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A Rank-Based Assignment Lottery for an Assignment Problem

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Abstract: For a traditional assignment problem with the same number of objects and agents, we introduce a new assignment lottery based on the notion of rank and analyze some of its properties. In particular, we prove that, like the Random Serial Dictatorship, it is ex post efficient and guarantees positive probability to each Pareto optimal deterministic assignment; moreover, the expected rank of this new assignment lottery, which is a measure of the social welfare, cannot be greater than the Random Serial Dictatorship's one and there exist assignment problems where it is strictly lower.

Keywords: assignment problems; assignment lotteries; rank

MSC: 91B15; 91B68

1. Introduction

In this paper, we consider an assignment problem described by a set N formed by n agents and a set O formed by the same number of indivisible objects. Each agent has a strict preference order over the objects. The goal is to allocate one distinct object to each agent based solely on the agents' preferences in such a way that some optimality properties are fulfilled. In the allocation process, monetary transfers are unavailable and there is no initial ownership of the objects. These models are also referred to as *house allocation problems* and differ from the so-called *housing markets* where each agent initially owns an object. They have several real-life applications; the objects may be, for example, seats at public schools, housing units or dormitory rooms, jobs, time slots for the use of a machine, or organs for transplant (see, [1–4]).

A deterministic assignment is a one-to-one function a from N to O which assigns exactly one object $a(i)$ to each agent $i \in N$; a permutation matrix $n \times n$ is associated with each deterministic assignment. Since deterministic assignments are typically not fair and moreover fairness and efficiency cannot typically be jointly maintained, randomization is introduced into these models and assignment lotteries are frequently encountered in place of deterministic ones. An assignment lottery φ is a probability distribution over the set $\mathcal{A}$ of all deterministic assignments. Every assignment lottery φ induces a unique random assignment that is a bistochastic $n \times n$ matrix P^{φ}, whose ij entry represents the probability that agent i obtains object o_j. On the other hand, by the Birkhoff–von Neumann theorem, any bistochastic matrix can be written as a (non-unique) convex combination of permutation matrices (i.e., deterministic assignments); as a consequence, each random assignment can be implemented through assignment lotteries.

One of the most popular assignment lotteries is the Random Serial Dictatorship (RSD henceforth). It works as follows. First, a priority order over agents is drawn uniformly at random and agents sequentially choose their most preferred object according to this priority order. Hence, the first agent receives his most preferred object, the second agent receives

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his most preferred object among the set of remaining objects, and so on until no object remains unassigned. Besides being easy to understand and practical to implement, RSD has several attractive features: it is ex post efficient, that is, it assigns positive probability only to Pareto optimal deterministic assignment; it is strategy-proof, that is, no agent can benefit by misreporting his preferences; and it is fair in the sense that agents with the same preference relations are treated equally. For these reasons, it is widely used in real-life applications, for instance, in university housing assignments.

In this paper, we focus on a further property which characterizes RSD and was first pointed out by Abdulkadiroğlu and Sönmez [1]. In fact, they proved that RSD is equivalent to another assignment lottery which they call Core from Random Endowments (CRE); the equivalence has to be intended in the sense that, for each assignment problem, CRE results in the same lottery over assignments as RSD. As evoked by its name, CRE works as follows. We consider an assignment problem $\mathcal{P}$. A deterministic assignment a is drawn with uniform distribution from the set $\mathcal{A}$; we interpret a as the initial endowment of the housing market $\mathcal{P}(a)$ associated with the assignment problem $\mathcal{P}$ and compute the core assignment of $\mathcal{P}(a)$. As shown by Roth and Postlewaite [5], this core assignment is unique and can be computed by using the Top Trading Cycle (TTC) algorithm. The Core from Random Endowments is the assignment lottery which results when this core assignment is selected as the solution of the assignment problem.

Our purpose is to introduce a new assignment lottery based on the same idea as the CRE but involving the notion of rank. Given an object $o \in O$ and an agent $i \in N$, $\text{Rank}^i o$ is the position of o in the preference relation of i. With a somewhat improper use of the term "rank", we define the rank of an assignment a as the sum of the rank of the objects that a assigns to each agent and the expected rank of an assignment lottery as the sum of the products between the rank of each deterministic assignment and its probability.

Since the number $\text{Rank}^i o$ can be viewed as a surrogate utility level that agent $i \in N$ assigns to the object o, the rank of an assignment clearly relates to a measure of the utilitarian social welfare associated with a, defined as the total utility of the agents. The rationale for introducing a new assignment lottery based on the rank is precisely based on this point; in fact, our main findings in this paper is that RSD, or equivalently CRE, is not optimal under the viewpoint of this measure. This, along with the construction of the new lottery, is explained in what follows.

CRE builds on a function C which associates each deterministic assignment $a \in \mathcal{A}$ with the unique core assignment $C(a)$ in the housing market $\mathcal{P}(a)$. The assignment $C(a)$ belongs to the upper contour set $\mathcal{A}_{\succeq a}$ associated with a, that is, the set of all deterministic assignments that are unanimously weakly preferred to a. However, it is not, in general, the minimum rank assignment in the set $\mathcal{A}_{\succeq a}$; indeed, there may be assignments in $\mathcal{A}_{\succeq a}$ whose rank is lower than the core assignment $C(a)$'s one. Based on this very simple point, we construct a new assignment lottery by replacing the function C with a correspondence F whose definition is based on the notion of rank. Precisely, F associates each deterministic assignment $a \in \mathcal{A}$ with the set of all deterministic assignments with minimum rank in $\mathcal{A}_{\succeq a}$.

We call Minimum Rank from Random Endowment (MRRE) this assignment lottery and prove that it preserves the same efficiency properties as RSD; in fact, it is ex post efficient (in particular, it assigns probability at least equal to $\frac{1}{n!}$ to each Pareto optimal assignment) but not ordinally efficient. However, it performs better than RSD under the viewpoint of the expected rank. Indeed, the expected rank of RSD cannot be lower than the expected rank of MRRE and there are assignment problems where such inequality is strict. Such a result is important because it addresses the efficiency of RSD under the viewpoint of social welfare.

The paper is structured into three sections. Section 2 illustrates the model along with the main solution concepts and optimality properties that are needed in the rest of the paper. In Section 3, we define the new assignment lottery and analyze its properties. The final section contains the conclusions.

2. Setup

We consider an *assignment problem*, that is, a triple $\mathcal{P} = (N, O, \succ)$ where $N = \{1, \ldots, n\}$ is a finite set of n agents, $O = \{o_1, \ldots, o_n\}$ is a finite set of indivisible objects with the same cardinality as N, and $\succ = (\succ_1, \ldots, \succ_n)$ is a profile of preferences. In particular, $\succ_i$ is the preference relation of agent i over the set O; we assume that $\succ_i$ is a linear order on the set O, that is, a complete, transitive, and antisymmetric relation. $o \succ_i o'$ means that agent i prefers object o to object o'. $\succeq_i$ denotes, as usual, the weak preference relation induced by $\succ_i$, that is

$$o \succeq_i o' \Leftrightarrow o \succ_i o' \text{ or } o = o' .$$

A solution for an assignment problem $\mathcal{P}$ consists in allocating one distinct object in O to each agent $i \in N$ in such a way that some optimality properties are satisfied. A **deterministic assignment** is a one-to-one function a from N to O; $a(i)$ is the object allocated to agent i under the assignment a. We denote by $\mathcal{A}$ the set of all deterministic assignments. Each deterministic assignment a can be associated with a permutation matrix $n \times n$, denoted by A^a, where $A^a_{ij} = 1$ if $a(i) = o_j$ and $A^a_{ij} = 0$ otherwise. Remember that a permutation matrix is a square matrix with entries in $\{0, 1\}$ that has exactly one entry equal to 1 in each row and each column and all 0 elsewhere.

A coalition is a nonempty subset S of N. For any coalition $S \subseteq N$ and any deterministic assignment $a \in \mathcal{A}$, we let $a(S) = \{a(i) \in O : i \in S\}$ be the set of objects that coalition S receives under the assignment a.

A coalition S blocks the assignment $a \in \mathcal{A}$ (equivalently, S is a blocking coalition for a) if agents in S can redistribute their objects among themselves and be better off. That is, there exists a redistribution b such that

a. $b(S) = a(S)$; and,
b. $b(i) \succ_i a(i)$ for every i in S.

An assignment $a \in \mathcal{A}$ is **stable** if no coalition can block it. We denote by $\mathcal{A}^S(\mathcal{P})$ the set of all stable assignments for the assignment problem $\mathcal{P}$.

The notion of stable assignment can be rephrased in terms of core assignments for a closely related instances of problems; in fact, a is stable for the assignment problem $\mathcal{P}$ if and only if a is a core assignment for the housing market $\mathcal{P}(a)$ associated with $\mathcal{P}$, where a itself is the initial endowment.

An assignment a is **Pareto optimal** if there does not exist an assignment b that weakly dominates a; that is, $b(i) \succeq_i a(i)$ for every $i \in N$ and $b(i) \succ_i a(i)$ for some $i \in N$. We denote by $\mathcal{A}^{PO}(\mathcal{P})$ the set of all Pareto optimal assignments for the assignment problem $\mathcal{P}$.

For the assignment problem $\mathcal{P}$, the set of stable assignments equals the set of Pareto optimal assignments.

Proposition 1. *Let $\mathcal{P} = (N, O, \succ)$ be an assignment problem. Then, $\mathcal{A}^{PO}(\mathcal{P}) = \mathcal{A}^S(\mathcal{P})$.*

Proof. If a is not Pareto optimal, there exists $b \in \mathcal{A}$ such that $b(i) \succeq_i a(i)$ for all $i \in N$ and $b(i) \succ_i a(i)$ for some $i \in N$.

We let $U = \{i \in N : b(i) = a(i)\}$ and $\overline{U} = N \setminus U$. It holds that $a(\overline{U}) = b(\overline{U})$ and $b(k) \succ_k a(k)$, for each $k \in \overline{U}$. Hence, a is not stable.

On the contrary, if a is not stable, there exist a coalition $S \subseteq N$ and an assignment b such that $b(S) = a(S)$ and $b(i) \succ_i a(i)$ for each $i \in S$.

We consider $\bar{b} = (b_S, a_{N \setminus S})$, where this notation stands for the assignment that gives $b(i)$ to each agent $i \in S$ and $a(i)$ to each agent $i \in N \setminus S$. It holds that $\bar{b}$ weakly dominates a and therefore a is not Pareto optimal. □

Given an object $o \in O$ and an agent $i \in N$, $\text{Rank}^i\ o$ denotes the rank of o according to the preference relation $\succ_i$ of agent i; that is, $\text{Rank}^i\ o = 1$ if o is the top choice of agent i, $\text{Rank}^i\ o = 2$ if o is his second choice, and so on. More formally, for each $i \in N$ and for each $o \in O$,

$$\text{Rank}^i\ o = |\{o' \in O : o' \succeq_i o\}|\ .$$

Of course, for each $o \in O$ and $i \in N$, it holds that $1 \leq \text{Rank}^i\ o \leq n$.

Given a deterministic assignment $a \in \mathcal{A}$, the rank of a is defined as follows:

$$\text{Rank}\ a = \sum_{i \in N} \text{Rank}^i\ a(i).$$

We remark that, for each $i \in N$, the function $u_i : O \longrightarrow \mathbb{R}$ defined by

$$u_i(o) = n - \text{Rank}^i o$$

is a utility function consistent with $\succ_i$; that is, for all $o, o' \in O$, $u_i(o) > u_i(o')$ if and only if $o \succ_i o'$.

Given a deterministic assignment $a \in \mathcal{A}$, we may consider the set of assignments b that weakly Pareto dominate a. This set is called the upper contour set of a and denoted by $\mathcal{A}_{\succeq a}$; formally, $\mathcal{A}_{\succeq a} = \{b \in \mathcal{A} : b(i) \succeq_i a(i),\ \text{for each } i \in N\}$.

$\mathcal{A}_{\succeq a}$ has a limpid visual interpretation in our model: it is formed by all the assignments that are on the left side of a in the preference table. Next example illustrates this point.

Example 1. *We consider an assignment problem with three agents and three objects; that is,* $N = \{1, 2, 3\}$ *and* $O = \{o_1, o_2, o_3\}$.

The agents' preferences are displayed in the next table.

Agent 1:	o_1	o_2	o_3
Agent 2:	o_3	o_1	o_2
Agent 3:	o_2	o_3	o_1

We consider the deterministic assignments $a_1 = (o_3, o_2, o_1)$ *and* $a_2 = (o_2, o_3, o_1)$. *It holds that*

$$\mathcal{A}_{\succeq a_1} = \mathcal{A}$$

and

$$\mathcal{A}_{\succeq a_2} = \{(o_2, o_3, o_1), (o_1, o_3, o_2)\}.$$

We remark that, for each $a \in \mathcal{A}$, $a \in \mathcal{A}_{\succeq a}$ and $\mathcal{A}_{\succeq a} = \{a\}$ if and only if a is stable.

We now move to the probabilistic counterpart of the setup considered so far. Randomization is commonplace in applications concerning resource allocation; for instance, it is frequently used to break ties among students applying for overdemanded public schools or to allocate dormitory rooms amongst college students. In such applications, any deterministic assignment of objects is likely to be unfair. In fact, the main rationale to

introduce randomization in assignment problems is that randomization can restore fairness (at least in the sense of equal treatments of equals).

An **assignment lottery** φ is a probability distribution over the set $\mathcal{A}$ of all deterministic assignments.

As usual, the support of the assignment lottery φ is given by

$$\text{Supp } \varphi = \{a \in \mathcal{A} : p(a) > 0\} ,$$

where $p(a)$ is the probability that φ associates with a.

Given a deterministic assignment $a \in \mathcal{A}$, we denote by π^a the (degenerate) assignment lottery that assigns Probability 1 to a.

For an assignment lottery φ, both the following equivalent descriptions are used throughout the paper:

$$\varphi = \sum_{a \in Supp\ \varphi} p(a)\pi^a = \begin{pmatrix} a_1 & \dots & a_k \\ p_1 & \dots & p_k \end{pmatrix},$$

where in the second description Supp $\varphi = \{a_1, \dots, a_k\}$ and $p_i = p(a_i) > 0$.

Every assignment lottery φ induces a **random assignment** P^φ, which is a bistochastic $n \times n$ matrix; the ij entry p_{ij} of P^φ is the probability that the object $o_j \in O$ is assigned to agent $i \in N$. A bistochastic matrix is a square matrix of nonnegative real entries where each row and each column sums to one.

By the Birkhoff–von Neumann theorem ([6]), any bistochastic matrix can be decomposed into a convex (not necessarily unique) combination of permutation matrices. As a consequence, random assignments can be actually implemented by some randomization over deterministic assignments. A lottery decomposition of a random assignment P is any representation of P in the following form:

$$P = \sum_{a \in \mathcal{A}} p(a)\pi^a .$$

We collect some definitions concerning assignment lotteries that are needed in the sequel.

Definition 1 (Expected rank for an assignment lottery)**.** *Given an assignment lottery* $\varphi = \sum_{a \in Supp\ \varphi} p(a)\pi^a$*, the expected rank of* φ *is defined by*

$$ExpRank\varphi = \sum_{a \in Supp\ \varphi} p(a)\, Rank\ a ,$$

where $p(a)$ *denotes the probability associated with the deterministic assignment* $a \in Supp\ \varphi$ *in the lottery* φ*.*

Definition 2 (Ex post Pareto optimality for an assignment lottery)**.** *An assignment lottery* φ *is ex post Pareto optimal if all the deterministic assignments in its support are Pareto optimal, that is,*

$$Supp\ \varphi \subseteq \mathcal{A}^{PO} .$$

As remarked before, the decomposition of a random assignment into assignment lotteries is not unique. This is illustrated in the next example, which also points out that such non-uniqueness may create some ambiguity when dealing with Pareto optimality of random assignments.

Example 2 (Example 2, page 162 in Abdulkadiroğlu and Sönmez, [2]). *We consider an assignment problem with four agents, that is,* $N = \{1, 2, 3, 4\}$. *Their preferences over the set of indivisible objects* $O = \{o_1, o_2, o_3, o_4\}$ *are displayed below.*

Agent 1:	o_1	o_2	o_3	o_4
Agent 2:	o_1	o_2	o_3	o_4
Agent 3:	o_2	o_1	o_4	o_3
Agent 4:	o_2	o_1	o_4	o_3

We consider the following random assignment:

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

P can be decomposed as a convex combination of permutation matrices (i.e., deterministic assignments) in two different ways as follows:

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The first decomposition involves the Pareto optimal deterministic assignments (o_1, o_2, o_4, o_3) *and* (o_3, o_4, o_2, o_1). *On the contrary, in the second decomposition of P, both the deterministic assignments are not Pareto optimal; indeed,* (o_1, o_4, o_2, o_3) *is weakly dominated by* (o_1, o_3, o_2, o_4) *and* (o_3, o_2, o_4, o_1) *is weakly dominated by* (o_3, o_1, o_4, o_2).

As a consequence of the previous example, the definition of ex post Pareto optimality for random assignments is formulated as follows.

Definition 3 (Ex post Pareto optimality for random assignments). *A random assignment P is ex post Pareto optimal at* $\succ$ *if it admits a lottery decomposition formed by ex post Pareto optimal assignments.*

The random assignment P in the previous example is ex post Pareto optimal since it admits a lottery decomposition that is formed by the ex post Pareto optimal assignments (o_1, o_2, o_4, o_3) and (o_3, o_4, o_2, o_1).

We remark that assignment lotteries are immune from the ambiguity illustrated in the previous example and, from this point of view, the two solution concepts of assignment lotteries and random lotteries cannot be considered as equivalent. In this paper, we focus on assignment lotteries which appear, in some sense, as a more primitive and less problematic solution concept.

The next two definitions are about a stronger notion of efficiency introduced by Bogomolnaia and Moulin [3] (see also [7,8]).

Definition 4 (First-order stochastic dominance). *A random first-order assignment P stochastically dominates the random assignment Q if, for each agent* $i \in N$ *and each object* $o_j \in O$, *the*

probability that i receives an object that is preferred to o_j is at least as large under P, with strictly inequality in at least one case. That is, for each $i \in N$ and for each $o_j \in O$,

$$\sum_{k:o_j \succ_i o_k} P_{ik} \geq \sum_{k:o_j \succ_i o_k} Q_{ik},$$

with strictly inequality in at least one case.

Definition 5 (Ordinally efficient random assignments)**.** *A random assignment P is ordinally efficient if it is not first-order stochastically dominated by any other random assignment.*

A solution procedure which is commonly used in real life applications for allocating indivisible objects is the Random Serial Dictatorship (RSD). In this procedure, every agent reports its preference order over the set of objects. Then, agents are randomly ordered with equal probability. The first agent in the realized order receives its most preferred object, the next agent receives its favorite object among the remaining ones, and so on. Formally, we let $\pi : \{1, 2, \ldots, n\} \to N$ be an ordering of the agent, that is, a permutation over N and Π be the set of all the orderings ($|\Pi| = n!$). We let $a(\pi)$ be the serial dictatorship assignment induced by π (that is, the assignment which gives the top choice to agent $\pi(1)$; the most preferred object among the remaining ones to agent $\pi(2)$ and so on) and $\varphi^{a(\pi)}$ be the assignment lottery that assigns Probability 1 to the assignment $a(\pi)$. The Random Serial Dictatorship (RSD) is the assignment lottery defined by

$$RSD = \sum_{\pi \in \Pi} \frac{1}{n!} \varphi^{a(\pi)}.$$

The random serial dictatorship is ex post efficient; its main flaw is, however, that it does not guarantee efficiency from the ex ante point of view. In fact, it can result in a random assignment that is first-order stochastically dominated, for all agents, by another random assignment (see, page 298 in [3] and the example in the proof of Proposition 3).

Abdulkadiroğlu and Sönmez [1] focused on another interesting property for RSD which links it to the core notion; in fact, they proved that RSD is equivalent to another assignment lottery that they call Core from Random Endowments (CRE). In CRE, a random initial endowment is selected and the unique core assignment of the associated housing market is taken as the solution. Formally, CRE is defined as follows:

$$CRE = \sum_{a \in \mathcal{A}} \frac{1}{n!} \varphi^{C(a)},$$

where C is a function which associates each assignment $a \in \mathcal{A}$ with the unique assignment in the core of the housing market $\mathcal{P}(a)$ associated with $\mathcal{P}$, where the assignment a is interpreted as the initial endowment. This unique core assignment can be found by employing the Top Trading Cycle (TTC) algorithm; hence, CRE can be expressed in an equivalent form as

$$CRE = \sum_{a \in \mathcal{A}} \frac{1}{n!} \varphi^{TTC(a)},$$

where TTC(a) is the function which associates each deterministic assignment $a \in \mathcal{A}$ with the unique core assignment found by the TTC algorithm.

The equivalence between RSD and CRE is considered as an evidence in support of the random serial dictatorship; moreover, it is theoretically sensible since it formally establishes the close relationship between models with an exogenously given initial endowment and models where agents collectively own the set of indivisible objects.

3. A New Assignment Lottery

Our aim in this section is to define a new assignment lottery for an assignment problem $\mathcal{P}$ which is based on the same idea as the Core from Random Endowments but accounts for the rank; roughly speaking, the function C which appears in the definition of the Core from Random Endowments is replaced with a correspondence F based on the rank notion.

Before providing its formal definition, next example illustrates the rationale behind the definition of the new assignment lottery.

Example 3. *Four objects are to be distributed to four agents on the basis of the preference profile displayed below.*

Agent 1:	o_3	o_4	$\mathbf{o_2}$	o_1
Agent 2:	o_3	$\mathbf{o_1}$	o_4	o_2
Agent 3:	o_2	o_4	o_1	$\mathbf{o_3}$
Agent 4:	o_2	o_1	o_3	$\mathbf{o_4}$

We consider the assignment $a = (o_2, o_1, o_3, o_4)$, in bold in the previous table. By applying the TTC algorithm, we obtain the assignment $b_1 = TTC(a) = Core(a) = (o_3, o_1, o_2, o_4)$ whose rank is equal to 8.

However, as it is clear from the previous preference table, the upper contour set $\mathcal{A}_{\succeq a}$ of a contains two assignments whose rank is lower than 8: $b_2 = (o_4, o_3, o_2, o_1)$ and $b_3 = (o_3, o_1, o_4, o_2)$. Indeed, it holds that $b_2, b_3 \in \mathcal{A}_{\succeq a}$ and

$$6 = Rank\ b_2 = Rank\ b_3 < Rank\ Core(a) = 8\,.$$

This example suggests two elementary but interesting features which emerge from the preference profile table.

a. Given a deterministic assignment a, the minimum rank assignment in its upper contour set is not unique in general; and
b. given a deterministic assignment a, the core assignment $C(a)$ is not the minimum rank assignment in the upper contour set $\mathcal{A}_{\succeq a}$ in general.

Based on the previous two points, our objective is to define a new assignment lottery which differs from RSD; we call it Minimum Rank from Random Endowment (MRRE). MRRE is founded on a correspondence F which associates each deterministic assignment $a \in \mathcal{A}$ with the set of assignments with minimum rank in the upper contour set of a. Formally,

$$\begin{aligned} &F : \mathcal{A} \to 2^{\mathcal{A}} \\ &F(a) = \{b \in \mathcal{A}_{\succeq a} : \text{Rank } b \leq \text{Rank } c, \text{ for each } c \in \mathcal{A}_{\succeq a}\}\,. \end{aligned}$$

Hence, given a deterministic assignment $a \in \mathcal{A}$, F selects the rank–minimizing assignments in the set $\mathcal{A}_{\succeq a}$.

The next proposition collects some properties of the correspondence F.

Proposition 2. *The correspondence F has the following properties:*

a. F is non-empty valued and not single-valued in general.
b. $F(a) = \{a\} \Leftrightarrow a \in \mathcal{A}^S$.
c. For each $a \in \mathcal{A}$, $F(a) \subseteq \mathcal{A}^S$.

Proof. Since $\mathcal{A}_{\succeq a}$ is a finite set for each $a \in \mathcal{A}$, $F(a)$ is non-empty. Moreover, the fact that F is not single-valued in general is clear from Example 3, where $b_2, b_3 \in F(a)$.

For point b, if a is stable, then $\mathcal{A}_{\succeq a} = \{a\}$. As a consequence, $F(a) = \{a\}$. On the contrary, if $F(a) = \{a\}$, then $\mathcal{A}_{\succeq a} = \{a\}$ and therefore a is stable.

As to point c, the proof is trivial when $a \in \mathcal{A}^S$ as a consequence of point b. We suppose now that $a \notin \mathcal{A}^S$ and let $b \in F(a)$. We suppose, by way of contradiction, that $b \notin \mathcal{A}^S$. Then, there is $S \subseteq N$ and c such that

i. $c(S) = b(S)$; and
ii. $c_i \succ_i b(i)$, for all $i \in S$.

We consider $\bar{c} = (c_S, b_{N\setminus S})$. It holds that $\bar{c} \in \mathcal{A}$. Moreover, $\bar{c} \in \mathcal{A}_{\succeq b}$ and, since $b \in \mathcal{A}_{\succeq a}$, it also holds that $\bar{c} \in \mathcal{A}_{\succeq a}$. Moreover,

$$\text{Rank } \bar{c} = \sum_{i \in S} \text{Rank } c(i) + \sum_{i \in N\setminus S} \text{Rank } b(i) < \sum_{i \in S} \text{Rank } b(i) + \sum_{i \in N\setminus S} \text{Rank } b(i) = \text{Rank } b\,,$$

which contradicts $b \in F(a)$. □

We define the Minimum Rank from Random Endowment assignment lottery, denoted by *MRRE*, as follows:

$$MRRE = \sum_{a \in \mathcal{A}} \frac{1}{n!} \sum_{b \in F(a)} \frac{1}{|F(a)|} \varphi^b\,.$$

The rest of the section is devoted to analyze the properties of the new assignment lottery MRRE. We prove that, like RSD, MRRE is ex post efficient but not ordinally efficient; moreover, it assigns to each stable assignment a probability greater than or equal to $\frac{1}{n!}$ (Proposition 3). Finally, and more importantly, MRRE outperforms RSD under the viewpoint of the expected rank; indeed, MR has an expected rank lower than RSD and there exist assignment problems where the inequality is strict (Proposition 4).

Proposition 3. *The assignment lottery MRRE is ex post efficient and assigns to each stable assignment $a \in \mathcal{A}^S$ a probability at least equal to $\frac{1}{n!}$. Moreover, MRRE is not ordinally efficient.*

Proof. To prove that MRRE is ex post efficient, it is enough to note that

$$\text{Supp } MRRE = \bigcup_{a \in \mathcal{A}} F(a) = \mathcal{A}^S\,.$$

Moreover, by point b in Proposition 2, it follows that the probability of each stable assignment is at least $\frac{1}{n!}$.

In order to prove that MRRE is not ordinally efficient, we consider the assignment problem $\widetilde{\mathcal{P}}$ with four agents used by Bogomolnaia and Moulin [3] to show that RSD is not ordinally efficient (see, page 298 in [3]). In such problem, four objects have to be assigned on the basis of the following preference profile:

Agent 1:	o_1	o_2	o_3	o_4
Agent 2:	o_1	o_2	o_3	o_4
Agent 3:	o_2	o_1	o_4	o_3
Agent 4:	o_2	o_1	o_4	o_3

Table 1 sums up the set of all deterministic assignments, the function C and the correspondence F for the assignment problem $\widetilde{\mathcal{P}}$. In parentheses, in the first column, there is the rank of each assignment.

Table 1. Assignments, function C, and correspondence F for the assignment problem $\widetilde{\mathcal{P}}$.

Assignment a	$C(a) = TTC(a)$	$F(a)$
$a_1 = (o_1, o_2, o_3, o_4)$ **(10)**	$a_1 = (o_1, o_2, o_3, o_4)$	$\{a_1\}$
$a_2 = (o_4, o_1, o_2, o_3)$ **(10)**	$a_{22} = (o_1, o_4, o_2, o_3)$	$\{a_{22}\}$
$a_3 = (o_3, o_4, o_1, o_2)$ **(10)**	$a_3 = (o_3, o_4, o_1, o_2)$	$\{a_3\}$
$a_4 = (o_2, o_3, o_4, o_1)$ **(10)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_8\}$
$a_5 = (o_2, o_1, o_3, o_4)$ **(10)**	$a_5 = (o_2, o_1, o_3, o_4)$	$\{a_5\}$
$a_6 = (o_4, o_2, o_1, o_3)$ **(12)**	$a_{22} = (o_1, o_4, o_2, o_3)$	$\{a_{22}\}$
$a_7 = (o_3, o_4, o_2, o_1)$ **(10)**	$a_7 = (o_3, o_4, o_2, o_1)$	$\{a_7\}$
$a_8 = (o_1, o_3, o_4, o_2)$ **(8)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_8\}$
$a_9 = (o_1, o_3, o_2, o_4)$ **(8)**	$a_{22} = (o_1, o_4, o_2, o_3)$	$\{a_{22}\}$
$a_{10} = (o_4, o_1, o_3, o_2)$ **(10)**	$a_{10} = (o_4, o_1, o_3, o_2)$	$\{a_{10}\}$
$a_{11} = (o_2, o_4, o_1, o_3)$ **(12)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_8\}$
$a_{12} = (o_3, o_2, o_4, o_1)$ **(10)**	$a_{12} = (o_3, o_2, o_4, o_1)$	$\{a_{12}\}$
$a_{13} = (o_1, o_2, o_4, o_3)$ **(10)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{15}\}$
$a_{14} = (o_3, o_1, o_2, o_4)$ **(8)**	$a_{17} = (o_3, o_2, o_1, o_4)$	$\{a_{17}\}$
$a_{15} = (o_4, o_3, o_1, o_2)$ **(10)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{15}\}$
$a_{16} = (o_2, o_4, o_3, o_1)$ **(12)**	$a_{17} = (o_3, o_2, o_1, o_4)$	$\{a_{17}\}$
$a_{17} = (o_3, o_2, o_1, o_4)$ **(10)**	$a_{17} = (o_3, o_2, o_1, o_4)$	$\{a_{17}\}$
$a_{18} = (o_4, o_3, o_2, o_1)$ **(10)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{15}\}$
$a_{19} = (o_1, o_4, o_3, o_2)$ **(10)**	$a_{19} = (o_1, o_4, o_3, o_2)$	$\{a_{19}\}$
$a_{20} = (o_2, o_1, o_4, o_3)$ **(10)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{15}\}$
$a_{21} = (o_4, o_2, o_3, o_1)$ **(12)**	$a_{21} = (o_4, o_2, o_3, o_1)$	$\{a_{21}\}$
$a_{22} = (o_1, o_4, o_2, o_3)$ **(10)**	$a_{22} = (o_1, o_4, o_2, o_3)$	$\{a_{22}\}$
$a_{23} = (o_3, o_1, o_4, o_2)$ **(8)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}\}$
$a_{24} = (o_2, o_3, o_1, o_4)$ **(10)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_8\}$

By Table 1, it is clear that the stable assignments for the assignment problem $\widetilde{\mathcal{P}}$ are the following:

$$\mathcal{A}^S = \{a_1, a_3, a_5, a_7, a_8, a_{10}, a_{12}, a_{15}, a_{17}, a_{21}, a_{22}, a_{23}\}.$$

For this assignment problem, it holds true that RSD equals MRRE; they both are given by

$$RSD = MRRE = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & a_8 & a_{10} & a_{12} & a_{15} & a_{17} & a_{21} & a_{22} & a_{23} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{4}{24} & \frac{1}{24} & \frac{1}{24} & \frac{4}{24} & \frac{4}{24} & \frac{1}{24} & \frac{4}{24} & \frac{1}{24} \end{pmatrix}.$$

Hence, both of them are ordinally dominated by the following assignment lottery:

$$\varphi = \begin{pmatrix} a_8 & a_{15} & a_{17} & a_{22} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

As a consequence, neither RSD nor MRRE is ordinally efficient. □

From the previous example, it also turns out that the assignment lottery MRRE does not minimize the expected rank in the set of all possible assignment lotteries. Indeed,

$$\text{ExpRank } \varphi < \text{ExpRank } MRRE\,.$$

The next proposition shows that MRRE has an expected rank lower than RSD.

Proposition 4. *For each assignment problem $\mathcal{P}$, it holds true that*

$$ExpRank\ MRRE \leq ExpRank\ RSD\,.$$

Moreover, there exist assignment problems where such inequality is strict.

Proof. We consider an assignment problem $\mathcal{P}$. By definition of F, it holds that, for each $a \in \mathcal{A}$,

$$\text{Rank } C(a) \geq \text{Rank } b, \text{ for each } b \in F(a)\,.$$

Since Rank b_1=Rank b_2 for each $b_1, b_2 \in F(a)$, we determine that, for each $a \in \mathcal{A}$,

$$|F(a)|\text{Rank } C(a) \geq \sum_{b \in F(a)} \text{Rank } b\,,$$

that is,

$$\text{Rank } C(a) \geq \sum_{b \in F(a)} \frac{\text{Rank } b}{|F(a)|}\,.$$

By summing over $a \in \mathcal{A}$, we obtain

$$\sum_{a \in \mathcal{A}} \text{Rank } C(a) \geq \sum_{a \in \mathcal{A}} \sum_{b \in F(a)} \frac{\text{Rank } b}{|F(a)|}\,.$$

But $C(a) \in \mathcal{A}^S$, $\forall\, a \in \mathcal{A}$, and $F(a) \subseteq \mathcal{A}^S$, $\forall\, a \in \mathcal{A}$; moreover, $C(\mathcal{A}) = F(\mathcal{A}) = \mathcal{A}^S$. Hence, the previous inequality can be written as

$$\sum_{a_j \in \mathcal{A}^S} k_j \text{Rank } a_j \geq \sum_{a_j \in \mathcal{A}^S} \sum_{a \in F^{-1}(a_j)} \frac{\text{Rank } a_j}{|F(a)|}\,,$$

where $k_j = |C^{-1}(a_j)|$. Dividing both sides by $n!$ and noting that

$$\frac{k_j}{n!} = p^{RSD}(a_j) \quad \text{and} \quad \frac{1}{n!} \sum_{a \in F^{-1}(a_j)} \frac{1}{|F(a)|} = p^{MRRE}(a_j)\,,$$

we obtain

$$\text{ExpRank } RSD \geq \text{ExpRank } MRRE\,.$$

The next example shows an assignment problem where the previous inequality is strict.

We consider the following assignment problem $\overline{\mathcal{P}}$ with four agents and four objects that have to be assigned on the basis of the following preference profile:

Agent 1:	o_3	o_4	o_2	o_1
Agent 2:	o_3	o_1	o_4	o_2
Agent 3:	o_2	o_4	o_1	o_3
Agent 4:	o_2	o_1	o_3	o_4

We need to determine the assignment lotteries RSD and MRRE.

Table 2 sums up the set of all deterministic assignments, the function C, and the correspondence F for this assignment problem. In parentheses, in the first column, there is the rank of each assignment.

Table 2. Assignments, function C, and correspondence F for the assignment problem $\mathcal{P}$.

Assignment a	$C(a) = TTC(a)$	$F(a)$
$a_1 = (o_1, o_2, o_3, o_4)$ **(16)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{23}, a_{18}\}$
$a_2 = (o_4, o_1, o_2, o_3)$ **(8)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{18}\}$
$a_3 = (o_3, o_4, o_1, o_2)$ **(8)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}\}$
$a_4 = (o_2, o_3, o_4, o_1)$ **(8)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{18}\}$
$a_5 = (o_2, o_1, o_3, o_4)$ **(13)**	$a_{14} = (o_3, o_1, o_2, o_4)$	$\{a_{23}, a_{18}\}$
$a_6 = (o_4, o_2, o_1, o_3)$ **(12)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{23}, a_{18}\}$
$a_7 = (o_3, o_4, o_2, o_1)$ **(7)**	$a_7 = (o_3, o_4, o_2, o_1)$	$\{a_7\}$
$a_8 = (o_1, o_3, o_4, o_2)$ **(8)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_8\}$
$a_9 = (o_1, o_3, o_2, o_4)$ **(10)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{18}\}$
$a_{10} = (o_4, o_1, o_3, o_2)$ **(9)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}\}$
$a_{11} = (o_2, o_4, o_1, o_3)$ **(12)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}, a_{18}\}$
$a_{12} = (o_3, o_2, o_4, o_1)$ **(9)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}\}$
$a_{13} = (o_1, o_2, o_4, o_3)$ **(9)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_{23}, a_{18}\}$
$a_{14} = (o_3, o_1, o_2, o_4)$ **(8)**	$a_{14} = (o_3, o_1, o_2, o_4)$	$\{a_{14}\}$
$a_{15} = (o_4, o_3, o_1, o_2)$ **(7)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{15}\}$
$a_{16} = (o_2, o_4, o_3, o_1)$ **(12)**	$a_7 = (o_3, o_4, o_2, o_1)$	$\{a_{23}, a_{18}\}$
$a_{17} = (o_3, o_2, o_1, o_4)$ **(12)**	$a_{14} = (o_3, o_1, o_2, o_4)$	$\{a_{23}\}$
$a_{18} = (o_4, o_3, o_2, o_1)$ **(6)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{18}\}$
$a_{19} = (o_1, o_4, o_3, o_2)$ **(12)**	$a_8 = (o_1, o_3, o_4, o_2)$	$\{a_{23}\}$
$a_{20} = (o_2, o_1, o_4, o_3)$ **(10)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}, a_{18}\}$
$a_{21} = (o_4, o_2, o_3, o_1)$ **(12)**	$a_{18} = (o_4, o_3, o_2, o_1)$	$\{a_{23}, a_{18}\}$
$a_{22} = (o_1, o_4, o_2, o_3)$ **(11)**	$a_7 = (o_3, o_4, o_2, o_1)$	$\{a_{18}\}$
$a_{23} = (o_3, o_1, o_4, o_2)$ **(6)**	$a_{23} = (o_3, o_1, o_4, o_2)$	$\{a_{23}\}$
$a_{24} = (o_2, o_3, o_1, o_4)$ **(11)**	$a_{15} = (o_4, o_3, o_1, o_2)$	$\{a_{18}\}$

By Table 2, it is clear that the stable assignments are

$$\mathcal{A}^S = \{a_7, a_8, a_{14}, a_{15}, a_{18}, a_{23}\}.$$

RSD for this assignment problem is given by

$$RSD = \begin{pmatrix} a_7 & a_8 & a_{14} & a_{15} & a_{18} & a_{23} \\ \frac{3}{24} & \frac{3}{24} & \frac{3}{24} & \frac{3}{24} & \frac{6}{24} & \frac{6}{24} \end{pmatrix}.$$

MRRE is given by

$$MRRE = \begin{pmatrix} a_7 & a_8 & a_{14} & a_{15} & a_{18} & a_{23} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{10}{24} & \frac{10}{24} \end{pmatrix}.$$

The expected rank of the two assignment lotteries can be easily computed. It holds that

$$\begin{aligned} \text{ExpRank } RSD &= \frac{3}{24}[\text{Rank } a_7 + \text{Rank } a_8 + \text{Rank } a_{14} + \text{Rank } a_{15}] + \frac{6}{24}[\text{Rank } a_{18} + \text{Rank } a_{23}] = \\ &= \frac{162}{24}. \\ \text{ExpRank } MRRE &= \frac{1}{24}[\text{Rank } a_7 + \text{Rank } a_8 + \text{Rank } a_{14} + \text{Rank } a_{15}] + \frac{10}{24}[\text{Rank } a_{18} + \text{Rank } a_{23}] = \\ &= \frac{150}{24}. \end{aligned}$$

Hence, for the assignment problem $\overline{\mathcal{P}}$, we have:

$$\text{ExpRank } RSD > \text{ExpRank } MRRE.$$

□

4. Conclusions

In this paper, we define a lottery for a traditional assignment problem $\mathcal{P}$ with strict preferences and the same number of agents and indivisible objects. Such lottery is based on the same construction as the Core from Random Endowments (CRE), which is an assignment lottery equivalent to RSD. In CRE, a deterministic assignment is randomly chosen with uniform distribution from the set $\mathcal{A}$ and associated with the unique core assignment $C(a)$ of the housing market problem $\mathcal{P}(a)$, where a is interpreted as the initial endowment. In the new assignment lottery, the assignment $C(a)$ is replaced with the set $F(a)$ formed by all the assignments in the upper contour set of a with minimum rank. The rationale under such replacement is that the deterministic assignment $C(a)$ is not, in general, the minimum rank assignment that can be reached from a.

We prove that the new assignment lottery, which we call Minimum Rank from Random Endowment (MRRE), preserves the same efficiency properties as RSD, that is, it is ex post efficient but not ordinally efficient. However, it performs better than RSD under the viewpoint of the expected rank. Indeed, the expected rank of MRRE cannot be greater than the expected rank of RSD. Since the rank of an object is closely related with a utility level (the lower the rank, the greater the utility), minimizing the expected rank of a lottery can be interpreted as maximizing the expected social welfare of the agents. Hence, the assignment lottery that we introduce can be interpreted as an alternative centralized solution that a decision maker may propose for an assignment problem with the aim of jointly maximizing some form of social welfare and implementing the solution from the profile of preferences over deterministic objects.

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