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Temporal Evolution of Small-Amplitude Internal Gravity Waves Generated by Latent Heating in an Anelastic Fluid Flow

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Abstract

A two-dimensional time-dependent model is presented for upward-propagating internal gravity waves generated by an imposed thermal forcing in a layer of fluid with uniform background velocity and stable stratification under the anelastic approximation. The configuration studied is representative of a situation with deep or shallow latent heating in the lower atmosphere where the amplitude of the waves is small enough to allow linearization of the model equations. Approximate asymptotic time-dependent solutions, valid for late time, are obtained for the linearized equations in the form of an infinite series of terms involving Bessel functions. The asymptotic solution approaches a steady-amplitude state in the limit of infinite time. A weakly nonlinear analysis gives a description of the temporal evolution of the zonal mean flow velocity and temperature resulting from nonlinear interaction with the waves. The linear solutions show that there is a vertical variation of the wave amplitude which depends on the relative depth of the heating to the scale height of the atmosphere. This means that, from a weakly nonlinear perspective, there is a non-zero divergence of vertical momentum flux, and hence, a non-zero drag force, even in the absence of vertical shear in the background flow.

Keywords: internal gravity waves; differential equations; nonlinear systems; asymptotic solution



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1. Introduction

Internal gravity waves in the atmosphere result from the combined effects of gravitation and buoyancy forces; they are generated primarily by topography [1–3] and convection in the lower atmosphere. The mechanisms for the generation of internal gravity waves by convection are less well understood than the topographic generation mechanisms, but it is generally accepted that latent heating from the earth's surface energy budget plays an important role in the process, e.g., [4–7].

It is well-known that the interactions between atmospheric waves and the background flow and the resulting divergence of vertical momentum flux contribute to large-scale low-frequency variations in the global circulation, which greatly affect climate and weather. General circulation models (GCMs) are able to correctly simulate these variations when there is sufficiently high resolution to correctly represent the contributions from the different types of waves, including planetary-scale Kelvin waves and Rossby-gravity waves [8] and smaller-scale internal gravity waves [9]. Historically, gravity wave drag parameterization

schemes have been used in model simulations where there is inadequate resolution to correctly represent internal gravity waves. These schemes include additional terms to represent the drag force that would have resulted from the unresolved gravity waves, and although many present day GCMs are able to resolve the waves, they are still in common use today.

The accuracy of parameterization schemes depends on our understanding of the mechanisms that generate the waves, their temporal evolution and their interactions with the background flow [10]. In terms of mathematical analyses, insight into wave–mean-flow interactions is obtained by studying weakly nonlinear models in which the leading-order terms in the solutions are obtained from linear theory.

In modelling the mechanisms for the generation of internal gravity waves, some simplified descriptions have been suggested for the interactions between latent heating, convection and waves. One possibility is that the waves are excited directly by a thermal forcing with no shear [4,10]; in this case, it is suggested that the dominant vertical wavelength of the waves is proportional to the buoyancy frequency of the background flow and approximately twice the depth of the heating [11–14]. Some numerical models, e.g., [15,16], have been developed based on this deep-heating mechanism. Analyses of measurements, e.g., [17], suggest a mechanism in which a thermal forcing generates convective cells in a low-altitude layer of unstable stratification in the atmospheric boundary layer, and the convection in turn generates gravity waves aloft in a similar manner to the mechanism by which obstacles such as mountains generate topographic gravity waves [4,10,18].

Sayed and Campbell [19] used a two-layer model to explore the “deep-heating” and the “obstacle-effect” mechanisms in configurations where they occur separately and in a configuration where the waves are generated by a combination of the two mechanisms. Their investigation was based on equations under the anelastic approximation, where the background density varies with altitude but there are no sound waves present [20,21]. The model comprises an upper layer with stable stratification and a lower layer with unstable stratification and a nonhomogeneous term in the energy conservation equation representing a thermal forcing with specified vertical location and depth.

The purpose of the present study is to investigate the deep-heating mechanism in a single layer of stably-stratified fluid where there are no convective cells, using a more realistic representation in which the amplitude of the gravity waves is allowed to vary with time as well as with altitude. This is motivated by the analyses of measurements given in observational studies that show temporal variations in gravity wave activity in the troposphere and stratosphere [22]. The aim is to derive exact analytical solutions or at least approximate asymptotic solutions, valid for late time, and use them to determine the effects, if any, that the nonlinear wave interactions would have on the background flow. While such effects are well-known for the case of gravity waves forced by an oscillatory lower boundary condition, the effects arising from gravity waves forced by latent heating have received less attention.

Linear time-dependent solutions for small-amplitude gravity waves forced by an oscillatory lower boundary condition are well-known. Booker and Bretherton [23] used a Laplace transform technique to derive approximate asymptotic solutions for the case of a background flow with vertical shear under the Boussinesq approximation. This is an approximation in which the background fluid density is set to a constant in all the terms in the equations except in the term representing the gravitational force in the vertical momentum conservation equation where it is taken to be a function of altitude. Under this approximation, the continuity equation can be written in its incompressible form and the energy conservation equation can be approximated by an equation written in terms of the fluid density (see, for example, [24]). This simplifies the mathematical problem and

makes it more tractable to analytical solutions but is only valid for a relatively shallow layer over which the vertical variation of the fluid density is taken to be “small”. With a vertically-sheared background flow, there may be a critical level where the background flow speed is equal to the wave phase speed and where nonlinear wave–mean-flow interactions occur. These interactions lead to a vertical flux of horizontal mean momentum that varies across the critical level. Nonlinear investigations show that this results in changes in the background velocity from wave absorption and eventually wave reflection [12].

In a fluid flow with no vertical shear and no critical level, it is well-known from the Eliassen–Palm Theorem [25] that the momentum flux is independent of altitude, and hence there are no wave-induced changes in the background velocity. In that case, waves forced by an oscillatory lower boundary condition simply continue to propagate upwards indefinitely unless there is a mechanism for their dissipation. Exact analytical expressions for the linear solutions have been obtained for the case of a constant background flow velocity with no dissipation under the Boussinesq approximation [26,27].

For gravity waves generated by a thermal forcing, however, it is found that there can be a nonzero momentum flux divergence and corresponding nonzero drag force and wave-induced changes in the background, even in the absence of vertical shear and a critical level. Sayed [28] used a Laplace transform technique similar to that of [26,27] and derived time-dependent exact solutions for gravity waves generated by a thermal forcing, also with a constant background flow velocity and under the Boussinesq approximation. The equations are linearized and further simplified by the long-wave approximation in which the vertical-to-horizontal aspect ratio is assumed to be small. Two alternative expressions are derived, each in terms of infinite series involving Bessel functions. Leading-order approximations are obtained for the vertical momentum flux, gravity wave drag and mean flow acceleration that would result from the wave interactions in a weakly nonlinear situation.

Here we follow a similar analytical procedure and derive asymptotic solutions, valid for late time, for gravity waves generated by a thermal forcing, with a constant background flow velocity, but for the more general problem defined by the anelastic approximation [29,30]. Making the anelastic approximation gives additional vertically-varying factors and terms that would have been omitted by the Boussinesq approximation, and it includes situations where the restrictions on the depth of the fluid layer and on the vertical variation of the fluid density can be relaxed. The anelastic model describes time evolution on a scale that is defined by the buoyancy frequency of the background fluid flow. This time scale allows the existence of internal gravity waves but filters out the faster acoustic waves that could be present in a model based on the fully compressible equations.

These time-dependent asymptotic solutions can be used as a starting point for further (nonlinear) analyses [30] and for comparison with numerical simulations. A recent study [29] involves the numerical solution of the nonlinear equations obtained under the anelastic approximation for gravity waves forced by a thermal forcing. The investigation [29] includes cases with a background flow with vertical shear and a critical level and cases with constant background flow velocity, like the case studied here. The solutions presented here are in agreement with the numerical solutions obtained with constant background flow velocity [29]. In particular, we find that the horizontal mean vertical momentum flux varies with altitude, and hence there is a nonzero gravity wave drag force and wave-induced mean flow, even though there is no vertical shear and no critical level.

2. Model Formulation

In this investigation, the propagation of internal gravity waves in a density-stratified fluid flow is represented by a system of nonlinear equations that describe the conservation of momentum, mass and energy in a two-dimensional rectangular configuration (see, for

example, [31]). The equations are nondimensional and written in terms of a horizontal coordinate x in the west-to-east direction, a vertical coordinate z and time t . These variables are defined in terms of the respective dimensional variables (with stars) by

$$x = \frac{x^*}{L_x}, \quad z = \frac{z^*}{L_z}, \quad t = \frac{Wt^*}{L_z} = \frac{Ut^*}{L_x}, \quad (1)$$

where L_x and L_z are, respectively, reference length scales in the x and z directions, which are on the order of magnitude of the horizontal and vertical wavelengths of the gravity waves, and U and W are reference velocity scales in the x and z directions.

The fluid flow is described in terms of the density ϱ_T , the potential temperature θ_T , and the x - and z -velocity components u_T and w_T by:

$$\begin{aligned} \frac{\partial \varrho_T}{\partial t} + \frac{\partial}{\partial x}(\varrho_T u_T) + \frac{\partial}{\partial z}(\varrho_T w_T) &= 0, \\ \varrho_T \left(\frac{\partial u_T}{\partial t} + u_T \frac{\partial u_T}{\partial x} + w_T \frac{\partial u_T}{\partial z} \right) &= -\frac{\partial p_T}{\partial x}, \\ \delta \varrho_T \left(\frac{\partial w_T}{\partial t} + u_T \frac{\partial w_T}{\partial x} + w_T \frac{\partial w_T}{\partial z} \right) &= -\frac{\partial p_T}{\partial z} - \varrho_T g, \\ \frac{1}{\theta_T} \left(\frac{\partial \theta_T}{\partial t} + u_T \frac{\partial \theta_T}{\partial x} + w_T \frac{\partial \theta_T}{\partial z} \right) &= \frac{F_T}{\varrho_T}, \end{aligned} \quad (2)$$

where $\delta = (L_z/L_x)^2$ is the square of the vertical-to-horizontal aspect ratio of the rectangular configuration. In the context of internal gravity waves, an appropriate reference vertical length scale L_z is on the order of magnitude of the atmospheric scale height, approximately 7 km, and is small relative to the reference horizontal length scale L_x ; thus, δ is considered to be a small (but non-zero) parameter. The effects of deep or shallow latent heating are represented by the nondimensional density-weighted forcing function F_T in the energy equation, which generates a perturbation to the background flow. This is expressed as $F_T = \varepsilon F(x, z, t)$, where F is a specified profile for the latent heat distribution. The parameter ε gives a measure of the amplitude of the forcing function and the disturbance that it generates, relative to magnitude of the corresponding unperturbed flow quantities.

Each of the total quantities is written as the sum of a horizontal mean component and a gravity wave quantity representing the perturbation that is generated by the thermal forcing:

$$\begin{aligned} u_T(x, z, t) &= \bar{u}(z) + \varepsilon u(x, z, t), \\ w_T(x, z, t) &= \varepsilon w(x, z, t), \\ \varrho_T(x, z, t) &= \bar{\rho}(z) + \varepsilon \rho(x, z, t), \\ p_T(x, z, t) &= \bar{p}(z) + \varepsilon p(x, z, t), \\ \theta_T(x, z, t) &= \bar{\theta}(z) + \varepsilon \theta(x, z, t). \end{aligned} \quad (3)$$

This investigation deals with the case where there is no vertical shear in the background flow, i.e., $\bar{u}$ is constant. In addition, the background flow is considered to be in hydrostatic balance, i.e.,

$$\frac{d\bar{p}}{dz} = -g\bar{\rho}, \quad (4)$$

with the background density and potential temperature given by

$$\bar{\rho}(z) = \rho_0 e^{-z/H}, \quad \bar{\theta}(z) = \theta_0 e^{\kappa z/H}, \quad (5)$$

where ρ_0 and θ_0 are constants and H is the density scale height. The constant $\kappa = 1 - 1/\gamma$, where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume.

The time scale for the propagation of the gravity wave perturbation is determined by the buoyancy frequency of the fluid flow, $N = (\frac{g}{\theta} \frac{d\theta}{dz})^{1/2}$. The buoyancy frequency measures the extent of the stability of the background flow to vertical displacements. The linear dispersion relation for a gravity wave gives the wave frequency as $\omega = \pm \frac{N^* k^*}{m^*}$, where k^* and m^* are the horizontal and vertical wavenumbers. This gives a reference time scale of $\frac{L_x}{N^* L_z}$ for the evolution of the wave. Thus, the reference horizontal velocity scale for the wave is $u^* = N^* L_z$ and the nondimensional amplitude parameter is $\varepsilon = \frac{u^*}{U} = \frac{N^* L_z}{U}$. This ratio allows us to consider ε a “small” parameter for the purpose of asymptotic analyses.

Under the anelastic approximation, the total density ρ_T in the continuity equation is replaced by the steady mean density $\bar{\rho}(z)$. This gives

$$\frac{\partial}{\partial x}(\bar{\rho} u_T) + \frac{\partial}{\partial z}(\bar{\rho} w_T) = 0. \quad (6)$$

This allows us to define a density-weighted streamfunction perturbation $\psi(x, z, t)$ by

$$\frac{\partial \psi}{\partial z} = -\bar{\rho} u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = \bar{\rho} w. \quad (7)$$

The corresponding vorticity perturbation $\zeta(x, z, t) = \frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x}$ can be written as

$$\zeta(x, z, t) = \frac{1}{\bar{\rho}} \left(\frac{\partial^2 \psi}{\partial z^2} + \delta \frac{\partial^2 \psi}{\partial x^2} - \frac{\bar{\rho}'}{\bar{\rho}} \frac{\partial \psi}{\partial z} \right) = \frac{1}{\bar{\rho}} \left(\nabla^2 \psi - \frac{\bar{\rho}'}{\bar{\rho}} \frac{\partial \psi}{\partial z} \right),$$

where ∇^2 is the nondimensional Laplacian operator and the primes denote differentiation with respect to z .

Substituting (3) into (2) gives nonlinear perturbation equations for u , w , θ and ρ . We then combine the x - and z -momentum equations by differentiating the x -momentum equation by z , differentiating the z -momentum equation by x and subtracting one from the other. Finally, we make the substitutions (7) and hence re-write the momentum equation and the energy equation, both in terms of the streamfunction ψ [30]. These steps lead to

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \zeta - \frac{g}{\bar{\theta}} \frac{\partial \theta}{\partial x} + \frac{\varepsilon}{\bar{\rho}} \left(\frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \zeta}{\partial x} \right) = 0, \quad (8)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \theta + \frac{\bar{\theta}'}{\bar{\rho}} \frac{\partial \psi}{\partial x} + \frac{\varepsilon}{\bar{\rho}} \left(\frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \theta}{\partial x} \right) = \frac{\bar{\theta}}{\bar{\rho}} F. \quad (9)$$

For small-amplitude waves, we consider that $\varepsilon \ll 1$ and retain the leading-order terms only, hence linearizing the perturbation equations. We examine the linear equations for $0 < t < \infty$ in a rectangular spatial domain given by

$$0 \leq x < 2\pi, \quad 0 < z < \infty \quad (10)$$

with a thermal forcing function of the form

$$F(x, z) = \hat{F}(z) e^{ikx} + \text{c.c.}, \quad (11)$$

where k is a positive integer and “c.c.” denotes the complex conjugate of the preceding term. In the linear formulation, this forcing term generates a horizontally periodic gravity

wave perturbation with horizontal wavenumber k . In general, the amplitude $\hat{F}(z)$ is a specified function that goes to zero as $z \rightarrow \infty$. In this analytical investigation, we set

$$\hat{F}(z) = e^{-b|z|} \quad (12)$$

to give a simple representation of vertically localized tropospheric heating. The parameter b determines the depth of the heating profile. We note that, in numerical solutions, other profiles may be used to more accurately model the form of localized heating that may be observed in the lower atmosphere; for example, $\hat{F}(z) = e^{-b|z-z_0|}$ [28,30] or $\hat{F}(z) = e^{-b(z-z_0)^2}$ [29], where z_0 is a specified altitude where the peak of the heating profile occurs. Here, since we have $z_0 = 0$, it means that we are only modeling the effects that occur above the peak.

For a more realistic formulation, we may consider a forcing function that is localized in the horizontal as well as the vertical direction. This could be used to model a situation where there is latent heating in the form of a “heat island” in the lower atmosphere and would generate a perturbation in the form of a horizontally localized wave packet comprising a spectrum of horizontal wavenumbers. In this investigation, we restrict our attention to the horizontally periodic form (11), which generates a monochromatic perturbation with a single horizontal wavenumber k .

With the horizontally periodic forcing function, the linear equations in the domain (10) are subject to periodic boundary conditions at $x = 0$ and $x = 2\pi$. In addition, we impose a zero boundary condition at $z = 0$ and the condition that only waves with upward group velocity are included in the solution, so that there is no incoming wave energy from infinite z .

The linear equations

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\nabla^2 \psi - \frac{\bar{\rho}'}{\bar{\rho}} \frac{\partial \psi}{\partial z} \right) - \frac{g \bar{\rho}}{\bar{\theta}} \frac{\partial \theta}{\partial x} = 0 \quad (13)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \theta + \frac{\bar{\theta}'}{\bar{\rho}} \frac{\partial \psi}{\partial x} = \frac{\bar{\theta}}{\bar{\rho}} F. \quad (14)$$

are combined to give a single equation for the streamfunction perturbation,

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left(\nabla^2 \psi - \frac{\bar{\rho}'}{\bar{\rho}} \frac{\partial \psi}{\partial z} \right) + N^2 \frac{\partial^2 \psi}{\partial x^2} = g \frac{\partial F}{\partial x}, \quad (15)$$

where $N = \left(\frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz} \right)^{1/2}$ is the buoyancy frequency of the background flow. A necessary condition for the existence of gravity waves is that $N^2 > 0$; the fluid is then said to be statically stable. With the specified form of $\bar{\theta}$ given in (5), $N^2 = g\kappa/H$, a positive constant.

3. Steady-Amplitude Solution

We first note that, with the thermal forcing (11) and with periodic boundary conditions at $x = 0$ and $x = 2\pi$, the linear Equation (15) has a steady-amplitude solution of the form

$$\psi(x, z) = \hat{\psi}(z) e^{ikx} + \text{c.c.}, \quad (16)$$

with amplitude $\hat{\psi}(z)$ satisfying the ordinary differential equation

$$\hat{\psi}'' + \frac{1}{H} \hat{\psi}' + \left(\frac{N^2}{\bar{u}^2} - \delta k^2 \right) \hat{\psi} = \frac{g F_0}{ik \bar{u}^2} e^{-bz}, \quad (17)$$

where the primes denote differentiation with respect to z . Equation (17) is a nonhomogeneous version of the Taylor–Goldstein equation [32,33] for internal gravity waves in an anelastic flow.

We define $\hat{\psi}(z) = e^{-\frac{z}{2H}}\phi(z)$ and substitute this into (17) to give

$$\phi'' + \left(\frac{N^2}{\bar{u}^2} - \frac{1}{4H^2} - \delta k^2 \right) \phi = \frac{gF_0}{ik\bar{u}^2} e^{(\frac{1}{2H}-b)z}. \quad (18)$$

The corresponding homogeneous equation has linearly-independent solutions proportional to $e^{\pm imz}$, where $m^2 = \frac{N^2}{\bar{u}^2} - \frac{1}{4H^2} - \delta k^2$. We only consider the situation where m^2 is positive, so that we obtain vertically oscillating wave functions; m^2 being negative would give “trapped” disturbances that would decay exponentially with altitude. We define m to have the same sign as $\bar{u}$. In that case, according to the group velocity argument of Booker and Bretherton [23], for a horizontal wavenumber $k > 0$, the solution with upward group velocity is e^{imz} .

A particular solution of (18) is

$$\phi_p(z) = \frac{gF_0 e^{(\frac{1}{2H}-b)z}}{ik\bar{u}^2 \left(\frac{N^2}{\bar{u}^2} + b^2 - \frac{b}{H} - \delta k^2 \right)}. \quad (19)$$

Applying the zero boundary condition at $z = 0$ then gives $\phi(z)$, from which we obtain the streamfunction perturbation,

$$\psi(x, z) = \frac{-gF_0}{ik\bar{u}^2 \left(\frac{N^2}{\bar{u}^2} + b^2 - \frac{b}{H} - \delta k^2 \right)} \left(e^{imz} e^{-\frac{z}{2H}} - e^{-bz} \right) e^{ikx} + \text{c.c.} \quad (20)$$

Figure 1a illustrates the solution (20) in a rectangular region defined by nondimensional variables $0 \leq x \leq 2\pi$ and $0 \leq z \leq 10$. For illustration and comparison, we set the same choice of parameters used for some of the numerical simulations presented in [29]: $k = 2$, $\delta = 0.2$, $H = 5$, $b = 2.5$, $\bar{u} = -1$, $N \approx 1.058$, $m \approx -0.456$. These values correspond to a region of approximate width 20 km and depth 14 km with reasonable values of the dimensional quantities, $g^* = 9.8 \text{ m s}^{-2}$, $H^* = 7 \text{ km}$ and $\bar{u}^* \approx 26.5 \text{ m s}^{-1}$ [29]. This gives linear gravity waves with horizontal wavelengths of up to 20 km (with $k = 1$), representative of small-scale waves in the lower atmosphere [34,35].

The contour plot shows that the streamfunction perturbation is the sum of two wave functions with different phases. The first is proportional to $\cos(kx + mz)$, with an exponentially decaying factor of $e^{-\frac{z}{2H}}$, and since m is negative for k positive, the phase lines have a positive slope of $-k/m$ in the xz -plane. The second term is proportional to $\cos(kx)$, with no oscillations in z , and an exponentially decaying factor of e^{-bz} . Since $H = 5$ and $b = 2.5 > 1/H > 1/(2H)$, this latter term decreases rapidly with increasing z so that the term proportional to $\cos(kx + mz)$ is dominant. Over the interval $0 \leq z \leq 10$, $\psi(x, z)$ reaches its maximum value in the lower half of the interval and then decreases as $z \rightarrow 10$. With the choice of parameter values used for this illustrative plot, there is close agreement with the contour plots of the late-time linear numerical solution obtained by [29].

Figure 1b illustrates the solution (20) in the same configuration and with all the parameter values being the same as in Figure 1a, except for the parameter b , which is now set to 0.2. In this case, the heating depth is the same as the density scale height H . As before, the solution is a sum of the two wave functions, but in this case the term proportional to $\cos(kx)$, with no oscillations in z , decreases more slowly with z than the term proportional to $\cos(kx + mz)$, and it is thus the dominant term in the upper half of the domain. This

gives a contour plot with phase lines that have a steeper (more positive slope) than those in Figure 1a.

For an additional comparison, we note that the corresponding steady solution under the Boussinesq approximation takes a form that is qualitatively similar to (20) but does not include the exponential factor of $e^{-\frac{z}{2H}}$ or any of the terms involving $1/H$ that appear in the denominator of (20). Having made the anelastic approximation here allows us to model the effects of the exponential variation of the background potential temperature and density, even with this simplified steady-amplitude representation.

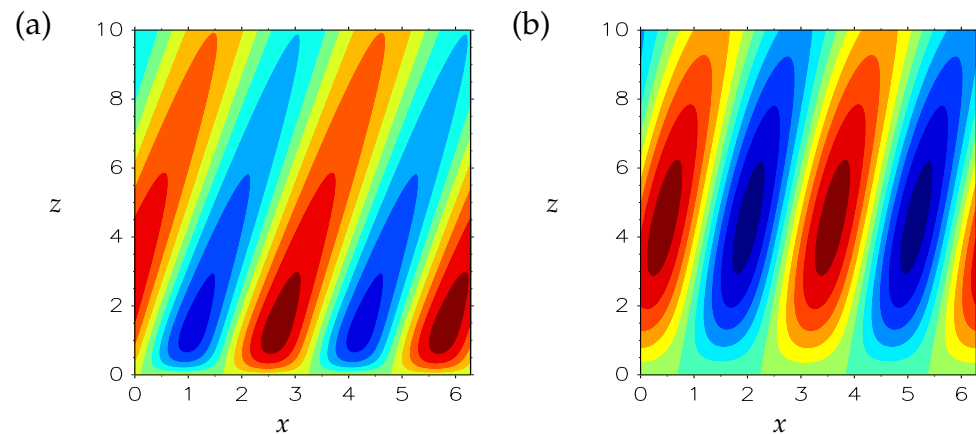


Figure 1. Contour plot of the wave streamfunction $\psi(x, z)$ according to Equation (20) with $k = 2$, $H = 5$ and (a) $b = 2.5 > 1/H$, (b) $b = 0.2 < 1/H$. Negative values are indicated in blue, positive values in red.

Finally, we note that, if instead of the horizontally periodic steady thermal forcing (11), we were to specify

$$F(x, z) = \hat{F}(z) e^{ik(x-ct)} + \text{c.c.}, \quad (21)$$

with k and c being real constants, we would obtain t -periodic solutions with phase speed c and with the amplitude $\hat{\psi}$ given by the same expression in (20) but with $\bar{u}$ replaced by $(\bar{u} - c)$ everywhere it appears in the expression.

4. Time-Dependent Solution

In what follows, the focus is on the situation where the fluid vorticity and temperature are initially unperturbed and the thermal forcing term generates a wave perturbation whose amplitude evolves with time. This gives an initial-boundary-value problem in which the vorticity perturbation and its time derivative are zero at $t = 0$ in the domain (10). For simplicity, we denote partial derivatives with respect to x , z and t by subscripts of the respective variables and write the initial conditions as

$$\delta\psi_{xx} + \psi_{zz} + \frac{1}{H}\psi_z = 0 \quad \text{and} \quad \delta\psi_{xxt} + \psi_{zzt} + \frac{1}{H}\psi_{zt} = 0 \quad (\text{at } t = 0). \quad (22)$$

In a linear configuration, the evolution of the perturbation is then given by the initial-boundary-value problem comprising the partial differential Equation (15) in the spatial domain (10), with $0 < t < \infty$, and with initial conditions (22), along with the boundary conditions that $\psi = 0$ at $z = 0$ and that there are solutions with upward group velocity only.

The form of the thermal forcing (11) implies that the solution of the linear initial-boundary-value problem takes the form

$$\psi(x, z, t) = \hat{\psi}(z, t) e^{ikx} + \text{c.c.}, \quad (23)$$

with a time-dependent amplitude $\hat{\psi}(z, t)$ given by

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 (\hat{\psi}_{zz} + \frac{1}{H}\hat{\psi}_z - \delta k^2 \hat{\psi}) - N^2 k^2 \hat{\psi} = ikg\hat{F}. \quad (24)$$

Defining the Laplace transform

$$\tilde{\psi}(z, s) = \mathcal{L}[\hat{\psi}(z, t)] = \int_0^\infty e^{-st} \hat{\psi}(z, t) dt,$$

where s is a complex variable, gives the transformed equation

$$\tilde{\psi}_{zz} + \frac{1}{H}\tilde{\psi}_z - \left(\frac{N^2 k^2}{(s + ik\bar{u})^2} + \delta k^2\right) \tilde{\psi} = \frac{ikgF_0}{s(s + ik\bar{u})^2} e^{-bz}, \quad (25)$$

with the boundary condition $\tilde{\psi}(0, s) = 0$. The solution satisfying the zero boundary condition at $z = 0$ and the condition of upward group velocity is of the same form as the steady-amplitude solution (20). with $ik\bar{u}$ replaced by $(s + ik\bar{u})$:

$$\tilde{\psi}(z, s) = -\frac{ikgF_0}{\left(b^2 - \frac{b}{H} - \delta k^2\right)} \frac{e^{-\left(\frac{N^2 k^2}{(s + ik\bar{u})^2} + \frac{1}{4H^2} + \delta k^2\right)^{\frac{1}{2}} z} e^{-\frac{z}{2H}} - e^{-bz}}{s\left((s + ik\bar{u})^2 - \frac{N^2 k^2}{(b^2 - \frac{b}{H} - \delta k^2)}\right)}. \quad (26)$$

Evaluating the inverse Laplace transform of (26) involves integrating over wave frequency s in complex s -space. We define the function

$$\lambda(s) = \left(\frac{N^2 k^2}{s^2} + \frac{1}{4H^2} + \delta k^2\right)^{1/2}$$

and the constant $\eta = \sqrt{b^2 - \frac{b}{H} - \delta k^2}$. Since λ is a function of the complex frequency, it describes how the different frequency modes of the time-dependent perturbation vary exponentially with altitude. The constant η gives a measure of the relative magnitude of b , the vertical decay rate of the thermal forcing, and $1/H$, the exponential decay rate of the background density.

We consider first the case where $b^2 - \frac{b}{H} - \delta k^2 > 0$. Making use of the first shifting theorem for Laplace transforms, we write

$$\hat{\psi}(z, t) = -\frac{ikgF_0}{\eta^2} e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ \frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}} - e^{-bz}}{(s - ik\bar{u})\left(s^2 - \frac{N^2 k^2}{\eta^2}\right)} \right\}. \quad (27)$$

In the corresponding Boussinesq problem [28], we are able to evaluate the inverse Laplace at this point by evaluating the residues at the singularities of the corresponding expression in the complex s plane. In the present case, with the anelastic approximation, because of the form of the coefficient $\lambda(s)$ in the exponent in the first term in the numerator, we are not able to obtain an exact expression, but we can derive an asymptotic approximation, valid for $t \gg 1$. To do so, we observe that, for $t \gg 1$, the dominant contribution to the

inverse Laplace transform integral comes from values of s for which $|s| \ll 1$. We can thus write $\lambda(s)$ in powers of s and approximate it by the leading-order term, for $|s| \ll 1$,

$$\lambda(s) = \frac{Nk}{s} \left(1 + \frac{1}{2} \left(\frac{\frac{1}{4H^2} + \delta k^2}{N^2 k^2} \right) s^2 - \frac{1}{8} \left(\frac{\frac{1}{4H^2} + \delta k^2}{N^2 k^2} \right)^2 s^4 + \dots \right) = \frac{Nk}{s} (1 + O(s^2)) \sim \frac{Nk}{s}, \quad (28)$$

The expression (27) has an essential singular point at $s = 0$, as well as three simple poles at $s_1 = ik\bar{u}$ and $s_{2,3} = \frac{\pm Nk}{\eta}$, and (27) can be written in terms of partial fractions as

$$\hat{\psi}(z, t) = -\frac{ikgF_0}{\eta^2} e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ C_1 \left(\frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}}}{s - ik\bar{u}} - \frac{e^{-bz}}{s - ik\bar{u}} \right) + C_2 \left(\frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}}}{s - \frac{Nk}{\eta}} - \frac{e^{-bz}}{s - \frac{Nk}{\eta}} \right) + C_3 \left(\frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}}}{s + \frac{Nk}{\eta}} - \frac{e^{-bz}}{s + \frac{Nk}{\eta}} \right) \right\}, \quad (29)$$

where

$$C_1 = -\frac{\eta^2}{k^2(N^2 + \bar{u}^2\eta^2)}, \quad C_{2,3} = \frac{\eta^2}{2Nk^2(N \mp i\bar{u}\eta)}.$$

To evaluate the inverse Laplace transforms of the terms $\frac{e^{-\lambda(s)z}}{s - s_k}$ in (29), we follow [26–28] and make use of the formula

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\frac{A}{s}}}{s + B} \right\} = e^{-Bt} e^{\frac{A}{B}} - \sum_{n=1}^{\infty} \left(\frac{\sqrt{A}}{B\sqrt{t}} \right)^n J_n(2\sqrt{At}), \quad (30)$$

where J_n is the Bessel function of the first type of order n . Using the leading-order asymptotic approximation $\lambda(s) \sim \frac{Nk}{s}$ and the Formula (30) gives

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}}}{s - ik\bar{u}} \right\} \sim e^{-\frac{z}{2H}} \left[e^{ik\bar{u}t} e^{-\lambda(ik\bar{u})z} - \sum_{n=1}^{\infty} \left(\frac{\sqrt{Nkz}}{-ik\bar{u}\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right], \quad (31)$$

where

$$\lambda(ik\bar{u}) = -\left(-\frac{N^2}{\bar{u}^2} + \frac{1}{4H^2} + \delta k^2 \right)^{1/2} = -i \left(\frac{N^2}{\bar{u}^2} - \frac{1}{4H^2} - \delta k^2 \right)^{1/2} = -im.$$

We note that the sign of the square root taken here ensures that the solution defines a wave with upward group velocity, with m chosen to have the same as the sign of $\bar{u}$. We also evaluate

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\lambda(s)z} e^{-\frac{z}{2H}}}{s \mp \frac{Nk}{\eta}} \right\} \sim e^{-\frac{z}{2H}} \left[e^{\pm \frac{Nk}{\eta} t} e^{-\lambda(\pm \frac{Nk}{\eta})z} - \sum_{n=1}^{\infty} \left(\frac{\sqrt{Nkz}}{\mp \frac{Nk\sqrt{t}}{\eta}} \right)^n J_n(2\sqrt{Nkzt}) \right] \quad (32)$$

where

$$\lambda\left(\pm \frac{Nk}{\eta}\right) = \pm \left(b - \frac{1}{2H} \right).$$

Thus, if $b(b - \frac{1}{H}) > \delta k^2$, the solution of Equation (24) is

$$\begin{aligned} \hat{\psi}(z, t) \sim & -\frac{ikgF_0}{\eta^2} \left\{ C_1 \left[e^{-\frac{z}{2H}} e^{imz} - e^{-bz} - e^{-ik\bar{u}t} e^{-\frac{z}{2H}} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nkz}}{k\bar{u}\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right] \right. \\ & + C_2 e^{-ik\bar{u}t} \left[e^{\frac{Nk}{\eta}t} e^{-bz} - e^{\frac{Nk}{\eta}t} e^{-bz} - e^{-\frac{z}{2H}} \sum_{n=1}^{\infty} \left(-\frac{\eta\sqrt{z}}{\sqrt{Nkt}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\ & \left. + C_3 e^{-ik\bar{u}t} \left[e^{-\frac{Nk}{\eta}t} e^{(b-\frac{1}{H})z} - e^{-\frac{Nk}{\eta}t} e^{-bz} - e^{-\frac{z}{2H}} \sum_{n=1}^{\infty} \left(\frac{\eta\sqrt{z}}{\sqrt{Nkt}} \right)^n J_n(2\sqrt{Nkzt}) \right] \right\}, \quad (33) \end{aligned}$$

valid for $t \gg 1$. The second line includes the difference of two identical exponentials, which is zero. The solution can be written as

$$\begin{aligned} \hat{\psi}(z, t) \sim & \alpha_1 \left(e^{-\frac{z}{2H}} e^{imz} - e^{-bz} \right) + \alpha_3 e^{-ik\bar{u}t} \left(e^{-\frac{Nk}{\eta}t} e^{(b-\frac{1}{H})z} - e^{-\frac{Nk}{\eta}t} e^{-bz} \right) \\ & + e^{-\frac{z}{2H}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \beta_n z^{\frac{n}{2}} t^{-\frac{n}{2}} J_n(2\sqrt{Nkzt}), \quad (34) \end{aligned}$$

where

$$\alpha_1 = \frac{igF_0}{k(N^2 + \bar{u}^2\eta^2)}, \quad \alpha_{2,3} = -\frac{igF_0}{2Nk(N \mp i\bar{u}\eta)},$$

and

$$\beta_n = -\alpha_1 \left(\frac{i\sqrt{Nk}}{k\bar{u}} \right)^n - \alpha_2 \left(-\frac{\eta}{\sqrt{Nk}} \right)^n - \alpha_3 \left(\frac{\eta}{\sqrt{Nk}} \right)^n, \quad n = 1, 2, \dots$$

This solution involves a sum of steady terms and time-dependent terms. To examine the behavior of the solution for large t , we note that the Bessel function has the asymptotic form, given in Formula 9.2.1 of [36],

$$J_n(\tau) \sim \sqrt{\frac{2}{\pi\tau}} \left\{ \cos \left(\tau - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) + \mathcal{O}(|\tau|^{-1}) \right\}, \quad |\tau| \gg 1. \quad (35)$$

and hence, (34) can be further approximated as

$$\begin{aligned} \hat{\psi}(z, t) \sim & \alpha_1 \left(e^{-\frac{z}{2H}} e^{imz} - e^{-bz} \right) + \alpha_3 e^{-ik\bar{u}t} e^{-\frac{Nk}{\eta}t} \left(e^{(b-\frac{1}{H})z} - e^{-bz} \right) \\ & + \frac{\beta_1}{(\pi^2 Nk)^{\frac{1}{4}}} e^{-\frac{z}{2H}} e^{-ik\bar{u}t} z^{\frac{1}{4}} t^{-\frac{3}{4}} \cos \left(2\sqrt{Nkzt} - \frac{3\pi}{4} \right), \quad (36) \end{aligned}$$

for $t \gg 1$. From this, we see that the time-dependent part of the solution comprises terms that decay exponentially in t and oscillate with a frequency of $k\bar{u}$, as well as terms that decay like $t^{-\frac{3}{4}}$ and oscillate as sinusoidal functions of $k\bar{u}t \pm 2\sqrt{Nkzt}$. All these terms go to zero as $t \rightarrow \infty$. Thus, as $t \rightarrow \infty$, the solution approaches

$$\phi_{\infty}(z) = \alpha_1 \left[e^{-\frac{z}{2H}} e^{imz} - e^{-bz} \right], \quad (37)$$

which is the amplitude of the steady solution (20).

An alternative asymptotic expression for the solution $\hat{\psi}(z, t)$ can be obtained by rewriting the solution in a different way, in which the coefficients of the Bessel functions are expressed in terms of $\frac{\sqrt{t}}{\sqrt{Nkz}}$ instead of $\frac{\sqrt{Nkz}}{\sqrt{t}}$. To do this, we follow Nijimbere and Camp-

bell [27]. We use the following generating function for Bessel functions that is given in Formula 9.1.41 of [36]:

$$e^{\frac{1}{2}Z(T-\frac{1}{T})} = \sum_{n=-\infty}^{\infty} T^n J_n(Z). \quad (T \neq 0)$$

For negative indices in the summation, we use the fact that $J_{-n}(Z) = (-1)^n J_n(Z)$ and obtain

$$\sum_{n=1}^{\infty} T^n J_n(Z) = e^{\frac{1}{2}Z(T-\frac{1}{T})} - \sum_{n=0}^{\infty} \left(\frac{-1}{T}\right)^n J_n(Z). \quad (38)$$

The solution (33) can then be rewritten by making use of the Formula (38), as

$$\begin{aligned} \hat{\psi}(z, t) = & \alpha_1 \left[-e^{-bz} + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{ik\bar{u}\sqrt{t}}{\sqrt{Nkz}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\ & + \alpha_2 \left[-e^{-bz} e^{-ik\bar{u}t} e^{\frac{Nk}{\eta}t} + e^{-\frac{z}{2H}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{\sqrt{Nkt}}{\eta\sqrt{z}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\ & + \alpha_3 \left[-e^{-bz} e^{-ik\bar{u}t} e^{-\frac{Nk}{\eta}t} + e^{-\frac{z}{2H}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(-\frac{\sqrt{Nkt}}{\eta\sqrt{z}} \right)^n J_n(2\sqrt{Nkzt}) \right]. \end{aligned} \quad (39)$$

This form of the solution is useful in obtaining insight into the behavior of the solution as $z \rightarrow \infty$ at finite t , which is not evident when the solution is expressed in the form (33). Based on the asymptotic behavior (35) of the Bessel function, we observe that the terms involving Bessel functions approach zero as $z \rightarrow \infty$ at finite time and thus $\hat{\psi} \rightarrow 0$ as $z \rightarrow \infty$ at finite time. This observation was also made in the problem studied by [26] and results from the fact that the waves propagate at finite speed, and hence they cannot reach infinite altitude within a finite interval of time.

Given the asymptotic approximation for $\hat{\psi}$, we can also obtain expressions for the velocity and potential temperature perturbations. For example, the vertical velocity perturbation takes the form

$$w(x, z, t) = \hat{w}(z, t) e^{ikx} + \text{c.c.} \quad (40)$$

with amplitude

$$\hat{w}(z, t) = ik\hat{\psi}/\bar{\rho}. \quad (41)$$

The potential temperature perturbation is given by (14) and takes the form

$$\theta(x, z, t) = \hat{\theta}(z, t) e^{ikx} + \text{c.c.}$$

where the amplitude $\hat{\theta}$ satisfies

$$\frac{\partial \hat{\theta}}{\partial t} + ik\bar{u}\hat{\theta} = \left(\frac{\bar{\theta}}{\bar{\rho}} F_0 e^{-bz} - \frac{ik}{\bar{\rho}} \left(\frac{d\bar{\theta}}{dz} \right) \hat{\psi} \right). \quad (42)$$

We solve this equation with an integrating factor of $e^{-ik\bar{u}t}$, using the asymptotic approximation (36) and the expression (5) for $\bar{\theta}(z)$ and $\bar{\rho}(z)$. We obtain

$$\begin{aligned} \hat{\theta}(z, t) \sim & \gamma_1 e^{(\kappa+1)\frac{z}{H}} e^{-bz} + \gamma_2 e^{(\kappa+1)\frac{z}{H}} \left(e^{-\frac{z}{2H}} e^{imz} - e^{-bz} \right) \\ & + \gamma_3 e^{-ik\bar{u}t} e^{(\kappa+1)\frac{z}{H}} e^{-\frac{Nk}{\eta}t} \left(e^{bz} e^{-\frac{z}{H}} - e^{-bz} \right) \\ & \gamma_4 e^{-ik\bar{u}t} e^{(\kappa+\frac{1}{2})\frac{z}{H}} \beta_1 z^{-\frac{1}{4}} t^{-\frac{1}{4}} \sin \left(2\sqrt{Nkzt} - \frac{3\pi}{4} \right). \end{aligned} \quad (43)$$

valid for $t \gg 1$, with constants

$$\gamma_1 = \frac{\theta_0 F_0}{ik\bar{u}\rho_0}, \quad \gamma_2 = \frac{\theta_0 \kappa \alpha_1}{\rho_0 H \bar{u}}, \quad \gamma_3 = \frac{\alpha_3 \kappa \theta_0 \eta}{i\rho_0 H N}, \quad \gamma_4 = -\frac{i\kappa \theta_0 \sqrt{k}}{\rho_0 H \sqrt{\pi N}}.$$

All the time-dependent terms go to zero as $t \rightarrow \infty$ and the solution approaches a steady-state $\hat{\theta}_\infty(z)$ given by the terms on the first line of (43).

The solutions derived above are valid for $b(b - \frac{1}{H}) > \delta k^2$ or

$$b > \frac{1}{2H} + \left(\frac{1}{4H^2} + \delta k^2 \right)^{1/2} > \frac{1}{H} > \frac{1}{2H},$$

i.e., where the rate of decay of the thermal forcing with altitude exceeds the rate of decay of the background density. This means that, in the steady-state components of (33) and (43), the terms proportional to the thermal forcing profile decrease more rapidly with altitude than the vertically oscillating terms.

Additional mathematical possibilities arise for the solution if $b(b - \frac{1}{H}) \leq \delta k^2$, i.e., if the depth of the layer of localized latent heating is approximately the same as or greater than the density scale height. If $b(b - \frac{1}{H}) = \delta k^2$, then the function $\tilde{\psi}(z, s)$ is given by the first line of (29) with only the contribution from the pole s_1 . Thus, the solution is

$$\hat{\psi}(z, t) \sim \frac{igF_0}{N^2 k} \left[e^{-\frac{z}{2H}} e^{imz} - e^{-bz} - e^{-ik\bar{u}t} e^{-\frac{z}{2H}} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nkz}}{k\bar{u}\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right], \quad (44)$$

valid for $t \gg 1$.

If $b(b - \frac{1}{H}) < \delta k^2$ then $s = \pm i\eta$, where $\eta = \left(\delta k^2 - b^2 - \frac{b}{H} \right)^{1/2}$, and the solution is

$$\begin{aligned} \hat{\psi}(z, t) \sim \alpha_1 \left(e^{-\frac{z}{2H}} e^{imz} - e^{-bz} \right) + \alpha_3 e^{-ik\bar{u}t} e^{i\frac{Nk}{\eta}t} \left(e^{bz} e^{-\frac{z}{2H}} - e^{-bz} \right) \\ + e^{-\frac{z}{2H}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \beta_n z^{\frac{n}{2}} t^{-\frac{n}{2}} J_n(2\sqrt{Nkzt}), \end{aligned} \quad (45)$$

valid for $t \gg 1$. If $0 < b(b - \frac{1}{H}) < \delta k^2$, then the streamfunction perturbation solution grows exponentially with z . The terms involving Bessel series go to zero as $t \rightarrow \infty$, but the solution does not approach a steady state; instead, it oscillates in t , with a phase speed of $\frac{N}{\eta} - \bar{u}$.

5. Wave-Induced Mean Flow

The linear solution, with streamfunction and potential temperature perturbations (33) and (43) gives us a first approximation to the gravity wave solution for a situation where the perturbation amplitude is very small relative to that of the background flow. However, the full problem is given by the nonlinear perturbation Equations (8) and (9), and interactions between the gravity wave modes in the nonlinear terms give rise to vertical fluxes of momentum and energy that lead to the development of higher horizontal wavenumber and zero wavenumber components at $O(\epsilon)$. Given the asymptotic expressions derived for the linear solution, we can obtain expressions for the momentum and heat flux divergence and the resulting changes in the background velocity and temperature with time.

The horizontal mean of each flow quantity is defined as the x -average over a horizontal wavelength $2\pi/k$ and denoted as

$$\overline{(\dots)} = \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} (\dots) dx. \quad (46)$$

The horizontal mean of the vertical flux of horizontal momentum or the Reynolds stress is

$$\mathcal{M}(z, t) = -\bar{\rho} \overline{uw} \quad (47)$$

and the horizontal mean of the vertical flux of energy is

$$\mathcal{H}(z, t) = -\bar{\rho} \overline{w\theta}. \quad (48)$$

In a weakly nonlinear formulation, the solutions of the nonlinear Equations (8) and (9) may be expressed as perturbation series in powers of ε ,

$$\psi(x, z, t) = \psi^{(0)}(x, z, t) + \varepsilon \psi^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2) \quad (49)$$

and

$$\theta(x, z, t) = \theta^{(0)}(x, z, t) + \varepsilon \theta^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2). \quad (50)$$

The leading-order functions

$$\psi^{(0)}(x, z, t) = \phi^{(0)}(z, t) e^{ikx} + \text{c.c.}$$

and

$$\theta^{(0)}(x, z, t) = G^{(0)}(z, t) e^{ikx} + \text{c.c.}$$

are the linear solutions with amplitude functions $\phi^{(0)}$ and $G^{(0)}$ given by (33) and (43). At the next order, each of the functions $\psi^{(1)}$ and $\theta^{(1)}$ comprises a zero-wavenumber component that is independent of x and a component proportional to $e^{\pm i2kx}$. The zero-wavenumber components give the dominant components of the wave-induced mean flow streamfunction, velocity and potential temperature.

The equations describing the changes over time of the wave-induced mean flow velocity $\bar{u}_0$ and potential temperature $\bar{\theta}_0$ are determined by taking the horizontal mean of the x -momentum equation and the energy equation in (2). This gives

$$\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon \frac{1}{\bar{\rho}} \overline{\left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right)} = -\varepsilon \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} \overline{uw}) = \varepsilon \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \left(\frac{\psi_x \psi_z}{\bar{\rho}} \right) = \varepsilon \frac{1}{\bar{\rho}} \frac{\partial \mathcal{M}}{\partial z}, \quad (51)$$

and

$$\frac{\partial \bar{\theta}_0}{\partial t} = -\varepsilon \overline{\left(u \frac{\partial \theta}{\partial x} + w \frac{\partial \theta}{\partial z} \right)} = -\varepsilon \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} \overline{w\theta}) = \varepsilon \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\psi_x \theta) = \varepsilon \frac{1}{\bar{\rho}} \frac{\partial \mathcal{H}}{\partial z}, \quad (52)$$

where, as defined before, the subscripts of x and z denote partial derivatives with respect to the respective variables. Using (33) and (43), we can obtain asymptotic expressions for the mean momentum flux and mean energy flux,

$$\mathcal{M} = \frac{ik}{\bar{\rho}} \phi^{(0)} \phi_z^{(0)*} + \text{c.c.} \sim 2k|\alpha_1|^2 \left[m - \left(b - \frac{1}{2H} \right) \sin mz \right] e^{-(b - \frac{1}{2H})z} + \mathcal{O}(t^{-1}) \quad (53)$$

and

$$\mathcal{H} = ik\phi^{(0)} G^{(0)*} + \text{c.c.} \sim -2k\alpha_1 e^{(\kappa+1)\frac{z}{H}} \left[\gamma_1 e^{-bz} + \gamma_2 e^{-(b - \frac{1}{H})z} \right] \sin mz + \mathcal{O}(t^{-1}), \quad (54)$$

where $\phi_z^{(0)*}$ and $G^{(0)*}$ denote the complex conjugates of $\phi_z^{(0)}$ and $G^{(0)}$, respectively.

These vertical profiles of fluxes can be contrasted with the corresponding profiles in the situation where gravity waves are forced by a horizontally periodic boundary condition, analogous to a topographic forcing, at the lower boundary of a rectangular domain in an environment with constant background velocity. In that case, for steady-amplitude waves

of the form $e^{-z/2H}e^{i(kx+imz)}$, the vertical fluxes of momentum and energy are independent of altitude z and $\mathcal{M}$ is given by just the first constant term on the right-hand side of (53). This is the Eliassen–Palm Theorem [25] for anelastic gravity waves.

However, in the current formulation, where the waves are generated by the thermal forcing term in the nonhomogeneous Equation (20), the solution also includes the term (19). This gives rise to the z -dependent terms in the asymptotic expressions (53) and (54), which oscillate with z and also vary exponentially. In addition, the time-dependent terms in (33) and (43) give the $O(t^{-1})$ terms in (53) and (54). These components of the solution have non-zero z -derivatives, and thus there is a nonzero vertical transport of momentum and energy within and above the thermal forcing region, even though there is no vertical shear or critical level.

Figure 2a,b show graphs of the momentum flux expression (53) as a function of z for the two configurations that were represented in Figure 1 with $b = 2.5$ and $b = 0.2$. In each case, the value of $\mathcal{M}$ has been scaled by dividing the expression (53) by the value of the constant first term in the expression. In Figure 2a, the depth of the heating layer is less than the density scale height and the waves generate a nonzero momentum flux divergence in the region above the peak of the heating profile; this decays to zero with altitude. In Figure 2b, there is a layer of deep heating, with depth greater than the density scale height, and there is a nonzero momentum flux divergence over the entire domain. In both cases, the momentum flux decreases with altitude from zero to negative values in the lower part of the domain; this means there is a negative drag force, which would result in a negative wave-induced mean flow component at that level. These conclusions support the results of nonlinear time-dependent numerical solutions [29] that show variations in the mean momentum flux and heat flux with altitude above the thermal forcing region that, over time, lead to a wave-induced mean flow.

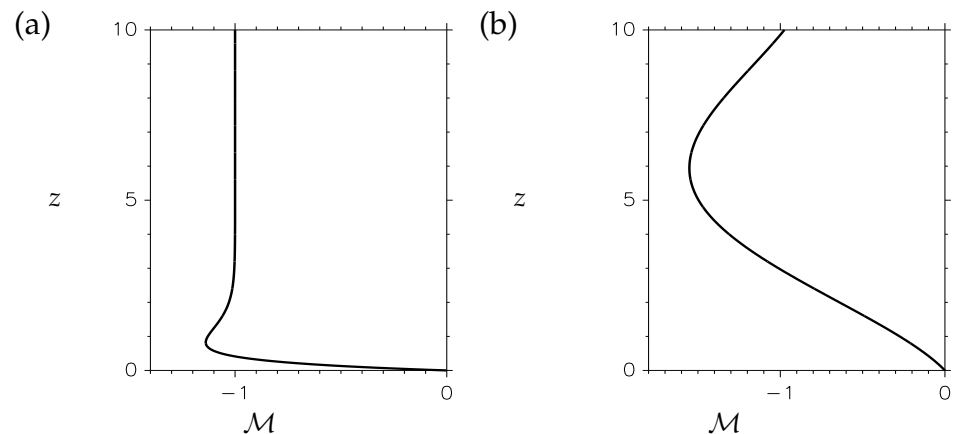


Figure 2. Vertical component of the horizontal-mean horizontal momentum flux $\mathcal{M}(z, t)$ according to Equation (53) with $k = 2$, $H = 5$ and (a) $b = 2.5 > 1/H$, (b) $b = 0.2 < 1/H$.

6. Discussion

We examined a linear time-dependent model for gravity waves excited by a thermal forcing in a layer of stably-stratified fluid with constant buoyancy frequency and constant background velocity under the anelastic approximation. We derived asymptotic analytical solutions, valid for late time, each comprising steady-amplitude terms and time-dependent terms involving Bessel functions. As $t \rightarrow \infty$, the time-dependent terms in (33) go to zero and the solution approaches the steady-amplitude solution (20).

We also obtained expressions for the leading-order components of the vertical momentum flux and heat flux that would result from weakly nonlinear wave interactions. These effects are nonzero, even in the absence of vertical shear in the background flow but are dependent on the background flow velocity and the heating profile. This is consistent

with nonlinear numerical solutions [29] and also consistent with some observations of situations where there is weak shear. For example, Oertel et al. [37] discuss upper tropospheric dynamical effects in a case based on airborne radar observations where there is weak vertical wind shear in the presence of localized diabatic heating and convective clouds. Measurements of mean momentum fluxes (e.g., [38]) show significant variability with spatial location and a strong dependence of the momentum flux on the background wind. The analysis presented here suggests such dependence.

These approximate asymptotic solutions can be considered as a starting point for nonlinear asymptotic analyses and numerical solutions, including the case with vertical shear [29,30], which allows us to model other wave interaction mechanisms and phenomena, such as critical layer absorption and reflection and gravity wave saturation, and including configurations with a horizontally localized thermal forcing function that can model a heat island [39]. This investigation can be further extended to a two-layer formulation, such as that of [19], where there is a layer with stable stratification and upward-propagating gravity waves above a lower layer with unstable stratification and convective cells. Other important effects, such as viscosity and heat conduction, can be included. The two-layer representation can be used to explore the mechanisms for convective gravity wave generation.

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