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# Advanced ILC Analysis of Switched Systems Subject to Non-Instantaneous Impulses Using Composite Fractional Derivatives

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## Abstract

This study deals with *P*-type iterative learning control (ILC) techniques for switched impulsive systems governed by composite fractional derivatives. The systems considered incorporate non-instantaneous impulses and an initial state offset, with the objective of accurately tracking time-varying reference trajectories over a finite time interval using a finite number of iterations. By implementing a *P*-type learning law integrated with an initial iteration mechanism, we derive sufficient conditions that guarantee the convergence of the tracking error. The effectiveness and robustness of the proposed control concepts are validated through a comprehensive illustrative example.

**Keywords:** impulsive systems; iterative learning control; convergence; composite fractional derivative; reference trajectory

**MSC:** 34A08; 93B05; 93C79; 47H10; 93C85

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## 1. Introduction

Differentiation and integration are extended to non-integer (fractional) orders in fractional calculus (FC), a branch of classical calculus. When compared to traditional calculus, which emphasises operations of integer order, FC offers a more flexible mathematical structure for describing difficult systems and processes (Ref. [1]). Memory effects, biological characteristics, and unusual behaviours in all kinds of scientific and engineering domains have all been successfully captured by this generalization. In the late 17th century, mathematicians like Leibniz and L'Hôpital conjectured about the meaning of derivatives of non-integer order, which is when FC first emerged. But until the 20th century, its real-world uses were mostly unknown. FC can be used in many different areas of the mathematical, engineering, and physical sciences; it has become an essential and popular mathematical tool. In modelling a wide range of phenomena, such as diffusion processes (Ref. [2]), impulsive dynamical systems (Ref. [3]), time-delay systems (Ref. [4]), and differential equation (DE) solutions (Refs. [5,6]), it is essential. Quantum mechanics (Ref. [7]), optimal control strategies (Ref. [8]), controllability analysis (Ref. [9]), thermoelasticity models (Ref. [10]),

and the study of chaotic synchronization systems (Ref. [11]) have all benefited from its successful application. For many real-world processes, FC provides a more realistic and accurate representation than traditional integer-order calculus.

In addition to the Riemann–Liouville (R-L) and Liouville–Caputo (L-C) fractional derivatives, Hilfer (Ref. [12]) developed a brand new derivative type that was a generalization of the R-L derivative or simply the composite (Hilfer) fractional derivative (CFD). The L-C and R-L are the most commonly used fractional formulations (Refs. [13,14]). Subashini et al. (Ref. [15]) focused on the existence results of fractionally ordered Hilfer integro-DEs. Kavitha et al. (Ref. [16]) recently provided a useful idea in a note regarding the approximate controllability of Hilfer fractional neutral differential inclusions with infinite delay, and Kavitha et al. (Ref. [17]) provide details on Sobolev-type fractional neutral differential inclusions of Clarke's subdifferential number. H. Gu and J. J. Trujillo (Ref. [18]) found that the evolution equation with CFD has an integral solution.

The following table (Table 1) summarises the differences among the R-L, L-C, and CFD:

**Table 1.** Comparison: R-L, L-C, and CFD.

| **Feature's** | **R-L Derivative (Ref. [13], Chapter 2, Page 69)** | **L-C Derivative (Ref. [13], Page 97)** | **CFD (Ref. [19], Page 852)** |
|---|---|---|---|
| Definition | $D^{\rho}_{RL}x(\tau) = \frac{d^n}{d\tau^n} I^{n-\rho}x(\tau)$ $\rho \in (0,1)$ | $D^{\rho}_{c}x(\tau) = I^{n-\rho}\frac{d^n x(\tau)}{d\tau^n}$ $\rho \in (0,1)$ | $\mathbb{H}^{\rho,\vartheta}x(\tau)$ $= I^{\vartheta(n-\rho)}\frac{d^n}{d\tau^n} I^{(1-\vartheta)(n-\rho)}x(\tau)$ $\rho \in (0,1)$ |
| Initial Conditions | Fractional integral-based | Integer-order derivative-based | Interpolates between R-L and Caputo |
| Used Case | Theoretical modelling | Physical simulations | Generalized modelling with memory interpolation |
| Memory Effect | Strong | Moderate | Tunable via $\vartheta \in [0,1]$ |

Conversely, there are numerous real-world occurrences and processes, like harvesting, drugs in the bloodstream, ecological change, the vibration of a car while driving, and bursting rhythmic models in pathology, where the model's state has a few sudden changes. The term "impulsive effects" refers to these modifications. Impulsive effects are classified in the literature into two categories: non-instantaneous impulses (NI) (the impulsive action begins at fixed points and remains active on a finite time interval) and instantaneous impulses (the duration of these changes is negligible in comparison with the total duration of the whole process). Pierri et al. (Ref. [20]) used semigroup analytic theory to study the existence of solutions to abstract semilinear DEs with NI. Hernandez and O'Regan (Ref. [21]) recently investigated the existence of solutions to a new class of impulsive abstract DEs involving NI. Liu et al. (Ref. [22]) recently investigated periodic solutions for time-varying impulsive differential systems. Chen et al. (Ref. [23]) investigated the existence of piecewise-continuous mild solutions to the initial value problems for a class of semilinear evolution equations with NI in Banach spaces. The existence of a PC-mild solution is determined using the properties of the Kuratowski noncompactness measure and the $k$-set contraction mapping. Additionally, the stability and uniform stability of the solutions of a system of nonlinear DEs with NI are studied in (Ref. [24]), and existence results on fractional functional DEs with NI are examined in (Refs. [25,26]). Existence for abstract DEs with NI is also observed in (Ref. [27]).

A switched system is a dynamic system with a finite number of subsystems and a switching law that specifies how they interact with one another. Switched systems

have received a lot of attention in control theory, and their applications include mobile robotics (Ref. [28]), vector fields (Ref. [29]), circuit systems (Ref. [30]), and signal processing (Ref. [31]), among others. From a control standpoint, the concept of multi-controller switching provides an effective mechanism for dealing with complex systems or those that contain significant uncertainties. While switched systems are important models for understanding a wide range of complex physical processes, they fail to include conditions that exhibit specific types of impulse-based dynamics (Ref. [32]).

Wang et al. (Ref. [33]) studied Hilfer-type fractional differential inclusions with non-local and delayed impulsive effects. They developed a mathematical concept for hybrid systems with switching and NI. Kumar et al. (Ref. [34]) studied switched Hilfer-type systems with NI and derived conditions for the existence and stability of solutions. This contributed to the theoretical development of impulsive fractional-order dynamics under switching environments. Dhayal et al. (Ref. [35]) investigated a type of Hilfer fractional stochastic switched dynamical system influenced by the Rosenblatt process and governed by NI, focusing on the development of fractional optimal control strategies, in which the existence of a single mild solution was established using fixed-point theory and the Mittag–Leffler function.

Furthermore, ILC is a control strategy that improves a system's transient performance by correcting the control input using historical (open-loop) or current (closed-loop) information and converges through iteration. Because ILC is simple, intelligent, and can operate independently of the system model, it has received a lot of attention as a tool for solving complex trajectory tracking problems, such as batch process control (Ref. [36]), manipulator tracking (Ref. [37]), and biomedical engineering (Ref. [38]). FC is a long-established discipline that studies non-integer-order derivatives and integrals of functions. This field has received a lot of attention in recent years (Ref. [39–42]) because scientific evidence shows that fractional-order characteristics exist in many disciplines, including viscoelasticity (Ref. [43]), circuit theory (Ref. [44]), diffusion (Ref. [45]), and so on.

To support the choice of the $P$-type ILC law, the following table (Table 2) compares commonly used ILC schemes.

**Table 2.** Comparison of ILC laws.

| ILC Type | Control Laws | Advantages |
|---|---|---|
| $P$-type | $u_{k+1}(\tau) = u_k(\tau) + Le_k(\tau)$ | Chosen for simplicity, memory compatibility, and robustness under impulses/switching (Ref. [46]). |
| $D$-type | $u_{k+1}(\tau) = u_k(\tau) + L\Delta e_k(\tau)$ | Sensitive to noise; not ideal for impulsive dynamics (Ref. [47]). |
| $I$-type | $u_{k+1}(\tau) = u_k(\tau) + L\int e_k(\tau)\, d\tau$ | May cause instability under switching or NI (Ref. [48]) |

Several researchers have investigated the integration of ILC with NI and fractional-order dynamics, but their investigations have been limited in scope. For instance, Cao et al. (Ref. [49]) proposed an ILC-based consensus strategy for multi-agent systems under finite-duration impulses and time-varying communication networks. While effective, their work focused on agent-based coordination and did not involve fractional-order dynamics or switched systems. Similarly, Liu (Ref. [50]) developed an ILC scheme that accounts for varying trial lengths and NI. Their method ensures convergence of the tracking error under irregular timing, yet the system considered remains of integer order, lacking memory effects inherent to fractional derivatives. On the other hand, Debbouche et al. (Ref. [51]) addressed the controllability of CFD fractional systems with NI using semigroup theory and fixed-point techniques. Although their study introduced CFD memory and delayed impulse effects, it did not consider iterative learning or switching dynamics.

Vivek et al. (Ref. [52]) investigated a class of quaternion-valued impulsive systems governed by Hilfer-type fractional dynamics, where they developed a quaternion-based ILC strategy and established convergence results using semigroup theory and mild solution concepts. Vivek et al. (Ref. [46]) analysed convergence in ILC for nonlinear pantograph-type equations with Hilfer impulsive dynamics, providing sufficient conditions under which the iterative error decays asymptotically. While these contributions offer valuable insights into fractional-order ILC design, they focus primarily on systems and impulses. Vivek et al. (Ref. [53]) proposed a P-type ILC law for impulsive pantograph equations involving the CFD, demonstrating stable convergence of the tracking error and impulsive effects. None of these works consider switched dynamics or the extended structure of NI. Motivated by this research gap, this study is the first to develop an ILC scheme for switched impulsive systems that simultaneously incorporates the CFD and NI. Both open-loop and closed-loop *P*-type learning laws are proposed, tailored for fractional-order switched impulsive systems. Fractional memory effects and switching dynamics are included to ensure convergence and robustness. An initial iteration correction is introduced, allowing adjustments to the initial state without resetting it to a fixed desired value. Sufficient convergence conditions are derived using CFD tools. These conditions guarantee zero tracking error despite the combined challenges of fractional dynamics, switching, and NI. A detailed numerical example verifies fast convergence, robustness to switching law changes, and high tracking accuracy with varying impulsive intervals.

## 2. System Description

Consider ILC for a switched fractional system with non-instantaneous impulses (SFNI) by the CFD, described as follows:

$$\begin{cases} \mathbb{H}^{\rho,\vartheta}_{\tau_i,\tau} x_k(\tau) = F_{\sigma(\tau)}(\tau, x_k(\tau), u_k(\tau)), \tau \in (\tau_i, s_i), i = 0, 1, \ldots, m, \\ x_k(\tau_i^+) = G_{\sigma(\tau)}(\tau_i^-, u_k(\tau_i^-)), i = 1, 2, \ldots, m, \\ x_k(\tau) = G_{\sigma(\tau)}(\tau, u_k(\tau_i^-)), \tau \in (s_{i-1}, \tau_i], i = 1, 2, \ldots, m, \\ \mathbb{I}^{1-\varpi}_{0^+} x_k(0) = x_0, \quad \mathbb{I}^{1-\varpi}_{\tau_i^+} x_k(\tau_i^+) = G_{\sigma(\tau)}(\tau_i, u_k(\tau_i^-)), i = 1, \ldots, m, \\ y_k(\tau) = H_{\sigma(\tau)}(\tau, x_k(\tau), u_k(\tau)), \text{almost every } \tau \in [0, b], \end{cases} \tag{1}$$

where $0 < \rho < 1, 0 \leq \vartheta \leq 1$, $\varpi = \rho + \vartheta - \rho\vartheta$, and $I = [0, b]$. $b$ is a pre-fixed number. $\mathbb{H}^{\rho,\vartheta}_{\tau_i,\tau}$ is a CFD of order $\rho$ and type $\vartheta$, where the lower limit $\tau_i, 0 = \tau_0 < s_0 < \tau_1 < s_1 < \ldots < \tau_m < s_m = b$. $\mathbb{I}^{1-\varpi}_{\tau_i}$ is the R-L integral of order $1 - \varpi$ with the lower limit at $\tau_i$. $\mathbb{I}^{1-\varpi}_{\tau_i} x(\tau_i^+) = \lim_{\tau \to \tau_i} \mathbb{I}^{1-\varpi}_{\tau_i} x(\tau)$. In addition we set $x(\tau_i^-) = x(\tau_i)$.

The switched signal $\sigma : I \to \{0, 1, \ldots, m\}$ is assumed to be known. It only changes its values at switching time $\tau_i$. That is to say (Ref. [34], Page 3),

$$\sigma(\tau) = i, \quad \tau_i \leq \tau < s_i, i = 0, 1, \ldots, m.$$

Therefore, by applying the above switching law in switched system (1), we get the following system:

$$\begin{cases} \mathbb{H}^{\rho,\vartheta}_{\tau_i,\tau} x_k(\tau) = F_i(\tau, x_k(\tau), u_k(\tau)), \tau \in (\tau_i, s_i), i = 0, 1, \ldots, m, \\ x_k(\tau_i^+) = G_i(\tau_i^-, u_k(\tau_i^-)), i = 1, 2, \ldots, m, \\ x_k(\tau) = G_i(\tau, u_k(\tau_i^-)), \tau \in (s_{i-1}, \tau_i], i = 1, 2, \ldots, m, \\ \mathbb{I}^{1-\varpi}_{0^+} x_k(0) = x_0, \quad \mathbb{I}^{1-\varpi}_{\tau_i^+} x_k(\tau_i^+) = G_i(\tau_i, u_k(\tau_i^-)), i = 1, 2, \ldots, m, \\ y_k(\tau) = H_i(\tau, x_k(\tau), u_k(\tau)), \text{almost every} \quad \tau \in I, \end{cases} \tag{2}$$

Moreover, $F_i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $G_i : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $H_i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. In general, $x_k(\cdot) \in \mathbb{R}^n$ is the state of the plant. $u_k(\cdot) \in \mathbb{R}^n$ and $y_k(\cdot) \in \mathbb{R}^n$ denote the control input and output, respectively.

Meanwhile, we impose the following assumptions:

**Assumption 1.** *$F_i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is strongly measurable for $\tau$ and is continuous for the first and second variables, and there exists a $\mathcal{L}_F > 0$ such that*

$$||F_i(\tau, x_k(\tau), u_k(\tau)) - F_i(\tau, \bar{x}_k(\tau), \bar{u}_k(\tau))|| \leq \mathcal{L}_F||x_k(\tau) - \bar{x}_k(\tau)|| + \mathcal{L}_F||u_k(\tau) - \bar{u}_k(\tau)||, \quad x_k, \bar{x}_k, u_k, \bar{u}_k \in \mathbb{R}, \quad \tau \in I. \tag{3}$$

**Assumption 2.** *$G_i : [s_{i-1}, \tau_i] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a $\mathcal{L}_i > 0$ such that*

$$||G_i(\tau, u_k(\tau)) - G_i(\tau, \bar{u}_k(\tau))|| \leq \mathcal{L}_i||u_k(\tau) - \bar{u}_k(\tau)||, \text{for } \tau \in [s_{i-1}, \tau_i], i = 1, 2, \ldots, m. \tag{4}$$

**Assumption 3.** *For $\beta_j > 0$, $j = 1, 2, 3, 4$,*

$$\begin{cases} \beta_1 \leq G_u := \frac{tialG(\tau, x_k(\tau), u_k(\tau))}{tialu_k(\tau)} \leq \beta_2, \\ \beta_3 \leq G_x := \frac{tialG(\tau, x_k(\tau), u_k(\tau))}{tialx_k(\tau)} \leq \beta_4, \end{cases}$$

*for every $\tau \in I$.*

Let $C(I, \mathbb{R}^n)$ be a Banach space of all continuous functions from $I$ into $\mathbb{R}^n$ with the norm $||x|| = \sup_{\tau \in I} ||x(\tau)||$ and the $\Lambda$-norm $||x||_\Lambda = e^{-\Lambda\tau}||x||$.

We define the Banach space of all piecewise continuous functions $PC_{1-\omega}(I, \mathbb{R}^n) = \{x : (\tau - \tau_i)^{1-\omega}x(\tau) \in C((\tau_i, \tau_{i+1}], \mathbb{R}^n), i = 0, 1, 2, \ldots, m$, and there exists $x(\tau_i^-)$ and $x(\tau_i^+), i = 1, 2, \ldots, m$ with $x(\tau_i^-) = x(\tau_i)\}$ endowed with the $PC_{1-\omega,\Lambda}$ (Ref. [46], Page 4),

$$||x||_{PC_{1-\omega,\Lambda}} := \sup_{\tau \in I} (\tau)^{1-\omega} e^{-\Lambda\tau}||x(\tau)|| \quad (\Lambda > 0).$$

For a detailed investigated of weighted space, please refer to (Refs. [19,34]).

According to (Ref. [13], Chapter 2, Equations 2.1.1 and 2.1.2, Page 69), the definitions of the R-L fractional integral are taken into consideration, as are the L-C fractional derivative (Ref. [13], Section 2.4, Page 97) and the CFD (Ref. [19], Page 852). They are not described in depth because there is so much literature on these classical formulations. All of these references are available to readers who are curious about the theoretical aspects.

In addition, detailed analysis of the Mittag–Leffler (M-L) (Ref. [54], Equations 1.1 and 1.2, Page 1) and Gronwall inequality (Ref. [55], Theorem A, Page 1075), Remark (Ref. [55], Corollary 3, Page 1078) is performed.

The following lemma provides the fractional integral form of the solution to system (2), which is essential for reformulating the system dynamics in the analysis of the *P*-type ILC law. This form enables direct handling of the error equation and simplifies the derivation of convergence conditions.

**Lemma 1.** *If $F : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an integrable function, then $F_i \in PC_{1-\omega}^{\vartheta(1-\rho)}(I, \mathbb{R}^n)$ is a solution of system (2) if and only if $x$ is a solution of the fractional integral equation for $i = 0, 1, \ldots, m$,*

$$x(\tau) = \begin{cases} \frac{x_0}{\Gamma(\omega)}\tau^{\omega-1} + \frac{1}{\Gamma(\rho)}\int_0^\tau (\tau-s)^{\rho-1}F_i(s,x_k(s),u_k(s))ds, \tau \in (0,s_0], i = 1,2,\ldots,m \\ G_i(\tau,u_k(\tau_i^-)), \quad \tau \in (s_{i-1},\tau_i], k = 1,2,\cdots,m \\ \frac{(\tau-s_i)^{\omega-1}}{\Gamma(\omega)}G_i(\tau_i,u_k(\tau_i^-)) + \frac{1}{\Gamma(\rho)}\int_0^\tau (\tau-s)^{\rho-1}F_i(s,x_k(s),u_k(s))ds, \tau \in (\tau_i,s_i], i = 1,2,\ldots,m. \end{cases} \tag{5}$$

**Proof.** For a detailed proof, the reader is referred to (Ref. [33], Lemma. 3, Page 925), as the proof is similar and therefore omitted here. □

## 3. *P*-Type ILC

For brevity, we define the following notations:

$$\begin{aligned} \Delta u_k(\tau) :=& u_{k+1}(\tau) - u_k(\tau), \\ \Delta x_k(\tau) :=& x_{k+1}(\tau) - x_k(\tau), \\ e_k(\tau) :=& y_d(\tau) - y_k(\tau), \end{aligned}$$

where $e_k(\tau)$ denotes the tracking error.

### *3.1. Open-Loop Case*

For the system described in (2), we first consider the following open-loop *P*-type ILC updating law with initial state learning:

$$x_{k+1}(0) = x_k(0) + Le_k(0), \quad u_{k+1}(\tau) = u_k(\tau) + \theta_1 e_k(\tau), \tag{6}$$

where $L$ and $\theta_1$ are unknown parameters to be determined.

**Theorem 1.** *Suppose that Assumptions 1–3 hold, and let $y_k(\cdot)$ denote the output of system (2). If the following conditions are satisfied:*

$$\begin{aligned} \max\{|1-\theta_1\beta_1 - L\beta_3|,\ |1-\theta_1\beta_1 - L\beta_4|,\ |1-\theta_1\beta_2 - L\beta_3|,\ |1-\theta_1\beta_2 - L\beta_4|\} \le \rho_0, \\ \max\{|1-\theta_1\beta_1|,\ |1-\theta_1\beta_2|\} \le \rho_1, \\ \max_{1\le i\le m}\{|1-\theta_1\beta_1| + \beta_4\mathcal{L}_i\theta_1,\ |1-\theta_1\beta_2| + \beta_4\mathcal{L}_i\theta_1\} \le \rho_2, \end{aligned}$$

*and*

$$\max\{\rho_0,\rho_1,\rho_2\} < 1, \tag{7}$$

*then for any arbitrary initial input $u_0(\tau)$, the open-loop P-type ILC updating law guarantees that the system output $y_k(\tau)$ converges to the desired trajectory $y_d(\tau)$, i.e.,*

$$\lim_{k\to\infty} y_k(\tau) = y_d(\tau), \quad \textit{for almost every } \tau \in I.$$

**Proof.** Let the tracking error at iteration $k+1$ be expressed as follows:

$$e_{k+1}(\tau) = e_k(\tau) - G_x(\xi)\Delta x_k(\tau) - G_u(\xi)\Delta u_k(\tau), \tag{8}$$

where $\xi(\tau) \in [\tau, x_k(\tau) + \psi x_k(\tau), u_k(\tau) + \psi u_k(\tau)]$ for $\psi \in [0,1]$ and almost every $\tau \in I$. By (Ref. [52], Equation 3.3, Page 7), we have

$$||e_{k+1}||_{PC_{1-\omega,\Lambda}} = ||1-\theta_1 G_u(\xi)||||e_k||_{PC_{1-\omega,\Lambda}} + \beta_4||\Delta x_k||_{PC_{1-\omega,\Lambda}} \tag{9}$$

In what follows, we prove $||e_{k+1}(\tau)|| \to 0$ as $k \to \infty$ for almost every $\tau \in I$.

**Case (i).**

For the case when $\tau \in (0, s_0]$, evaluate $||\Delta x_k||_{PC_{1-\omega,\Lambda}}$ as follows,

$$\begin{aligned}||\Delta x_k(\tau)|| &= ||x_{k+1}(\tau) - x_k(\tau)|| \\ &\leq \frac{||\Delta x_0||}{\Gamma(\omega)}\tau^{\omega-1} + \frac{1}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}[F_i(\tau, x_{k+1}(\tau), u_{k+1}(\tau)) - F_i(\tau, x_k(\tau), u_k(\tau))]ds.\end{aligned}$$

Multiplying both sides of the above inequality by $\tau^{1-\omega}$ results in the following inequality:

$$||\Delta x_k(\tau)||\tau^{1-\omega} \leq \frac{||\Delta x_k(0)||\tau^{1-\omega}\tau^{\omega-1}}{\Gamma(\omega)} + \frac{\tau^{1-\omega}}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}[\mathcal{L}_F||\Delta x_k(\tau)|| + \mathcal{L}_F||\Delta u_k(\tau)||]ds, \tag{10}$$

which implies

$$\begin{aligned}||\Delta x_k(\tau)||\tau^{1-\omega} &\leq \frac{||\Delta x_k(0)||}{\Gamma(\omega)} + \frac{\tau^{1-\omega}}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}\frac{\mathcal{L}_F}{s^{1-\omega}}||\Delta x_k(\tau)||ds \\ &+ \frac{\tau^{1-\omega}}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}\frac{\mathcal{L}_F}{s^{1-\omega}}||\Delta u_k(\tau)||ds \\ &\leq \frac{||L||\,||e_k(0)||}{\Gamma(\omega)} + \frac{\mathcal{L}_F}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}\mathcal{L}_F||\Delta x_k(\tau)||ds \\ &+ \frac{1}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}\mathcal{L}_F\theta_1||e_k(\tau)||ds,\end{aligned} \tag{11}$$

With the definition of $PC_{1-\omega,\Lambda}$ and the fact that $\int_0^\tau (\tau - s)^{\rho-1}e^{\Lambda s}ds \leq \frac{e^{\Lambda s}\Gamma(\rho)}{\Lambda^\rho}$, the integral can be evaluated as below (Ref. [56], Page 3213):

$$\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}||e_k(s)||ds \leq ||e_k||_{PC_{1-\omega,\Lambda}}\frac{e^{\Lambda\tau}\Gamma(\rho)}{\Lambda^\rho}. \tag{12}$$

Substituting Equation (12) into inequality (11), we get

$$\begin{aligned}||\Delta x_k(\tau)||\tau^{1-\omega} &\leq \frac{||L||\,||e_k(0)||}{\Gamma(\omega)} + \frac{\mathcal{L}_F}{\Gamma(\rho)}\int_0^\tau (\tau - s)^{\rho-1}s^{1-\omega}||\Delta x_k(\tau)||ds \\ &+ \frac{\mathcal{L}_F\theta_1}{\Lambda^\rho}e^{\Lambda\tau}||e_k(\tau)||_{PC_{1-\omega,\lambda}}.\end{aligned}$$

Multiplying both sides by $e^{-\Lambda\tau}$ and (Ref. [57], Remark. 1, Page 126), the inequality simplifies to

$$||\Delta x_k(\tau)||_{PC_{1-\omega,\lambda}} \leq \Big[\frac{||L||\,||e_k(0)||_\Lambda}{\Gamma(\omega)} + \frac{\mathcal{L}_F\theta_1}{\Lambda^\rho}||e_k||_{PC_{1-\omega,\Lambda}}\Big]E_\rho(\mathcal{L}_F\tau^{1+\vartheta(\rho-1)}). \tag{13}$$

Substituting the above inequality Equation (13) into Equation (9), we obtain

$$\begin{aligned}||e_{k+1}||_{PC_{1-\omega,\Lambda}} &\leq ||1 - \theta_1 G_u(\xi)||||e_k||_{PC_{1-\omega,\Lambda}} \\ &+ \beta_4\Big(\Big[\frac{||L||\,||e_k(0)||_\Lambda}{\Gamma(\omega)} + \frac{\mathcal{L}_F\theta_1}{\Lambda^\rho}||e_k||_{PC_{1-\omega,\Lambda}}\Big]E_\rho(\mathcal{L}_F\tau^{1+\vartheta(\rho-1)})\Big) \\ &\leq ||1 - \theta_1 G_u(\xi)||\,||e_k||_{PC_{1-\gamma,\Lambda}} \\ &+ \beta_4\frac{||L||\,||e_k(0)||_\Lambda E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)})}{\Gamma(\omega)} + \frac{\beta_4\mathcal{L}_F\theta_1}{\Lambda^\rho}||e_k||_{PC_{1-\omega,\Lambda}}E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}),\end{aligned}$$

which implies that

$$||e_{k+1}||_{PC_{1-\varpi,\Lambda}} \leq \Big[||1-\theta_1 G_u(\xi)|| + \frac{\beta_4 \mathcal{L}_F \theta_1}{\Lambda^\rho} E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)})\Big] ||e_k||_{PC_{1-\varpi,\Lambda}} + \frac{||L||\,||e_k(0)||_\Lambda \beta_4}{\Gamma(\varpi)} E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}). \tag{14}$$

By inequality (7), there exists a sufficiently large $\Lambda$ such that

$$|1-\theta_1 G_u(\xi)| + \frac{\beta_4 \mathcal{L}_F \theta_1}{\Lambda^\rho} E_\rho(\mathcal{L}_F T^{1+\vartheta(\rho-1)}) \leq \rho_1,$$
$$|1-\theta_1 G_u(\xi)| + \frac{\beta_4 \mathcal{L}_F \theta_1}{\Lambda^\rho} E_\rho(\mathcal{L}_F T^{1+\vartheta(\rho-1)}) < 1.$$

Therefore, from inequality (14), we have $\lim_{k\to\infty} ||e_k||_{PC_{1-\varpi,\Lambda}} = 0$.

**Case (ii).**

We prove that $||e_{k+1}|| \to 0$ as $k \to \infty$ for all $\tau \in [s_{i-1}, \tau_i], i = 1, 2, \ldots, m$.

$$\begin{aligned} ||\Delta x_k||_{PC_{1-\varpi,\Lambda}} &= ||x_{k+1}(\tau) - x_k(\tau)||_{PC_{1-\varpi,\Lambda}} \\ &\leq ||G_i(\tau, u_{k+1}(\tau)) - G_i(\tau, u_k(\tau))||_{PC_{1-\varpi,\Lambda}} \\ &\leq \mathcal{L}_i ||u_{k+1}(\tau) - u_k(\tau)||_{PC_{1-\varpi,\Lambda}} \\ &\leq \mathcal{L}_i ||\Delta u_k||_{PC_{1-\varpi,\Lambda}}. \end{aligned} \tag{15}$$

Now substituting inequality (15) into Equation (9) we have

$$\begin{aligned} ||e_{k+1}||_{PC_{1-\varpi,\Lambda}} &\leq ||1-\theta_1 G_u(\xi)||\,||e_k||_{PC_{1-\varpi,\Lambda}} + \beta_4 \mathcal{L}_i ||\Delta u_k||_{PC_{1-\varpi,\Lambda}} \\ &||1-\theta_1 G_u(\xi)||\,||e_k||_{PC_{1-\varpi,\Lambda}} + \beta_4 \mathcal{L}_i \theta_1 ||e_k||_{PC_{1-\varpi,\Lambda}} \\ &\leq \rho_2 ||e_k||_{PC_{1-\varpi,\Lambda}}, \end{aligned}$$

which implies that $\lim_{k\to\infty} ||e_k||_{PC_{1-\varpi,\Lambda}} = 0$.

**Case (iii).**

We prove that $||e_{k+1}|| \to 0$ as $k \to \infty$ for $\tau \in [\tau_i, s_i], i = 1, 2, \ldots, m$.

$$\begin{aligned} ||\Delta x_k(\tau)|| &\leq \frac{(\tau - s_i)^{1-\varpi}}{\Gamma(\varpi)} G_i(\tau_i, u(\tau_i^-)) \\ &+ \frac{1}{\Gamma(\rho)} \int_{s_i}^{\tau} [F_i(\tau, x_{k+1}(\tau), u_{k+1}(\tau)) - F_i(\tau, x_k(\tau), u_k(\tau))] ds \\ &\leq \Big[\frac{\mathcal{L}_i \theta_1 ||L||\,||e_k(0)||_\Lambda}{\Gamma(\varpi)} + \frac{\mathcal{L}_F \theta_1}{\Lambda^\rho} ||e_k||_{PC_{1-\varpi,\Lambda}}\Big] E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}). \end{aligned} \tag{16}$$

Substituting the above inequality Equation (16) into Equation (9) we have

$$\begin{aligned} ||e_{k+1}||_{PC_{1-\varpi,\Lambda}} &\leq ||1-\theta_1 G_u(\xi)||\,||e_k||_{PC_{1-\varpi,\Lambda}} \\ &+ \beta_4 \Big[\mathcal{L}_i \frac{||L||\theta_1||e_k(0)||_\Lambda}{\Gamma(\varpi)} + \frac{\mathcal{L}_F \theta_1}{\Lambda^\rho} ||e_k||_{PC_{1-\varpi,\Lambda}}\Big] E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}) \\ ||e_{k+1}||_{PC_{1-\varpi,\Lambda}} &\leq \rho_1 ||e_k||_{PC_{1-\varpi,\Lambda}} \\ &+ \beta_4 \Big[\frac{\mathcal{L}_i ||L||\theta_1||e_k(0)||_\Lambda}{\Gamma(\varpi)} + \frac{\mathcal{L}_F \theta_1}{\Lambda^\rho} ||e_k||_{PC_{1-\varpi,\Lambda}}\Big] E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}). \end{aligned} \tag{17}$$

It follows from inequality (17) that $\lim_{k\to\infty} ||e_{k+1}||_{PC_{1-\omega,\Lambda}} = 0$. This yields $\lim_{k\to\infty} ||e_k|| = 0$. The proof is completed. □

### 3.2. Closed-Loop Case

In this section, we consider the following closed-loop $P$-type ILC updating law:

$$\begin{cases} x_{k+1}(0) = x_k(0) + Le_k(0), \\ u_{k+1}(\tau) = u_k(\tau) + \theta_2 e_{k+1}(\tau), \end{cases} \tag{18}$$

where $L$ and $\theta_2$ are unknown parameters to be determined.

**Theorem 2.** *Assumptions 1–3 are satisfied. Let $y_k(\cdot)$ be the output system (2). If*

$$\begin{aligned} \max\left\{\left|\frac{1+L\beta_3}{1+\theta_2\beta_1}\right|, \left|\frac{1-L\beta_3}{1+\theta_2\beta_2}\right|, \left|\frac{1-L\beta_4}{1+\theta_2\beta_1}\right|, \left|\frac{1-L\beta_4}{1+\theta_2\beta_4}\right|\right\} &\leq \rho_3, \\ \max_{1\leq i\leq m}\left\{\frac{1}{|1-\beta_4\mathcal{L}_i\theta_2|\cdot|1+\theta_2\beta_1|}, \frac{1}{|1-\beta_4\mathcal{L}_i\theta_2|\cdot|1+\theta_2\beta_2|}\right\} &\leq \rho_4, \\ \max\left\{\frac{1}{|1+\theta_2\beta_1|}, \frac{1}{|1+\theta_2\beta_2|}\right\} &\leq \rho_5, \end{aligned}$$

*and* $\max\{\rho_3, \rho_4, \rho_5\} < 1$, *then for arbitrary initial input $u_0(\cdot)$, the closed-loop P-type ILC updating law (18) guarantees that*

$$\lim_{k\to\infty} y_k(\tau) = y_d(\tau), \textit{almost every} \quad \tau \in I.$$

**Proof.** Obviously,

$$\begin{aligned} ||e_{k+1}||_{PC_{1-\omega,\Lambda}} &\leq \left|\frac{1}{1+\theta_2 G_u(\xi)}\right| ||e_k||_{PC_{1-\omega,\Lambda}} + \beta_4 ||\Delta x_k||_{PC_{1-\omega,\Lambda}} \\ &\leq \frac{||e_k||_{PC_{1-\omega,\Lambda}}}{1+\theta_2 G_u(\xi)} + \beta_4 ||\Delta x_k||_{PC_{1-\omega,\Lambda}} \end{aligned} \tag{19}$$

Similarly to the proof of Theorem 1, we consider the following three possible cases.

**Case (i).**

As for $||\Delta x_k(\tau)||_{PC_{1-\omega,\Lambda}}$, we take Equation (13) into inequality (19) and obtain

$$\begin{aligned} ||e_{k+1}||_{PC_{1-\omega,\Lambda}} \leq{}& \frac{||e_k||_{PC_{1-\omega,\Lambda}}}{1+\theta_2 G_u(\xi)} \\ &+ \beta_4\Big(\Big[\frac{||L||\,||e_k(0)||_\Lambda}{\Gamma(\omega)} + \frac{\mathcal{L}_F\theta_1}{\Lambda^\rho}||e_k||_{PC_{1-\omega,\Lambda}}\Big]E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)})\Big). \end{aligned}$$

We can get $\lim_{k\to\infty} ||e_k||_{PC_{1-\omega,\Lambda}} = 0$.

**Case (ii).**

Substituting Equation (15) into inequality (19), we get

$$\begin{aligned} ||e_{k+1}||_{PC_{1-\omega,\Lambda}} &\leq \left|\frac{1}{1+\theta_2 G_u(\xi)}\right| ||e_k||_{PC_{1-\omega,\Lambda}} + \beta_4\left|\frac{G_x(\xi)}{1+G_u(\xi)\theta_2}\right|\mathcal{L}_i||\Delta u_k||_{PC_{1-\omega,\Lambda}} \\ &\leq \left|\frac{1}{1+G_u(\xi)\theta_2}\right| ||e_k||_{PC_{1-\omega,\Lambda}} + \left|\frac{G_x(\xi)}{1+G_u(\xi)\theta_2}\right|\mathcal{L}_i\beta_4\theta_2||e_{k+1}||_{PC_{1-\omega,\Lambda}}, \end{aligned}$$

which yields

$$||e_{k+1}||_{PC_{1-\varpi,\Lambda}} \leq \frac{||e_k||_{PC_{1-\varpi,\Lambda}}}{||1-\beta_4\mathcal{L}_i\theta_2||\,||1+\theta_2||G_u(\xi)}.$$

**Case (iii).**

Substituting inequality (16) into Equation (19), we have

$$\begin{aligned}||e_{k+1}||_{PC_{1-\varpi,\Lambda}} \leq & \frac{||e_k||_{PC_{1-\varpi,\Lambda}}}{||1+\theta_2 G_u(\xi)||} \\ & + \beta_4\Big[||L||\frac{\mathcal{L}_i\theta_1||e_k(0)||_\Lambda}{\Gamma(\varpi)} + \frac{\mathcal{L}_F\theta_1}{\Lambda^\rho}||e_k||_{PC_{1-\varpi,\Lambda}}\Big]E_\rho(\mathcal{L}_F b^{1+\vartheta(\rho-1)}).\end{aligned}$$

The rest of the proof is similar to that in Theorem 1. So we omit the details. □

## 4. An Example

Consider the following SFNI using the CFD of order $(\rho,\vartheta) = (0.6, 0.4)$, $\varpi = 0.76$. The switching signal is defined by $\sigma(\tau) = i$, where $i = 0, 1, 2$ and $\tau \in [0, 1]$, with

$$\sigma(\tau) = \begin{cases} 0, & \tau \in [0, 0.3), \\ 1, & \tau \in [0.3, 0.6), \\ 2, & \tau \in [0.6, 1.0]. \end{cases}$$

Then the system is governed by

$$\mathbb{H}^{0.6,0.4}_{\tau_i,\tau}x_k(\tau) = \begin{cases} -0.2x_k(\tau) + \dfrac{\sin(x_k(\tau))}{1+|x_k(\tau)|} + u_k(\tau), & \text{if } \sigma(\tau) = 0, \\ -0.4x_k^2(\tau) + u_k^2(\tau), & \text{if } \sigma(\tau) = 1, \\ -0.3x_k(\tau) + \cos(\tau)u_k(\tau), & \text{if } \sigma(\tau) = 2, \end{cases} \quad \tau \in (\tau_i, s_i), \tag{20}$$

$$x_k(\tau_i^+) = \begin{cases} x_k(\tau_i^-) + \ln(1+|u_k(\tau_i^-)|), & \text{if } \sigma(\tau) = 0, \\ x_k(\tau_i^-) + \sin(\tau)\ln(1+|u_k(\tau_i^-)|), & \text{if } \sigma(\tau) = 1, \\ x_k(\tau_i^-) + \dfrac{u_k(\tau_i^-)}{1+|\tau_i|}, & \text{if } \sigma(\tau) = 2, \end{cases} \tag{21}$$

$$x_k(\tau) = \begin{cases} x_k(\tau_i^+), & \text{if } \sigma(\tau) = 0, \\ x_k(\tau_i^+) + \sin(\tau)\cdot\ln(1+|u_k(\tau_i^-)|), & \text{if } \sigma(\tau) = 1, \\ x_k(\tau_i^+) + \dfrac{u_k(\tau_i^-)}{1+|\tau|}, & \text{if } \sigma(\tau) = 2, \end{cases} \quad \tau \in (s_{i-1}, \tau_i], \tag{22}$$

$$\mathbb{I}^{0.24}_{0^+}x_k(0) = 0, \qquad \mathbb{I}^{0.24}_{\tau_i^+}x_k(\tau_i^+) = x_k(\tau_i^+), \quad \text{for } i = 1, 2, \ldots, m. \tag{23}$$

The output function corresponding to each switching mode is defined by

$$y_k(\tau) = \begin{cases} x_k(\tau) + \cos(u_k(\tau)), & \text{if } \sigma(\tau) = 0, \\ x_k^2(\tau) + u_k(\tau), & \text{if } \sigma(\tau) = 1, \\ x_k(\tau)\sin(\tau u_k(\tau)), & \text{if } \sigma(\tau) = 2. \end{cases} \tag{24}$$

Assumptions 1–3 are satisfied. The original reference trajectory is given by

$$y_d(\tau) = \cos(\pi\tau) + \tau \quad \tau \in [0, 1]. \tag{25}$$

Figure 1 shows the tracking performance of all iterations, with convergence at the 15-th iteration. The final iteration $y_{15}(\tau)$ overlaps the reference curve $y_d(\tau)$. Figure 2 illustrates the bar graph of the maximum tracking error versus iteration. The error drops below $10^{-3}$ at $k = 15$ and the switching law $\sigma(\tau)$ with three distinct modes, producing a switching profile over the interval $[0, 1]$ in Figure 3. All output errors vanish beyond this point, validating the effectiveness of the designed ILC scheme. The maximum tracking error satisfies

$$||y_k(\tau) - y_d(\tau)|| \approx 0.000897, \quad \text{for} \quad k \geq 15.$$

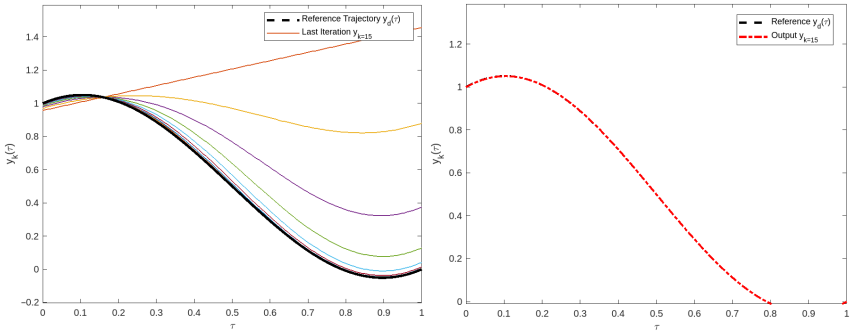



**Figure 1.** The system output.

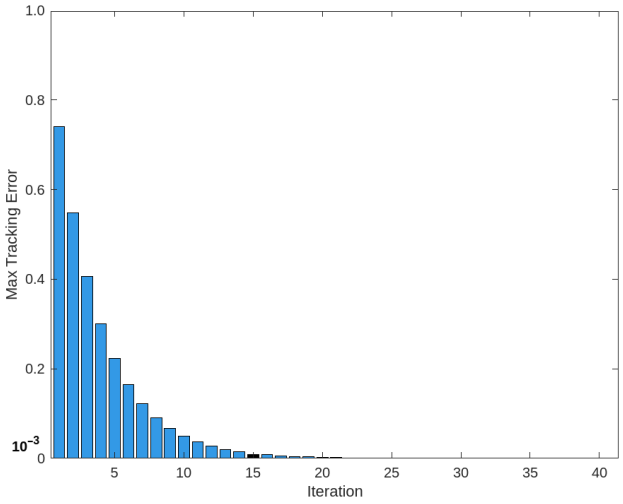



**Figure 2.** The tracking performance.

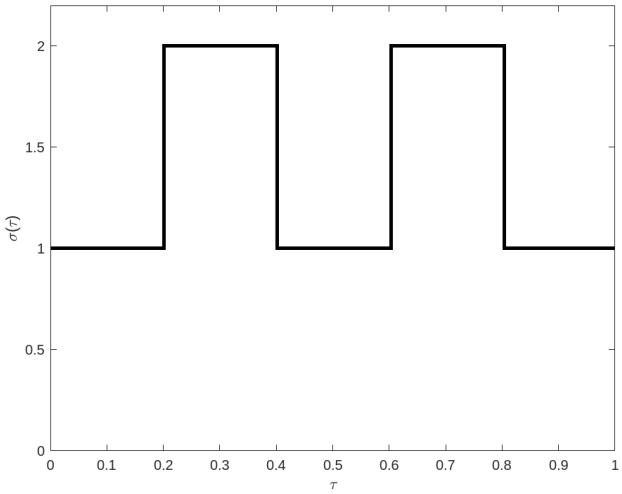



**Figure 3.** Random switched rule $\sigma(\tau)$ for Equation (20).

## 5. Conclusions

Considering the broad applicability of composite fractional calculus in modeling systems with memory and hereditary characteristics, this paper has addressed the ILC problem for switched impulsive systems governed by the CFD with non-instantaneous impulses. By formulating the system within an appropriate functional concept, sufficient conditions for robust convergence of the tracking error have been derived for both open-loop and closed-loop $P$-type learning laws incorporating an initial iteration mechanism. Numerical simulations on a illustrative switched impulsive system have verified the effectiveness and robustness of the proposed approach, showing fast error decay under varying switching laws and impulse intervals. The results improve the theoretical foundation of ILC for composite fractional-order switched systems and provide practical insights for systems with switching impulses.

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