z). \tag{25}$$

Inserting the above value of $\psi(z)$ into (21), invoking Equation (23), and comparing coefficients with like powers of $\mathcal{K}(z)$ yields the system of algebraic equations:

$$
\begin{aligned}
&8\,\nu^4 C_2 + 6\,\nu^2 {C_2}^2 + {C_2}^3 = 0,\\
&120\,\nu^5 C_1 - 2400\,\nu^5 C_2 + 600\,\nu^3 C_1 C_2 - 1290\,\nu^3 {C_2}^2 + 225\,\nu\, C_1 {C_2}^2 - 90\,\nu\, {C_2}^3 = 0,\\
&C_1 - \nu^5 C_1 - 15\,\nu^3 C_0 C_1 - 45\,\nu\, {C_0}^2 C_1 - \mu\, C_1 + \alpha\nu\, C_1 = 0,\\
&3000\,\nu^5 C_2 + 360\,\nu^3 C_0 C_2 + 120\,\nu^3 {C_1}^2 - 1380\,\nu^3 C_1 C_2 + 990\,\nu^3 {C_2}^2 + 180\,\nu\, C_0 {C_2}^2\\
&+ 180\,\nu\, {C_1}^2 C_2 - 360\,\nu^5 C_1 - 225\,\nu\, C_1 {C_2}^2 = 0,\\
&390\,\nu^5 C_1 - 1710\,\nu^5 C_2 + 90\,\nu^3 C_0 C_1 - 810\,\nu^3 C_0 C_2 - 255\,\nu^3 {C_1}^2 + 1005\,\nu^3 C_1 C_2 - 240\,\nu^3 {C_2}^2\\
&+ 45\,\nu\, {C_1}^3 + 270\,\nu\, C_0 C_1 C_2 - 180\,\nu\, C_0 {C_2}^2 - 180\,\nu\, {C_1}^2 C_2 = 0,\\
&- 180\,\nu^5 C_1 + 422\,\nu^5 C_2 - 180\,\nu^3 C_0 C_1 + 570\,\nu^3 C_0 C_2 + 165\,\nu^3 {C_1}^2 - 225\,\nu^3 C_1 C_2 + 90\,\nu\, {C_0}^2 C_2\\
&+ 90\,\nu\, C_0 {C_1}^2 - 270\,\nu\, C_0 C_1 C_2 - 45\,\nu\, {C_1}^3 + 2\,\mu\, C_2 - 2\,\alpha\nu\, C_2 - 2\,C_2 = 0,\\
&31\,\nu^5 C_1 - 32\,\nu^5 C_2 + 105\,\nu^3 C_0 C_1 - 120\,\nu^3 C_0 C_2 - 30\,\nu^3 {C_1}^2 + 45\,\nu\, {C_0}^2 C_1 - 90\,\nu\, {C_0}^2 C_2\\
&+ 2\,C_2 - \alpha\nu\, C_1 + 2\,\alpha\nu\, C_2 - C_1 - 90\,\nu\, C_0 {C_1}^2 + \mu\, C_1 - 2\,\mu\, C_2 = 0.
\end{aligned} \tag{26}
$$

Using the Maple program, solving (26) yields two sets of solutions, given as

$$C_0 = -\frac{1}{3}\nu^2, \quad C_1 = 4\nu^2, \quad C_2 = -4\nu^2, \tag{27}$$

and

$$C_0 = C_0, \quad C_1 = 2\nu^2, \quad C_2 = -2\nu^2. \tag{28}$$

Thus, Kudryashov's technique produces two analytic solutions for (3), as

$$u_1(t,x,y) = -\frac{1}{3}\nu^2 + 4\nu^2\left(\frac{1}{1+\mathrm{e}^z}\right) - 4\nu^2\left(\frac{1}{1+\mathrm{e}^z}\right)^2, \tag{29}$$

and

$$u_2(t,x,y) = C_0 + 2\nu^2\left(\frac{1}{1+\mathrm{e}^z}\right) - 2\nu^2\left(\frac{1}{1+\mathrm{e}^z}\right)^2, \tag{30}$$

where C_0 is a constant and $z = y - \nu x - (\mu - \alpha\nu)t$.

Solutions of (21) via the (G'/G) expansion method

We now seek solutions of the nonlinear ODE (21) using the (G'/G) expansion method [18]. Assume that the solutions of (21) are of the form

$$\psi(z) = \sum_{i=0}^{M} A_i \left(\frac{G'(z)}{G(z)}\right)^i, \tag{31}$$

where G satisfies the second-order ODE

$$G''(z) + \lambda G'(z) + \beta G(z) = 0, \tag{32}$$

where λ and β have arbitrary real values; A_i, $i = 1, \ldots, M$ are constants; and M is obtained by the balancing procedure. The balancing procedure applied to (21), which gives $M = 2$ [32]. Thus, from (31), we have

$$\psi(z) = A_0 + A_1\left(\frac{G'}{G}\right) + A_2\left(\frac{G'}{G}\right)^2, \tag{33}$$

where A_0, A_1, and A_2 are found. By substituting (33) into (21), invoking (32), and collecting terms with like powers of (G'/G) yields a system of algebraic equations in A_0, A_1, and A_2. Solving these equations with the help of Mathematica, we obtain

$$A_0 = -\frac{1}{3}\lambda^2\nu^2, \quad A_1 = -4\lambda\nu^2, \quad A_2 = -4\nu^2. \tag{34}$$

As a consequence, we have the following three categories of solutions for the gFKdVp Equation (3):

(i) When $K = \lambda^2 - 4\beta > 0$, we get hyperbolic function solution

$$u_3(t,x,y) = -\frac{1}{3}\lambda^2\nu^2 - 4\lambda\nu^2\left(\Delta_1\frac{L\cosh(\Delta_1 z) + N\sinh(\Delta_1 z)}{L\sinh(\Delta_1 z) + N\cosh(\Delta_1 z)} - \frac{\lambda}{2}\right) - 4\nu^2\left(\Delta_1\frac{L\cosh(\Delta_1 z) + N\sinh(\Delta_1 z)}{L\sinh(\Delta_1 z) + N\cosh(\Delta_1 z)} - \frac{\lambda}{2}\right)^2, \tag{35}$$

where $\Delta_1 = \sqrt{K}/2$, $z = y - \nu x - (\mu - \alpha\nu)t$, L and N are arbitrary constants.

(ii) When $K = \lambda^2 - 4\beta < 0$, we get a trigonometric function solution

$$u_4(t,x,y) = -\frac{1}{3}\lambda^2\nu^2 - 4\lambda\nu^2\left(\Delta_2\frac{-L\sin(\Delta_2 z) + N\cos(\Delta_2 z)}{L\cos(\Delta_2 z) + N\sin(\Delta_2 z)} - \frac{\lambda}{2}\right) - 4\nu^2\left(\Delta_2\frac{-L\sin(\Delta_2 z) + N\cos(\Delta_2 z)}{L\cos(\Delta_2 z) + N\sin(\Delta_2 z)} - \frac{\lambda}{2}\right)^2, \tag{36}$$

where $\Delta_2 = \sqrt{-K}/2$, $z = y - \nu x - (\mu - \alpha\nu)t$, and L and N are arbitrary constants.

(iii) Finally, $K = \lambda^2 - 4\beta = 0$ produces a rational function solution

$$u_5(t,x,y) = -\frac{1}{3}\lambda^2\nu^2 - 4\lambda\nu^2\left(\frac{N}{Nz + L} - \frac{\lambda}{2}\right) - 4\nu^2\left(\frac{N}{Nz + L} - \frac{\lambda}{2}\right)^2, \tag{37}$$

where $z = y - \nu x - (\mu - \alpha\nu)t$, L and N are arbitrary constants.

4.2. Symmetry Reductions and Solutions Using $\mathcal{S}_1$

We consider time translation symmetry $\mathcal{S}_1 = F^1(y - t)\partial/\partial t$ of (3) with $F^1(y - t) = 1$. The associated Lagrange system produces three invariants $p = x$, $q = y$, $\mathcal{X}(x,y) = u$, which transform Equation (3) into the nonlinear PDE

$$\mathcal{X}_{ppppp} + a\mathcal{X}\mathcal{X}_{ppp} + a\mathcal{X}_p\mathcal{X}_{pp} + 3a\mathcal{X}^n\mathcal{X}_p + \mathcal{X}_q = 0. \tag{38}$$

Equation (38) has two point symmetries $\mathcal{Z}_1 = \partial/\partial p$, $\mathcal{Z}_2 = \partial/\partial q$. Using the invariants of $\mathcal{Z}_1 = \partial/\partial p$, Equation (38) is transformed into the nonlinear ODE

$$\mathcal{Y}''''(p) + a\mathcal{Y}(p)\mathcal{Y}'''(p) + a\mathcal{Y}'(p)\mathcal{Y}''(p) + 3a\mathcal{Y}^n(p)\mathcal{Y}'(p) = 0. \tag{39}$$

Using the (G'/G)-expansion method on Equation (39) with $n = 2$, we obtain

(i) a hyperbolic function solution

$$u_6(t,x,y) = -\lambda^2 - 4\lambda\left(\Delta_1\frac{L\cosh(\Delta_1 x) + N\sinh(\Delta_1 x)}{L\sinh(\Delta_1 x) + N\cosh(\Delta_1 x)} - \frac{\lambda}{2}\right) - 4\left(\Delta_1\frac{L\cosh(\Delta_1 x) + N\sinh(\Delta_1 x)}{L\sinh(\Delta_1 x) + N\cosh(\Delta_1 x)} - \frac{\lambda}{2}\right)^2, \tag{40}$$

where $K = \lambda^2 - 4\beta > 0$, $\Delta_1 = \sqrt{K}/2$, and L and N are arbitrary constants;

(ii) a trigonometric function solution

$$u_7(t,x,y) = -\lambda^2 - 4\lambda\left(\Delta_2\frac{-L\sin(\Delta_2 x)+N\cos(\Delta_2 x)}{L\cos(\Delta_2 x)+N\sin(\Delta_2 x)} - \frac{\lambda}{2}\right) - 4\left(\Delta_2\frac{-L\sin(\Delta_2 x)+N\cos(\Delta_2 x)}{L\cos(\Delta_2 x)+N\sin(\Delta_2 x)} - \frac{\lambda}{2}\right)^2, \tag{41}$$

where $K = \lambda^2 - 4\beta < 0$, $\Delta_2 = \sqrt{-K}/2$, and L and N are arbitrary constants;

(iii) a rational function solution

$$u_8(t,x,y) = -\lambda^2 - 4\lambda\left(\frac{N}{Nx+L} - \frac{\lambda}{2}\right) - 4\left(\frac{N}{Nx+L} - \frac{\lambda}{2}\right)^2, \tag{42}$$

where $K = \lambda^2 - 4\beta = 0$, and L and N are arbitrary constants.

4.3. Symmetry Solutions Using $\mathcal{S}_2$ and $\mathcal{S}_3$

The space translation symmetries $\mathcal{S}_2$ and $\mathcal{S}_3$ both yield the constant solution $u(t,x,y) = C$, where C is a constant.

4.4. Symmetry Reductions and Solutions Using $\mathcal{S}_4$

We now make use of the symmetry $\mathcal{S}_4$ obtained when $n = 2$ from Equation (3), i.e.,

$$\mathcal{S}_4 = 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.$$

The symmetry $\mathcal{S}_4$ provides us with three invariants

$$r = \frac{x}{t^{\frac{1}{5}}},\ z = y - t,\ \phi(r,z) = t^{\frac{2}{5}}u,$$

and using these invariants, Equation (3) reduces to

$$\phi_{rrrrr} + a\phi\phi_{rrr} + a\phi_r\phi_{rr} + 3a\phi^2\phi_r - \frac{2}{5}\phi - \frac{1}{5}r\phi_r = 0. \tag{43}$$

Equation (43) has the symmetry

$$\Gamma = f^2(z)\frac{\partial}{\partial r} + f^1(z)\frac{\partial}{\partial z},$$

where $f^1(z)$ and $f^2(z)$ are arbitrary functions. Letting $f^1(z) = f^2(z) = 1$, the symmetry $\Gamma = \partial/\partial r + \partial/\partial z$ provides $U(\xi) = \phi(r,z)$ as an invariant solution, where $\xi = z - r$. Thus, the nonlinear PDE (43) transforms to a nonlinear ODE

$$U''''''(\xi) + aU(\xi)U'''(\xi) + aU'(\xi)U''(\xi) + 3aU^2(\xi)U'(\xi) + \frac{2}{5}U(\xi) - \frac{1}{5}\xi U'(\xi) = 0. \tag{44}$$

Power series solution

We employ the power series expansion method [33] to obtain analytic solutions of (44). We presume that the power series form of solution (44) is

$$U(\xi) = \sum_{n=0}^{\infty} W_n\xi^n, \tag{45}$$

where W_0, W_1, $W_2, \ldots$ are the constants to be determined. The derivatives of $U(\xi)$ from Equation (45) are given by

$$\begin{aligned} U'(\xi) &= \sum_{n=0}^{\infty}(n+1)W_{n+1}\xi^n,\ U''(\xi) = \sum_{n=0}^{\infty}(n+2)(n+1)W_{n+2}\xi^n, \\ U'''(\xi) &= \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)W_{n+3}\xi^n, \\ U''''(\xi) &= \sum_{n=0}^{\infty}(n+5)(n+4)(n+3)(n+2)(n+1)W_{n+5}\xi^n. \end{aligned} \tag{46}$$

Substituting the above expressions into the nonlinear ODE (44), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+5)(n+4)(n+3)(n+2)(n+1)W_{n+5}\xi^n - \frac{1}{5}\xi\left(\sum_{n=0}^{\infty}(n+1)W_{n+1}\xi^n\right) \\ &+ \frac{2}{5}\sum_{n=0}^{\infty}W_n\xi^n + a\left(\sum_{n=0}^{\infty}W_n\xi^n\right)\left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)W_{n+3}\xi^n\right) \\ &+ a\left(\sum_{n=0}^{\infty}(n+1)W_{n+1}\xi^n\right)\left(\sum_{n=0}^{\infty}(n+2)(n+1)W_{n+2}\xi^n\right) \\ &+ 3a\left(\sum_{n=0}^{\infty}W_n\xi^n\right)\left(\sum_{n=0}^{\infty}W_n\xi^n\right)\left(\sum_{n=0}^{\infty}(n+1)W_{n+1}\xi^n\right) = 0. \end{aligned} \tag{47}$$

Evaluating the above Equation (47) for $n = 0$ and $n = 1$, we get the recurrence formula as

$$\begin{aligned} W_{n+5} = A\Bigg\{ &\left(a\sum_{k=0}^{n}(n-k+3)(n-k+2)(n-k+1)W_kW_{n-k+3}\right) \\ &+\left(a\sum_{k=0}^{n}(k+1)(n-k+2)(n-k+1)W_{k+1}W_{n-k+2}\right) \\ &+\left(3a\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)W_iW_{k-i}W_{n-k+1} - \frac{1}{5}W_n[n-2]\right)\Bigg\}, \end{aligned}$$

where $A = -1/(n+5)(n+4)(n+3)(n+2)(n+1)$. The above mathematical equation describes a function in terms of its previous values, allowing for the calculation of values for a range of inputs based on a known starting point. Here W_0, W_1, W_2, W_3 and W_4 are arbitrary constants along with

$$\begin{aligned} W_5 &= \frac{1}{120}\left(-6aW_0W_3 - 2aW_1W_2 - 3aW_0^2W_1 - \frac{2}{5}W_0\right), \\ W_6 &= \frac{1}{720}\left(\frac{1}{5}W_1 - 21aW_1W_4 - 12aW_2W_3 - 6aW_1^2W_2 - \frac{2}{5}W_1\right). \end{aligned}$$

Thus, the solution to nonlinear ODE (44) yields

$$U(\xi) = W_0 + W_1\xi + W_2\xi^2 + W_3\xi^3 + W_4\xi^4 + W_5\xi^5 + W_6\xi^6 + \sum_{n=2}^{\infty}W_{n+5}\xi^{n+5}. \tag{48}$$

Consequently, the power series solution to the gFKdVp Equation (3) is

$$
\begin{aligned}
u_9(t,x,y) = t^{-\frac{2}{5}}\Bigg\{ & W_0 + W_1\left(y-t-xt^{-\frac{1}{5}}\right) + W_2\left(y-t-xt^{-\frac{1}{5}}\right)^2 + W_3\left(y-t-xt^{-\frac{1}{5}}\right)^3 \\
& + W_4\left(y-t-xt^{-\frac{1}{5}}\right)^4 - \frac{1}{120}\left(6aW_0W_3 + 2aW_1W_2 + 3aW_0^2W_1 + \frac{2}{5}W_0\right) \\
& \times \left(y-t-xt^{-\frac{1}{5}}\right)^5 + \frac{1}{720}\left(\frac{1}{5}W_1 - 21aW_1W_4 - 12aW_2W_3 - 6aW_1^2W_2\right. \\
& \left. -\frac{2}{5}W_1\right)\left(y-t-xt^{-\frac{1}{5}}\right)^6 - \frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)} \\
& \times \sum_{n=2}^{\infty}\Bigg\{\left(a\sum_{k=0}^{n}(n-k+3)(n-k+2)(n-k+1)W_kW_{n-k+3}\right) \\
& + \left(a\sum_{k=0}^{n}(k+1)(n-k+2)(n-k+1)W_{k+1}W_{n-k+2}\right) \\
& + \left(3a\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)W_iW_{k-i}W_{n-k+1} - \frac{1}{5}W_n[n-2]\right)\Bigg\} \\
& \times \left(y-t-xt^{-\frac{1}{5}}\right)^{n+5}\Bigg\}.
\end{aligned}
$$

4.5. Results and Discussion

In this work, we obtained a number of analytic solutions to the gFKdVp Equation (3) by invoking the Lie group method together with Kudryashov's, (G'/G) expansion, and the power series solution methods. In this subsection, we give various graphical representations to demonstrate the physical look of the analytic solutions of the Equation (3). Thus, to view the dynamical behavior of the solutions of Equation (3), we assign different values to the constants in the solutions. For instance, Figure 1 depicts the dynamical behavior of the solution (29) in the 3D, density, and 2D plots. The parameters involved in the 3D plot and density plot are given different values, including, $y = 0, \nu = 1, \mu = 1$, and $\alpha = 1$, where the variables t and x range from -10 to 10. Hence, for the 2D plot, we choose $t = 0.5$ and $y = 0.3$, where x ranges from -10 to 10.

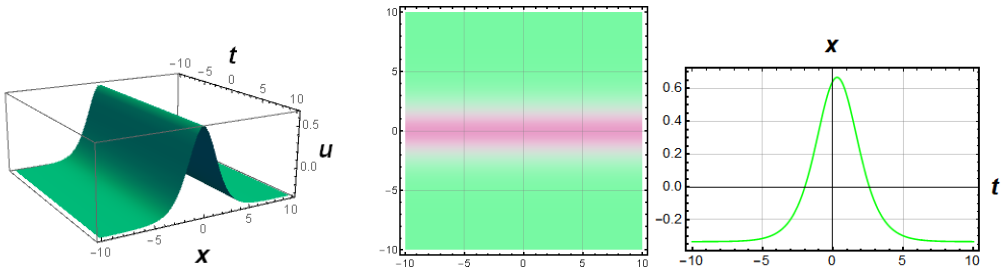

Figure 1. Hyperbolic solution profile of (29) in 3D, density, and 2D.

Moreover, the solution (30) is shown in 3D, density, and 2D plots in Figure 2. The parameters are given different values, including, $C_0 = 1, y = 0, \nu = 1.5, \mu = 1.9$ and $\alpha = 1$, where the variables t and x range from -10 to 10 for the 3D plot and density plot. Hence, we choose $t = 0.5$ and $y = 0.3$, where x ranges from -10 to 10 for the 2D plot.

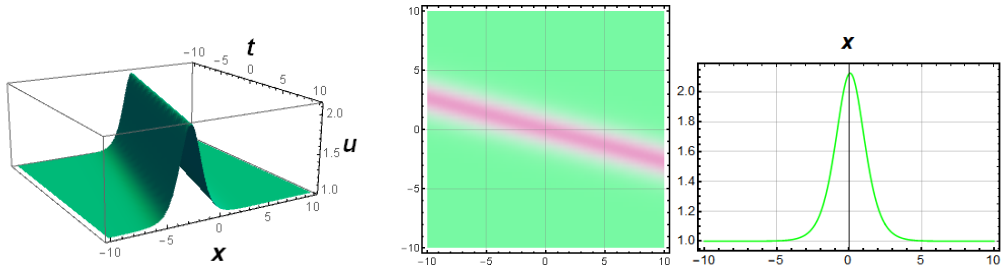

Figure 2. Profile of hyperbolic solution (30) in 3D, density, and 2D.

Figure 3 depicts the dynamical behavior of the hyperbolic function solution (35) with 3D and density plots with dissimilar values for the parameters of $y = 1$, $\beta = 0.02$, $\mu = 1, \gamma = 2, \nu = 0.6, \alpha = 0.9, L = 1, N = 1.8$, where the variables x and t range from -10 to 10. For the 2D plot, all the parameters from th e3D and density plots remain the same, except for t, in this case $t = 1$.

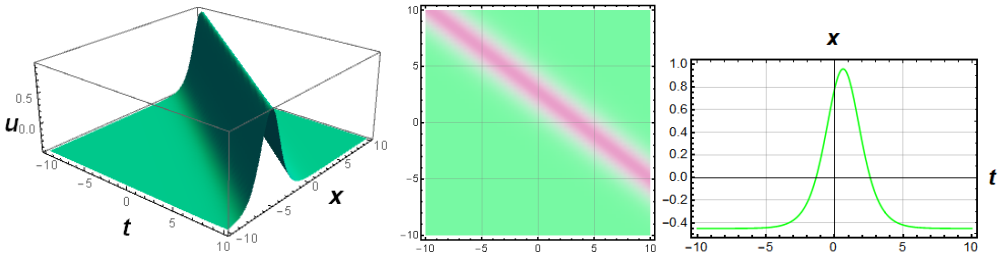

Figure 3. Profile of hyperbolic solution (35) in 3D, density, and 2D.

Figure 4 shows the physical look of the rational function solution (37) and the parameters $\mu = 0.0001, \gamma = 5.009, \nu = 0.2, \alpha = 0.001, L = 1, N = 1.8$, with $y = 1$, and x and t ranging from -200 to 200. For the 2D plot, these parameters remain the same as for 3D and density plots, except for t, which in this case is $t = 1$.

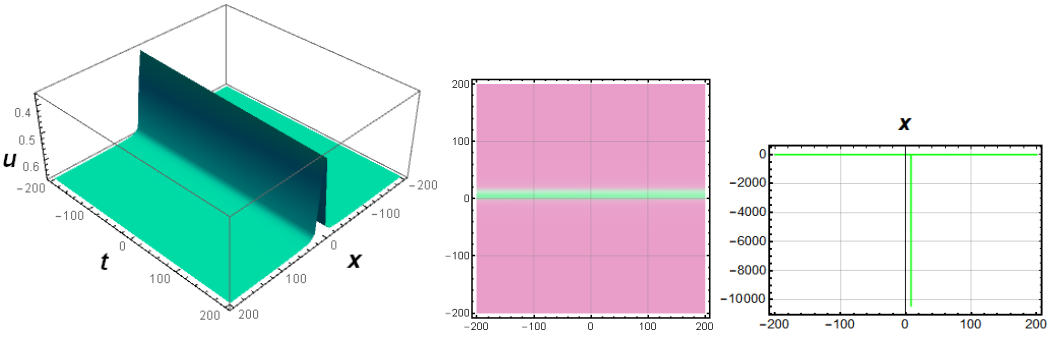

Figure 4. Profile of rational solution (37) in 3D, density, and 2D.

It should be noted from the above results and discussions that by employing different methods to find solutions to the gFKdVp Equation (3), one can obtain solutions of different types that have different physical meanings.

5. Concluding Remarks

To summarize, we used Lie group analysis as well as various other methodologies, including Kudryashov's method, the (G'/G) expansion method, and the power series expansion method, to find analytic solutions to the gFKdVp Equation (3). The solutions were obtained in the form of exponential, trigonometric, hyperbolic, and rational functions. These obtained solutions can be of interest to physicists who are working in this area of research. To demonstrate the dynamical behavior of the obtained solutions, a graphical

depiction of the solutions for certain parameter values were also given. The novel solutions developed in this paper may have a substantial influence on future research. At the end, we constructed the conservation law, which provides insight into conserved physical quantities such as energy and momentum, among others, by employing the general multiplier technique and Ibragimov's method. In future work, one can use the conservation laws obtained here to construct more analytic solutions to the gFKdVp Equation (3).

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