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Two Positive Solutions for Elliptic Differential Inclusions

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Abstract: The existence of two positive solutions for an elliptic differential inclusion is established, assuming that the nonlinear term is an upper semicontinuous set-valued mapping with compact convex values having subcritical growth. Our approach is based on variational methods for locally Lipschitz functionals. As a consequence, a multiplicity result for elliptic Dirichlet problems having discontinuous nonlinearities is pointed out.

Keywords: differential inclusion; second-order elliptic equation; critical point theory

MSC: 35R70; 35J60; 58E05

1. Introduction

This paper deals with the existence of two positive solutions for the following elliptic differential inclusion:

$$\begin{cases} -\Delta u \in G(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 3$, having a smooth boundary $\partial\Omega$, Δ is the classical Laplace operator and $G : \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued mapping with convex compact values. Usually, problems of type (1) are investigated assuming G is either lower semicontinuous or upper semicontinuous. Indeed, in the first case, Michael's selection theorem allows us to study differential inclusion through the equation with selection as its nonlinear term, while in the second case, fixed-point theorems for set-valued maps are applied (see, for instance [1] (Introduction p. 93)). We cite here the seminal work of S.A. Marano, where, by applying the Ky Fan fixed-point theorem and assuming G is upper semicontinuous, the existence of at least one solution for (1) was obtained (see [2] (Theorem 2.1)). As a consequence of this result, Marano obtained the existence of at least one solution for elliptic differential equations with discontinuous nonlinearities, where it is worth noticing that the set of discontinuity points may also be uncountable (see also [3]). Elliptic differential Dirichlet problems have been widely investigated (see, for instance, the papers of Stuart and Toland, such as [4]). We also cite the papers of Mi ([5,6]) for the research of positive solutions for a singular Dirichlet p-Laplacian elliptic problem and singular elliptic systems, respectively, through the lower and upper solutions method. Additionally, we can refer to [7,8] for the study of positive solutions for fractional differential systems. In particular, we emphasize the fundamental contribution of the seminal work of Chang [9] and the powerful paper of Ambrosetti and Badiale [10], where such arguments have been studied by variational methods. We also mention the very interesting paper of Marano and Motreanu [11], and we refer to Chapter 6.2, p. 321 in [12] for a complete overview of this field.

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The aim of this paper is to present an existence and multiplicity result for differential inclusion (1). Specifically, we establish the existence of two positive solutions, assuming G is an upper semicontinuous map with compact convex values and has subcritical growth. We observe that in [2], in order to obtain a non-zero solution, it must be assumed in addition that $0 \notin G(0)$, which we do not require; in [11], multiple solutions are obtained with a sublinear growth, while here, the growth may also be superlinear; in [10,13], the set of discontinuity points is assumed to be countable, while here, it may also be uncountable. To give an idea of this, we now report a consequence of our main result.

Theorem 1. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be an upper semicontinuous set-valued mapping with compact convex values. Assume that*

(i) *there are $s \in [1,2[$ and $q \in]2, \frac{2N}{N-2}[$ and two positive constants a_s and a_q, such that*

$$0 \leq \min G(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \quad \forall t \in \mathbb{R}.$$

(ii) *there exist two constants, $m > 2$ and $l > 0$, such that*

$$0 < m \int_0^t \min G(\xi)d\xi \leq t \min G(t) \quad \text{for all } t \geq l.$$

(iii)

$$\limsup_{t \to 0^+} \frac{\int_0^t \min G(\xi)d\xi}{t^2} = +\infty.$$

Then, there exists $\hat{R} > 0$ such that for each $R < \hat{R}$, denoting by B_R the open ball of center 0 and radius R, problem

$$\begin{cases} -\Delta u \in G(u) & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

admits at least two positive solutions.

The meaning of a solution is that of a generalized weak solution defined in Section 2 which returns the classical one, that is $u \in W^{2,2}(B_R) \cap W_0^{1,2}(\Omega)$ such that $-\Delta u(x) \in G(u(x))$ for a.a. $x \in B_R$.

The main result of this paper is Theorem 3, which establishes, under essentially the same assumptions of G as previously mentioned, the existence of a precise interval for the positive real parameter λ such that the problem

$$\begin{cases} -\Delta u \in \lambda G(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

admits at least two positive solutions. Moreover, a consequence is pointed out (see Theorem 4), and an application to Dirichlet problems with equations having discontinuous nonlinearities is established. Finally, two examples of applications are given (see Examples 1 and 2).

The reader can refer to the seminal books [14,15] for an overview of the non-smooth analysis theory and to [1,12] for basic notions of set-valued analysis. Finally, as references for boundary value problems with multivalued right-hand sides, we mention the papers [16–19].

The paper is organized as follows. Section 2 is devoted to some preliminaries and auxiliary results of set-valued and non-smooth analysis, which are useful to define the variational framework for problem (2). Additionally, we recall an abstract critical point theorem (see [20] ([Theorem 2.10])), that is Theorem 2, which is the main tool in our investigation to obtain our main result, presented in Section 3, together with its consequences and applications.

2. Basic Notions of Set-Valued and Non-Smooth Analysis

We recall here some notions from set-valued analysis, while for a general overview, we refer to the seminal book [1]. Let X, Y be two topological spaces. Given a set-valued mapping $G : X \to 2^Y$, we say that G is upper semicontinuous if, for any open set $A \subseteq Y$, the set $G^+(A) = \{x \in X : G(x) \subseteq A\}$ is open in X. We point out the following lemma (see, for instance, [17] (Lemma 2.1)).

Lemma 1. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping with compact convex values. The following propositions are equivalent:*

(a) *G is upper semicontinuous;*
(b) *the single-valued mappings $\min G, \max G : \mathbb{R} \to \mathbb{R}$ are l.s.c. and u.s.c., respectively.*

Our approach is variational and it is based on the non-smooth analysis theory as developed by Clarke (see [14]). Let X be a Banach space with norm $\|\cdot\|_X$ and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair $(X^*; X)$. We recall that a functional $\Psi : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exists a neighborhood U of x, $U \subset X$and $L_U > 0$ such that

$$|\Psi(z) - \Psi(y)| \leq L_U \|z - y\|_X \quad \text{for all } z, y \in U .$$

Moreover, if $\Psi : X \to \mathbb{R}$ is a locally Lipschitz functional, the *generalized directional derivative in the sense of Clarke* of Ψ at the point x in the direction $y \in X$ is defined by

$$\Psi^\circ(x; y) = \limsup_{z \to x,\, t \to 0^+} \frac{\Psi(z + ty) - \Psi(z)}{t} .$$

The *(Clarke) generalized gradient* of Ψ at $x \in X$ is defined by

$$\partial\Psi(x) = \{z \in X^* : \langle z, y \rangle \leq \Psi^\circ(x; y) \ \forall y \in X\} .$$

The next proposition lists some properties of the generalized directional derivative and generalized gradient that are useful in the subsequent sections.

Proposition 1 (See [14]). *Let $\Psi, \Phi : X \to \mathbb{R}$ be two locally Lipschitz functions. Then, for every $x, y \in X$, the following conditions hold:*

(1) $\Psi^\circ(x; cy) = c\Psi^\circ(x; y)$ *with* $c > 0$;
(2) $(\Psi + \Phi)^\circ(x; y) \leq \Psi^\circ(x; y) + \Phi^\circ(x; y)$ *or, equivalently,* $\partial(\Psi + \Phi)(x) \subseteq \partial\Psi(u) + \partial\Phi(u)$;
(3) $(-\Psi)^\circ(x; y) = \Psi^\circ(x; -y)$;
(4) *the function $(x, y) \longmapsto \Psi^\circ(x; y)$ is upper semicontinuous;*
(5) *if X is finite-dimensional, $\partial\Psi$ is upper semicontinuous in X;*
(6) $\Psi^\circ(x; y) = \max\{\langle \xi, y \rangle, \ \xi \in \partial\Psi(x)\}$.

Clearly, the generalized directional derivative extends the classical directional derivative. To be precise, one has

Lemma 2 ([14]). *Let $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional. Then, Φ is locally Lipschitz and*

$$\langle \Phi'(x), y \rangle = \Phi^\circ(x; y) \quad \text{for all} \quad x, y \in X .$$

Finally, we say that $x \in X$ is a critical point of Ψ when $0 \in \partial\Psi(x)$, namely $\Psi^\circ(x; y) \geq 0$ for all $y \in X$ (see [9]).
For a full treatment of these topics, we refer to [15].

Now, consider the problem

$$\begin{cases} -\Delta u \in \lambda G(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, is a bounded open set, with a smooth boundary $\partial\Omega$, λ is a positive real parameter and $G : \mathbb{R} \to 2^{\mathbb{R}}$ is an upper semicontinuous set-valued mapping with compact convex values. We say that a function $u \in W_0^{1,2}(\Omega)$ is a generalized weak solution of (2) if there is a function $w \in L^2(\Omega)$ such that

(i) $w(x) \in G(u(x))$ *a.e.* in Ω
(ii) u is a weak solution of the problem

$$\begin{cases} -\Delta u = \lambda w & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, we have

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega w v dx \quad \forall v \in W_0^{1,2}(\Omega).$$

Clearly, when G is a single-valued mapping, the previous definition coincides with the usual definition of a weak solution. We also observe that, in a general case, since $\partial\Omega$ is smooth enough (see, for instance, [21] (Theorem 9.25), [22] (Theorem 9.15)), we have $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $-\Delta u(x) \in \lambda G(u(x))$ for a.a. $x \in \Omega$.
We assume that

(H) *there are* $a > 0$ *and* $0 \leq s \leq \frac{N+2}{N-2}$ *such that*

$$0 \leq f_G(t) \leq a(1 + |t|^s) \quad \forall t \in \mathbb{R},$$

where $f_G(t) = \min G(t)$ for all $t \in \mathbb{R}$. In particular, we observe that from Lemma 1, $f_G : \mathbb{R} \to \mathbb{R}$ is a measurable function. Moreover, put $F_G : \mathbb{R} \to \mathbb{R}$, the function defined by $F_G(\xi) = \int_0^\xi f_G(t)\, dt \ \ \forall\ \xi \in \mathbb{R}$. Clearly, F_G is a locally Lipschitz function. Moreover, we define $f_G^-(t) = \lim\limits_{\delta \to 0^+} \operatorname{ess\,inf}\limits_{|h|<\delta} f_G(t+h)$ and $f_G^+(t) = \lim\limits_{\delta \to 0^+} \operatorname{ess\,sup}\limits_{|h|<\delta} f_G(t+h)$.

Remark 1. *Thanks to Lemma 1,* $\min G$ *and* $\max G$ *are measurable selections and any convex combination of these functions is still a measurable selection. However, we choose as a particular selection the minimum of G since its growth behaviour does not impose restrictions on the growth of the set-valued mapping G, which can also have supercritical growth. Therefore, our choice is motivated by the aim of considering a more general case that is possible.*

Now, we introduce the variational setting for our problem, which involves Dirichlet boundary conditions, but we observe that it can be adapted to other boundary conditions, such as Neumann, mixed, or periodic conditions.

We take as Banach space X the Sobolev space $W_0^{1,2}(\Omega)$, endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \quad \forall u \in X. \tag{3}$$

Next, we define the functionals $\Phi, \Psi, I_\lambda : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2, \tag{4}$$

$$\Psi(u) = \int_\Omega F_G(u(x)) dx, \tag{5}$$

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{6}$$

for all $u \in X$. Notice that $\Phi \in C^1(X, \mathbb{R})$ and

$$\Phi'(u)(v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \text{for all } u, v \in X.$$

Moreover, standard computations show that the functional Ψ is well-defined and sequentially weakly continuous on X. Further, it is a locally Lipschitz functional on X (see [9]).

We point out the following proposition, which is based on a result of Chang (see [9] (Corollary p. 111)).

Proposition 2. *If $u \in X$ is a critical point of functional I_λ defined by (6), then u is a generalized weak solution for problem (2).*

Proof. Let $u \in X$ be a critical point of I_λ. We have $0 \in \partial I_\lambda(u)$, which means $I_\lambda^\circ(u)(v) \geq 0 \; \forall v \in X$. Therefore, we have

$$0 \leq (\Phi - \lambda\Psi)^\circ(u; v) \leq \Phi'(u; v) + \lambda(-\Psi)^\circ(u; v) = \Phi'(u; v) + \lambda(\Psi)^\circ(u; -v) \text{ for all } v \in X.$$

Hence, $\Phi'(u; -v) \leq \lambda(\Psi)^\circ(u; -v)$ for all $v \in X$, that is $\Phi'(u; s) \leq \lambda(\Psi)^\circ(u; s)$ for all $s \in X$.
Therefore, we have

$$\Phi'(u) \in \lambda\partial\Psi(u). \tag{7}$$

Now, ref. [9] (Corollary p. 111) ensures that Ψ is locally Lipschitz on X and we have

$$\partial\Psi(u) \subseteq [f_G^-(u(x)), f_G^+(u(x))] \quad a.e.\ in\ \Omega,$$

where the latest must be read as follows: for all $L \in \partial\Psi(u) \subseteq X^*$, there is $w \in L^2(\Omega)$ such that $L(v) = \lambda \int_\Omega wv dx$ for all $v \in X$ and $w(x) \in [f_G^-(u(x)), f_G^+(u(x))]$ a.e. in Ω. Therefore, from (7), there is $w \in L^2(\Omega)$ such that $\Phi'(u)(v) = \int_\Omega wv dx$ for all $v \in X$ and $w(x) \in [f_G^-(u(x)), f_G^+(u(x))]$ a.e. in Ω.
In other words, $u \in X$ is a weak solution of

$$\begin{cases} -\Delta u = \lambda w & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, in order to obtain the conclusion, we claim

$$[f_G^-(u(x)), f_G^+(u(x))] \subseteq G(u(x)) \qquad a.e.\ in\ \Omega.$$

Indeed, for all $\delta > 0$, we have $\operatorname{ess\,inf}_{|h|<\delta} f_G(t+h) \geq \inf_{|h|<\delta} \min G(t+h)$ for which, taking into account that $\min G$ is u.s.c., we have $f_G^-(t) = \lim_{\delta \to 0^+} \operatorname{ess\,inf}_{|h|<\delta} f_G(t+h) \geq \min G(t)$. In a similar way, it follows $f_G^+(t) \leq \max G(t)$, and hence, our claim is proven.

At this point, we have the conclusion, since $w(x) \in G(u(x))$ for a.a. $x \in \Omega$. □

Finally, we recall that a functional I_λ satisfies the Palais–Smale condition (briefly the (PS)-condition) if, for any sequence $\{u_n\} \subseteq X$, such that $\{I_\lambda(u_n)\}$ is bounded and

$$I_\lambda^\circ(u_n; v - u_n) + \varepsilon_n \|v - u_n\| \geq 0, \; \forall v \in X, \; n \in \mathbb{N}, \tag{8}$$

where $\varepsilon_n \to 0^+$, has a convergent subsequence.
We conclude this section by pointing out that, in order to assure the existence of multiple solutions to problem (2), our main tool will be the following abstract critical point theorem

for locally Lipschitz continuous functions (see Theorem 2.10 in [20]). We recall that this result is based on the Ambrosetti–Rabinowitz theorem ([23]).

Theorem 2. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Suppose that there exist $r \in \mathbb{R}$ and $\hat{u} \in X$ with $0 < \Phi(\hat{u}) < r$ such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \tag{9}$$

and for each $\lambda \in \Lambda^{r,\hat{u}} := \left]\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}\right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ fulfills the (PS)-condition and it is unbounded from below.
Then, for each $\lambda \in \Lambda^{r,\hat{u}}$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Elliptic Differential Inclusions

In this section, $G : \mathbb{R} \to 2^{\mathbb{R}}$ is an upper semicontinuous set-valued mapping with compact convex values such that the following condition holds:
(H_1) *there are $s \in [1, 2[$ and $q \in]2, \frac{2N}{N-2}[$ and two positive constants a_s and a_q such that*

$$0 \leq f_G(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \quad \forall t \in \mathbb{R},$$

where f_G is defined as in Section 2. Clearly, condition (H_1) implies the condition (H) given in Section 2, so that Proposition 2 holds.
Denoted by $2^* = \frac{2N}{N-2}$, the critical Sobolev exponent, it is well known that we have the continuous embedding

$$\|u\|_{2^*} \leq C\|u\| \quad \text{for all } u \in X, \tag{10}$$

where the constant C, given by

$$C = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\,\Gamma\left(1 + \frac{N}{2}\right)}\right)^{\frac{1}{N}}, \tag{11}$$

is the best constant, see Talenti [24], and Γ stands for the Gamma function. By Hölder's inequality and (10) we obtain

$$\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}} \leq |\Omega|_N^{\frac{2^*-p}{p2^*}} \|u\|_{2^*} \leq |\Omega|_N^{\frac{2^*-p}{p2^*}} C\|u\| \tag{12}$$

for all $u \in X$ and for all $p \in [1, 2^*[$, where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. As in Section 2, put $F_G(\xi) = \int_0^\xi f_G(t)dt$ for all $\xi \in \mathbb{R}$ and

$$\Phi(u) = \frac{\|u\|^2}{2}, \quad \Psi(u) = \int_\Omega F_G(u)dx \quad \text{and} \quad I_\lambda = \Phi(u) - \lambda\Psi(u) \tag{13}$$

for all $u \in X$ and for $\lambda > 0$. In addition, we set

$$R(x) = \sup\{\delta : B(x, \delta) \subseteq \Omega\}$$

for all $x \in \Omega$ and $R = \sup_{x \in \Omega} R(x)$ for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. Furthermore, put

$$K = \frac{R^2}{2} \frac{1}{2^N - 1} \frac{1}{2\left(C|\Omega|_N^{\frac{1}{N}}\right)^2} \tag{14}$$

and

$$K_\delta = \frac{1}{K}\frac{1}{2\left(C|\Omega|^{\frac{1}{N}}\right)^2}\frac{\delta^p}{F_G(\delta)},$$
$$K_\gamma = \frac{1}{2\left(C|\Omega|^{\frac{1}{N}}\right)^2}\frac{1}{\frac{a_s}{s}\gamma^{s-p}+\frac{a_q}{q}\gamma^{q-p}}. \tag{15}$$

for two positive constants γ and δ, respectively.

The main result in this section reads as follows.

Theorem 3. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be an upper semicontinuous set-valued mapping with compact convex values satisfying condition (H_1). Assume that there exist two positive constants γ, δ, with $\delta < \gamma$, such that*

$$\frac{a_s}{s}\gamma^{s-2}+\frac{a_q}{q}\gamma^{q-2} < K\frac{F_G(\delta)}{\delta^2} \tag{16}$$

and suppose that there exist two constants $m > 2$ and $l > 0$ such that

$$0 < mF_G(\xi) \le \xi f_G(\xi) \quad \text{for all } \xi \ge l. \tag{17}$$

Then, for each $\lambda \in]K_\delta, K_\gamma[$, problem (2) admits at least two distinct positive generalized weak solutions.

Proof. Let $\lambda \in]K_\delta, K_\gamma[$ be fixed. From (15) and (16), we easily see that the interval $]K_\delta, K_\gamma[$ is not empty. We want to apply Theorem 2. First, we mention that the Ambrosetti–Rabinowitz condition stated in (17) implies that the functional I_λ is unbounded from below and satisfies the Palais–Smale condition (see, for example, [25]). So, we only need to show that inequality (9) is satisfied. To this end, put

$$r = \frac{|\Omega|^{\frac{2}{2^*}}}{2C^2}\gamma^2 \tag{18}$$

and note that the growth condition in (H_1) implies

$$F_G(t) \le \frac{a_s}{s}|t|^s + \frac{a_q}{q}|t|^q \quad \text{for all } t \in \mathbb{R}. \tag{19}$$

Taking into account (12), (18) and (19), we have

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{a_s}{s}\|u\|_s^s+\frac{a_q}{q}\|u\|_q^q\right)}{r}\\
&\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{2^*-s}{2^*}}\|u\|^s+\frac{a_q}{q}C^q|\Omega|_N^{\frac{2^*-q}{2^*}}\|u\|^q\right)}{r}\\
&\le \frac{\left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{2^*-s}{2^*}}(2r)^{\frac{s}{2}}+\frac{a_q}{q}C^q|\Omega|_N^{\frac{2^*-q}{2^*}}(2r)^{\frac{q}{2}}\right)}{r}\\
&= 2C^2|\Omega|_N^{\frac{2^*-2}{2^*}}\left(\frac{a_s}{s}\left(\frac{2C^2r}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{s-2}{2}}+\frac{a_q}{q}\left(\frac{2C^2r}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{q-2}{2}}\right)\\
&= 2C^2|\Omega|_N^{\frac{2}{N}}\left(\frac{a_s}{s}\gamma^{s-2}+\frac{a_q}{q}\gamma^{q-2}\right)\\
&= \frac{1}{K_\gamma},
\end{aligned}$$

where

$$\gamma = \left(\frac{2C^2 r}{|\Omega|_N^{\frac{2}{2^*}}} \right)^{\frac{1}{2}}.$$

This implies that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{1}{\lambda}. \tag{20}$$

In order to prove the other inequality, let

$$v_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \frac{2\delta}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \delta & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

We easily see that $v_\delta \in X$. Moreover, one has

$$\begin{aligned} \Phi(v_\delta) &= \frac{1}{2} \int_{B(x_0,R) \setminus B\left(x_0, \frac{1}{2}R\right)} \frac{(2\delta)^2}{R^2} dx \\ &= \frac{1}{2} \frac{(2\delta)^2}{R^2} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \left(R^N - \left(\frac{1}{2} R \right)^N \right) \\ &= \frac{2\left(2^N - 1\right)}{2^N} R^{N-2} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \delta^2 \end{aligned} \tag{21}$$

and

$$\Psi(v_\delta) \geq \int_{B\left(x_0, \frac{R}{2}\right)} F_G(\delta) dx = F_G(\delta) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \frac{R^N}{2^N}.$$

Combining these estimates yields

$$\begin{aligned} \frac{\Psi(v_\delta)}{\Phi(v_\delta)} &\geq \frac{R^2}{2} \frac{1}{2^N - 1} \frac{F_G(\delta)}{\delta^2} \\ &= 2C^2 |\Omega|_N^{\frac{2}{N}} K \frac{F_G(\delta)}{\delta^2} = \frac{1}{K_\delta} > \frac{1}{\lambda}. \end{aligned} \tag{22}$$

From (20) and (22), we obtain

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}. \tag{23}$$

We only need to show that $\Phi(v_\delta) < r$. Setting

$$\hat{k} = \left(\frac{2\left(2^N - 1\right)}{2^N} R^{N-2} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \frac{2C^2}{|\Omega|_N^{\frac{2}{2^*}}} \right)^{\frac{1}{2}},$$

we get from (21) that

$$\Phi(v_\delta) = \hat{k}^2 \frac{|\Omega|_N^{\frac{2}{2^*}}}{2C^2} \delta^2.$$

Recall that $\delta < \gamma$ and we are going to show that $\hat{k}\delta < \gamma$.
First, we observe that

$$\begin{aligned}\frac{1}{\tilde{k}^2} &= \frac{2^N}{2(2^N-1)}\frac{1}{R^{N-2}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{|\Omega|_N^{\frac{2}{2^*}}}{2C^2} \\ &= \frac{R^2}{2}\frac{1}{2^N-1}\frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{2^N}{R^N}|\Omega|_N \\ &= \frac{|\Omega|_N}{\left|B\left(x_0,\frac{R}{2}\right)\right|}K \geq K.\end{aligned} \tag{24}$$

Now, we apply (16) in combination with the growth condition in (19) to obtain

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^2} < K\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2}. \tag{25}$$

Arguing by contradiction, we assume that $\hat{k}\delta \geq \gamma$. Then, this fact together with $\delta < \gamma$ and (24) gives

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^2} \geq \frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\hat{k}^2\delta^2} \geq \frac{1}{\hat{k}^2}\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2} \geq K\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2},$$

which is a contradiction to (25). Hence, $\hat{k}\delta < \gamma$ and this implies $\Phi(v_\delta) < r$.

Now, taking (23) also into account, we are in the position to apply Theorem 2, which says that I_λ admits two distinct non-zero critical points u_1, u_2. Therefore, Proposition 2 ensures that they are solutions of problem (2). So, in particular, there are $w_i \in L^2(\Omega)$, $i = 1, 2$, such that u_i are weak solutions of

$$\begin{cases} -\Delta u = \lambda w_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $w_i(x) \in G(u_i(x)) \subseteq [0, +\infty[$ a.e. in Ω, $i = 1, 2$, owing to (H_1). Hence, the strong maximum principle guarantees that u_i, $i = 1, 2$, are positive in Ω, and the conclusion is achieved. □

Now put

$$\hat{\lambda} = \frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}}\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s}. \tag{26}$$

We point out the following consequences of Theorem 3.

Theorem 4. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be an upper semicontinuous set-valued mapping with compact convex values satisfying conditions (H_1) and (17). Assume that*

$$\limsup_{t\to 0^+}\frac{F_G(t)}{t^2} = +\infty. \tag{27}$$

Then, for each $\lambda \in]0, \hat{\lambda}[$, problem (2) admits at least two distinct positive generalized weak solutions.

Proof. Let $\lambda \in]0, \hat{\lambda}[$ be fixed. We easily see that $\hat{\lambda} = \sup_\gamma K_\gamma$, and so there is $\gamma > 0$ such that $\lambda < K_\gamma$. On the other side, since

$$\limsup_{t\to 0^+} 2C^2K|\Omega|^{\frac{2}{N}}\frac{F_G(t)}{t^2} = +\infty,$$

there exists $\delta < \gamma$ such that

$$2C^2K|\Omega|^{\frac{2}{N}}\frac{F_G(t)}{t^2} > \frac{1}{\lambda}.$$

Hence, $\lambda \in]K_\delta, K_\gamma[$, and so condition (16) is fulfilled. Therefore, the statement of the corollary follows from Theorem 3. □

Remark 2. *Theorem 1 in the Introduction is a consequence of Theorem 4. It is enough to pick*

$$\hat{R} = \left(\frac{\Gamma(1+\frac{N}{2})}{2C^2\pi}\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s}\right)^{1/2}$$

and observe that $\hat{\lambda} > 1$ *if and only if* $R < \hat{R}$.

Now, we give an application of previous results to Dirichlet problems with discontinuous nonlinearities.

Theorem 5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be almost every continuous function satisfying the following:* (H_2) *there are* $s \in [1, 2[$ *and* $q \in]2, \frac{2N}{N-2}[$ *and two positive constants* a_s *and* a_q *such that*

$$0 \le f(t) \le a_s|t|^{s-1} + a_q|t|^{q-1} \quad \forall t \in \mathbb{R}.$$

Put $F(t) = \int_0^t f(s)ds$, $t \in \mathbb{R}$, *and assume that there exist two constants* $m > 2$ *and* $l > 0$ *such that*

$$0 < mF(\xi) \le \xi f(\xi) \quad \text{for all } \xi \ge l, \tag{28}$$

and

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} = +\infty. \tag{29}$$

We denote by

$$D_f = \{t \in \mathbb{R} : f \text{ is not continuous at t}\}.$$

and we require

$$\inf_{D_f} f > 0. \tag{30}$$

Then, for each $\lambda \in]0, \hat{\lambda}[$, *problem*

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{31}$$

admits at least two distinct positive weak solutions.

Proof. First, we observe that the assumption "f is a.e. continuous in $\mathbb{R}$" signifies $|D_f| = 0$. Moreover, assumption (H_2) implies that f is a locally bounded function. So, F is well defined, and it is a locally Lipschitz function. Moreover, we can define f^-, f^+ as in Section 2. Consider $\partial F(t)$, $t \in \mathbb{R}$. Moreover, put $L(t) = \{a \in \mathbb{R} : \langle a, t\rangle \in \partial F(t)\}$. Since $L(t)$ and ∂F are homeomorphic sets, from Propositions 2.1.5 (d) and 2.1.2 (a) in [14], L is an upper semicontinuous set-valued mapping with convex and compact values. Moreover, from Example 1 in [9] we have $L(t) \subseteq [f^-(t), f^+(t)]$ a.e. in $\mathbb{R}$. So, in particular, one has

$L(t) = f(t)$ for a.a. $t \in \mathbb{R}$ since $|D_f| = 0$. Moreover, taking into account that $\min L(t) = f(t)$ for a.a. $t \in \mathbb{R}$, simple computations show that L satisfies the assumptions (H_1), (17) and (27) of Theorem 4. Hence, problem

$$\begin{cases} -\Delta u \in \lambda L(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive generalized weak solutions. Let u be one of them. There is a function $w \in L^2(\Omega)$ such that

(j) $w(x) \in L(u(x))$ *a.e.* in Ω

(jj)

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega w v dx \quad \forall v \in W_0^{1,2}(\Omega).$$

Since $\partial\Omega$ is smooth enough, from usual regularity results, it follows that $u \in W^{2.2}(\Omega)$, and from (jj), one has

$$-\Delta u(x) = \lambda w(x) \tag{32}$$

for a.a. $x \in \Omega$. Now, we claim that $|u^{-1}(D_f)|_N = 0$. Arguing by a contradiction, assume $|u^{-1}(D_f)|_N > 0$. From one side, Lemma 1 by De Giorgi, Buttazzo, and Dal Maso [26] ensures that $-\Delta u(x) = 0$ for a.a. $x \in u^{-1}(D_f)$. On the other side, (j) ensures $w(x) \in L(u(x))$ for a.a. $x \in u^{-1}(D_f)$, for which $w(x) = f(u(x))$ for a.a. $x \in u^{-1}(D_f)$. Therefore, from (30), we obtain $w(x) > 0$ for a.a. $x \in u^{-1}(D_f)$. Hence, we have $0 = -\Delta u(x) = \lambda w(x) > 0$ for a.a. $x \in u^{-1}(D_f)$ and this is absurd, so our claim is proven.

Hence, we have

$$w(x) = f(u(x)) \tag{33}$$

for a.a. $x \in \Omega$, and from (jj) follows

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega f(u(x)) v dx \quad \forall v \in W_0^{1,2}(\Omega).$$

This completes the proof. □

We now give two examples of applications of Theorem 5. The first has a nonlinearity with only one discontinuity point, while the nonlinearity on the second one has an uncountable set of discontinuity points.

Example 1. *The following problem has been introduced in [13]. Put*

$$g(s) = \begin{cases} 1 & \text{if } s \le 0 \\ 0 & \text{if } s > 0 \end{cases} \tag{34}$$

and consider the problem

$$\begin{cases} -\Delta u = u^{p-1} + bg(u-a) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{35}$$

with $2 < p < \frac{2N}{N-2}$ *and* $a, b > 0$. *Theorem 5 ensures that there is*

$$\hat{b} = \left(\frac{1}{2C^2|\Omega|^{\frac{2}{N}}}\right)^{\frac{p-1}{p-2}} \left(\frac{p}{p-2}\right)^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

such that for each $a > 0$ *and for each* $b \in]0, \hat{b}[$, *problem* (35) *admits two positive weak solutions.*

Indeed, by choosing $p = q$ and $s = 1$, all assumptions of Theorem 5 are verified. In particular, we have

$$0 \leq |u|^{q-1} + bg(u-a) \leq |u|^{q-1} + b$$

for all $u \in \mathbb{R}$. Hence, Theorem 5 ensures that the problem

$$\begin{cases} -\Delta u = \lambda\Big(u^{p-1} + bg(u-a)\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions for each $\lambda \in]0, \hat{\lambda}[$ where

$$\hat{\lambda} = \frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}} \left(\frac{1}{b}\right)^{\frac{p-2}{p-1}} (p)^{\frac{1}{p-1}} \left(\frac{1}{p-2}\right)^{\frac{1}{p-1}} \frac{p-2}{p-1},$$

so that $\hat{\lambda} > 1$ if and only if $b < \hat{b}$.

Example 2. *Put*

$$g(s) = \begin{cases} 0 & \text{if} \quad s \in C \\ 1 & \text{if} \quad s \in \mathbb{R} \setminus C, \end{cases} \tag{36}$$

where C is the Cantor set, and consider the problem

$$\begin{cases} -\Delta u = u^{p-1} + bg(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{37}$$

with $2 < p < \frac{2N}{N-2}$ and $b > 0$. Theorem 5 ensures that there is

$$\hat{b} = (2C^2|\Omega|_N^{\frac{2}{N}})^{\frac{p-1}{p-2}} \left(\frac{p}{p-2}\right)^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

such that for each $b \in]0, \hat{b}[$, problem (37) admits two positive weak solutions.
Indeed, by choosing $p = q$ and $s = 1$, all assumptions of Theorem 5 are verified. Hence, Theorem 5 ensures that the problem

$$\begin{cases} -\Delta u = \lambda\Big(u^{p-1} + bg(u)\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions for each $\lambda \in]0, \hat{\lambda}[$, where

$$\hat{\lambda} = \frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}} \left(\frac{1}{b}\right)^{\frac{p-2}{p-1}} (p)^{\frac{1}{p-1}} \left(\frac{1}{p-2}\right)^{\frac{1}{p-1}} \frac{p-2}{p-1},$$

so that $\hat{\lambda} > 1$ if and only if $b < \hat{b}$.

Remark 3. *Consider the following problem involving a hemivariational inequality*

$$\begin{cases} -\displaystyle\int_\Omega \nabla u \cdot \nabla v dx + \lambda \int_\Omega F^\circ(u(x); v(x)) dx \geq 0 \quad \forall v \in W_0^{1,2}(\Omega) \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{38}$$

where F° stands for Clarke's generalized directional derivative of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(z)dz$, $z \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is a locally essentially bounded function. If we assume (H_2), (27) and (28), and then for each $\lambda \in]0, \hat{\lambda}[$, problem (38) admits

at least two positive solutions. Indeed, arguing as in the first part of the proof of Theorem 5, condition (i) and (ii) are obtained. Hence, taking into account that $w(x) \in L(u(x))$ if and only if $w(x)v(x) \leq F^{\circ}(u(x); v(x))$, one has

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} wv dx \leq \lambda \int_{\Omega} F^{\circ}(u(x); v(x)) dx \quad \forall v \in W_0^{1,2}(\Omega),$$

which is the conclusion.

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