

Article

# Decompositions of the $\lambda$ -Fold Complete Mixed Graph into Mixed 6-Stars

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**Abstract:** Graph and digraph decompositions are a fundamental part of design theory. Probably the best known decompositions are related to decomposing the complete graph into 3-cycles (which correspond to Steiner triple systems), and decomposing the complete digraph into orientations of a 3-cycle (the two possible orientations of a 3-cycle correspond to directed triple systems and Mendelsohn triple systems). Decompositions of the  $\lambda$ -fold complete graph and the  $\lambda$ -fold complete digraph have been explored, giving generalizations of decompositions of complete simple graphs and digraphs. Decompositions of the complete mixed graph (which contains an edge and two distinct arcs between every two vertices) have also been explored in recent years. Since the complete mixed graph has twice as many arcs as edges, an isomorphic decomposition of a complete mixed graph into copies of a sub-mixed graph must involve a sub-mixed graph with twice as many arcs as edges. A partial orientation of a 6-star with two edges and four arcs is an example of such a mixed graph; there are five such mixed stars. In this paper, we give necessary and sufficient conditions for a decomposition of the  $\lambda$ -fold complete mixed graph into each of these five mixed stars for all  $\lambda > 1$ .

**Keywords:** graph decomposition; mixed graph; orientations of stars;  $\lambda$ -fold complete mixed graph



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## 1. Introduction

### 1.1. Outline

In this paper, we give necessary and sufficient conditions for the existence of mixed 6-star decompositions of the  $\lambda$ -fold complete mixed graph. In Section 1.2, we give well-known definitions of *graph* and *digraph*. Based on the form of these definitions, we define the less well-known objects of *fuzzy graph* and *mixed graph*. A broad description of applications of these is mentioned. The main result of this paper concerns a mixed graph decomposition, so in Section 1.3 we define *decompositions* of graphs/multigraphs, digraphs/multidigraphs, and mixed graphs/multi-mixed graphs, and relate these concepts to each other. Section 2 is the main body of the paper and includes a description of the proof technique, background results, and a statement and proof of the main result. In Section 2.1, we explain the primary proof technique (“difference methods”) used in a large part of the construction, which establishes the validity of our main result. In Section 2.2, we elaborate on the construction method and described a modified difference method that is also employed in our construction. Section 2.3 includes a background result and a lemma containing necessary conditions for the decomposition of interest. In Section 2.4, we give several theorems that combine to give necessary and sufficient conditions for our main result, namely a decomposition of the  $\lambda$ -fold complete mixed graph into mixed 6-stars. Section 3 discusses the applications of graph decompositions in general, the application of the proof technique to other possible mixed graph structures, and suggests some future directions for additional research.

## 1.2. Graph Definitions

We largely follow the graph theoretic definitions set out in [1]. A (simple) *graph* is a pair of (finite) sets  $G = (V, E)$ , where the elements of  $E = E(G)$  are 2-element subsets of  $V = V(G)$ . The elements of  $V$  are called *vertices* (though the term *nodes* is also common) and the elements of  $E$  are called *edges*. Edge  $\{u, v\}$  is said to *join* vertices  $u$  and  $v$ ; we will often denote edge  $\{u, v\}$  as  $uv$ . The complete graph on  $v$  vertices, denoted  $K_v$ , is the graph with vertex set  $V$ , where  $|V| = v$  and  $E = \{\{u, v\} \mid u, v \in V, u \neq v\}$ . A *cycle* of length  $v$ , denoted  $C_v$ , is the graph with vertex set  $V = \{v_1, v_2, \dots, v_n\}$  and edge set  $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$ . A *star* on  $v + 1$  vertices, denoted  $S_v$ , is a graph with vertex set  $V = \{v_0, v_1, \dots, v_n\}$  and edge set  $E = \{\{v_0, v_i\} \mid i = 1, 2, \dots, n\}$ . A *multigraph* is a pair  $(V, E)$  where (finite) set  $V$  is the vertex set and  $E$  is a (finite) multiset (that is, a finite collection that allows elements to appear a repeated number of times), where the elements of  $E$  are multisets of two, not necessarily distinct, elements of  $V$ . An element of  $E$  of the form  $\{v, v\}$  is a *loop* on  $v$ . The  $\lambda$ -fold complete graph on  $v$  vertices, denoted  $\lambda K_v$ , is the multigraph with vertex set  $V$ , where  $|V| = v$  and  $E$  contains every edge  $\{u, v\}$ , where  $u \neq v$ , exactly  $\lambda$  times.

Applications have been part of the reason for the pursuit of graph theory since its origins. One of the most famous graph theory problems (likely due to its difficulty and wide-ranging applications) is the traveling salesman problem. Introduced to find a minimal weighted spanning tree in a given weighted graph, applications of the problem appear in genome mapping, aiming telescopes, guiding machines through a sequence of tasks, and organizing data. In addition, the difficulty of the problem has had an impact on the study of computational complexity and the study of algorithms; for details on this, see [2]. The “assignment problem” (also called the “scheduling problem”) involves finding a maximal matching in a bipartite graph and can be applied to matching a group of workers, each with a certain set of skills, with a group of jobs, each of which requires a certain set of skills [3] (see Section 7.2), ref. [4] (see Section 16.1 and Problem 16.2). The areas of application of graph theory are growing rapidly in the 21st century. In particular, graph neural networks (or “deep learning on graphs”) have received remarkable levels of attention. Applications include computer vision, language recognition, automated planning, social networks, bioinformatics, and cybersecurity [5] (page vii).

A (simple) *digraph* (or *directed graph*) is a pair of (finite) sets,  $D = (V, A)$ , together with two maps,  $\text{init} : A \rightarrow V$  and  $\text{ter} : A \rightarrow V$ . Again, the elements of  $V$  are called vertices. The elements of  $A$  are called *arcs* (or *directed edges*). For arc  $a \in A$ , we call  $\text{init}(a)$  the *initial vertex* of  $a$  and  $\text{ter}(a)$  the *terminal vertex* of  $a$ , and we denote  $a$  as the ordered pair  $(\text{init}(a), \text{ter}(a))$ . The complete digraph on  $v$  vertices, denoted  $D_v$ , is the graph with vertex set  $V$ , where  $|V| = v$  and  $A = \{\{u, v\} \mid u, v \in V, u \neq v\}$ . A *multidigraph* is a pair,  $D = (V, A)$ , where (finite) set  $V$  is the vertex set and  $A$  is a (finite) multiset, where the elements of  $A$  are arcs of the form  $(u, v)$ , where  $u, v \in V$ . The  $\lambda$ -fold complete digraph on  $v$  vertices, denoted  $\lambda D_v$ , is the multidigraph with vertex set  $V$ , where  $|V| = v$  and  $A$  contains every arc  $(u, v)$ , where  $u \neq v$ , exactly  $\lambda$  times. A digraph (or multidigraph),  $D$ , is an *orientation* of an (undirected) graph (or multigraph),  $G$ , if  $V(D) = V(G)$ , and for each  $uv \in E(G)$  we have either  $(u, v) \in A(D)$  or  $(v, u) \in A(D)$  (but not both, unless this results from multiple appearances of edge  $uv$  in  $E(G)$ ).

A *mixed graph* is a triple of finite sets  $M = (V, E, A)$ , where  $V$  is a set of vertices,  $E$  is a set of edges (as defined above for graph  $(V, E)$ ), and  $A$  is a set of arcs (as defined above for digraph  $(V, A)$ ). A *multi-mixed graph* is a triple  $M = (V, E, A)$ , where  $V$  is a set of vertices,  $E$  is a multiset of edges (as defined above for multigraph  $(V, E)$ ), and  $A$  is a multiset of arcs (as defined above for multidigraph  $(V, A)$ ). We note that Harary and Palmer, who first introduced mixed graphs in 1966, defined a mixed graph as containing “both ordinary and oriented lines” [6]. They followed a convention of combining two anti-parallel arcs into a single edge. We deviate from this practice in our definition and allow the presence of both an edge and two anti-parallel arcs between a pair of vertices (as has become more common in recent years). The complete mixed graph on  $v$  vertices, denoted  $M_v$ , is the

mixed graph  $M_v = (V, E, A)$ , where  $|V| = v$ , edge set  $E$  contains each edge  $\{u, v\}$ , where  $u \neq v$ , exactly once, and arc set  $A$  contains every arc  $(u, v)$ , where  $u \neq v$ , exactly once. The  $\lambda$ -fold complete multi-mixed graph on  $v$  vertices, denoted  $\lambda M_v$ , is the multi-mixed graph with vertex set  $V$ , where  $|V| = v$ , multiset  $E$  contains every edge  $uv$ , where  $u \neq v$ , exactly  $\lambda$  times, and multiset  $A$  contains every arc  $(u, v)$ , where  $u \neq v$ , exactly  $\lambda$  times. A multi-mixed graph,  $M$ , is a *partial orientation* of an (undirected) graph (or multigraph),  $G$ , if  $V(M) = V(G)$ ,  $E(M) \subset E(G)$ , and for every edge  $e = uv \in E(G) \setminus E(M)$  we have either  $(u, v) \in A(M)$  or  $(v, u) \in A(M)$  (but not both, unless this results from multiple appearances of edge  $e = uv$  in  $E(G)$ ).

Of additional interest is the idea of a fuzzy graph. An outgrowth of fuzzy logic (which has recently found a huge number of applications, particularly in the areas of machine learning and artificial intelligence), the theory of fuzzy graphs involves structures that do not fall into the binary relationships of “element of” and “not an element of”, but instead involves a degree of membership (which can be interpreted as a probability of membership). We rely on [7] for definitions and descriptions of this topic, though we modify the notation slightly for consistency with our other definitions. For a given set,  $X$ , a *fuzzy subset* of  $X$  (or simply a “fuzzy set”) is a function,  $\mu$ , mapping  $X \rightarrow [0, 1]$ . A *fuzzy graph* is a quadruple  $F = (V, E, \sigma, \mu)$ , where  $V$  is a finite set of vertices,  $E$  is a set of edges (as defined above for graph  $(V, E)$ ), and  $\sigma$  and  $\mu$  are functions, where  $\sigma : V \rightarrow [0, 1]$  and  $\mu : E \rightarrow [0, 1]$ , which satisfy the condition  $\mu(xy) \leq \min\{\sigma(x), \sigma(y)\}$  for all  $xy \in E$ . Fuzzy set  $\sigma$  is the *fuzzy vertex set* of  $F$  and fuzzy set  $\mu$  is the *fuzzy edge set* of  $F$ . Applications of fuzzy graph theory include cluster analysis and decision analysis in statistical science, pattern classification, neural networks, and social science [7] (page 1). Recent examples of applications of fuzzy graph theory include the use of the concept of the “Randic index” in a connected system applied to Indonesian tourism [8], and the use of Cartesian products, compositions, and unions of “picture fuzzy graphs” applied to railway networks and medical science models [9].

### 1.3. Decomposition Definitions

A *g-decomposition* of graph  $G$  is a set of subgraphs of  $G$ ,  $\gamma = \{g_1, g_2, \dots, g_n\}$ , where  $g_i \cong g$  for  $i \in \{1, 2, \dots, n\}$ ,  $E(g_i) \cap E(g_j) = \emptyset$  for  $i \neq j$ , and  $\cup_{i=1}^n E(g_i) = E(G)$ . The  $g_i$  are called *blocks* of the decomposition. When  $G$  is a complete graph, the  $g$ -decomposition is often called a *graph design*. A  $K_3$ -decomposition of  $K_v$  is a Steiner triple system of order  $v$ , and it is well known that such a triple system exists if and only if  $v \equiv 1$  or  $3 \pmod{6}$  (see [10] for references). Cycle graph designs were a topic of intense interest until the early 2000s, when necessary and sufficient conditions for a  $C_m$ -decomposition of  $K_v$  were given for all  $m$  and  $v$  [11,12]. An easier and self-contained construction for odd cycle systems is given in [13]. Necessary and sufficient conditions for decompositions of  $K_v$  into copies of (not necessarily isomorphic) stars are given in [14]. To illustrate the potential application of graph decompositions, consider a (hypothetical) machine that runs samples three at a time. The machine can only make comparisons between samples that are run together in the machine (it cannot be calibrated from run to run, say). If one desires to compare a collection of samples of size  $v$ , can this be done optimally (that is, by comparing every pair of samples exactly once)? By representing the samples as vertices of a graph, comparisons of a pair of samples as an edge, and a run of the machine as a  $K_3$ , an optimal solution is equivalent to a Steiner triple system of order  $v$ , and the subgraphs in the decomposition give the runs of the machine in the optimal solution. A  $g$ -decomposition of multigraph  $G$  is defined similar to that of a  $g$ -decomposition of a graph, with the edges of  $G$  repeated an appropriate number of times in the multiset of the  $g_i$ . Necessary and sufficient conditions for a  $K_3$ -decomposition of  $\lambda K_v$  are given in [15]. Notice that the three-at-a-time comparison application can be solved, if exactly  $\lambda$  comparisons per pair of samples is required, by considering a  $K_3$ -decomposition of  $\lambda K_v$ .

A *d-decomposition* of digraph  $D$  is a set of subdigraphs of  $D$ ,  $\gamma = \{d_1, d_2, \dots, d_n\}$ , where  $d_i \cong d$  for  $i \in \{1, 2, \dots, n\}$ ,  $A(d_i) \cap A(d_j) = \emptyset$  for  $i \neq j$ , and  $\cup_{i=1}^n A(d_i) = A(G)$ . The  $d_i$  are called *blocks* of the decomposition. When  $D$  is a complete digraph, the  $d$ -

decomposition is often called a *digraph design*. There are two orientations of  $K_3 = C_3$ , a 3-circuit (in which case the three arcs are oriented in the “same” direction, say clockwise) and the transitive triple (say, with two clockwise-oriented arcs and one counterclockwise-oriented arc). A decomposition of  $D_v$  into 3-circuits is a Mendelsohn triple system of order  $v$ , and a decomposition of  $D_v$  into transitive triples is a directed triple system of order  $v$ . Each of these exists if and only if  $v \equiv 0$  or  $1 \pmod{3}$ ,  $v \geq 3$ , except in the case of  $v = 6$  for a Mendelsohn triple system [16,17]. A  $d$ -decomposition of multidigraph  $D$  is defined similar to that of a  $d$ -decomposition of a digraph, with the arcs of  $D$  repeated an appropriate number of times in the multiset of the  $d_i$ . Decompositions of  $\lambda D_v$  into Mendelsohn triples are considered in [18] and decompositions of  $\lambda D_v$  into transitive triples are considered in [19]. The literature on graph designs and graph decompositions (and of digraph decompositions) is extensive [10,20], especially in the case of triple systems [21].

An  $m$ -decomposition of mixed graph  $M$  is a set of sub-mixed graphs of  $M$ ,  $\gamma = \{m_1, m_2, \dots, m_n\}$ , where  $m_i \cong m$  for  $i \in \{1, 2, \dots, n\}$ ,  $E(m_i) \cap E(m_j) = \emptyset$  and  $A(m_i) \cap A(m_j) = \emptyset$  for  $i \neq j$ ,  $\cup_{i=1}^n E(m_i) = E(M)$ , and  $\cup_{i=1}^n A(m_i) = A(M)$ . The  $m_i$  are called *blocks* of the decomposition. When  $M$  is a complete mixed graph, the  $m$ -decomposition is often called a *mixed graph design*. Since the complete mixed graph,  $M_v$ , has twice as many arcs as edges, then for an  $m$ -decomposition of  $M_v$  we must have that  $m$  also has twice as many arcs as edges. This gives a necessary condition for the existence of a mixed graph design. For example, each partial orientation of  $C_3$  with two arcs and one edge could yield a decomposition of  $M_v$ . Such a decomposition is called a *mixed triple system* of order  $v$ . Necessary and sufficient conditions for the existence of the various types of mixed triple systems of order  $v$  are given in [22]. A partial orientation of the star  $S_3$  with two arcs and one edge could yield a decomposition of  $\lambda M_v$ . Necessary and sufficient conditions for the existence of a decomposition of  $\lambda M_v$  is given for each such partial orientation of  $S_3$  in [23]. Of particular interest to the current work is the decomposition of  $M_v$  into partial orientations of  $S_6$  with four arcs and two edges; such mixed graph designs are classified in [24]. An  $m$ -decomposition of multi-mixed graph  $M$  is defined similar to that of an  $m$ -decomposition of a mixed graph, with the arcs of  $M$  repeated an appropriate number of times in the multiset of the  $m_i$ . The purpose of this paper is to classify decompositions of  $\lambda M_v$  into partial orientations of  $S_6$  with four arcs and two edges for all  $\lambda > 1$ . In this way, the research gap between decompositions of simple mixed graphs and decompositions of multi-mixed graphs is filled in the case of partial orientations of  $S_6$ . There are five partial orientations of  $S_6$  with four arcs and two edges, and these are given in Figure 1.

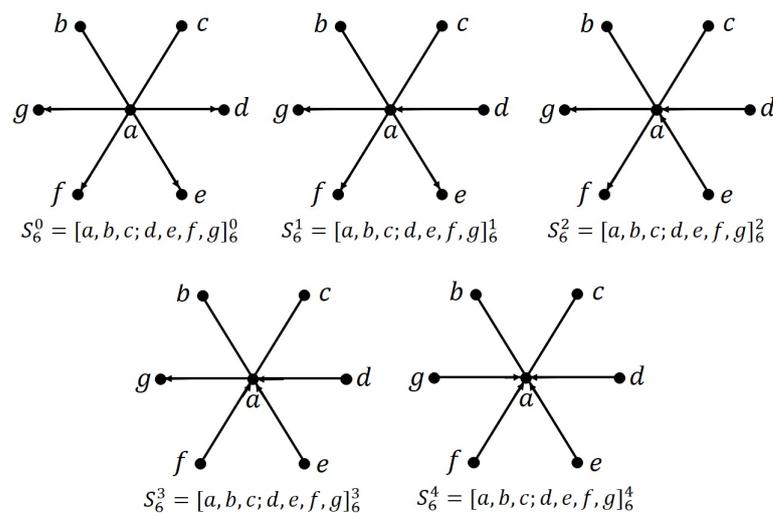


Figure 1. The five partial orientations of  $S_6$  with four arcs and two edges and the notation we use to represent them.

## 2. Decompositions of the $\lambda$ -Fold Complete Mixed Graph into Mixed 6-Stars

### 2.1. Difference Methods

For the complete mixed graph  $M_v = (V, E, A)$ , we take  $V = \{0, 1, 2, \dots, v-1\}$ . Define the *edge difference* associated with edge  $xy$  as  $\min\{(x-y) \pmod{v}, (y-x) \pmod{v}\}$ . Define the *arc difference* associated with arc  $(x, y)$  as  $(y-x) \pmod{v}$ . Notice that the total set of edge differences associated with  $E$  is  $\{1, 2, \dots, \lfloor (v-1)/2 \rfloor\}$ , and the total set of arc differences associated with  $A$  is  $\{1, 2, \dots, v-1\}$ . Under the permutation  $\pi = (0, 1, 2, \dots, v-1)$ , the orbit of the edge  $xy$  includes all edges of  $M_v$ , with associated edge difference  $\min\{(x-y) \pmod{v}, (y-x) \pmod{v}\}$ . Similarly, under permutation  $\pi$ , the orbit of arc  $(x, y)$  includes all arcs of  $M_v$ , with associated arc differences  $(y-x) \pmod{v}$ . Suppose  $\{b_1, b_2, \dots, b_k\}$ , where  $b_i \cong m$  for  $i \in \{1, 2, \dots, k\}$ , is a collection of sub-mixed graphs of  $M_v$ , such that every edge difference  $1, 2, \dots, \lfloor (v-1)/2 \rfloor$  (respectively, arc difference  $1, 2, \dots, v-1$ ) is associated with exactly one edge (respectively, arc) in the edge sets (respectively, arc sets) of the  $b_i$ . Then, if we take all of the images of the  $b_i$  under the powers of  $\pi$ , we obtain an  $m$ -decomposition of  $M_v$ , with  $\gamma = \{\pi^i(b_1), \pi^i(b_2), \dots, \pi^i(b_k) \mid i = 0, 1, \dots, v-1\}$ . The set  $\{b_1, b_2, \dots, b_k\}$  is a set of *base blocks* for the  $m$ -decomposition of  $M_v$  under  $\pi$ . Similarly, if multiset  $\{b_1, b_2, \dots, b_k\}$  contains edges and arcs that give each edge difference and each arc difference exactly  $\lambda$  times, then the images of the  $b_i$  under the powers of  $\pi$  give an  $m$ -decomposition of  $\lambda M_v$ . This is the construction technique often used in the proof of the main result of this paper when  $v$  is odd.

### 2.2. Automorphisms of Decompositions

An *automorphism* of a decomposition,  $\gamma = \{g_1, g_2, \dots, g_n\}$ , of graph  $G = (V, E)$  is a permutation of  $V$  that fixes set  $\gamma$ . Automorphisms of a digraph decomposition and of a mixed graph decomposition (as well as the multigraph/multidigraph/multi-mixed graph versions) are similarly defined. A decomposition admitting the automorphism  $\pi$  of Section 2.1 is a *cyclic* decomposition. Cyclic automorphisms and difference methods are a frequently used method of construction for various decompositions. For example, a cyclic Steiner triple system of order  $v$  exists for all possible orders  $v \equiv 1$  or  $3 \pmod{6}$ , except for  $v = 9$ . The construction of such systems, along with an explanation of the difference methods used, is given in [10].

A decomposition of one of the classes of graphs discussed, where  $V = \{\infty, 0, 1, 2, \dots, v-2\}$ , which admits the automorphism  $\rho = (\infty)(0, 1, 2, \dots, v-2)$ , is a *rotational* decomposition. Rotational Steiner triple systems were the first class of objects studied in terms of the question: "For what orders does a graph decomposition admit a given type of permutation as an automorphism?" [25]. The use of a rotational permutation in the construction of a decomposition is related to the idea of difference methods, as discussed in Section 2.1. However, the differences are associated with the cycle of length  $v-1$ , and edges and/or arcs to/from the fixed point  $\infty$  must be taken into consideration. When using a rotational permutation, a set of base blocks would need each edge/arc difference to be present an appropriate number of times (either 1 or  $\lambda$ ), and an edge/arc with  $\infty$  as an end/terminal end/initial end would need to be present an appropriate number of times (either 1 or  $\lambda$  each). We often use rotational permutations in the proof of the main result of this paper when  $v$  is even.

### 2.3. Background Result and a Lemma

The necessary and sufficient conditions for the existence of an  $S_6^i$ -decomposition of  $M_v$  are as follows [24].

**Theorem 1.** An  $S_6^i$ -decomposition of  $M_v$  exists if and only if  $v \geq 9$ ,

1. if  $i \in \{0, 4\}$  then  $v \equiv 1 \pmod{4}$ , and
2. if  $i \in \{1, 2, 3\}$  then  $v \equiv 0$  or  $1 \pmod{4}$ .

The next lemma gives necessary conditions for the existence of an  $S_6^i$ -decomposition, where  $0 \leq i \leq 4$ , of  $\lambda M_v$  when  $\lambda$  is odd.

**Lemma 1.** For  $\lambda$  odd and  $0 \leq i \leq 4$ , if an  $S_6^i$ -decomposition of  $\lambda M_v$  exists then  $v \equiv 0$  or  $1 \pmod{4}$ .

**Proof.** First, notice that for  $\lambda$  odd and  $v \equiv 2 \pmod{4}$ , say  $v = 4k + 2$ , the number of edges of  $\lambda M_v$  is  $\lambda \binom{v}{2} = \lambda(4k + 1)(2k + 1)$ , which is odd. In this case, no  $S_6^i$ -decomposition of  $\lambda M_v$  exists since  $S_6^i$  has two edges.

Next, for  $\lambda$  odd and  $v \equiv 3 \pmod{4}$ , say  $v = 4k + 3$ , the number of edges of  $\lambda M_v$  is  $\lambda \binom{v}{2} = \lambda(4k + 3)(2k + 1)$ , which is odd. In this case, no  $S_6^i$ -decomposition of  $\lambda M_v$  exists.

Therefore, if an  $S_6^i$ -decomposition of  $\lambda M_v$  exists then  $v \equiv 0$  or  $1 \pmod{4}$ , as claimed.  $\square$

#### 2.4. The Decomposition Theorem and Proof

Since  $S_6^i$  has seven vertices for each  $i$ , then of course a necessary condition for an  $S_6^i$ -decomposition of any mixed graph on  $v$  vertices is  $v \geq 7$ . In several of the following constructions, we take  $\{0, 1, 2, \dots, v - 1\}$  as the vertex set of  $\lambda M_v$ . We will also use the permutation  $\pi = (0, 1, 2, \dots, v - 1)$  in several of the constructions. We now give necessary and sufficient conditions for the existence of an  $S_6^i$ -decomposition of  $\lambda M_v$  in each case (with the exception of the small cases  $v = 8$  and  $v = 10$  when  $\lambda = 2$  and  $i = 1$ , which we leave open).

**Theorem 2.** An  $S_6^0$ -decomposition of  $\lambda M_v$  exists if and only if  $v \geq 7$  and

1.  $v \equiv 0 \pmod{2}$  and  $\lambda \equiv 0 \pmod{4}$ , or
2.  $v \equiv 1 \pmod{4}$  and  $\lambda \geq 1$ , or
3.  $v \equiv 3 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ .

**Proof.** Each vertex of  $\lambda M_v$  has out-degree  $\lambda(v - 1)$  and each vertex of  $S_6^0$  has out-degree  $0 \pmod{4}$  so, if an  $S_6^0$ -decomposition of  $\lambda M_v$  exists, then  $\lambda(v - 1) \equiv 0 \pmod{4}$  is necessary. This observation, along with Lemma 1, gives the necessary conditions. We now show these necessary conditions are sufficient.

**Case 1.** Suppose  $v \equiv 0 \pmod{4}$ ,  $v \geq 8$ , say  $v = 4k$  where  $k \geq 2$ , and  $\lambda \equiv 0 \pmod{4}$ . Consider the blocks:

$$\{2 \times [0, 2k - 2, 2k - 1; 2k, 4k - 3, 4k - 2; 4k - 1]_6^0, 2 \times [0, 2k + 1, 4k - 1; 1, 2, 3, 2k]_6^0\}$$

$$\cup \{[0, 4k - 1, 2k; 1, 2, 4k - 3; 4k - 2]_6^0, [0, 2k + 2, 2k; 1, 2, 3, 4k - 1]_6^0,$$

$$[0, 1, 2k - 2; 3, 4k - 3, 4k - 2, 4k - 1]_6^0\}$$

$$\cup \{4 \times [0, 2 + 2i, 3 + 2i; 4 + 2i, 5 + 2i, 2k + 1 + 2i, 2k + 2 + 2i]_6^0 \text{ for } i = 0, 1, \dots, k - 3\}.$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^0$ -decomposition of  $4M_v$ . By taking  $\lambda/4$  copies of the blocks of such a decomposition, we obtain an  $S_6^0$ -decomposition of  $\lambda M_v$ .

**Case 2.** Suppose  $v \equiv 1 \pmod{4}$ . Then by Theorem 1 an  $S_6^0$ -decomposition of  $M_v$  exists by Theorem 1. For any  $\lambda \geq 1$ , we take  $\lambda$  copies of the blocks of such a decomposition and this gives an  $S_6^0$ -decomposition of  $\lambda M_v$ .

**Case 3.** Suppose  $v \equiv 2 \pmod{4}$ ,  $v \geq 10$ , say  $v = 4k + 2$  where  $k \geq 2$ , and  $\lambda \equiv 0 \pmod{4}$ . Consider the blocks

$$\{2 \times [0, 1, 2k - 2; 4k - 2, 4k - 1, 4k, 4k + 1]_6^0, 2 \times [0, 2k - 1, 2k + 2; 1, 2, 4, 2k + 1]_6^0\}$$

$$\cup \{[0, 1, 2k + 3; 2, 3, 4k, 4k + 1]_6^0, [0, 2k - 2, 2k; 3, 2k + 1, 4k - 2, 4k]_6^0,$$

$$[0, 2k + 1, 2k; 1, 2, 3, 4k - 2]_6^0, [0, 1, 2k - 2; 3, 4, 2k + 1, 4k - 1]_6^0$$

$$[0, 2k + 1, 2k + 2; 1, 4, 4k - 1, 4k + 1]_6^0\}$$

$$\cup\{4 \times [0, 2 + 2i, 3 + 2i; 5 + 2i, 6 + 2i, 2k + 2 + 2i, 2k + 3 + 2i]_6^0 \text{ for } i = 0, 1, \dots, k - 3\}.$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^0$ -decomposition of  $4M_v$ . By taking  $\lambda/4$  copies of the blocks of such a decomposition, we obtain an  $S_6^0$ -decomposition of  $\lambda M_v$ .

**Case 4a.** Suppose  $v = 7$ . Consider the blocks

$$\{[0, 1, 6; 2, 3, 4, 5]_6^0, [0, 4, 5; 1, 2, 3, 6]_6^0, [0, 2, 3; 1, 4, 5, 6]_6^0\}.$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^0$ -decomposition of  $2M_7$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^0$ -decomposition of  $\lambda M_7$ .

**Case 4b.** Suppose  $v \equiv 3 \pmod{4}$ ,  $v \geq 11$ , say  $v = 4k + 3$  where  $k \geq 2$ , and  $\lambda \equiv 2 \pmod{4}$ . Consider the blocks

$$\begin{aligned} &\{[0, 4k + 2, 4k + 1; 1, 2, 3, 4]_6^0, [0, 2k + 1, 2k + 2; 1, 2, 4k + 1, 4k + 2]_6^0\} \\ &\cup\{[0, 1 + 2i, 2 + 2i; 3 + 4i, 4 + 4i, 5 + 4i, 6 + 4i]_6^0 \text{ for } i = 0, 1, \dots, k - 1\} \\ &\cup\{[0, 3 + 2i, 4 + 2i; 5 + 4i, 6 + 4i, 7 + 4i, 8 + 4i]_6^0 \text{ for } i = 0, 1, \dots, k - 2\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^0$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^0$ -decomposition of  $\lambda M_v$ .  $\square$

Since  $\lambda M_v$  is self-converse and the converse of  $S_6^0$  is  $S_6^4$ , then an  $S_6^4$ -decomposition of  $\lambda M_v$  exists if and only if an  $S_6^0$ -decomposition of  $\lambda M_v$  exists. So, the conditions for the existence of an  $S_6^0$ -decomposition of  $\lambda M_v$ , given in Theorem 2, are also the conditions for the existence of an  $S_6^4$ -decomposition of  $\lambda M_v$ .

**Theorem 3.** An  $S_6^1$ -decomposition of  $\lambda M_v$  exists if and only if  $v \geq 7$  and

1.  $v \equiv 0$  or  $1 \pmod{4}$  and  $\lambda \geq 1$ , where  $v \neq 8$  when  $\lambda = 1$ , or
2.  $v \equiv 2 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ , or
3.  $v \equiv 3 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ ,

with the possible exceptions of  $v = 8$  and  $v = 10$  when  $\lambda = 2$ , which we leave open.

**Proof.** Theorem 1 and Lemma 1 give the necessary conditions. We now show these necessary conditions are sufficient.

**Case 1a.** Suppose  $v = 8$ . Consider the blocks

$$\begin{aligned} &\{[\infty, 0, 1; 2, 3, 4, 5]_6^1, [0, 1, 2; 3, 4, 5, 6]_6^1, [0, 2, 3; 5, 6, 1, 4]_6^1, \\ &[0, 4, 3; \infty, 5, 2, 6]_6^1, [0, 6, 2; \infty, 3, 1, 5]_6^1, [0, 6, \infty; 4, 3, 1, 2]_6^1\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 7)$ , form an  $S_6^1$ -decomposition of  $3M_8$ . Consider the blocks

$$\begin{aligned} &\{[\infty, 0, 1; 2, 3, 4, 5]_6^1, 2 \times [0, \infty, 1; 6, 3, 4, 5]_6^1, [0, 1, 5; \infty, 2, 3, 4]_6^1, [0, 2, 4; \infty, 3, 5, 6]_6^1, \\ &[0, 1, 3; \infty, 2, 5, 6]_6^1, [0, 2, 3; 5, \infty, 1, 6]_6^1, [0, 2, 3; 5, 1, 4, 6]_6^1\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 7)$ , form an  $S_6^1$ -decomposition of  $4M_8$ . Consider the blocks

$$\begin{aligned} &\{[\infty, 0, 1; 2, 3, 4, 5]_6^1, [0, \infty, 1; 6, 2, 3, 4]_6^1, [0, \infty, 2; 6, 3, 4, 5]_6^1, [0, \infty, 3; 4, 1, 2, 5]_6^1, \\ &[0, 1, 2; \infty, 3, 4, 5]_6^1, [0, 2, 3; \infty, 4, 5, 6]_6^1, [0, 1, 3; \infty, 2, 4, 6]_6^1, \\ &[0, 1, 2; \infty, 3, 5, 6]_6^1, [0, 2, 3; 5, \infty, 1, 6]_6^1, [0, 1, 3; 5, \infty, 1, 6]_6^1\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 7)$ , form an  $S_6^1$ -decomposition of  $5M_8$ . Since any  $\lambda \geq 3$  can be written as a sum of a multiple of 3 and a multiple of 4 (except for 5), then an  $S_6^1$ -decomposition of  $\lambda M_8$ , where  $\lambda \geq 3$ , exists and can be constructed by taking appropriate numbers of copies of decompositions of  $3M_8$  and  $4M_8$  (with the exception of  $\lambda = 5$ , which we have dealt with separately).

Case 1b. Suppose  $v \equiv 0$  or  $1 \pmod{4}$ ,  $v \geq 9$ . Then by Theorem 1 an  $S_6^1$ -decomposition of  $M_v$  exists by Theorem 1. For any  $\lambda \geq 1$ , we take  $\lambda$  copies of the blocks of such a decomposition and this gives an  $S_6^1$ -decomposition of  $\lambda M_v$ .

Case 2a. Suppose  $v = 10$ . Consider the blocks

$$\begin{aligned} & \{[\infty, 0, 1; 2, 3, 4, 5]_6^1, [0, 1, 2; 8, 3, 4, 5]_6^1, [0, 2, 3; \infty, 1, 4, 5]_6^1, [0, 3, 4; \infty, 1, 2, 5]_6^1, \\ & [0, 6, 4; \infty, 2, 3, 5]_6^1, [0, 1, 2; 5, 6, 7, 8]_6^1, [0, 1, 4; 3, 6, 7, 8]_6^1, \\ & [0, 1, 2; 6, \infty, 7, 8]_6^1, [0, \infty, 3; 7, 6, 1, 2]_6^1, [0, \infty, 5; 6, 4, 7, 8]_6^1\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 8)$ , form an  $S_6^1$ -decomposition of  $4M_{10}$ . Consider the blocks

$$\begin{aligned} & \{2 \times [\infty, 0, 1; 2, 3, 4, 5]_6^1, [0, 1, 2; 8, 3, 4, 5]_6^1, [0, 2, 3; \infty, 1, 4, 5]_6^1, [0, 3, 4; \infty, 1, 2, 5]_6^1, \\ & [0, 6, 4; \infty, 2, 3, 5]_6^1, [0, 1, 2; \infty, 6, 7, 8]_6^1, [0, 1, 4; 3, 6, 7, 8]_6^1, [0, 1, 2; 6, 3, 7, 8]_6^1, \\ & [0, \infty, 1; 7, 2, 3, 4]_6^1, [0, \infty, 8; 1, 2, 3, 4]_6^1, [0, 2, 3; 1, 8, 4, 5]_6^1, \\ & [0, 2, 3; 7, 1, 4, 5]_6^1, [0, 3, 4; 2, 1, 6, 7]_6^1, [0, 4, 5; 3, 1, 6, 7]_6^1\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 8)$ , form an  $S_6^1$ -decomposition of  $6M_{10}$ . Since any even  $\lambda \geq 4$  can be written as a sum of a multiple of 4 and a multiple of 6, then an  $S_6^1$ -decomposition of  $\lambda M_8$ , where  $\lambda \geq 4$  is even, exists and can be constructed by taking appropriate numbers of copies of decompositions of  $4M_8$  and  $6M_8$ .

Case 2b. Suppose  $v = 18$ . Consider the blocks

$$\begin{aligned} & [0, 1, \infty; 16, 2, 3, 4]_6^1, [0, 1, \infty; 16, 2, 3, 4]_6^1, [0, 2, 3; 12, 6, 7, \infty]_6^1, \\ & [0, 2, 3; 9, 8, 10, \infty]_6^1, [0, 4, 5; 8, 9, 10, 11]_6^1, [0, 4, 5; \infty, 11, 12, 13]_6^1, \\ & [0, 6, 7; \infty, 12, 13, 14]_6^1, [0, 7, 8; 3, 15, 16, 5]_6^1, [0, 6, 8; 11, 7, 15, 16]_6^1. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 16)$ , form an  $S_6^1$ -decomposition of  $2M_{18}$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^1$ -decomposition of  $\lambda M_{18}$ .

Case 2c. Suppose  $v \equiv 2 \pmod{8}$ , say  $v = 8k + 2$  where  $k \geq 3$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{2 \times [0, \infty, 8k; 8k - 3, 1, 2, 3]_6^1, 2 \times [0, 2, 3; \infty, 5, 6, 7]_6^1, [0, 4, 5; 8k - 9, 8, 9, \infty]_6^1, \\ & [0, 5, 6; 8k - 9, 8, 12, \infty]_6^1, [0, 4, 6; 8k - 10, 9, 11, 12]_6^1, [0, 7, 8k - 6; 8k - 12, 13, 17, 21]_6^1, \\ & [0, 8, 8k - 7; 8k - 13, 14, 15, 22]_6^1, [0, 9, 8k - 8; 8k - 17, 18, 19, 22]_6^1, \\ & [0, 10, 8k - 9; 8k - 19, 17, 20, 24]_6^1, [0, 11, 8k - 10; 8k - 15, 16, 21, 24]_6^1, \\ & [0, 12, 8k - 11; 8k - 22, 15, 19, 23] \} \cup \{[0, 13 + 4i, 8k - 12 - 4i; 7 + 8i, \\ & 31 + 8i, 26 + 8i, 30 + 8i]_6^1, [0, 14 + 4i, 8k - 13 - 4i; 8 + 8i, 28 + 8i, 25 + 8i, 29 + 8i]_6^1, \\ & [0, 15 + 4i, 8k - 14 - 4i; 1 + 8i, 28 + 8i, 29 + 8i, 32 + 8i]_6^1, [0, 16 + 4i, 8k - 15 - 4i; 6 + 8i, \\ & 27 + 8i, 30 + 8i, 31 + 8i]_6^1 \text{ for } i = 0, 1, \dots, k - 4\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 8k)$ , form an  $S_6^1$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^1$ -decomposition of  $\lambda M_v$ .

**Case 3.** Suppose  $v \equiv 3 \pmod{4}$ ,  $v \geq 7$ , say  $v = 4k + 3$  where  $k \geq 1$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{[0, 4k + 2, 4k + 1; 4k - 1, 1, 2, 3]_6^1, [0, 2k + 1, 2k + 2; 4k - 1, 2, 4k + 1, 4k + 2]_6^1\} \\ & \cup \{[0, 3 + 2i, 4 + 2i; 4k - 4 - 4i, 8 + 4i, 9 + 4i, 10 + 4i]_6^1 \text{ for } i = 0, 1, \dots, k - 2 \text{ where} \\ & \quad i \neq (2k - 4)/3 \text{ and } i \neq (k - 3)/2\} \cup \{[0, 4k + 2, 2; 4k, 1, 5, 6]_6^1\} \\ & \cup \left\{ \left[ 0, \frac{4k + 1}{3}, \frac{8k + 5}{3}; \frac{4k + 4}{3}, \frac{8k + 8}{3}, \frac{8k + 11}{3}, \frac{8k + 14}{3} \right]_6^1 \text{ if } i = \frac{2k - 4}{3} \right\} \\ & \cup \left\{ [0, k, k + 1; 2k, 2k + 2, 2k + 1, 2k + 4]_6^1 \text{ if } i = \frac{k - 3}{2} \right\} \\ & \cup \{[0, 3 + 2i, 4 + 2i; 4k - 2 - 4i, 6 + 4i, 7 + 4i, 8 + 4i]_6^1 \text{ for } i = 0, 1, \dots, k - 2 \text{ where} \\ & \quad i \neq (2k - 3)/3 \text{ and } i \neq (k - 2)/2\} \\ & \cup \left\{ \left[ 0, \frac{4k + 3}{3}, \frac{8k + 3}{3}; \frac{4k + 6}{3}, \frac{8k + 6}{3}, \frac{8k + 9}{3}, \frac{8k + 12}{3} \right] \text{ if } i = (2k - 3)/3 \right\} \\ & \cup \{[0, k + 1, k + 2; 2k, 2k + 2, 2k + 1, 2k + 4]_6^1 \text{ if } i = (k - 2)/2\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^1$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^1$ -decomposition of  $\lambda M_v$ .

**Case 4.** Suppose  $v \equiv 6 \pmod{8}$ ,  $v \geq 14$ , say  $v = 8k + 6$  where  $k \geq 1$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{2 \times [0, \infty, 8k + 4; 8k + 1, 1, 2, 3]_6^1, 2 \times [0, 2, 3; \infty, 5, 6, 7]_6^1, [0, 4, 5; 8k - 5, 8, 9, \infty]_6^1, \\ & [0, 5, 6; 8k - 5, 8, 12, \infty]_6^1, [0, 4, 6; 8k - 6, 9, 11, 12]_6^1\} \cup \{[0, 7 + 4i, 8k - 2 - 4i; 8 + 8i, 16 + 8i, \\ & \quad 13 + 8i, 17 + 8i]_6^1, [0, 8 + 4i, 8k - 3 - 4i; 7 + 8i, 19 + 8i, 14 + 8i, 18 + 8i]_6^1, \\ & [0, 9 + 4i, 8k - 4 - 4i; 6 + 8i, 15 + 8i, 18 + 8i, 19 + 8i]_6^1, [0, 10 + 4i, 8k - 5 - 4i; \\ & \quad 1 + 8i, 16 + 8i, 17 + 8i, 20 + 8i]_6^1 \text{ for } i = 0, 1, \dots, k - 2\}. \end{aligned}$$

These blocks, along with their images under the powers of permutation  $(\infty)(0, 1, \dots, 8k + 4)$ , form an  $S_6^1$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^1$ -decomposition of  $\lambda M_v$ .  $\square$

Since  $\lambda M_v$  is self-converse and the converse of  $S_6^1$  is  $S_6^3$ , then an  $S_6^3$ -decomposition of  $\lambda M_v$  exists if and only if an  $S_6^1$ -decomposition of  $\lambda M_v$  exists. So, the conditions for the existence of an  $S_6^1$ -decomposition of  $\lambda M_v$ , given in Theorem 3, are also the conditions for the existence of an  $S_6^3$ -decomposition of  $\lambda M_v$  (with the exceptions of  $v = 8$  and  $v = 10$  when  $\lambda = 2$ ).

**Theorem 4.** An  $S_6^2$ -decomposition of  $\lambda M_v$  exists if and only if  $v \geq 7$  and

1.  $v \equiv 0$  or  $1 \pmod{4}$  and  $\lambda \geq 1$ , where  $v \neq 8$  in the case  $\lambda = 1$ , or
2.  $v \equiv 2 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ , or
3.  $v \equiv 3 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ .

**Proof.** Theorem 1 and Lemma 1 give the necessary conditions. We now show these necessary conditions are sufficient.

Case 1a. Suppose  $v \equiv 0$  or  $1 \pmod{4}$ ,  $v \neq 8$ . Then by Theorem 1 an  $S_6^2$ -decomposition of  $M_v$  exists by Theorem 1. For any  $\lambda \geq 1$ , we take  $\lambda$  copies of the blocks of such a decomposition and this gives an  $S_6^2$ -decomposition of  $\lambda M_v$ .

Case 1b. Suppose  $v = 8$ . Take the blocks of an  $S_6^2$ -decomposition of  $2M_7$  (which exists by Case 1a), where the vertices of  $2M_7$  are  $0, 1, \dots, 6$ . To this, add the images of  $[\infty, 0, 1; 2, 3, 4, 5]_6^2$  under the permutation  $(\infty)(0, 1, \dots, 6)$ . This gives an  $S_6^2$ -decomposition of  $2M_8$ , where the vertex set is  $\{\infty, 0, 1, \dots, 6\}$ . Next, consider the blocks:

$$\begin{aligned} & \{[\infty, 0, 1; 2, 3, 4, 5]_6^2, [0, 1, 2; 3, 4, 5, 6]_6^2, [0, 2, 3; 5, 6, 1, 4]_6^2, \\ & [0, 1, 3; 2, 4, \infty, 6]_6^2, [0, \infty, 3; 5, 4, 1, 2]_6^2, [0, 1, 2; 3, \infty, 5, 6]_6^2\}. \end{aligned}$$

These blocks, along with their images under permutation  $(\infty)(0, 1, \dots, 6)$ , form an  $S_6^2$ -decomposition of  $3M_8$ . Since any  $\lambda \geq 2$  can be written as a sum of a multiple of 2 and a multiple of 3, then an  $S_6^2$ -decomposition of  $\lambda M_8$ , where  $\lambda \geq 2$ , exists and can be constructed by taking appropriate numbers of copies of decompositions of  $2M_8$  and  $3M_8$ .

Case 2a. Suppose  $v = 10$ . Take the blocks of an  $S_6^2$ -decomposition of  $2M_9$  (which exists by Case 1a), where the vertices of  $M_9$  are  $0, 1, \dots, 8$ . To this, add the images of  $[\infty, 0, 1; 2, 3, 4, 5]_6^2$  under the permutation  $(\infty)(0, 1, \dots, 8)$ . This gives an  $S_6^2$ -decomposition of  $2M_{10}$ , where the vertex set is  $\{\infty, 0, 1, \dots, 8\}$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^2$ -decomposition of  $\lambda M_{10}$  for all even  $\lambda$ .

Case 2b. Suppose  $v = 18$ . Consider the blocks

$$\begin{aligned} & \{[0, 1, \infty; 16, 15, 3, 4]_6^2, [0, 1, \infty; 16, 15, 3, 4]_6^2, [0, 2, 3; 12, 11, 7, \infty]_6^2, \\ & [0, 2, 3; 9, 7, 8, \infty]_6^2, [0, 4, 5; 8, 7, 9, 11]_6^2, [0, 4, 5; \infty, 6, 12, 13]_6^2, \\ & [0, 6, 7; \infty, 5, 13, 14]_6^2, [0, 7, 8; 3, 2, 16, 5]_6^2, [0, 6, 8; 11, 10, 15, 16]_6^2\}. \end{aligned}$$

These blocks, along with their images under permutation  $(\infty)(0, 1, \dots, 16)$ , form an  $S_6^2$ -decomposition of  $2M_{18}$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^2$ -decomposition of  $\lambda M_{18}$  for all even  $\lambda$ .

Case 2c. Suppose  $v \equiv 2 \pmod{8}$ , say  $v = 8k + 2$  where  $k \geq 3$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{2 \times [0, \infty, 8k; 8k - 3, 8k - 2, 1, 2]_6^2, 2 \times [0, 2, 3; \infty, 8k - 4, 6, 7]_6^2, [0, 4, 5; 8k - 9, 8k - 7, 9, \infty]_6^2, \\ & [0, 5, 6; 8k - 9, 8k - 7, 12, \infty]_6^2, [0, 4, 6; 8k - 10, 8k - 8, 11, 12]_6^2, \\ & [0, 7, 8k - 6; 8k - 12, 8k - 16, 13, 21]_6^2, [0, 8, 8k - 7; 8k - 13, 8k - 14, 14, 22]_6^2, \\ & [0, 9, 8k - 8; 8k - 17, 8k - 18, 18, 22]_6^2, [0, 10, 8k - 9; 8k - 19, 8k - 16, 20, 24]_6^2, \\ & [0, 11, 8k - 10; 8k - 15, 8k - 20, 16, 24]_6^2, [0, 12, 8k - 11; 8k - 22, 8k - 14, 19, 23]_6^2\} \\ & \cup \{[0, 13 + 4i, 8k - 12 - 4i; 7 + 8i, 8k - 29 - 8i, 31 + 8i, 26 + 8i]_6^2 \text{ for } i = 0, 1, \dots, k - 4\} \\ & \cup \{[0, 14 + 4i, 8k - 13 - 4i; 8 + 8i, 8k - 27 - 8i, 25 + 8i, 29 + 8i]_6^2 \\ & \quad \text{for } i = 0, 1, \dots, k - 4 \text{ and } i \neq (k - 7)/2\} \\ & \cup \{[0, 2k, 6k + 1; 4k - 20, 4k + 4, 4k, 4k + 1]_6^2 \text{ if } i = (k - 7)/2\} \\ & \cup \{[0, 15 + 4i, 8k - 14 - 4i; 1 + 8i, 8k - 27 - 8i, 29 + 8i, 32 + 8i]_6^2 \\ & \quad \text{for } i = 0, 1, \dots, k - 4, \text{ and } i \neq (k - 7)/2\} \\ & \cup \{[0, 2k + 1, 6k; 4k - 27, 4k - 3, 4k + 1, 4k]_6^2 \text{ if } i = (k - 7)/2\} \\ & \cup \{[0, 16 + 4i, 8k - 15 - 4i; 6 + 8i, 8k - 29 - 8i, 27 + 8i, 31 + 8i]_6^2 \\ & \quad \text{for } i = 0, 1, \dots, k - 4, \text{ and } i \neq (k - 7)/2\} \end{aligned}$$

$$\cup \{[0, 2k + 2, 6k - 1; 4k - 22, 4k - 2, 4k - 1, 4k + 2]_6^2 \text{ if } i = (k - 7)/2\}.$$

These blocks, along with their images under the powers of permutation  $(\infty)(0, 1, \dots, 8k + 4)$ , form an  $S_6^2$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^2$ -decomposition of  $\lambda M_v$ .

**Case 3.** Suppose  $v \equiv 3 \pmod{4}$ ,  $v \geq 7$ , say  $v = 4k + 3$  where  $k \geq 1$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{[0, 4k + 2, 4k + 1; 4k - 1, 4k, 1, 2]_6^2, [0, 2k + 1, 2k + 2; 4k - 1, 4k + 2, 2, 4k + 1]_6^2, \\ & [0, 1, 2; 4k, 4k - 3, 4k + 2, 5]_6^2\} \cup \{[0, 3 + 2i, 4 + 2i; 4k - 4 - 4i, 4k - 5 - 4i, 9 + 4i, 10 + 4i]_6^2 \\ & \text{for } i = 0, 1, \dots, k - 2 \text{ where } i \neq (2k - 4)/3\} \\ & \cup \{[0, (4k + 1)/3, (8k + 5)/3; (4k + 4)/3, (4k - 5)/3, (8k + 8)/3, (8k + 11)/3]_6^2 \\ & \text{if } i = (2k - 4)/3\} \\ & \cup \{[0, 3 + 2i, 4 + 2i; 4k - 2 - 4i, 4k - 3 - 4i, 7 + 4i, 8 + 4i]_6^2 \\ & \text{for } i = 0, 1, \dots, k - 2 \text{ where } i \neq (2k - 3)/3\} \\ & \cup \{[0, (4k + 3)/3, (8k + 3)/3; (4k + 6)/3, (4k - 3)/3, (8k + 6)/3, (8k + 9)/3]_6^2 \\ & \text{if } i = (2k - 3)/3\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $\pi$ , form an  $S_6^2$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^2$ -decomposition of  $\lambda M_v$ .

**Case 4.** Suppose  $v \equiv 6 \pmod{8}$ ,  $v \geq 14$ , say  $v = 8k + 6$  where  $k \geq 1$ , and  $\lambda \equiv 0 \pmod{2}$ . Consider the blocks

$$\begin{aligned} & \{2 \times [0, \infty, 8k + 4; 8k + 1, 8k + 2, 1, 2]_6^2, 2 \times [0, 2, 3; \infty, 8k, 6, 7]_6^2, [0, 5, 6; 8k - 3, 8k - 5, 12, \infty]_6^2 \\ & [0, 4, 5; 8k - 5, 8k - 3, 9, \infty]_6^2, [0, 4, 6; 8k - 6, 8k - 4, 11, 12]_6^2\} \\ & \{[0, 7 + 4i, 8k - 2 - 4i; 8 + 8i, 8k - 11 - 8i, 13 + 8i, 17 + 8i]_6^2 \\ & \text{for } i = 0, 1, \dots, k - 2 \text{ and } i \neq (k - 3)/2\} \\ & \cup \{[0, 2k + 1, 6k + 4; 4k - 4, 4k, 4k + 1, 4k + 4]_6^2 \text{ if } i = (k - 3)/2\} \\ & \cup \{[0, 8 + 4i, 8k - 3 - 4i; 7 + 8i, 8k - 14 - 8i, 14 + 8i, 18 + 8i]_6^2 \\ & \text{for } i = 0, 1, \dots, k - 2 \text{ and } i \neq (k - 4)/2\} \\ & \cup \{[0, 2k, 6k + 5; 4k - 9, 4k + 7, 4k + 3, 4k + 2]_6^2 \text{ if } i = (k - 4)/2\} \\ & \cup \{[0, 9 + 4i, 8k - 4 - 4i; 6 + 8i, 8k - 10 - 8i, 18 + 8i, 19 + 8i]_6^2 \\ & \text{for } i = 0, 1, \dots, k - 2 \text{ and } i \neq (k - 2)/2\} \\ & \cup \{[0, 2k + 5, 6k; 4k - 5, 4k - 2, 4k + 7, 4k + 11] \text{ if } i = (k - 2)/2\} \\ & \cup \{[0, 10 + 4i, 8k - 5 - 4i; 1 + 8i, 8k - 11 - 8i, 17 + 8i, 20 + 8i]_6^2 \text{ for } i = 0, 1, \dots, k - 2\}. \end{aligned}$$

These blocks, along with their images under powers of permutation  $(\infty)(0, 1, \dots, 8k + 4)$ , form an  $S_6^2$ -decomposition of  $2M_v$ . By taking  $\lambda/2$  copies of the blocks of such a decomposition, we obtain an  $S_6^2$ -decomposition of  $\lambda M_v$ .  $\square$

In conclusion, Theorems 2–4 (along with the observations about converses) give necessary and sufficient conditions for an  $S_6^2$ -decomposition of  $\lambda M_v$  for each  $0 \leq i \leq 4$ , with the exceptions of the small cases  $v = 8$  and  $v = 10$  when  $i \in \{1, 3\}$  and  $\lambda = 2$ .

### 3. Discussion

In this paper, a classification of mixed 6-star decompositions of the complete  $\lambda$ -fold mixed graph is given for all  $\lambda > 1$ .

Graph decompositions have applications in coding theory, crystallography, radio astronomy, radio location, communication networks, and other fields [20]. Graph designs have their beginning in the design and analysis of statistical experiments, but now also have applications in tournament scheduling, mathematical biology, algorithmic design, and cryptography [26]. There has been a flurry of recent activity in applications of graph theory in general to neural networks [5]. The study of mixed graph decompositions and mixed graph designs is relatively new, but it is expected that they will have similar applications. For example, a network of roads can be represented by a mixed graph, where the edges represent two-way roads and arcs represent one-way roads. Notice that, in the event of a two-way main road connecting two locations where there are one-way access roads on either side of the main road, these connections can be represented by an edge and two anti-parallel arcs joining the locations. This example shows that there is potential for applications of mixed graphs in network theory, where now only graphs and digraphs play a role [27] (notice Chapters 2 and 3 on graphs and digraphs).

Since the area of mixed graph decompositions is little studied, there is significant potential for future research. The approach presented here, as explained in Sections 2.1 and 2.2, could be applied, for example, to additional mixed star decompositions of complete mixed graphs and  $\lambda$ -fold complete mixed graphs. Since a mixed star in such a decomposition must have twice as many arcs as edges, then this requires a partially oriented mixed star,  $S_{3n}$ , with  $2n$  arcs. There are  $2n + 1$  such partial orientations of  $S_{3n}$ . Constructions of  $S_{3n}$ -decompositions of  $M_v$  (and of  $\lambda M_v$ ) should lend themselves, at least in part, to the approach used here. Decompositions of complete bipartite mixed graphs (and  $\lambda$ -fold versions thereof) are unsolved; in fact, they could form part of a recursive construction for decompositions of  $M_v$  and  $\lambda M_v$ . The complete mixed graph with a hole also offers an opportunity for various decompositions. When considering these different types of complete mixed graphs, or any other type (in which any two vertices are either not adjacent or are joined by one edge and two distinct arcs), there is the possibility for a decomposition into partial orientations of stars.

We have used cyclic and rotational permutations in our approach to the constructions of Section 2. This motivates the study of star decompositions of  $M_v$  (and  $\lambda M_v$ ), which admit certain permutations as automorphisms. In addition to cyclic and rotational permutations, Steiner triple systems have been studied for the existence of  $k$ -rotational automorphisms (such automorphisms consist of one fixed point and  $k$  disjoint cycles of the same length; therefore a “rotational” automorphism is a special case of a  $k$ -rotational automorphism, since it is just a 1-rotational automorphism) [25]. The question of  $k$ -rotational mixed star designs is unaddressed (except for the parts of the special case  $k = 1$ , which appears in this work). A reverse permutation on a set of even size  $2n$  consists of  $n$  disjoint transpositions, and on an odd set of size  $2n + 1$  consists of one fixed point and  $n$  disjoint transpositions. Steiner triple systems admitting a reverse automorphism have been considered [28]. A bicyclic permutation is one consisting of two disjoint cycles, and a tricyclic permutation is one consisting of three disjoint cycles. Future research on mixed star designs (closely related to the topic of this paper) could include the conditions under which they admit reverse, bicyclic, or tricyclic automorphisms. Many of the other design theory ideas, such as automorphism groups, packings, coverings, and embeddings, are available for study in the setting of mixed star designs. The area is fertile for further exploration.

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## List of Symbols

The following table includes the symbols used in this paper and their meaning.

$G = (V, E)$	Graph with vertex set $V$ and edge set $E$
$\{u, v\} = uv$	Edge joining vertices $u$ and $v$
$K_v$	Complete graph on $v$ vertices
$C_v$	Cycle of length $v$
$S_v$	Star on $v + 1$ vertices
$\lambda K_v$	The $\lambda$ -fold complete graph on $v$ vertices
$D = (V, A)$	Digraph with vertex set $V$ and arc set $A$
$\text{init}(a)$	Initial vertex of arc $a$
$\text{ter}(a)$	Terminal vertex of arc $a$
$(u, v)$	The arc with initial vertex $u$ and terminal vertex $v$
$D_v$	The complete digraph on $v$ vertices
$\lambda D_v$	The $\lambda$ -fold complete digraph on $v$ vertices
$M = (V, E, A)$	Mixed graph with vertex set $V$ , edge set $E$ , and arc set $A$
$M_v$	The complete mixed graph on $v$ vertices
$\lambda M_v$	The $\lambda$ -fold complete mixed graph
$F = (V, E, \sigma, \mu)$	Fuzzy graph with vertex set $V$ , edge set $E$ , fuzzy edge set $\sigma$ , and fuzzy vertex set $\mu$
$\gamma = \{g_1, \dots, g_n\}$	A $g$ -decomposition of graph $G$
$\gamma = \{d_1, \dots, d_n\}$	A $d$ -decomposition of digraph $D$
$\gamma = \{m_1, \dots, m_n\}$	A $m$ -decomposition of mixed graph $M$

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