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New Harmonic Number Series

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Abstract: Based on a recent representation of the psi function due to Guillera and Sondow and independently Boyadzhiev, new closed forms for various series involving harmonic numbers and inverse factorials are derived. A high point of the presentation is the rediscovery, by much simpler means, of a famous quadratic Euler sum originally discovered in 1995 by Borwein and Borwein. In addition, the following series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)\binom{n+z}{n}}$, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)\binom{n+z}{n}}$, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)(n+3)\binom{n+z}{n}}$, as well as the harmonic and odd harmonic number series associated with them are evaluated.

Keywords: harmonic number; Riemann zeta function; binomial coefficient; Euler sum

MSC: 30B50; 33E20

1. Introduction

Our main purpose in this note is to discover the harmonic number series associated with the following identity:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{n+z}{n}}=\zeta(2)-H_z^{(2)},\quad z\in\mathbb{C}\setminus Z^-, \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function and $H_z^{(2)}$ is a second order harmonic number (both definitions are given below). At $z=0$ identity (1) subsumes the solution to the Basel problem and we will see that its derivative includes the well-known relation between the Apéry constant and a classical Euler sum, namely,

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\,\zeta(3), \tag{2}$$

as a special case, H_j being the jth harmonic number. The series (2) was discovered by Euler in 1775, has a long history, and has been rediscovered several times (see the historical notes in Berndt's classic [1] [Chapter 9, Entry 9]). Euler sums are prominent research objects and there is a whole branch in number theory dealing with their relations and evaluations. We refer to the articles [2–11] for further reading. Borwein and Borwein [5] applied Parseval's identity to a Fourier series and the contour integral to a generating function to establish the following interesting identity

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}=\frac{11}{17}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{11}{4}\,\zeta(4), \tag{3}$$

to which we will return later in the text.

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Harmonic numbers H_α and odd harmonic numbers O_α are defined for $0 \neq \alpha \in \mathbb{C} \setminus \mathbb{Z}^-$ by the recurrence relations

$$H_\alpha = H_{\alpha-1} + \frac{1}{\alpha} \qquad \text{and} \qquad O_\alpha = O_{\alpha-1} + \frac{1}{2\alpha-1},$$

with $H_0 = 0$ and $O_0 = 0$. Harmonic numbers are connected to the psi or digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ through the fundamental relation

$$H_\alpha = \psi(\alpha+1) + \gamma, \tag{4}$$

with γ being the Euler–Mascheroni constant and $\psi(z)$ given by

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+z} \right).$$

Generalized harmonic numbers $H_\alpha^{(m)}$ and odd harmonic numbers $O_\alpha^{(m)}$ of order $m \in \mathbb{C}$ are defined by

$$H_\alpha^{(m)} = H_{\alpha-1}^{(m)} + \frac{1}{\alpha^m} \qquad \text{and} \qquad O_\alpha^{(m)} = O_{\alpha-1}^{(m)} + \frac{1}{(2\alpha-1)^m},$$

with $H_0^{(m)} = 0$ and $O_0^{(m)} = 0$ so that $H_\alpha = H_\alpha^{(1)}$ and $O_\alpha = O_\alpha^{(1)}$. The recurrence relations imply that if $\alpha = n$ is a non-negative integer, then

$$H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \qquad \text{and} \qquad O_n^{(m)} = \sum_{j=1}^{n} \frac{1}{(2j-1)^m}.$$

Generalized harmonic numbers are linked to the polygamma functions $\psi^{(r)}(z)$ of order r defined by

$$\psi^{(r)}(z) = \frac{d^r}{dz^r}\psi(z) = (-1)^{r+1} r! \sum_{j=0}^{\infty} \frac{1}{(j+z)^{r+1}},$$

through

$$H_z^{(r)} = \zeta(r) + \frac{(-1)^{r-1}}{(r-1)!}\psi^{(r-1)}(z+1),$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad s \in \mathbb{C}, \Re(s) > 1.$$

Other representations of $\psi^{(r)}(z)$ and summation identities can be found in the articles by Coffey [12,13], Alzer et al. [14], and Sofo and Choi [15]. At rational arguments $z = p/q$ (generalized) harmonic numbers $H_{p/q}^{(r)}$ may be evaluated using the corresponding digamma or polygamma values. As was shown by Kölbig [16], or Choi and Cvijović [17], the polygamma function $\psi^{(r-1)}(p/q+1)$ can be explicitly evaluated terms of polylogarithms or other special functions. For instance, we have

$$H_{-3/4} = -\frac{\pi}{2} - 3\ln 2 \qquad \text{and} \qquad H_{-1/6}^{(3)} = -2\sqrt{3}\pi^3 - 90\zeta(3).$$

Some other values relevant for this study are given in Lemma 1.

The motivation for writing this paper comes form the authors' recent encounters with two representations for the digamma function $\psi(z)$. The first such encounter was the rediscovery of an old and seemingly forgotten representation of Nörlund and resulted in

the paper [18]. The second encounter deals with (1). Indeed, identity (1), stated without proof in Sofo and Srivastava [19] [Equation (2.13)], is a consequence of the following representation of the psi function:

$$\psi(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \ln(z+k). \tag{5}$$

This representation holds for all $z \in \mathbb{C}$ with $\Re(z) > 0$. It was derived by Guillera and Sondow [20] [Theorem 5.1] and was rediscovered by Boyadzhiev [21] [Identity (5.18)].

We begin by offering a short proof of identity (1) using (5). As consequences we will derive closed forms for various series involving inverse factorials and (odd) harmonic numbers. Of particular importance will be Theorem 4 from which we will be able to deduce a simple proof of (3). We will also evaluate the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+z}{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\binom{n+z}{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)\binom{n+z}{n}},$$

and derive the harmonic and odd harmonic number series associated with them.

2. Proof of Identity (1)

For completeness and a better readability we first prove identity (1).

Theorem 1. *For all $z \in \mathbb{C} \setminus Z^-$ the following identity holds:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{n+z}{n}} = \zeta(2) - H_z^{(2)}.$$

Proof. Differentiating the representation (5) gives

$$\begin{aligned}
\psi^{(1)}(z) = \zeta(2) - H_{z-1}^{(2)} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{z+k} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} k \binom{n}{k} \frac{(-1)^{k-1}}{z+k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{\binom{n+z-1}{n}};
\end{aligned}$$

and hence (1). Note that we used

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}k}{z+k} = \frac{1}{\binom{n+z}{n}},$$

which is a particular case of the Frisch identity ([22,23]). □

3. Required Identities

We will make frequent use of the following basic identity [24]:

$$\sum_{k=1}^{m} \frac{1}{\binom{k+n}{k}} = \frac{1}{n-1} - \frac{n}{n-1} \frac{1}{\binom{m+n}{m+1}}, \quad 0, 1 \neq n \in \mathbb{C}. \tag{6}$$

and the identities stated in the following lemmata.

Lemma 1. *We have*

$$H_{k-1/2} = 2\,O_k - 2\,\ln 2, \tag{7}$$

$$H^{(2)}_{k-1/2} = -2\,\zeta(2) + 4\,O^{(2)}_k, \tag{8}$$

$$H^{(3)}_{-1/2} = -6\,\zeta(3), \tag{9}$$

$$H^{(4)}_{-1/2} = -14\,\zeta(4), \tag{10}$$

$$H^{(m+1)}_{k-1/2} - H^{(m+1)}_{-1/2} = 2^{m+1} O^{(m+1)}_k. \tag{11}$$

Lemma 2. *For integers u and v, we have*

$$\binom{u-1/2}{v} = \binom{2u}{u}\binom{u}{v} 2^{-2v} \binom{2(u-v)}{u-v}^{-1}, \tag{12}$$

$$\binom{u}{1/2} = \frac{2^{2u+1}}{\pi}\binom{2u}{u}^{-1}, \tag{13}$$

$$\binom{u}{1/2-v} = \frac{(-1)^{v-1}}{v}\binom{u+v}{v}^{-1}\binom{2(u+v)}{u+v}^{-1}\binom{2(v-1)}{v-1} 2^{2u+2}, \tag{14}$$

$$\binom{u+1/2}{v} = \binom{u}{v}^{-1}\binom{2u+1}{2v} 2^{-2v}\binom{2v}{v}, \tag{15}$$

$$\binom{u+1/2}{v} = \frac{(-1)^{v-u-1}}{2^{2v-1}}\,\frac{2u+1}{v-u}\binom{v}{u}^{-1}\binom{2u}{u}\binom{2(v-u-1)}{v-u-1}, \quad v > u, \tag{16}$$

$$\binom{-3/2}{u} = (-1)^u\,(2u+1)\,2^{-2u}\binom{2u}{u}. \tag{17}$$

Proof. These identities are readily derived using the following well-known Gamma function identities:

$$\Gamma\left(u+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2u}}\binom{2u}{u}\Gamma(u+1), \quad \Gamma\left(-u+\frac{1}{2}\right) = (-1)^u 2^{2u}\binom{2u}{u}^{-1}\frac{\sqrt{\pi}}{\Gamma(u+1)},$$

together with the definition of the generalized binomial coefficients:

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}.$$

□

4. Results

In this section, we state new closed forms for series involving inverse factorials. More results of this nature have been produced, among others, by Sofo [25,26], Xu and his collaborators [27–29], Boyadzhiev [30,31] and by the authors [18].

Theorem 2. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2(n+m)}{n+m}\binom{n+m}{m}} = \frac{1}{\binom{2m}{m}}\left(3\,\zeta(2) - 4\,O^{(2)}_m\right). \tag{18}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} = \frac{\pi^2}{2}. \tag{19}$$

Proof. Write $m - 1/2$ for z in (1) to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{n+m-1/2}{n}} = \zeta(2) - H^{(2)}_{m-1/2}, \tag{20}$$

from which (18) follows on account of Lemmas 1 and 2. □

Theorem 3. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{H_{n+z}}{n^2 \binom{n+z}{n}} = H_z\Big(\zeta(2) - H^{(2)}_z\Big) + 2\Big(\zeta(3) - H^{(3)}_z\Big). \tag{21}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\,\zeta(3), \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)} = 2\,\zeta(3) - \zeta(2), \tag{23}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)(n+2)} = \zeta(3) - \frac{3}{4}\,\zeta(2) + \frac{1}{4}. \tag{24}$$

Proof. Differentiate (1) with respect to z to obtain

$$\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n^2 \binom{n+z}{n}} = 2\Big(\zeta(3) - H^{(3)}_z\Big), \tag{25}$$

and use (1) again to rewrite the second term on the left hand side of (25). Identity (23) corresponds to an evaluation of (21) at $z = 1$ followed by the use of

$$H_{n+1} = H_n + \frac{1}{n+1}, \tag{26}$$

and the decomposition

$$\frac{1}{n^2(n+1)^2} = \frac{1}{n^2} + \frac{1}{(n+1)^2} - 2\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

□

Corollary 1. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n}\, O_{n+m}}{n^2 \binom{2(n+m)}{n+m} \binom{n+m}{m}} = \frac{1}{\binom{2m}{m}}\Big(O_m\Big(3\,\zeta(2) - 4\,O^{(2)}_m\Big) + \Big(7\,\zeta(3) - 8\,O^{(3)}_m\Big)\Big). \tag{27}$$

Proof. Write $m - 1/2$ for z in (25) and use Lemmas 1 and 2. □

Theorem 4. *If $z \in \mathbb{C} \setminus Z^-$, then*

$$\sum_{n=1}^{\infty} \frac{(H_{n+z} - H_z)^2 + H^{(2)}_{n+z} - H^{(2)}_z}{n^2 \binom{n+z}{z}} = 6\,\zeta(4) - 6\,H^{(4)}_z. \tag{28}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^2} = 6\,\zeta(4), \tag{29}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{2^{2n}}{\binom{2n}{n}} \left(O_n^2 + O_n^{(2)}\right) = \frac{\pi^2}{4}. \tag{30}$$

Proof. Differentiate (25) with respect to z. □

Remark 1. *The symmetry relation*

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} + \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p)\zeta(q) + \zeta(p+q)$$

makes it easy to calculate

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}\left(\zeta(p)^2 + \zeta(2p)\right),$$

so that, in particular,

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{7}{4}\,\zeta(4);$$

and using this in (29) *therefore gives*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\,\zeta(4). \tag{31}$$

Thus (29) *allows a much easier determination of the sum* (31) *which was originally evaluated by Borwein and Borwein [5] through Fourier series and contour integration.*

Lemma 3. *If* $z \in \mathbb{C} \setminus \mathbb{Z}^-$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+z}{n}} = z\left(H_z^{(2)} - \zeta(2)\right) + 1. \tag{32}$$

Proof. Sum (1) over z from 1 to r, using (6) and the fact that [32] [Equation (3.1)]

$$\sum_{z=1}^{r} H_z^{(2)} = (r+1)H_r^{(2)} - H_r,$$

to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\binom{n+r}{r}} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} - 1 - (r-1)\zeta(2) + rH_r^{(2)}.$$

But

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) + 1 - \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 2 - \zeta(2),
\end{aligned}$$

which when substituted into the previous sum gives (32) after replacing r with z. □

Remark 2. *Identity* (32) *was also derived by Sofo and Srivastava [19] [Equation (2.17)].*

Theorem 5. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)\binom{2(n+m)}{n+m}\binom{n+m}{m}} = \frac{1}{\binom{2m}{m}}\left(\left(m-\frac{1}{2}\right)\left(4O_m^{(2)} - 3\zeta(2)\right)+1\right). \tag{33}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)\binom{2n}{n}} = \frac{\pi^2}{4}+1, \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)^2\binom{2(n+1)}{n+1}} = \frac{3}{2}-\frac{\pi^2}{8}, \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)^2(n+2)\binom{2(n+2)}{n+2}} = \frac{23}{36}-\frac{\pi^2}{16}. \tag{36}$$

Theorem 6. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{H_{n+z}}{n(n+1)\binom{n+z}{n}} = \left(\zeta(2) - H_z^{(2)}\right)(1-zH_z) + H_z - 2z\left(\zeta(3) - H_z^{(3)}\right). \tag{37}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = \frac{\pi^2}{6}, \tag{38}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)^2} = \zeta(2) - \zeta(3). \tag{39}$$

Proof. Differentiate (32) with respect to z to obtain

$$\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n(n+1)\binom{n+z}{n}} = \zeta(2) - H_z^{(2)} - 2z\left(\zeta(3) - H_z^{(3)}\right), \tag{40}$$

and hence, (37) upon using (32) again to rewrite the second sum on the left hand side of (40). Identity (39) is an evaluation of (37) at $z = 1$, where we used (26) and the fact that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)^3} &= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3}\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^3}.\end{aligned}$$

□

Remark 3. *Identity* (38) *was also recorded by Chu in [33].*

Remark 4. *Subtraction of* (39) *from* (23) *gives the Euler sum*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2(n+1)^2} = 3\zeta(3) - 2\zeta(2). \tag{41}$$

Corollary 2. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+m}}{n(n+1)\binom{2(n+m)}{n+m}\binom{n+m}{m}} = \frac{1}{2\binom{2m}{m}}\left(3\zeta(2) - 4O_m^{(2)}\right)(1 - O_m(2m-1)) + \frac{O_m}{\binom{2m}{m}} - \frac{(2m-1)}{2\binom{2m}{m}}\left(7\zeta(3) - 8O_m^{(3)}\right). \tag{42}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_n}{n(n+1)\binom{2n}{n}} = \frac{\pi^2}{4} + \frac{7}{2}\zeta(3), \tag{43}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+1}}{n(n+1)^2\binom{2(n+1)}{n+1}} = \frac{5}{2} - \frac{7}{4}\zeta(3). \tag{44}$$

Proof. Write $m - 1/2$ for z in (40) and use Lemmas 1 and 2. □

Theorem 7. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}}{n(n+1)\binom{n+z}{z}} = 4\left(\zeta(3) - H_z^{(3)}\right) - 6z\left(\zeta(4) - H_z^{(4)}\right). \tag{45}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n(n+1)} = 4\,\zeta(3), \tag{46}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}\left(O_n^2 + O_n^{(2)}\right)}{n(n+1)\binom{2n}{n}} = \frac{\pi^4}{8} + 7\,\zeta(3). \tag{47}$$

Proof. Differentiate (40) with respect to z. □

Remark 5. *Since Xu showed that [27] [Equation (2.30)]*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n(n+1)} = \zeta(3),$$

identity (46) yields

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+1)} = 3\,\zeta(3); \tag{48}$$

an identity that was also reported by Nimbran and Sofo in [11] [Identity (2.24)].

Lemma 4. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\binom{n+z}{n}} = \frac{1}{2}z(z+1)\left(H_z^{(2)} - \zeta(2)\right) + \frac{z}{2} + \frac{1}{4}. \tag{49}$$

Proof. We first derive the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)\binom{n+z}{n}} = \frac{1}{2}z(z-1)\left(\zeta(2) - H_z^{(2)}\right) - \frac{z}{2} + \frac{3}{4}, \tag{50}$$

by summing both sides of (32) over z from 1 to r and replacing r with z in the final identity. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4},$$

since

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+1} + \frac{1}{n+2}\right).$$

Note also that [32] [Equation (3.2)]:

$$2\sum_{z=1}^{r} zH_z^{(2)} = r(r+1)H_r^{(2)} + H_r - r. \tag{51}$$

Identity (49) is obtained by subtracting (50) from (32). □

Theorem 8. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)(n+2)\binom{2(n+m)}{n+m}\binom{n+m}{m}} \\ = \frac{1}{2\binom{2m}{m}}\left(\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(4O_m^{(2)} - 3\zeta(2)\right) + m\right).\end{aligned} \tag{52}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)(n+2)\binom{2n}{n}} = \frac{\pi^2}{16}, \tag{53}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)^2(n+2)\binom{2(n+1)}{n+1}} = 1 - \frac{3\pi^2}{32}, \tag{54}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{n(n+1)^2(n+2)^2\binom{2(n+2)}{n+2}} = \frac{14}{9} - \frac{5\pi^2}{32}. \tag{55}$$

Proof. Set $z = m - 1/2$ in (49). □

Theorem 9. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n+z}}{n(n+1)(n+2)\binom{n+z}{n}} = \left(z + \frac{1}{2} - \frac{1}{2}z(z+1)H_z\right)\left(\zeta(2) - H_z^{(2)}\right) \\ + \left(\frac{z}{2} + \frac{1}{4}\right)H_z - z(z+1)\left(\zeta(3) - H_z^{(3)}\right) - \frac{1}{2}.\end{aligned} \tag{56}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)(n+2)} = \frac{\pi^2}{12} - \frac{1}{2}, \tag{57}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)^2(n+2)} = \frac{\pi^2}{12} + \frac{1}{2} - \zeta(3). \tag{58}$$

Proof. Differentiate (49) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n(n+1)(n+2)\binom{n+z}{n}} = \left(z + \frac{1}{2}\right)\left(\zeta(2) - H_z^{(2)}\right) \\ - z(z+1)\left(\zeta(3) - H_z^{(3)}\right) - \frac{1}{2}.\end{aligned} \tag{59}$$

Identity (58) is an evaluation of (56) at $z = 1$, where we also used (26) and the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^3(n+2)} = \frac{5}{4} - \zeta(3),$$

since

$$\frac{1}{n(n+1)^3(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \frac{1}{(n+1)^3}.$$

□

Theorem 10. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty} &\frac{2^{2n} O_{n+m}}{n(n+1)(n+2)\binom{2(n+m)}{n+m}\binom{n+m}{m}} \\ &= \frac{1}{2\binom{2m}{m}}\left(m - O_m\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\right)\left(3\zeta(2) - 4O_m^{(2)}\right) \\ &\quad - \frac{1}{2\binom{2m}{m}}\left(\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(7\zeta(3) - 8O_m^{(3)}\right) - mO_m + \frac{1}{2}\right).\end{aligned} \tag{60}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_n}{n(n+1)(n+2)\binom{2n}{n}} = \frac{7}{8}\zeta(3) - \frac{1}{4}, \tag{61}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} O_{n+1}}{n(n+1)^2(n+2)\binom{2(n+1)}{n+1}} = \frac{\pi^2}{32} - \frac{21}{16}\zeta(3) + \frac{11}{8}. \tag{62}$$

Proof. Set $z = m - 1/2$ in (59) and use Lemmas 1 and 2. □

Theorem 11. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}}{n(n+1)(n+2)\binom{n+z}{z}} &= H_z^{(2)} - \zeta(2) - 2(2z+1)\left(H_z^{(3)} - \zeta(3)\right) \\ &\quad + 3z(z+1)\left(H_z^{(4)} - \zeta(4)\right).\end{aligned} \tag{63}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n(n+1)(n+2)} = 2\zeta(3) - \zeta(2), \tag{64}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}\left(O_n^2 + O_n^{(2)}\right)}{n(n+1)(n+2)\binom{2n}{n}} = \frac{\pi^4}{32} - \frac{\pi^2}{8}. \tag{65}$$

Proof. Differentiate (59) with respect to z. □

Lemma 5. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)\binom{n+z}{n}} &= \frac{1}{12} z(z+1)(z+2)\left(H_z^{(2)} - \zeta(2)\right) \\ &\quad + \frac{z^2}{12} + \frac{5z}{24} + \frac{1}{18}.\end{aligned} \tag{66}$$

Proof. Sum (50) over z using (6), (51) and [32] [Equation (3.6)]

$$\sum_{z=1}^{r} z^2 H_z^{(2)} = \frac{r(r+1)(2r+1)}{6} H_r^{(2)} - \frac{1}{6} H_r + \frac{r}{3} - \frac{r^2}{6},$$

to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)\binom{n+z}{n}} = \frac{1}{6} z(z-1)(z-2)\left(H_z^{(2)} - \zeta(2)\right) + \frac{1}{6} z^2 - \frac{7}{12} z + \frac{11}{18}. \tag{67}$$

Identity (66) is obtained by adding (67) and (49) and subtracting (50). □

Theorem 12. *If m is a non-negative integer, then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)(n+2)(n+3)\binom{2(n+m)}{n+m}\binom{n+m}{m}} \\ &\quad = \frac{1}{12\binom{2m}{m}} \left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right)\left(m + \frac{3}{2}\right)\left(4O_m^{(2)} - 3\zeta(2)\right) \\ &\qquad + \frac{1}{4\binom{2m}{m}} \left(\frac{m^2}{3} + \frac{m}{2} - \frac{1}{9}\right). \end{aligned} \tag{68}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)(n+2)(n+3)\binom{2n}{n}} = \frac{\pi^2}{64} - \frac{1}{36}, \tag{69}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n(n+1)^2(n+2)(n+3)\binom{2(n+1)}{n+1}} = \frac{29}{72} - \frac{5\pi^2}{128}, \tag{70}$$

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{n(n+1)^2(n+2)^2(n+3)\binom{2(n+2)}{n+2}} = \frac{65}{72} - \frac{35\pi^2}{384}. \tag{71}$$

Proof. Set $z = m - 1/2$ in (66). □

Theorem 13. *If $z \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{H_{n+z}}{n(n+1)(n+2)(n+3)\binom{n+z}{n}} \\ &\quad = \left(\frac{1}{12} z(z+1)(z+2) H_z - \frac{z(z+2)}{4} - \frac{1}{6}\right)\left(H_z^{(2)} - \zeta(2)\right) \\ &\qquad + \frac{1}{6} z(z+1)(z+2)\left(H_z^{(3)} - \zeta(3)\right) + \left(\frac{z^2}{12} + \frac{5z}{24} + \frac{1}{18}\right) H_z - \frac{z}{6} - \frac{5}{24}. \end{aligned} \tag{72}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)(n+2)(n+3)} = \frac{\pi^2}{36} - \frac{5}{24}. \tag{73}$$

Proof. Differentiate (66) to obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_{n+z} - H_z}{n(n+1)(n+2)(n+3)\binom{n+z}{n}} &= -\left(\frac{z(z+2)}{4} + \frac{1}{6}\right)\left(H_z^{(2)} - \zeta(2)\right) \\ &\quad + \frac{1}{6} z(z+1)(z+2)\left(H_z^{(3)} - \zeta(3)\right) - \frac{z}{6} - \frac{5}{24}, \end{aligned} \tag{74}$$

and hence, (72) by a repeated use of (66). □

Theorem 14. *If m is a non-negative integer, then*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^{2n}O_{n+m}}{n(n+1)(n+2)(n+3)\binom{2(n+m)}{n+m}\binom{n+m}{m}}&=\frac{1}{2\binom{2m}{m}}\left(\frac{1}{6}\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)O_m-\frac{m^2}{4}-\frac{m}{4}+\frac{1}{48}\right)\left(4O_m^{(2)}-3\zeta(2)\right)\\&+\frac{1}{12\binom{2m}{m}}\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\left(8O_m^{(3)}-7\zeta(3)\right)\\&+\frac{1}{2\binom{2m}{m}}\left(\frac{1}{36}\left(6m^2+9m-2\right)O_m-\frac{m}{6}-\frac{1}{8}\right).\end{aligned}\tag{75}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{2^{2n}O_n}{n(n+1)(n+2)(n+3)\binom{2n}{n}}=\frac{7}{32}\zeta(3)-\frac{\pi^2}{192}-\frac{1}{16},\tag{76}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}O_{n+1}}{n(n+1)^2(n+2)(n+3)\binom{2(n+1)}{n+1}}=\frac{137}{288}+\frac{\pi^2}{48}-\frac{35}{64}\zeta(3).\tag{77}$$

Proof. Write $m-1/2$ for z in (74) and use Lemmas 1 and 2. □

Theorem 15. *If $z\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(H_{n+z}-H_z)^2+H_{n+z}^{(2)}-H_z^{(2)}}{n(n+1)(n+2)(n+3)\binom{n+z}{z}}&=\frac{1}{2}(z+1)\left(H_z^{(2)}-\zeta(2)\right)\\&-\left(z(z+2)+\frac{2}{3}\right)\left(H_z^{(3)}-\zeta(3)\right)\\&+\frac{1}{2}z(z+1)(z+2)\left(H_z^{(4)}-\zeta(4)\right)+\frac{1}{6}.\end{aligned}\tag{78}$$

In particular,

$$\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n(n+1)(n+2)(n+3)}=\frac{2}{3}\zeta(3)-\frac{\pi^2}{12}+\frac{1}{6},\tag{79}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}\left(O_n^2+O_n^{(2)}\right)}{n(n+1)(n+2)(n+3)\binom{2n}{n}}=\frac{\pi^4}{128}-\frac{\pi^2}{32}-\frac{7}{48}\zeta(3)+\frac{1}{24}.\tag{80}$$

Proof. Differentiate (74) with respect to z. □

5. Conclusions

This paper is the second part of our work on series with inverse factorials based on series representations for the psi function. We have derived closed forms for various such series and for harmonic number series associated with them. One particular result has proved to be very useful to offer an alternative (and simpler) proof of a famous quadratic Euler sum due to Borwein and Borwein. In our future research, we hope to extend the results to other classes of harmonic number series.

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