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Review

Interval Quadratic Equations: A Review

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Abstract: In this study, we tackle the subject of interval quadratic equations and we aim to accurately determine the root enclosures of quadratic equations, whose coefficients constitute interval variables. This study focuses on interval quadratic equations that contain only one coefficient considered as an interval variable. The four methods reviewed here in order to solve this problem are: (i) the method of classic interval analysis used by Elishakoff and Daphnis, (ii) the direct method based on minimizations and maximizations also used by the same authors, (iii) the method of quantifier elimination used by Ioakimidis, and (iv) the interval parametrization method suggested by Elishakoff and Miglis and again based on minimizations and maximizations. We will also compare the results yielded by all these methods by using the computer algebra system *Mathematica* for computer evaluations (including quantifier eliminations) in order to conclude which method would be the most efficient way to solve problems relevant to interval quadratic equations.

Keywords: interval quadratic equations; interval coefficients; interval variables; uncertainty; uncertain variables; real roots; sharp enclosures; classic interval analysis; direct method; quantifier elimination; interval parametrization; symbolic computations

MSC: 26C10; 12D10; 65G40; 03C10

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1. Introduction

In *A History of Mathematics* [1], Uta C. Merzbach and Carl B. Boyer stated that "Egyptian algebra had been much concerned with linear equations, but the Babylonians evidently found these too elementary for much attention" [1] (p. 28) and further that "The solution of a three-term quadratic equation seems to have exceeded by far the algebraic capabilities of the Egyptians, but Otto Neugebauer in 1930 disclosed that such equations had been handled effectively by the Babylonians in some of the oldest problem texts." [1] (p. 29).

As was known apparently from the 9th century, the classic quadratic equation

$$ax^2 + bx + c = 0 \quad \text{evidently with} \quad a \neq 0, \tag{1}$$

where a, b and c are its coefficients and x denotes an unknown, has two roots given by the well-known formulas

$$x_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

Here we assume that $x_1 < x_2$ in the case of distinct real roots. This is the case when $b^2 - 4ac > 0$.

According to Steven Strogatz [2] (Section 10), " THE QUADRATIC FORMULA is the Rodney Dangerfield of algebra. Even though it's one of the all-time greats, it don't get no respect". This paper remedies the above situation in a certain sense and it discusses the interval quadratic equation, respectfully reviewing the subject and providing some new solutions. We deal with an interval version of the quadratic equation with one coefficient in Equation (1) being an interval variable. Interval quadratic equations have also been treated multiple times in the literature. For example, Buckley and Eslami [3] treated issues related to fuzzy quadratic equations using neural networks. Mamehrashi [4] solved dual interval quadratic equations with the interval extended zero method introduced by Sevastjanov and Dymova [5]. Landowski [6] compared the results yielded by classic interval analysis following Moore's principles [7] with the RDM method, which stands for Relative Distance Measure. In fact, Landowski's method is a similar way to parametrize an interval to what Elishakoff and Miglis introduced as interval parametrization method [8]. The results by Hansen and Walster [9], Alolyan [10] and Piegat and Pluciński [11] are also of interest in interval quadratic equations.

In this article, we will focus on solving interval quadratic equations of the form (1) in which one of the three coefficients (a, b or c) is an interval variable. In this case, we will determine the lower bounds $\underline{x}_{1,2}$ and upper bounds $\overline{x}_{1,2}$ of its roots $x_{1,2}$, thus obtaining their enclosures in closed form. The four methods reviewed and used here for the determination of these bounds $\underline{x}_{1,2}$ and $\overline{x}_{1,2}$ are

- The method of classic interval analysis used by Elishakoff and Daphnis;
- The direct method based on minimizations and maximizations also used by the same authors;
- The method of quantifier elimination used by Ioakimidis;
- The interval parametrization method suggested by Elishakoff and Miglis and again based on minimizations and maximizations.

Beyond the analytical formulas derived by these four methods, we will also apply them to three practical problems and we will compare their efficiency in the computation of the sought bounds and, therefore, of the enclosures of the roots $x_{1,2}$ as well.

More explicitly, the present paper is organized as follows: Section 1 is the present introductory section to interval quadratic equations of the form (1) with one interval parameter. Sections 2–4 concern the determination of the bounds of the two roots $x_{1,2}$ of the interval quadratic Equation (1) when its interval coefficient is the coefficient a, the coefficient b, and the coefficient c, respectively. Section 5 concerns applications to some generalized Babylonian problems involving interval quadratic equations. Section 6 concerns an analogous application to a classical but practical as well kinematics problem concerning a bus and a pedestrian trying to catch the bus. Section 7 concerns an elementary simply-supported beam problem concerning its bending moment with a generalization of the method of quantifier elimination to the deflection of the beam. Finally, Section 8 concerns the conclusions drawn from the present review on interval quadratic equations.

2. Interval Coefficient *a* in the Quadratic Equation

First, we report the results furnished by classic interval analysis [7]. Here we consider the simplest case with the coefficient (parameter) a in Equation (1) being an interval variable, that is [12] (p. 1026)

$$a \in A := [\,\underline{a}, \overline{a}\,] \tag{3}$$

whereas the other two coefficients (parameters) b and c are deterministic quantities. As usual in interval analysis, in Equation (3) and in general in this paper, an underlined symbol (here $\underline{a}$) designates the left endpoint of an interval (here the closed interval $[\,\underline{a}, \overline{a}\,]$ of the coefficient a of the quadratic Equation (1) whereas an overlined symbol (here $\overline{a}$) denotes the right endpoint of the same interval.

Elishakoff and Daphnis [12] (Subsection 3.1, p. 1026, Equations (14)–(17)) list the following formulas for the four endpoints $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the intervals $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ of the two real roots x_1 and x_2 respectively (or bounds of these roots) displayed in

Equation (2) under the obvious assumption that the interval coefficient a in Equation (3) is either positive or negative, but different from zero ($a \neq 0$):

$$\underline{x}_1 = \frac{-b - \sqrt{b^2 - 4\underline{a}c}}{2\underline{a}}, \tag{4}$$

$$\overline{x}_1 = \frac{-b - \sqrt{b^2 - 4\overline{a}c}}{2\overline{a}}, \tag{5}$$

$$\underline{x}_2 = \frac{-b + \sqrt{b^2 - 4\overline{a}c}}{2\overline{a}}, \tag{6}$$

$$\overline{x}_2 = \frac{-b + \sqrt{b^2 - 4\underline{a}c}}{2\underline{a}}. \tag{7}$$

Recently, Ioakimidis applied the computational technique of quantifier elimination [13,14] to several problems related to interval analysis [7] including the determination of sharp enclosures of the two real roots $x_{1,2}$ in Equation (2) of the interval quadratic Equation (1) [15]. In the same reference [15] (Section 5), he also studied the above case, where the coefficient (parameter) a is an interval variable as in Equation (3), whereas the coefficients (parameters) b and c are deterministic quantities.

Ioakimidis [15] (Section 5, pp. 20–25) considered two different subcases: (i) $a > 0$ and (ii) $a < 0$ for the interval coefficient a, which must be different from zero ($a \neq 0$) as was already mentioned.

In the first subcase, where $a > 0$, by using the method of quantifier elimination [13,14] Ioakimidis [15] (Section 5, pp. 20–23) obtained exactly the same Formulas (4)–(7) (already derived by Elishakoff and Daphnis [12] (Subsection 3.1, p. 1026, Equations (14)–(17))) for the four endpoints (bounds of the roots $x_{1,2}$) $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$. In this first subcase, where $a > 0$, the related assumptions are

$$\begin{aligned} \mathcal{A}_{a,a>0} \coloneqq a > 0 \wedge \underline{a} > 0 \wedge \overline{a} > 0 \wedge b \neq 0 \wedge c \neq 0 \wedge \underline{a} \leq a \leq \overline{a} \wedge \underline{a} \leq \overline{a} \\ \wedge\, b^2 - 4ac > 0 \wedge b^2 - 4\underline{a}c > 0 \wedge b^2 - 4\overline{a}c > 0. \end{aligned} \tag{8}$$

On the other hand, next, in the second subcase, where $a < 0$, we make the assumptions [15] (p. 25, Equation (120))

$$\begin{aligned} \mathcal{A}_{a,a<0} \coloneqq a < 0 \wedge \underline{a} < 0 \wedge \overline{a} < 0 \wedge b \neq 0 \wedge c \neq 0 \wedge \underline{a} \leq a \leq \overline{a} \wedge \underline{a} \leq \overline{a} \\ \wedge\, b^2 - 4ac > 0 \wedge b^2 - 4\underline{a}c > 0 \wedge b^2 - 4\overline{a}c > 0. \end{aligned} \tag{9}$$

In both subcases ($a > 0$ or $a < 0$), our assumptions (either the assumptions $\mathcal{A}_{a,a>0}$ in Equation (8) or the assumptions $\mathcal{A}_{a,a<0}$ in Equation (9)) can be simplified to equivalent assumptions $\mathcal{A}^*_{a,a>0}$ or $\mathcal{A}^*_{a,a<0}$, respectively, but with fewer conjunctive terms (here, inequalities). In this case, although the resulting quantifier-free formulas will not change remaining correct, however, the required CPU times in the computer significantly increase during some, but not all, of the required quantifier eliminations since the computer algebra system (here, *Mathematica* [16]) has to derive the missing inequalities. For example, the assumptions $\mathcal{A}_{a,a>0}$ defined in Equation (8) can be replaced by the simpler assumptions

$$\mathcal{A}^*_{a,a>0} := a > 0 \wedge \underline{a} > 0 \wedge b \neq 0 \wedge c \neq 0 \wedge \underline{a} \leq a \leq \overline{a} \wedge b^2 - 4\overline{a}c > 0 \tag{10}$$

containing fewer conjunctive terms (here, inequalities), but they are equivalent to the initial assumptions $\mathcal{A}_{a,a>0}$ as is easily verified since $\underline{a} \leq a \leq \overline{a}$. Then we obtain the same quantifier-free formulas, essentially the bounds (endpoints) (4)–(7), but for the bounds (4) and (7) (only for these bounds) in much more required CPU times (about 29 times and 41 times more CPU time, respectively).

Now under the above assumptions $\mathcal{A}_{a,a<0}$ defined in Equation (9) the method of quantifier elimination yields [15] (Section 5, p. 25):

$$\underline{x}_1 = \frac{-b + \sqrt{b^2 - 4\overline{a}c}}{2\overline{a}}, \tag{11}$$

$$\overline{x}_1 = \frac{-b + \sqrt{b^2 - 4\underline{a}c}}{2\underline{a}}, \tag{12}$$

$$\underline{x}_2 = \frac{-b - \sqrt{b^2 - 4\underline{a}c}}{2\underline{a}}, \tag{13}$$

$$\overline{x}_2 = \frac{-b - \sqrt{b^2 - 4\overline{a}c}}{2\overline{a}}. \tag{14}$$

At this point, we remark that a second and perhaps simpler approach of working in the present case, i.e., in the case of a negative interval coefficient $a \in A$ (with $a < 0$), is simply to transform the initial interval quadratic Equation (1) under consideration (here with $a < 0$) to an equivalent interval quadratic equation having the same distinct real roots $x_{1,2}$, but now with $a > 0$. This change can easily be achieved through a multiplication of the initial interval quadratic Equation (1) by -1. This approach is studied in more detail below in Section 4 concerning an interval coefficient $c \in C$ in the interval quadratic Equation (1). This multiplication (by -1) is clarified in Equations (174) of Section 4.

The above results (11)–(14) coincide with the results presented by Elishakoff and Daphnis [12] (p. 1026), i.e., with Equations (4)–(7) here, provided that we omit the need to assume the validity of the inequality $x_1 < x_2$. Indeed, we see that the formulas found by Ioakimidis [15] (Section 5, pp. 20–23, 25) differ when the interval coefficient a in the interval quadratic Equation (1) is positive or negative. On the contrary, the formulas found by Elishakoff and Daphnis [12] (p. 1026) using classic interval analysis [7] do not distinguish between these two cases ($a > 0$ and $a < 0$) for the interval coefficient a. In the case $a < 0$, this situation would lead to the inequality $x_2 < x_1$ instead of $x_1 < x_2$.

More explicitly, in the case of two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1) (of course always with $a \neq 0$ and a discriminant $\Delta := b^2 - 4ac > 0$ for the existence of two distinct real roots $x_{1,2}$), if we wish the validity of the inequality $x_1 < x_2$, then we must use (i) the first (left) pair of Formula (2) for these roots $x_{1,2}$ if $a > 0$ and (ii) the second (right) pair of Formula (2) for the same roots if $a < 0$. Elishakoff and Daphnis [12] (Subsection 3.1, p. 1026) used only the first (left) pair of Formula (2) in their results [12] (Subsection 3.1, p. 1026, Equations (14)–(17)). Therefore, these results yield the inequalities (i) $x_1 < x_2$ if $a > 0$ and (ii) $x_2 < x_1$ if $a < 0$. On the contrary, Ioakimidis [15] (Section 5, pp. 20–25) preferred to separately study the two cases (i) $a > 0$ and (ii) $a < 0$ by using the method of quantifier elimination [13,14] and the computer algebra system *Mathematica* [16]. Therefore, the results of Ioakimidis [15] (Section 5, pp. 20–25) coincide with those of Elishakoff and Daphnis [12] (Subsection 3.1, p. 1026), which have also been displayed in Equations (4)–(7) here, only when $a > 0$ as has already been mentioned.

We now turn to interval parametrization analysis introduced by Elishakoff and Miglis [8]. We consider the same problem with the same interval for the interval coefficient (parameter) a as given in Equation (3) and the other two coefficients (parameters) b and c remaining deterministic quantities.

Now we need to introduce two auxiliary quantities in order to parametrize the interval $A := [\underline{a}, \overline{a}]$ of the interval coefficient $a \in A$. These quantities are (i) the average value (the midpoint or center) of the interval A, here denoted by the symbol a_{ave} and (ii) the deviation value (the radius) of the same interval A here denoted by the symbol a_{dev}, that is

$$a_{\text{ave}} \coloneqq \frac{\underline{a} + \overline{a}}{2}, \tag{15}$$

$$a_{\text{dev}} \coloneqq \frac{\overline{a} - \underline{a}}{2}. \tag{16}$$

Now, by using these equations, Equations (15) and (16), we can rewrite the interval coefficient a as

$$a = a_{\text{ave}} + a_{\text{dev}}\, t \tag{17}$$

with

$$t \in T := [-1, 1]. \tag{18}$$

As a consequence, by using Equation (17) we can express the two roots $x_{1,2}$ (with $x_1 < x_2$ if $a > 0$) in the first pair of Equation (2) of the present interval quadratic Equation (1) as

$$x_1 = \frac{-b - \sqrt{b^2 - 4(a_{\text{ave}} + a_{\text{dev}}\, t)c}}{2(a_{\text{ave}} + a_{\text{dev}}\, t)}, \tag{19}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4(a_{\text{ave}} + a_{\text{dev}}\, t)c}}{2(a_{\text{ave}} + a_{\text{dev}}\, t)}. \tag{20}$$

Of course, if $a < 0$, it is preferable to use the completely analogous formulas

$$x_1 = \frac{-b + \sqrt{b^2 - 4(a_{\text{ave}} + a_{\text{dev}}\, t)c}}{2(a_{\text{ave}} + a_{\text{dev}}\, t)}, \tag{21}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4(a_{\text{ave}} + a_{\text{dev}}\, t)c}}{2(a_{\text{ave}} + a_{\text{dev}}\, t)}, \tag{22}$$

which are simply based on the second pair of roots $x_{1,2}$ of Equation (1) displayed in Equations (2) (now with the $+$ and $-$ signs in front of the square roots reversed). In this way, the inequality $x_1 < x_2$ will hold true again (but now with $a < 0$).

Finally, we can proceed to minimizations and maximizations (of course under the continuous validity of the constraint (18) for the present parameter t, that is $t \in T := [-1, 1]$ or $-1 \leq t \leq 1$) and, hence, determine the enclosures of the two roots $x_{1,2}$. For concrete interval quadratic Equations (1) these computations can be directly performed by using the commands `Minimize` and `Maximize` of *Mathematica* [16] or the analogous commands of another powerful computer algebra system.

Of course, there is no need to adopt the parametrization (17) and (18) for the derivation of the bounds of the two roots $x_{1,2}$ of the interval quadratic Equation (1) although, surely, this parametrization is a very simple and convenient one and this is the sole reason that it has been adopted here.

For example, Elishakoff and Miglis [8] adopted a parametrization based on the trigonometric function $\sin t$, more explicitly, in our present case concerning the uncertain parameter a

$$a = a_{\text{ave}} + a_{\text{dev}} \sin t \tag{23}$$

with

$$t \in T^* := \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{24}$$

but additional parametrizations are also completely acceptable and, therefore, can also be adopted. For example, we can use the rather strange parametrizaton

$$a = a_{\text{ave}} + a_{\text{dev}}\, t^5 \tag{25}$$

now with t^5 instead of t and again with parameter $t \in T := [-1, 1]$.

Another possible and useful parametrization of the present interval $A := [\underline{a}, \overline{a}]$ of the interval coefficient $a \in A$ of the interval quadratic Equation (1) is the elementary parametrization

$$a = \underline{a} + (\overline{a} - \underline{a})\, t \tag{26}$$

used in the aforementioned RDM (Relative Distance Measure) method and already applied to the interval quadratic equation by Landowski [6]. The above parametrization makes direct use of the bounds $\underline{a}$ and $\overline{a}$ of the interval coefficient a and, evidently, now $t \in T^{**} := [0, 1]$ as far as the parameter t is concerned with $a = \underline{a}$ for $t = 0$ and $a = \underline{a} + (\overline{a} - \underline{a}) \cdot 1 = \overline{a}$ for $t = 1$ as is directly verified.

Additionally, Elishakoff and Daphnis [12] (Section 3.1, p. 1026) also used the direct approach in their two numerical examples, which are also studied here just below. In the direct approach, we do not need to introduce a parameter t for the parametrization of the interval coefficient a in the interval quadratic Equation (1), but we simply work with a itself as a parameter, exactly as Elishakoff and Daphnis did in their numerical examples [12] (Section 3.1, p. 1026) by using the `minimize` and `maximize` commands of the computer algebra system *Maple* for the computation of the minimum and the maximum, respectively, of a function.

On the other hand, Ioakimidis used the computer algebra system *Mathematica* [16] for the same task, but in the general case of parametric interval quadratic equations (1) with $a > 0$ [15] (Section 5, pp. 21–23, commands c57, c59, c61 and c64). Evidently, the minimization−maximization results obtained in this way by Ioakimidis by using the direct method coincide with Equations (4)–(7) derived (i) by Elishakoff and Daphnis [12] (Section 3.1, p. 1026) by using the method of interval analysis [7] and later (ii) by Ioakimidis in the same reference [15] (Section 5, pp. 21–23) by using the alternative method of quantifier elimination [13,14]. Therefore, they constitute a verification of these results.

We will now proceed to some numerical computations in two numerical examples concerning the uncertain interval coefficient (parameter) a and we will compare the results obtained via the four techniques presented above. For illustration, at first, let us consider the following example, which was proposed by Elishakoff and Daphnis [12] (p. 1026, Equation (18)):

$$a \in A := [\underline{a}, \overline{a}] = [5, 6], \quad b = 10, \quad c = 1 \quad \text{with} \quad a_{\text{ave}} = 5.5 \quad \text{and} \quad a_{\text{dev}} = 0.5. \tag{27}$$

By using the method of classic interval analysis [7] Elishakoff and Daphnis [12] obtained from Equations (4)–(7) by also using the numerical values (27) the following numerical results [12] (p. 1026, Equations (19) and (20)):

$$\underline{x}_1 = -1.894427, \quad \overline{x}_1 = -1.559816, \tag{28}$$

$$\underline{x}_2 = -0.106850, \quad \overline{x}_2 = -0.105573. \tag{29}$$

Next, as far as the direct approach is concerned, as has been already mentioned, it comprises the straightforward minimization and maximization of the two roots $x_{1,2}$ of the interval quadratic Equation (1) in the first pair of Equations (2) (here with $a > 0$) with respect to the interval coefficient a given by the first of Equations (27) by utilizing, e.g., the `minimize` and `maximize` commands of the computer algebra system *Maple*. This method, which was proposed and used by Elishakoff and Daphnis [12] (p. 1026), yields the following results [12] (p. 1026, Equations (21) and (22)):

$$\underline{x}_1 = -1.894427 \quad (\text{at } a = \underline{a} = 5), \quad \overline{x}_1 = -1.559816 \quad (\text{at } a = \overline{a} = 6), \tag{30}$$

$$\underline{x}_2 = -0.106850 \quad (\text{at } a = \overline{a} = 6), \quad \overline{x}_2 = -0.105573 \quad (\text{at } a = \underline{a} = 5), \tag{31}$$

which coincide with Equations (28) and (29), respectively. In the parentheses, Elishakoff and Daphnis report the values of the uncertain interval coefficient a where the above extrema take place.

On the other hand, the method of quantifier elimination [13,14] was applied by Ioakimidis [15] (p. 24, Equations (115) and (116)) to the present uncertainty problem concerning the uncertain roots $x_{1,2}$ of the interval quadratic Equation (1). From the related quantifier-free formulas and the numerical values in Equations (27) for the three coefficients

a, b, and c both Equations (28) and (29) derived by Elishakoff and Daphnis [12] (p. 1026) by using classic interval analysis and, almost equivalently, both Equations (30) and (31) also derived by Elishakoff and Daphnis [12] (p. 1026), but now by using the direct approach, were again finally obtained and, therefore, verified as well.

Now we consider the interval parametrization method introduced by Elishakoff and Miglis [8], but here with the use of Equation (17) with a parameter $t \in T := [-1, 1]$, Equation (18), in the parametrization. Then by performing minimizations and maximizations, but now with respect to the parameter t instead of the uncertain coefficient a itself in the direct method (as we did previously based on this method), because of the validity of Equations (27), (15) and (16) from Equations (19) and (20) we obtain the four bounds

$$\underline{x}_1 = -1.894427 \quad (\text{at } t = -1), \quad \overline{x}_1 = -1.559816 \quad (\text{at } t = 1), \tag{32}$$

$$\underline{x}_2 = -0.106850 \quad (\text{at } t = 1), \quad \overline{x}_2 = -0.105573 \quad (\text{at } t = -1). \tag{33}$$

Therefore, in the present numerical example, all four methods yield the same results for the four endpoints $\underline{x}_{1,2}$ and $\overline{x}_{1,2}$ of the intervals of the roots x_1 and x_2 of the interval quadratic Equation (1). Obviously, these endpoints $\underline{x}_{1,2}$ and $\overline{x}_{1,2}$ are also lower and upper bounds, respectively, of the roots $x_{1,2}$.

Let us now consider the following similar numerical example, but where the interval coefficient (or interval parameter) a is now a negative-valued interval variable, that is

$$a \in A := [\underline{a}, \overline{a}] = [-6, -5], \quad b = 10, \quad c = 1 \quad \text{now with} \quad a_{\text{ave}} = -5.5 \quad \text{and} \quad a_{\text{dev}} = 0.5. \tag{34}$$

This example was also proposed and studied by Elishakoff and Daphnis [12] (p. 1026, Equation (23)).

On the basis of the above numerical values (34), the direct approach here based on the first pair of Equations (2) for the roots $x_{1,2}$ gives the following results, which were already obtained by Elishakoff and Daphnis [12] (p. 1026, Equations (24) and (25)):

$$\underline{x}_1 = 1.761294 \quad (\text{at } a = \underline{a} = -6), \quad \overline{x}_1 = 2.095445 \quad (\text{at } a = \overline{a} = -5), \tag{35}$$

$$\underline{x}_2 = -0.095445 \quad (\text{at } a = \overline{a} = -5), \quad \overline{x}_2 = -0.094627 \quad (\text{at } a = \underline{a} = -6). \tag{36}$$

However, because here $a < 0$, it is much better to use the second pair of Equations (2) for the same roots $x_{1,2}$, that is, essentially, simply to reverse the formulas used for these two roots $x_{1,2}$. Then the previous and already correct bounds computed by Elishakoff and Daphnis and displayed in Equations (35) and (36) take the slightly modified forms

$$\underline{x}_1 = -0.095445 \quad (\text{at } a = \overline{a} = -5), \quad \overline{x}_1 = -0.094627 \quad (\text{at } a = \underline{a} = -6), \tag{37}$$

$$\underline{x}_2 = 1.761294 \quad (\text{at } a = \underline{a} = -6), \quad \overline{x}_2 = 2.095445 \quad (\text{at } a = \overline{a} = -5) \tag{38}$$

of course, now with $x_1 < x_2$ as is generally assumed in the present paper.

Next, by using the method of classic interval analysis [7] Elishakoff and Daphnis obtained from Equations (4)–(7) and the numerical values (34) the following numerical results [12] (p. 1026, Equations (26) and (28)):

$$\underline{x}_1 = 1.761294, \quad \overline{x}_1 = 2.095445, \tag{39}$$

$$\underline{x}_2 = -0.095445, \quad \overline{x}_2 = -0.094627. \tag{40}$$

Note that the enclosures of both roots $x_{1,2}$ in Equations (39) and (40) coincide with those in Equations (35) and (36), respectively. However, on the other hand, here we have $\underline{x}_1 > \underline{x}_2$ and, similarly, $\overline{x}_1 > \overline{x}_2$ contrary to the previous example. This happens since in the present example we have $a < 0$ (instead of $a > 0$ in the previous example) and in spite of this change of sign of the interval coefficient a, the above results (39) and (40) were based on the formulas valid for the case where $a > 0$, that is on Equations (4)–(7) and also on the first pair of Formula (2) for the roots $x_{1,2}$ of the interval quadratic Equation (1).

Now by using the method of quantifier elimination [13,14], from Equations (11)–(14) and the numerical values (34) we obtain

$$\underline{x}_1 = -0.095445, \quad \overline{x}_1 = -0.094627, \tag{41}$$

$$\underline{x}_2 = 1.761294, \quad \overline{x}_2 = 2.095445. \tag{42}$$

These results were also computed by Ioakimidis [15] (p. 25, Equations (128) and (129)) (but with $x_1 < x_2$ since it was explicitly assumed that $a < 0$), who directly used *Mathematica* [16] for their derivation.

Finally, we consider the interval parametrization method of Elishakoff and Miglis [8]. By using this method from Equations (21) and (22) together with Equations (15)–(18) and the numerical values (34) after minimizations and maximizations with respect to the parameter $t \in T := [-1, 1]$ we obtain

$$\underline{x}_1 = -0.095445 \quad (\text{at } t = 1), \quad \overline{x}_1 = -0.094627 \quad (\text{at } t = -1), \tag{43}$$

$$\underline{x}_2 = 1.761294 \quad (\text{at } t = -1), \quad \overline{x}_2 = 2.095445 \quad (\text{at } t = 1). \tag{44}$$

The results in this example are again identical for all four methods reviewed here even though the method of classic interval analysis due to its general character (that is, essentially, the use of the first pair of Formula (2)) does not lead to the satisfaction the inequality $x_1 < x_2$ for the two roots $x_{1,2}$, contrary to what happened in the first example and contrary to what happens in the present second example as well with the other three methods. Here, in order to get this inequality satisfied (although this is not important) by using the method of classic interval analysis, the two subcases considered by Ioakimidis [15] (Section 5, pp. 20–25) with the method of quantifier elimination need to be separately studied with this method—the method of classic interval analysis—as well, namely by distinguishing the cases (i) $a > 0$ and (ii) $a < 0$ of the interval coefficient a (with $a \neq 0$). In fact, if $a < 0$, this seems to be a very easy task simply by using the second pair of Formula (2) for the two roots $x_{1,2}$ of the interval quadratic Equation (1) instead of the first pair of the same Formula (2).

3. Interval Coefficient *b* in the Quadratic Equation

We start by reporting the results obtained via the method of classic interval analysis [7], now considering the case where the coefficient b of the interval quadratic Equation (1) is an interval variable, more explicitly, in this section,

$$b \in B := [\underline{b}, \overline{b}] \tag{45}$$

whereas the other two coefficients a and c of the same quadratic Equation (1), are crisp parameters.

Using Equations (1) and (45) with classic interval analysis [7], Elishakoff and Daphnis [12] (Subsection 3.2, p. 1027, Equations (30)–(33)) obtained the following formulas for the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two roots $x_{1,2}$ of the interval quadratic Equation (1) under the obvious condition that the coefficient a must be either positive or negative, but, undoubtedly, different from zero ($a \neq 0$):

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{46}$$

$$\overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{47}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{48}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}. \tag{49}$$

Note that the last two of the above four formulas, Equations (48) and (49), concerning the bounds $\underline{x}_2$ and $\overline{x}_2$, respectively, of the second root x_2 of the interval quadratic Equation (1) simultaneously include both endpoints $\underline{b}$ and $\overline{b}$ of the interval (45) for the interval coefficient b of the same equation. Hence, because $\underline{b} < \overline{b}$ in a nontrivial interval of the form (45), $B := [\underline{b}, \overline{b}]$, these two formulas cannot provide the exact interval for the second root x_2, but they overestimate this interval. This means that the lower bound $\underline{x}_2$ is not the greatest lower bound (infimum) $\inf x_2$ of x_2 and, analogously, the upper bound $\overline{x}_2$ is not the least upper bound (supremum) $\sup x_2$ of x_2. This situation leads to the overestimation of the interval of x_2 as will also be observed in the first numerical example below.

Of course, it is clear that the endpoints $\underline{x}$ and $\overline{x}$ of an interval $X := [\underline{x}, \overline{x}]$ in classic interval analysis [7] concern the greatest lower bound (infimum) and the least upper bound (supremum), respectively, of the interval variable $x \in X$. Here we will restrict our attention to the simple case where both coefficients a (crisp coefficient) and $b \in B := [\underline{b}, \overline{b}]$ (interval coefficient) of the interval quadratic Equation (1) take positive values, $a > 0$ and $b > 0$ (hence, $\underline{b} > 0$ and $\overline{b} > 0$ as well), but the third coefficient c (also a crisp coefficient) in the same equation may take either positive or negative values. Here we also constantly assume that the discriminant $\Delta := b^2 - 4ac$ of this quadratic equation, Equation (1), is positive. Hence, this equation has two distinct real roots $x_{1,2}$ and we assume that $x_1 < x_2$. Then, since we already assumed that $a > 0$, it is the first pair of roots of Equation (1) displayed in Equation (2) that must be used in the present case. Hence, under the present conditions we have the pair of roots

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{\Delta}}{2a} \tag{50}$$

with $a > 0$, $b > 0$ (our assumptions for these coefficients here) and $\Delta := b^2 - 4ac > 0$ as well. Now we are completely ready to proceed to the computation of the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots x_1 and x_2 of Equation (1) that determine the corresponding intervals X_1 and X_2, respectively, with

$$x_1 \in X_1 := [\underline{x}_1, \overline{x}_1], \quad x_2 \in X_2 := [\underline{x}_2, \overline{x}_2]. \tag{51}$$

Here this task will be performed without any essential difficulty by using classic interval analysis.

Now, at first, we can mention two very well-known and surely obvious elementary properties of the interval variables $x \in X := [\underline{x}, \overline{x}]$ and $y \in Y := [\underline{y}, \overline{y}]$. These properties state that [17] (pp. 11–12)

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \quad \text{and} \quad -X = [-\overline{x}, -\underline{x}]. \tag{52}$$

For the interval coefficient b of Equation (1) by using the second of the above properties (52) we have

$$b \in B := [\underline{b}, \overline{b}] \quad \text{and, hence,} \quad -b \in -B = [-\overline{b}, -\underline{b}]. \tag{53}$$

Next, since we assumed that $b > 0$, for the lower and upper bounds $\underline{\Delta}$ and $\overline{\Delta}$, respectively, of the discriminant $\Delta := b^2 - 4ac > 0$ of the same quadratic equation, Equation (1), we obviously have

$$\underline{\Delta} = \underline{b}^2 - 4ac, \quad \overline{\Delta} = \overline{b}^2 - 4ac. \tag{54}$$

However, in the Formulas (50) for the two roots $x_{1,2}$, we observe the appearance of the square root $\sqrt{\Delta}$ of the discriminant Δ of the interval quadratic Equation (1), which has been already assumed to be continuously positive: $\Delta > 0$. Here we take into account the obvious property [17] (p. 40, Equation (5.8))

$$\sqrt{X} = \left[\sqrt{\underline{x}}, \sqrt{\overline{x}}\right] \quad \text{for} \quad \underline{x} \geq 0 \tag{55}$$

for the square root $\sqrt{X}$ of an interval here of the interval X. In our case, for the interval (the enclosure) D of the discriminant $\Delta := b^2 - 4ac > 0$ (with $\Delta \in D$) the above property (55) takes the form

$$\sqrt{D} = \left[\sqrt{\underline{\Delta}}, \sqrt{\overline{\Delta}}\,\right] \quad \text{here with} \quad \underline{\Delta} > 0. \tag{56}$$

Next, by taking into account the above result (56), because of the second of Equation (52) we also have

$$-\sqrt{D} = \left[-\sqrt{\overline{\Delta}}, -\sqrt{\underline{\Delta}}\,\right] \quad \text{here with} \quad \underline{\Delta} > 0. \tag{57}$$

We are now ready to use Equation (50) for the determination of the lower and upper bounds of the roots $x_{1,2}$ of Equation (1) displayed in these equations. The corresponding interval forms of Equation (50) are

$$x_1 \in X_1 := \frac{-B - \sqrt{B^2 - 4ac}}{2a} = \frac{-B - \sqrt{D}}{2a}, \quad x_2 \in X_2 := \frac{-B + \sqrt{B^2 - 4ac}}{2a} = \frac{-B + \sqrt{D}}{2a} \tag{58}$$

because, evidently, $D = B^2 - 4ac$. Now we can add the intervals $-B$ and $-\sqrt{D}$ in Equations (53) (second equation there) and (57) by employing the first of the properties (52), which concerns the rule for the addition of two intervals. Then, for the corresponding uncertain quantity $-b - \sqrt{\Delta}$, we find that

$$-b - \sqrt{\Delta} \in -B - \sqrt{D} = \left[-\overline{b} - \sqrt{\overline{\Delta}}, -\underline{b} - \sqrt{\underline{\Delta}}\,\right] = \left[-\overline{b} - \sqrt{\overline{b}^2 - 4ac}, -\underline{b} - \sqrt{\underline{b}^2 - 4ac}\,\right], \tag{59}$$

where Equations (54) were also taken into consideration. Next, taking into account that $\Delta := b^2 - 4ac$ and after a division by the positive quantity $2a$ we finally find for the interval of the first root x_1 that

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \in X_1 := \left[\frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}\right]. \tag{60}$$

Hence, we have proved both Formulas (46) and (47) for the bounds $\underline{x}_1$ and $\overline{x}_1$ of the first root x_1 by using the method of classic interval analysis of course under the present assumptions that $a > 0$ and $b > 0$. At this point we can also remark that the Formulas (46) and (47) for these two bounds proved here are obvious if we think that the lower bound $\underline{x}_1$ of x_1 is obtained when both quantities b and $\sqrt{b^2 - 4ac}$ take their maximum values because of the minus signs in front of these quantities.

Quite similarly, we can also work in the present case ($a > 0$ and $b > 0$) with the bounds of the second root x_2 of the interval quadratic Equation (1) (again with $x_1 < x_2$). Now we simply have to use the second of Equation (58) for this root x_2. We add the intervals $-B$ and $\sqrt{D}$ in Equation (53) (second equation there) and (56), respectively. Then for the related uncertain quantity $-b + \sqrt{\Delta}$ we find that

$$-b+\sqrt{\Delta}\in -B+\sqrt{D}=\left[-\overline{b}+\sqrt{\underline{\Delta}},\,-\underline{b}+\sqrt{\overline{\Delta}}\,\right]=\left[-\overline{b}+\sqrt{\underline{b}^2-4ac},\,-\underline{b}+\sqrt{\overline{b}^2-4ac}\,\right]. \tag{61}$$

Next, we work exactly as previously for the first root x_1. We take into account that $\Delta := b^2 - 4ac$ and again after a division by the positive quantity $2a$ we finally find for the interval of the present second root x_2 that

$$x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}\in X_2:=\left[\frac{-\overline{b}+\sqrt{\underline{b}^2-4ac}}{2a},\,\frac{-\underline{b}+\sqrt{\overline{b}^2-4ac}}{2a}\right]. \tag{62}$$

In this way, we have been able to prove the Formulas (48) and (49) concerning the bounds $\underline{x}_2$ and $\overline{x}_2$ of the second root x_2 of the interval quadratic Equation (1) again by employing the method of classic interval analysis and, of course, again under the present assumptions that $a > 0$ and $b > 0$.

As was already mentioned, the same formulas, (48) and (49), although they are mathematically correct and valid, as is clear from the above proof, nevertheless, provide conservative bounds $\underline{x}_2$ and $\overline{x}_2$ for the root x_2. This unpleasant outcome is simply due to the fact that these formulas include both endpoints $\underline{b}$ and $\overline{b}$ of the interval (the enclosure) B of the interval parameter $b \in B := [\underline{b}, \overline{b}]$ and, actually, it is impossible for the uncertain parameter b to simultaneously take both of these values, that is $\underline{b}$ and $\overline{b}$. On the other hand, sharp bounds for this root, the root x_2 of Equation (1), can easily be found by using competitive methods (such as the method of quantifier elimination [13,14] below) instead of the method of classic interval analysis [7]. Additionally, as will be observed in the first numerical example of this section below, sharp bounds $\underline{x}_2$ and $\overline{x}_2$ for the same root x_2 can also be computed by the method of classic interval analysis, but now by using the Fagnano alternative formula [18] for this root x_2, that is the formula [12] (Section 2, p. 1025, Equation (9))

$$x_2=\frac{2c}{-b-\sqrt{b^2-4ac}} \tag{63}$$

instead of the classical Sridhara formula in the second of Equations (50). This is the approach suggested by Elishakoff and Daphnis [12] and the obtained bounds are sharp (that is without overestimations) for the root x_2 although, unfortunately, now they cease to be sharp for the root x_1 if it is determined by the related Fagnano formula [12] (Section 2, p. 1025, Equation (9))

$$x_1=\frac{2c}{-b+\sqrt{b^2-4ac}}. \tag{64}$$

On the contrary, the present bounds (using the classical Sridhara formula for this root x_1) are sharp.

In fact, in all of these formulas for the roots the uncertain parameter b appears twice and this causes the "dependency effect". This may lead to conservative, but mathematically correct, bounds.

Now we turn to the method of quantifier elimination [13,14] and we consider the same coefficient (parameter) b to lie in the interval B ($b \in B$) already defined in Equation (45) whereas the other two coefficients (parameters) a and c are deterministic quantities. Here we also make the additional assumptions that the coefficient a is positive, $a > 0$, and, moreover, $x_1 < x_2$. This approach, based on quantifier elimination, was adopted by Ioakimidis [15] (Section 4, pp. 11–20), who used the computer algebra system *Mathematica* [16] for quantifier eliminations [13,14]. Moreover, Ioakimidis made the following assumptions [15] (Section 4, p. 11, Equation (46)):

$$\begin{aligned}\mathcal{A}_{b,\,a>0} \coloneqq{}& a>0\wedge \underline{b}\le b\le \overline{b}\wedge \underline{b}\ne 0\wedge \overline{b}\ne 0\wedge \underline{b}<\overline{b}\\ &\wedge\left[c<0\vee\left(c>0\wedge 4a<\frac{b^2}{c}\wedge 4a<\frac{\underline{b}^2}{c}\wedge 4a<\frac{\overline{b}^2}{c}\right)\right].\end{aligned} \tag{65}$$

The above assumptions $\mathcal{A}_{b,a>0}$ beyond the assumed positivity of the coefficient a, $a > 0$, also include two different cases for the coefficient c: (i) $c < 0$ and (ii) $c > 0$ with c assumed different from zero. In the case where $c > 0$, the same assumptions $\mathcal{A}_{b,a>0}$ also include the existence of a positive discriminant $\Delta := b^2 - 4ac$ (hence, $b \neq 0$). This positivity, $\Delta > 0$, assures the existence of two distinct real roots $x_{1,2}$ in Equations (2) of the interval quadratic Equation (1) and, evidently, it is necessary only in the case where $c > 0$, but, obviously, it is not required in the case where $c < 0$ because it was assumed that $a > 0$ and, therefore, in this case, $4ac < 0$ and, next, $\Delta := b^2 - 4ac > 0$.

In the present case where $a > 0$, at first we consider the aforementioned subcase where $c < 0$ with the method of quantifier elimination [13,14] on the basis of the interval (45) for the uncertain coefficient (parameter) b of Equation (1) and the following assumptions [15] (Section 4, p. 11, Equation (47)):

$$\mathcal{A}_{b,a>0,c<0} := a > 0 \wedge c < 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} \neq 0 \wedge \overline{b} \neq 0 \wedge \underline{b} < \overline{b}. \tag{66}$$

These assumptions directly result from the original assumptions $\mathcal{A}_{b,a>0}$ defined in Equation (65). In this subcase, $c < 0$, by using the method of quantifier elimination [13,14] Ioakimidis found the following QFFs (quantifier-free formulas) [15] (Section 4, pp. 11–20, Equations (52), (60), (68) and (77)) concerning the bounds of the roots $x_{1,2}$:

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{67}$$

$$\overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{68}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{69}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}. \tag{70}$$

Here these QFFs are written simply as expressions of the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1) here with b as an uncertain coefficient.

The above results (here QFFs, quantifier-free formulas) obtained by Ioakimidis [15] (Section 4, pp. 11–20, Equations (52), (60), (68) and (77)) by the method of quantifier elimination and displayed in Equations (67)–(70) with $a > 0$ and $c < 0$ can be compared with the corresponding but more general results (46)–(49) having been obtained by Elishakoff and Daphnis [12] (Subsection 3.2, p. 1027, Equations (30)–(33)), who used the method of classic interval analysis [7]. Such a comparison reveals that the results for the bounds $\underline{x}_1$, Equations (46) and (67), and $\overline{x}_1$, Equations (47) and (68), for the first root x_1 coincide, but, unfortunately, this is not true with the results for the bounds $\underline{x}_2$, Equations (48) and (69), and $\overline{x}_2$, Equations (49) and (70), for the second root x_2. This disagreement seems to be simply due to the fact that classic interval analysis quite frequently leads to conservative intervals and not to the narrowest possible intervals; thus we have interval overestimations. However, by no means does this mean that Equations (48) and (49) are incorrect; it simply means that these equations do not provide the greatest lower bound (infimum) $\inf x_2$ and the least upper bound (supremum) $\sup x_2$ of the second root x_2, respectively, of Equation (1). These quantities, $\inf x_2$ and $\sup x_2$, are provided by Equations (69) and (70), respectively, but, of course, only under the validity of the assumptions $\mathcal{A}_{b,a>0,c<0}$ defined in Equation (66).

Next, again under the assumption $a > 0$ we also consider the second and somewhat more difficult subcase where $c > 0$ again by using the method of quantifier elimination [13,14] based on the interval (45) for the uncertain coefficient (parameter) b and, additionally, the somewhat modified assumptions [15] (Section 4, p. 11, Equation (48))

$$\begin{aligned}\mathcal{A}_{b,\,a>0,\,c>0} \coloneqq a > 0 \wedge c > 0 \wedge \underline{b} \le b \le \overline{b} \wedge \underline{b} \ne 0 \wedge \overline{b} \ne 0 \wedge \underline{b} < \overline{b} \\ \wedge\, 4a < \frac{b^2}{c} \wedge 4a < \frac{\underline{b}^2}{c} \wedge 4a < \frac{\overline{b}^2}{c}.\end{aligned} \tag{71}$$

The above nine assumptions $\mathcal{A}_{b,\,a>0,\,c>0}$ directly result from the original assumptions $\mathcal{A}_{b,\,a>0}$ defined in Equation (65) (of course, again for $a > 0$) and, evidently, with the required positivity of the discriminant $\Delta := b^2 - 4ac$ ($\Delta > 0$) explicitly included in the same assumptions $\mathcal{A}_{b,\,a>0,\,c>0}$ with respect to the uncertain coefficient b itself as well as with respect to its lower and upper bounds $\underline{b}$ and $\overline{b}$, respectively, exactly as was the case in the original assumptions $\mathcal{A}_{b,\,a>0}$ defined in Equation (65).

In this subcase, $a > 0$ and $c > 0$, and under the validity of the above assumptions $\mathcal{A}_{b,\,a>0,\,c>0}$ the method of quantifier elimination yields the following more complicated QFFs (quantifier-free formulas) derived by Ioakimidis [15] (Section 4, pp. 11–20, Equations (54), (62), (70) and (79)) with $x_1 < x_2$:

$$\underline{x}_1 = \begin{cases} \dfrac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \\[2ex] \dfrac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0, \end{cases} \tag{72}$$

$$\overline{x}_1 = \begin{cases} \sqrt{\dfrac{c}{a}} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} < 0, \\[2ex] \dfrac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} > 0, \\[2ex] \dfrac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \end{cases} \tag{73}$$

$$\underline{x}_2 = \begin{cases} -\sqrt{\dfrac{c}{a}} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} < 0, \\[2ex] \dfrac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} > 0, \\[2ex] \dfrac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \end{cases} \tag{74}$$

$$\overline{x}_2 = \begin{cases} \dfrac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \underline{b} < 0, \\[2ex] \dfrac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \underline{b} > 0. \end{cases} \tag{75}$$

Here these QFFs are again written simply as expressions of the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1) here, but with b as an uncertain coefficient exactly as in the previous case where $a > 0$ and $c < 0$ instead of $a > 0$ and $c > 0$ here.

Additionally, we can also use the direct method by performing minimizations and maximizations to the formulas for the roots $x_{1,2}$ in Equation (2) in order to compute the bounds for these roots and, therefore, the related intervals. The direct method was successfully used by Ioakimidis [15] (Section 4, pp. 13–19) and it led to exactly the same results (here QFFs) computed by the method of quantifier elimination and displayed in

Equations (67)–(70) and (72)–(75) of course under exactly the same assumptions $\mathcal{A}_{b,\,a>0,\,c<0}$ defined in Equation (66) and $\mathcal{A}_{b,\,a>0,\,c>0}$ defined in Equation (71), respectively.

Evidently, several additional subcases of the present problem related to the distinct real roots $x_{1,2}$ (here continuously with $x_1 < x_2$) of the interval quadratic Equation (1) can also be considered and the related QFFs (quantifier-free formulas) essentially concerning the bounds of these roots can be derived.

For example, let us consider the first subcase where

$$a > 0, \quad c > 0, \quad b < 0 \quad \text{and} \quad \Delta := b^2 - 4ac > 0 \tag{76}$$

with the latter inequality holding true for the existence of two distinct real roots $x_{1,2}$ (with $x_1 < x_2$).

Then $\underline{b} < 0$ and $\overline{b} < 0$ as well with $\underline{b} < \overline{b}$. In this subcase, we make and use the related assumptions

$$\begin{aligned} \mathcal{A}_{b,a>0,c>0,b<0} :=\, & a > 0 \wedge c > 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} < 0 \wedge \overline{b} < 0 \wedge \underline{b} < \overline{b} \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0. \end{aligned} \tag{77}$$

This subcase differs from the previous one having led to the QFFs (72)–(75) since now we made the additional assumption that the uncertain coefficient b is negative, which makes the problem simpler.

By using the above assumptions $\mathcal{A}_{b,\,a>0,\,c>0,\,b<0}$ and by proceeding to quantifier eliminations following the same approach that was described in detail by Ioakimidis [15] (Section 4, pp. 11–20) we derived the following formulas for the bounds of the roots $x_{1,2}$:

$$\underline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{78}$$

$$\overline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{79}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{80}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}. \tag{81}$$

It is easily verified that these formulas are in agreement with the more general formulas displayed in Equations (72)–(75) provided that we select to use the present case there, that is $\underline{b} < 0$ and $\overline{b} < 0$.

Quite similarly, we can consider the second related subcase where

$$a > 0, \quad c > 0, \quad b > 0 \quad \text{and} \quad \Delta := b^2 - 4ac > 0 \tag{82}$$

now with $b > 0$ and with the latter inequality again holding true for the existence of two distinct real roots $x_{1,2}$ (again with $x_1 < x_2$). Then $\underline{b} > 0$ and $\overline{b} > 0$ as well with $\underline{b} < \overline{b}$. In this second subcase, we make and use the following related assumptions:

$$\begin{aligned} \mathcal{A}_{b,a>0,c>0,b>0} \coloneqq\, & a > 0 \wedge c > 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} > 0 \wedge \overline{b} > 0 \wedge \underline{b} < \overline{b} \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0. \end{aligned} \tag{83}$$

Evidently, the last three assumptions assure the existence of a positive discriminant Δ and, hence, two distinct real roots $x_{1,2}$ (here with $x_1 < x_2$). Of course, the present subcase differs from the more general subcase having led to the QFFs (72)–(75) because now we made the additional assumption that the uncertain coefficient b is positive. Naturally, this assumption makes the problem simpler.

By using the above assumptions $\mathcal{A}_{b,a>0,c>0,b>0}$ and by proceeding to quantifier eliminations following the same approach that was described in detail by Ioakimidis [15] (Section 4, pp. 11–20) we derived the following modified formulas for the bounds of the roots $x_{1,2}$ (again with $x_1 < x_2$):

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{84}$$

$$\overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{85}$$

$$\underline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{86}$$

$$\overline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}. \tag{87}$$

It is again easily verified that these formulas are in agreement with the more general formulas displayed in Equations (72)–(75) provided that we select to use the present case, i.e., $\underline{b} > 0$ and $\overline{b} > 0$.

Quite similarly, we can also consider the third and somewhat more strange related subcase where

$$a > 0, \quad c > 0, \quad \underline{b} < 0, \quad \overline{b} > 0 \quad \text{and} \quad \Delta := b^2 - 4ac > 0 \tag{88}$$

now with the interval parameter b taking both negative and positive values and again with the last inequality concerning the discriminant Δ assumed to hold true again for the existence of two distinct real roots $x_{1,2}$ (again with $x_1 < x_2$). In this third subcase, we make and use the related assumptions

$$\begin{aligned}\mathcal{A}_{b,a>0,c>0,\underline{b}<0,\overline{b}>0} \coloneqq\, & a > 0 \wedge c > 0 \wedge \underline{b} \le b \le \overline{b} \wedge \underline{b} < 0 \wedge \overline{b} > 0 \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0.\end{aligned} \tag{89}$$

By using the previous assumptions $\mathcal{A}_{b,a>0,c>0,\underline{b}<0,\overline{b}>0}$ and again proceeding to quantifier eliminations [13,14] following the same approach that was described in detail by Ioakimidis [15] (Section 4, pp. 11–20) we derived the following formulas for the bounds of the roots $x_{1,2}$ (continuously under the assumption $x_1 < x_2$):

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{90}$$

$$\overline{x}_1 = \sqrt{\frac{c}{a}}, \tag{91}$$

$$\underline{x}_2 = -\sqrt{\frac{c}{a}}, \tag{92}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}. \tag{93}$$

In this subcase under the above assumptions $\mathcal{A}_{b,a>0,c>0,\underline{b}<0,\overline{b}>0}$ defined in Equation (89), it can be again directly verified that the above Formulas (90)–(93) for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ are in agreement with the more general and, clearly, more complicated, formulas displayed in Equations (72)–(75) for the same bounds concerning the more general case $a > 0$ and $c > 0$. Of course, this happens provided that we select to use the present case, i.e., $\underline{b} < 0$ and $\overline{b} > 0$, in these more general formulas.

Another and, most likely, more interesting remark concerns the assumptions made for the derivation of the formulas for the present four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$. In fact, the above nine assumptions $\mathcal{A}_{b,a>0,c>0,b<0}$ in Equation (77) and $\mathcal{A}_{b,a>0,c>0,b>0}$ in Equation (83) could also be written in two simpler but essentially equivalent forms. More explicitly, the assumptions $\mathcal{A}_{b,a>0,c>0,b<0}$ in Equation (77) could also be written in the much simpler form (now with only six assumptions instead of initial nine)

$$\mathcal{A}^*_{b,a>0,c>0,b<0} := a > 0 \wedge c > 0 \wedge \underline{b} \le b \le \overline{b} \wedge \overline{b} < 0 \wedge \underline{b} < \overline{b} \wedge \overline{b}^2 - 4ac > 0, \tag{94}$$

where the three assumptions

$$\underline{b} < 0, \quad b^2 - 4ac > 0 \quad \text{and} \quad \underline{b}^2 - 4ac > 0 \tag{95}$$

are now omitted. This is reasonable and completely possible and leads to exactly the same bounds. In fact, since $\overline{b} < 0$ and $\underline{b} < \overline{b}$ in Equation (77), it is clear that $\underline{b} < 0$ as well and, hence, this assumption can safely be omitted as is the case in Equation (94) and contrary to the case in Equation (77). Similarly, again because $\overline{b} < 0$ and $\underline{b} < \overline{b}$, it is clear that $\underline{b}^2 > \overline{b}^2$ and, hence, $\underline{b}^2 - 4ac > \overline{b}^2 - 4ac$. Therefore, the assumption $\overline{b}^2 - 4ac > 0$ in Equation (77) is sufficient and the additional assumption $\underline{b}^2 - 4ac > 0$ can safely be omitted exactly as is the case in Equation (94) and contrary to the case in Equation (77). Analogously, it can easily be observed that the assumption $b^2 - 4ac > 0$ can also safely be omitted. Hence, all three assumptions (95) can be omitted as is really the case in Equation (94) ,which only has six assumptions.

Quite analogous is the case with the assumptions $\mathcal{A}_{b,a>0,c>0,b>0}$ defined in Equation (83). These assumptions can also be written in the simpler but essentially equivalent form

$$\mathcal{A}^*_{b,a>0,c>0,b>0} := a > 0 \wedge c > 0 \wedge \underline{b} \le b \le \overline{b} \wedge \underline{b} > 0 \wedge \underline{b} < \overline{b} \wedge \underline{b}^2 - 4ac > 0 \tag{96}$$

as can easily be verified with only six assumptions (conjunctive terms) instead of nine previously.

Clearly, the simplified assumptions $\mathcal{A}^*_{b,a>0,c>0,b<0}$ in Equation (94) are simpler than the equivalent initial (complete) assumptions $\mathcal{A}_{b,a>0,c>0,b<0}$ in Equation (77) and, analogously, the simplified assumptions $\mathcal{A}^*_{b,a>0,c>0,b>0}$ in Equation (96) are simpler than the equivalent initial (complete) assumptions $\mathcal{A}_{b,a>0,c>0,b>0}$ in Equation (83). However, on the other hand, the use of the initial (complete) assumptions permits significantly smaller computational times for some bounds during quantifier eliminations.

In the previous section devoted to the interval coefficient a, by using the method of quantifier elimination [13,14] we have considered both cases: (i) $a > 0$, where the Formulas (4)–(7) hold true, as well as (ii) $a < 0$, where the Formulas (11)–(14) hold true, but in both cases under the requirement that the interval quadratic Equation (1) has two distinct real roots $x_{1,2}$ with $x_1 < x_2$. In the present section, so far we have studied the use of the same method, quantifier elimination, but we have restricted our attention just to the case where $a > 0$ in the interval quadratic Equation (1). Below we will also consider the second case, $a < 0$, in the same equation, Equation (1), continuously assuming that $x_1 < x_2$. Obviously, the case $a = 0$ is correctly excluded in the interval quadratic Equation (1). At this point it should also be mentioned that Ioakimidis [15] (Section 4, pp. 11–20) has restricted his attention to the first case, $a > 0$, when the case of an interval coefficient $b \in B := [\underline{b}, \overline{b}]$ was studied and, similarly, for the case of an interval coefficient $c \in C := [\underline{c}, \overline{c}]$ to be studied in the next section (also in both cases $a > 0$ and $a < 0$). Therefore, the results obtained below in the case $a < 0$ with an interval coefficient $b \in B$ in Equation (1) by using the method of quantifier elimination [13,14] constitute new applications of this computer algebra method to the interval quadratic Equation (1).

Now assuming that $a < 0$ at first we consider the case $c > 0$ of course with b being an interval coefficient, $b \in B := [\underline{b}, \overline{b}]$, exactly as previously in the case $a > 0$. Here our assumptions have the form

$$\mathcal{A}_{b,a<0,c>0} := a < 0 \land c > 0 \land \underline{b} \le b \le \overline{b} \land \underline{b} \ne 0 \land \overline{b} \ne 0 \land \underline{b} < \overline{b}. \tag{97}$$

Here the assumptions $a < 0$ and $c > 0$ assure the existence of a positive discriminant $\Delta := b^2 - 4ac$ and, therefore, of two distinct real roots $x_{1,2}$ (with $x_1 < x_2$) of the interval quadratic Equation (1). Moreover, the closed-form Formulas (2) for these roots are not of interest here (they are not used at all) because we apply the method of quantifier elimination exactly as happened in the corresponding previous results by Ioakimidis [15] (Section 4, pp. 11–20) in the already studied case where $a > 0$.

The whole computational approach is completely analogous to that having been followed by Ioakimidis [15] (Section 4, pp. 11–20) in the case where $a > 0$ and, hence, it will not be repeated here. Only the assumptions made, here the above assumptions $\mathcal{A}_{b,a<0,c>0}$ defined in Equation (97), are different because now $a < 0$. The resulting formulas for the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two roots $x_{1,2}$ (with $x_1 < x_2$) have the forms

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{98}$$

$$\overline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{99}$$

$$\underline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{100}$$

$$\overline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}. \tag{101}$$

Quite similarly, we can study the case where $a < 0$ and $c < 0$ of course again with b an interval coefficient, $b \in B := [\underline{b}, \overline{b}]$, exactly as previously in the case $a < 0$ and $c > 0$. Here our assumptions have the form

$$\begin{aligned}\mathcal{A}_{b,a<0,c<0} \coloneqq\, & a < 0 \land c < 0 \land \underline{b} \le b \le \overline{b} \land \underline{b} \ne 0 \land \overline{b} \ne 0 \land \underline{b} < \overline{b} \\ & \land b^2 - 4ac > 0 \land \underline{b}^2 - 4ac > 0 \land \overline{b}^2 - 4ac > 0.\end{aligned} \tag{102}$$

Here the last three assumptions assure the existence of a positive discriminant $\Delta := b^2 - 4ac$ and, hence, of two distinct real roots $x_{1,2}$ (with $x_1 < x_2$) of the interval quadratic Equation (1). Moreover, again the closed-form Formula (2) for these roots are not of interest here simply because we apply the method of quantifier elimination exactly as in the previous cases already studied by this method.

The whole computational approach is again similar to that having been followed by Ioakimidis [15] (Section 4, pp. 11–20) in the case where $a > 0$ and, hence, it will not be repeated here. Only the assumptions made, here the above assumptions $\mathcal{A}_{b,a<0,c<0}$ defined in Equation (102), are different. The resulting formulas for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$ (with $x_1 < x_2$) have the forms

$$\underline{x}_1 = \begin{cases} \dfrac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0 \lor [\overline{b} > 0 \land (\underline{b} < 0 \lor \underline{b} + \overline{b} \le 0)], \\ \dfrac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0 \land \underline{b} > 0, \end{cases} \tag{103}$$

$$\overline{x}_1 = \begin{cases} \sqrt{\frac{c}{a}} & \text{for} \quad \overline{b} > 0 \wedge (\underline{b} < 0 \vee \underline{b} + \overline{b} \leq 0), \\ \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} > 0, \\ \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \end{cases} \tag{104}$$

$$\underline{x}_2 = \begin{cases} -\sqrt{\frac{c}{a}} & \text{for} \quad \overline{b} > 0 \wedge (\underline{b} < 0 \vee \underline{b} + \overline{b} \leq 0), \\ \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0 \wedge \underline{b} > 0, \\ \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \end{cases} \tag{105}$$

$$\overline{x}_2 = \begin{cases} \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} < 0, \\ \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} & \text{for} \quad \overline{b} > 0. \end{cases} \tag{106}$$

The above formulas for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ in the case $a < 0$ and $c < 0$ are somewhat complicated. Simpler formulas can be derived by using subcases of this case, i.e., $a < 0$ and $c < 0$.

As a first subcase we consider that with $a < 0$, $c < 0$, and $b < 0$. In this subcase, we make the assumptions

$$\begin{aligned} \mathcal{A}_{b,a<0,c<0,b<0} \coloneqq\, & a < 0 \wedge c < 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} < 0 \wedge \overline{b} < 0 \wedge \underline{b} < \overline{b} \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0. \end{aligned} \tag{107}$$

The obtained four bounds by using the method of quantifier elimination [13,14] have the forms

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{108}$$

$$\overline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{109}$$

$$\underline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{110}$$

$$\overline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}. \tag{111}$$

As a second subcase we consider that with $a < 0$, $c < 0$ and $b > 0$. In this subcase, we make the assumptions

$$\begin{aligned} \mathcal{A}_{b,a<0,c<0,b>0} \coloneqq\, & a < 0 \wedge c < 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} > 0 \wedge \overline{b} > 0 \wedge \underline{b} < \overline{b} \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0. \end{aligned} \tag{112}$$

The obtained bounds again by using the method of quantifier elimination [13,14] have the forms

$$\underline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a}, \tag{113}$$

$$\overline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{114}$$

$$\underline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{115}$$

$$\overline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}. \tag{116}$$

Next, as a third and final subcase we consider that with $a < 0$, $c < 0$, $\underline{b} < 0$ and $\overline{b} > 0$ now with the interval parameter b taking both negative and positive values. Here we make and use the related assumptions

$$\begin{aligned}\mathcal{A}_{b,a<0,c<0,\underline{b}<0,\overline{b}>0} \coloneqq\, & a < 0 \wedge c < 0 \wedge \underline{b} \leq b \leq \overline{b} \wedge \underline{b} < 0 \wedge \overline{b} > 0 \\ & \wedge b^2 - 4ac > 0 \wedge \underline{b}^2 - 4ac > 0 \wedge \overline{b}^2 - 4ac > 0\end{aligned} \tag{117}$$

again with the last three inequalities holding true for the existence of two distinct real roots $x_{1,2}$.

By using the above assumptions $\mathcal{A}_{b,a<0,c<0,\underline{b}<0,\overline{b}>0}$ and again proceeding to quantifier eliminations [13,14] following the same approach that was described in detail by Ioakimidis [15] (Section 4, pp. 11–20) we derived the following formulas for the four bounds of the roots $x_{1,2}$ (with $x_1 < x_2$):

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a}, \tag{118}$$

$$\overline{x}_1 = \sqrt{\frac{c}{a}}, \tag{119}$$

$$\underline{x}_2 = -\sqrt{\frac{c}{a}}, \tag{120}$$

$$\overline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a}. \tag{121}$$

The above three sets of bounds, i.e., (i) the bounds (108)–(111) in the first subcase, where $a < 0$, $c < 0$ and $b < 0$, (ii) the bounds (113)–(116) in the second subcase, where $a < 0$, $c < 0$ and $b > 0$ and (iii) the bounds (118)–(121) in the third subcase, where $a < 0$, $c < 0$, $\underline{b} < 0$ and $\overline{b} > 0$ were also verified by using the direct method with minimizations and maximizations with respect to the interval coefficient b under the assumptions (107), (112), and (117), respectively. These minimizations and maximizations were again performed by using the `Minimize` and `Maximize` commands of *Mathematica* [16]. Moreover, the same bounds are also easily verified to be in agreement with the more general bounds (103)–(106), which concern the more general case where $a < 0$ and $c < 0$ essentially without assumptions on the positivity or negativity of the interval parameter b contrary to the bounds (108)–(111), (113)–(116) and (118)–(121), where such assumptions were made.

At this point, we should also add that exactly similar remarks to those based on the simplified assumptions $\mathcal{A}^*_{b,a>0,c>0,b<0}$ and $\mathcal{A}^*_{b,a>0,c>0,b>0}$ defined in Equations (94) and (96), respectively, and concerning the case with $a > 0$ also hold true in the present case with $a < 0$ exactly as previously in the case with $a > 0$.

Finally, we can also mention (exactly as we already did in the previous section, Section 2) that a second and, most likely, simpler approach of working in the present case of a negative coefficient a ($a < 0$) is simply to transform the initial interval quadratic Equation (1) under consideration (here with $a < 0$) to an equivalent interval quadratic equation that has exactly the same distinct real roots $x_{1,2}$, but now with $a > 0$. Naturally, this change can easily be made through a multiplication of the initial interval quadratic Equation (1) by -1. This possibility is studied in more detail below in Section 4 that concerns an interval coefficient $c \in C := [\underline{c}, \overline{c}]$ in the interval quadratic Equation (1). This multiplication (by -1) is briefly described in Equations (174) in Section 4.

Let us consider the same problem, but now using the interval parametrization analysis proposed by Elishakoff and Miglis [8] here with the same interval $B := [\underline{b}, \overline{b}]$ for the uncertain coefficient (parameter) b given by Equation (45) as an interval coefficient. In this problem, the other two coefficients (parameters) a and c of the interval quadratic Equation (1) remain deterministic quantities.

Exactly as in the previous section for an uncertain coefficient a with $a \in A := [\underline{a}, \overline{a}]$, here we also need to introduce two auxiliary quantities in order to parametrize the interval $B := [\underline{b}, \overline{b}]$ of the interval coefficient b in Equation (45). These quantities are (i) the average value (the midpoint or center) of the interval B, here denoted by the symbol b_{ave} and (ii) the deviation value (the radius) of the same interval B here denoted by the symbol b_{dev}, that is

$$b_{\text{ave}} \coloneqq \frac{\underline{b} + \overline{b}}{2}, \tag{122}$$

$$b_{\text{dev}} \coloneqq \frac{\overline{b} - \underline{b}}{2}. \tag{123}$$

Now, by using these equations, Equations (122) and (123), we can rewrite the uncertain (interval) coefficient b in the parametric form

$$b = b_{\text{ave}} + b_{\text{dev}}\, t \tag{124}$$

again with

$$t \in T := [-1, 1]. \tag{125}$$

As a consequence, by using Equation (124) we can express the two roots $x_{1,2}$ of the present interval quadratic Equation (1) displayed in the first pair of Equations (2) (with $x_1 < x_2$ if $a > 0$) as

$$x_1 = \frac{-(b_{\text{ave}} + b_{\text{dev}}\, t) - \sqrt{(b_{\text{ave}} + b_{\text{dev}}\, t)^2 - 4ac}}{2a}, \tag{126}$$

$$x_2 = \frac{-(b_{\text{ave}} + b_{\text{dev}}\, t) + \sqrt{(b_{\text{ave}} + b_{\text{dev}}\, t)^2 - 4ac}}{2a}. \tag{127}$$

Evidently, if $a < 0$, it is preferable to use the completely analogous formulas

$$x_1 = \frac{-(b_{\text{ave}} + b_{\text{dev}}\, t) + \sqrt{(b_{\text{ave}} + b_{\text{dev}}\, t)^2 - 4ac}}{2a}, \tag{128}$$

$$x_2 = \frac{-(b_{\text{ave}} + b_{\text{dev}}\, t) - \sqrt{(b_{\text{ave}} + b_{\text{dev}}\, t)^2 - 4ac}}{2a}. \tag{129}$$

which are now simply based on the second pair of roots $x_{1,2}$ of Equation (1) displayed in Equation (2) (now with the $+$ and $-$ signs in front of the square roots reversed). In this way, the inequality $x_1 < x_2$ will hold true again (but now with $a < 0$).

Finally, we can proceed to minimizations and maximizations (of course here under the continuous validity of the constraint (125) for the present parameter t, that is $t \in T := [-1, 1]$ or, equivalently, $-1 \le t \le 1$) and, therefore, determine the intervals of the two roots $x_{1,2}$ e.g., by using the commands `Minimize` and `Maximize` of *Mathematica* [16]. Of course, as was already mentioned in Section 2, there is no need to use the present parametrization (124) and (125) for the derivation of the present bounds of the two roots $x_{1,2}$ of the interval quadratic Equation (1).

Now that the expressions of the bounds were obtained for each method considered above, we can evaluate them using some numerical examples to see whether they all lead to the same results.

At first, let us consider the following example values of the three parameters a, b and c:

$$a = 5, \quad b \in B := [\underline{b}, \overline{b}] = [10, 11], \quad c = 1. \tag{130}$$

This first numerical example was initially proposed and studied by Elishakoff and Daphnis [12] (Subsection 3.2, p.1027, Equations (34)), who used the method of classic interval analysis [7] for the computation of the bounds of the roots $x_{1,2}$. Next, the same numerical example was also studied by Ioakimidis [15] (Subsection 4.6, pp. 19–20), who used the method of quantifier elimination [13,14].

In this numerical example, Elishakoff and Daphnis obtained the following four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$ by using the method of classic interval analysis [7], the Formulas (46)–(49) for these bounds, which are valid in the present case, where $a > 0$, $c > 0$ and $b > 0$, as well as the numerical values (130) [12] (Subsection 3.2, p. 1027, Equations (35) and (36)):

$$\underline{x}_1 = -2.104988, \quad \overline{x}_1 = -1.894427, \tag{131}$$

$$\underline{x}_2 = -0.205573, \quad \overline{x}_2 = 0.004988. \tag{132}$$

On the other hand, by using the direct method together with the same numerical values (130) Elishakoff and Daphnis found the following bounds [12] (Subsection 3.2, p. 1027, Equations (37) and (38)):

$$\underline{x}_1 = -2.104988 \quad (\text{at } b = \overline{b} = 11), \quad \overline{x}_1 = -1.894427 \quad (\text{at } b = \underline{b} = 10), \tag{133}$$

$$\underline{x}_2 = -0.105573 \quad (\text{at } b = \underline{b} = 10), \quad \overline{x}_2 = -0.095012 \quad (\text{at } b = \overline{b} = 11). \tag{134}$$

These results, (133) and (134), obtained by the direct method for the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two roots $x_{1,2}$ coincide with the previous results (131) and (132) obtained by the method of classic interval analysis [7], but only as far as the first root x_1 is concerned, see Equations (131) and (133). With respect to the second root x_2 the results obtained by the method of classic interval analysis and displayed in Equation (132) are conservative. The exact results for the related bounds $\underline{x}_2$ and $\overline{x}_2$ are those obtained by the direct method and displayed in Equation (134). For the related intervals we have the inclusion relation

$$x_2 \in [-0.105573, -0.095012] \subset [-0.205573, 0.004988] \tag{135}$$

as is clear from Equations (134) and (132), respectively, concerning these two intervals of the root x_2.

Next, Elishakoff and Daphnis also considered the alternative Fagnano's representation [18] for the roots $x_{1,2}$ of the classic quadratic Equation (1) in the same example. This representation is based on the formulas [12] (Section 2, p. 1025, Equation (9))

$$x_{1,2} = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}, \tag{136}$$

but the obtained results for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ were the sharpest possible (no interval overestimations) only for the root x_2 (in the present notation) with the bounds (134) [12] (Subsection 3.2, p. 1027, Equations (39)–(42)). On the other hand, Elishakoff and Daphnis also employed appropriate modified Fagnano's formulas for the same bounds [12]

(Subsection 3.2, pp. 1027–1028) based on the modified Fagnano's formulas [12] [Subsection 3.2, p. 1027, Equation (43) for the roots $x_{1,2}$:

$$x_{1,2} = \frac{2c}{-b\left(1 \mp \sqrt{1 - \frac{4ac}{b^2}}\right)}. \tag{137}$$

It seems that the whole difficulty in the present first numerical example and in general in similar numerical examples lies in the fact that classic interval analysis [7] leads to conservative bounds when interval subtractions have to be computed, but not when this is the case for interval additions. This situation is related to the "dependency effect" in classic interval analysis [7] leading to overestimation of intervals; see, e.g., the paper by Elishakoff, Gabriele, and Wang [19] (Section 2, pp. 1205–1207). Therefore, it is strongly recommended that subtractions of intervals should be avoided when using the method of classic interval analysis. For the present problem, a simple device for the remedy of this undesirable situation was proposed again by Elishakoff and Daphnis [12] (Section 2, p. 1025). This device simply consists in the simultaneous use of one of the classic Formula (2) (also called Sridhara's formulas [12]) and of one of Fagnano's Formula (136) for the two unequal roots $x_{1,2}$ of Equation (1). More explicitly, here we can use the following two formulas:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{2c}{-b - \sqrt{b^2 - 4ac}}. \tag{138}$$

Therefore, because both of the intervals, $-b$ and $-\sqrt{b^2 - 4ac}$, appearing in both of these formulas have the same sign (because $b > 0$ here), we have already avoided subtractions, and, hence, we have also avoided subsequent overestimations of the intervals of the two roots $x_{1,2}$ or, equivalently, the computation of conservative lower bounds $\underline{x}_1$ and /or $\underline{x}_2$ and upper bounds $\overline{x}_1$ and /or $\overline{x}_2$ for the same roots $x_{1,2}$, respectively.

Now, as far as these bounds are concerned, at first, the bounds $\underline{x}_1$ and $\overline{x}_1$ of the first root x_1 given by the first of Equation (138) (these bounds computed by using classic interval analysis [7]) are given by Equations (46) and (47) derived by Elishakoff and Daphnis [12] (Subsection 3.2, p. 1027, Equations (30) and (31)), that is

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} = -2.104988, \quad \overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} = -1.894427 \tag{139}$$

with the resulting exact (sharp) bounds (131), as already mentioned. Quite similarly, the bounds $\underline{x}_2$ and $\overline{x}_2$ of the second root x_2 now given by the second of Equation (138) (these bounds were again computed by using classic interval analysis [7]) are given by the following formulas, which correspond to those derived by Elishakoff and Daphnis [12] (Subsection 3.2, p. 1027, Equations (39) and (40)):

$$\underline{x}_2 = \frac{2c}{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}} = -0.105573, \quad \overline{x}_2 = \frac{2c}{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}} = -0.095012. \tag{140}$$

Evidently, the above Formula (140) for the two bounds $\underline{x}_2$ and $\overline{x}_2$ can also be written (through appropriate simplifications of the denominators of the fractions in these formulas) in the completely equivalent forms

$$\underline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = -0.105573, \quad \overline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} = -0.095012. \tag{141}$$

From Equations (139) and (140) (or, preferably, (141)) we observe that these bounds $\underline{x}_2$ and $\overline{x}_2$ coincide with the bounds already having been computed by using the direct method and displayed in Equations (133) and (134). Hence, our conclusion is that the method based on classic interval analysis [7] can also lead to the exact (sharp) bounds exactly as the direct method does, but, of course, provided that the sources of overestimation (here simply interval subtractions by using the rules of classic interval analysis [7]) are avoided.

Next, by using the Formulas (84)–(87) of the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$, which are valid in the present case (with $a > 0$, $c > 0$, and $b > 0$) and were derived by the method of quantifier elimination [13,14], we obtain the same numerical results (133) and (134) already computed by the direct method or, equivalently, again the same numerical results (139) and (141) also already computed, but now by the modified method of classic interval analysis [7]. The same numerical results for the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ have also been computed by Ioakimidis [15] (Subsection 4.6, pp. 19–20, Equations (86)) again on the basis of the method of quantifier elimination, but in this reference by using QFFs (quantifier-free formulas), which are based only on the positivity assumption $a > 0$ for the coefficient a without analogous assumptions for the coefficient c and for the bounds $\underline{b}$ and $\overline{b}$ of the interval coefficient b during the derivation of these QFFs.

At this point we should also remark that the Formulas (84)–(87) for the aforementioned bounds derived with the method of quantifier elimination coincide with the corresponding Formulas (139) and (141) derived with the modified method of classic interval analysis.

Finally, let us consider the fourth of the present methods: the interval parametrization method. In this method, by using Equations (128) and (127), of course together with Equation (130) for the present numerical values and performing the appropriate minimizations and maximizations (again by employing the `Minimize` and `Maximize` commands of *Mathematica* [16]) with respect to the adopted parameter $t \in T := [-1, 1]$ we obtain the exact (sharp) bounds (133) and (134) of the roots $x_{1,2}$ already obtained by the direct method and, additionally, by the modified method of classic interval analysis [7] as well as by the method of quantifier elimination [13,14]. More explicitly, the interval parametrization method yields the bounds

$$\underline{x}_1 = -2.104988 \quad (\text{at } t = \overline{t} = 1), \quad \overline{x}_1 = -1.894427 \quad (\text{at } t = \underline{t} = -1), \tag{142}$$

$$\underline{x}_2 = -0.105573 \quad (\text{at } t = \underline{t} = -1), \quad \overline{x}_2 = -0.095012 \quad (\text{at } t = \overline{t} = 1). \tag{143}$$

Obviously, these results (the same bounds and, therefore, the same intervals as well for the two roots $x_{1,2}$ by using the present methods) were expected because both the direct method and the interval parametrization method are essentially equivalent minimization–maximization methods although here the direct method has the coefficient b as the uncertain variable for the minimization– maximization to be performed whereas the interval parametrization method has the parameter t as such a variable.

Similarly, the method of quantifier elimination also leads to exact results, that is, to the two greatest lower bounds (infima, inf) $\underline{x}_{1,2}$ and also to the two least upper bounds (suprema, sup) $\overline{x}_{1,2}$ of the roots $x_{1,2}$ of the interval quadratic Equation (1). Additionally, analogous is the case with the modified method of classic interval analysis proposed by Elishakoff and Daphnis [12] (Section 2, p. 1025) and based on both the classic (Sridhara's) and Fagnano's formulas for the roots $x_{1,2}$ of the quadratic Equation (1).

Let us now study the following second numerical example, which is similar to the first numerical example and has also been proposed and studied by Elishakoff and Daphnis [12] (Subsection 3.2, p. 1028). In this numerical example, the uncertain coefficient b of the interval quadratic Equation (1) is a negative-valued interval [12] (Subsection 3.2, p. 1028, Equation (48)) instead of a positive-valued interval in the previous numerical example. Here we assume the values

$$a = 5, \quad b \in B := [\underline{b}, \overline{b}] = [-11, -10], \quad c = 1. \tag{144}$$

At first, by using the above values (144) of the three coefficients a, b and c Elishakoff and Daphnis [12] (Subsection 3.2, p. 1028) computed the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$ of Equation (1) through the application of the direct approach by employing the first pair of Equation (2) as the formulas for the two roots $x_{1,2}$ (because $a = 5 > 0$) and also the appropriate minimizations and maximizations with respect to the uncertain coefficient (uncertain parameter) b (with $b < 0$ in the present second numerical example). The related results derived by Elishakoff and Daphnis are [12] (Subsection 3.2, p. 1028, Equations (49) and (50))

$$\underline{x}_1 = 0.095012 \quad (\text{at } b = \underline{b} = -11), \quad \overline{x}_1 = 0.105573 \quad (\text{at } b = \overline{b} = -10), \tag{145}$$

$$\underline{x}_2 = 1.894427 \quad (\text{at } b = \overline{b} = -10), \quad \overline{x}_2 = 2.104988 \quad (\text{at } b = \underline{b} = -11). \tag{146}$$

Next, by appropriately using the method of classic interval analysis [7] (but now with $b < 0$) as well as the numerical values (144) Elishakoff and Daphnis [12] (Subsection 3.2, p. 1028, Equations (51) and (52)) found the bounds

$$\underline{x}_1 = -0.004988, \quad \overline{x}_1 = 0.205573, \tag{147}$$

$$\underline{x}_2 = 1.894427, \quad \overline{x}_2 = 2.104988. \tag{148}$$

Therefore, classic interval analysis [7] leads to the exact (sharp) bounds, but only for the second root x_2 whereas conservative bounds have been obtained for the first root x_1. For this reason classic interval analysis should be used without the need to proceed to interval subtractions exactly as in the previous first example. This was achieved again by using the modification of the approach based on the use of both the classical (Sridhara's) formulas for one root, here the root x_2, and Fagnano's formulas for the other root, here the root x_1. This approach is completely analogous to the approach used in the previous numerical example, but now it is used for different roots, more explicitly here (i) the root x_1 instead of the root x_2 for Fagnano's formula and (ii) the root x_2 instead of the root x_1 for Sridhara's formula. Working in this way, Elishakoff and Daphnis computed the exact (sharp) bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ by using classic interval analysis in its present modified form. More explicitly, by using the present modified method of classic interval analysis Elishakoff and Daphnis found the exact (sharp) bounds [12] (Subsection 3.2, p. 1028, Equations (53) and (54))

$$\underline{x}_1 = \frac{2c}{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}} = 0.095012, \quad \overline{x}_1 = \frac{2c}{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}} = 0.105573, \tag{149}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} = 1.894427, \quad \overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = 2.104988 \tag{150}$$

in complete agreement with the corresponding bounds (145) and (146) having also been computed by Elishakoff and Daphnis, but by using the direct method for the derivation of these bounds instead of the method of classic interval analysis [7].

Next, as far as the method of quantifier elimination [13,14] is concerned, the present second numerical example was successfully studied by Ioakimidis [15] (Subsection 4.6, p. 20). Naturally, this method also leads to the exact (sharp) bounds displayed in Equations (145) and (146) and also in Equations (149) and (150) [15] (Subsection 4.6, p. 20, Equations (91) and (92)). These results by Ioakimidis by using the method of quantifier elimination [13,14] have been based on the general QFFs (quantifier-free formulas) valid for a positive value of the coefficient a as is also here the case with $a = 5$, first Equation (144).

Here by using the method of quantifier elimination it is surely much simpler to directly use the bounds (78)–(81) obtained by this method and valid in the present case with $a > 0$,

$c > 0$ and $b < 0$ (exactly as is here the case, Equation (144)), as well as the numerical values displayed in Equation (144). Then we get the following numerical results:

$$\underline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} = 0.095012, \quad \overline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} = 0.105573, \tag{151}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} = 1.894427, \quad \overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = 2.104988. \tag{152}$$

Of course, we observe that the above four bounds coincide with the bounds (145) and (146) computed by using the direct approach and with the bounds (149) and (150) computed by the modified method of classic interval analysis.

Next, Equations (128) and (127) obtained via interval parametrization after the necessary minimizations and maximizations with the help of the `Minimize` and `Maximize` commands of the computer algebra system *Mathematica* [16] give the following values for the bounds of course on the basis of the numerical values (144):

$$\underline{x}_1 = 0.095012 \quad (\text{at } t = \underline{t} = -1), \quad \overline{x}_1 = 0.105573 \quad (\text{at } t = \overline{t} = 1), \tag{153}$$

$$\underline{x}_2 = 1.894427 \quad (\text{at } t = \overline{t} = 1), \quad \overline{x}_2 = 2.104988 \quad (\text{at } t = \underline{t} = -1). \tag{154}$$

Let us now consider yet another and somewhat different example. In this example, the lower bound $\underline{b}$ of the interval coefficient b has a negative value whereas its upper bound $\overline{b}$ has a positive one. In this example, we assume that

$$a = 5, \quad b \in B := [\underline{b}, \overline{b}] = [-11, 11], \quad c = 1. \tag{155}$$

In the present example, we can split the interval b into two subintervals since, obviously, the discriminant $\Delta := b^2 - 4ac$ is required to be positive for the existence of two distinct real roots $x_{1,2}$ in the interval quadratic Equation (1). This means that $b^2 - 4ac > 0$ and, therefore, $b^2 > 4ac = 20$ here. Then, Equation (155) takes the following modified form with respect to the interval variable b:

$$a = 5, \quad b \in \{[\underline{b}_1, \overline{b}_1] = [-11, -2\sqrt{5}]\} \cup \{[\underline{b}_2, \overline{b}_2] = [2\sqrt{5}, 11]\}, \quad c = 1. \tag{156}$$

In these equations, for convenience, we have also accepted a zero discriminant $\Delta = 0$ (beyond a positive discriminant $\Delta > 0$), that is, finally, we have assumed that the discriminant Δ is simply non-negative: $\Delta \geq 0$.

For the computation of the four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$ at first we used the direct method, which is based on the first pair of the roots $x_{1,2}$ in the Formula (2) for these roots (since $a = 5 > 0$). Next, we performed the minimization and the maximization of these roots $x_{1,2}$ with respect to the interval parameter b here continuously with the help of the two relevant `Minimize` and `Maximize` commands of the computer algebra system *Mathematica* [16]. Here this approach yielded the following bounds of the two roots $x_{1,2}$:

$$\underline{x}_1 = -2.104988 \quad (\text{at } b = \overline{b} = 11), \quad \overline{x}_1 = 0.447214 \quad (\text{at } b = -b^* = -2\sqrt{5}), \tag{157}$$

$$\underline{x}_2 = -0.447214 \quad (\text{at } b = b^* = 2\sqrt{5}), \quad \overline{x}_2 = 2.104988 \quad (\text{at } b = \underline{b} = -11), \tag{158}$$

where we have also defined the new symbol $b^* := \sqrt{20} = 2\sqrt{5} = 4.472136$. Naturally, the above four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ were computed under the assumption $\underline{b} \leq b \leq \overline{b}$, that is $-11 \leq b \leq 11$ in the present example and this assumption has been necessary because of the second of Equations (155), which concerns the interval of the interval parameter b. On the other hand, the use of the additional assumption $\Delta := b^2 - 4ac \geq 0$ that assures a non-negative discriminant Δ does not change at all the above bounds (157) and (158). This happens simply because the computations in *Mathematica* in minimizations and maximizations concern only real variables and, therefore, the real roots $x_{1,2}$ of Equation (1) as well.

As a second method for the computation of the same bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$, and $\overline{x}_2$ we used the method of quantifier elimination [13,14]. Here this method leads to the Formulas (90)–(93), which are valid in the present case where $a = 5 > 0$, $c = 1 > 0$, $\underline{b} = -11 < 0$ and $\overline{b} = 11 > 0$. Then we find the following numerical results for the sought bounds:

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} = -2.104988, \quad \overline{x}_1 = \sqrt{\frac{c}{a}} = 0.447214, \tag{159}$$

$$\underline{x}_2 = -\sqrt{\frac{c}{a}} = -0.447214, \quad \overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = 2.104988. \tag{160}$$

Of course, instead of the Formulas (90)–(93), we could also have appropriately employed the more general Formulas (72)–(75) , which are valid in the more general case $a > 0$ and $c > 0$. Evidently, these formulas lead to exactly the same bounds (159) and (160). Moreover, we observe that these bounds are in complete agreement with the previous bounds (157) and (158) having been computed by the direct method.

As a third method for the computation of the same bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$, and $\overline{x}_2$ we used the interval parametrization method. In this method, by using Equations (128) and (127), of course together with Equations (155) concerning the present numerical values and performing the appropriate minimizations and maximizations (again by using the `Minimize` and `Maximize` commands of *Mathematica* [16]) with respect to the adopted parameter $t \in T := [-1, 1]$ we again obtain the bounds (157) and (158) of the roots $x_{1,2}$, which were already computed both by the direct method and, additionally, by the method of quantifier elimination. More explicitly, the interval parametrization method yields the following four bounds:

$$\underline{x}_1 = -2.104988 \quad (\text{at } t = \overline{t} = 1), \qquad \overline{x}_1 = 0.447214 \quad \left(\text{at } t = t^* = -\frac{2\sqrt{5}}{11} = -0.406558\right), \tag{161}$$

$$\underline{x}_2 = -0.447214 \quad \left(\text{at } t = t^* = \frac{2\sqrt{5}}{11} = 0.406558\right), \qquad \overline{x}_2 = 2.104988 \quad (\text{at } t = \underline{t} = -1) \tag{162}$$

with the new symbol t^* defined here as $t^* = \sqrt{20}/11 = 2\sqrt{5}/11 = 0.406558$. We observe that these four bounds are in complete agreement with the corresponding bounds having been computed by the direct method as well as by the method of quantifier elimination. Moreover, we should mention that the simple assumptions $-1 \leq t \leq 1$ are sufficient for the computation of the above bounds with the additional assumption of a non-negative discriminant, $\Delta \geq 0$, having no influence on them exactly as has been the case in the direct method.

Finally, the method of classic interval analysis [7] is also applicable to the present numerical example, but for its application we should work with both intervals $[\underline{b}_1, \overline{b}_1]$ and $[\underline{b}_2, \overline{b}_2]$ in Equations (156) with respect to the interval parameter b. Next, for each of these two intervals, we should appropriately employ both the Sridhara and the Fagnano formulas so that we can get exact (sharp) bounds. Therefore, this mixed approach is somewhat complicated in the present numerical example and its detailed application was not made.

4. Interval Coefficient c in the Quadratic Equation

In this section, we consider the third and last case where it is the coefficient (parameter) c of the interval quadratic Equation (1), which is an uncertain parameter and, in this case, an interval parameter. On the other hand, the other two parameters, the coefficients a and b

of the same interval quadratic equation, Equation (1), are deterministic quantities. The present interval parameter c is defined as

$$c \in C := [\underline{c}, \overline{c}]. \tag{163}$$

By using the interval quadratic Equation (1), the above interval (163) for the uncertain coefficient c and classic interval analysis [7] Elishakoff and Daphnis [12] (Subsection 3.3, pp. 1028–1029, Equations (56)–(59)) found the following formulas for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$, and $\overline{x}_2$ of the roots $x_{1,2}$ (here assumed real and distinct) of the interval quadratic Equation (1) respecting the conditions that the deterministic coefficient a must be either positive or negative, but, evidently, different from zero:

$$\underline{x}_1 = \frac{-b - \sqrt{b^2 - 4a\underline{c}}}{2a}, \tag{164}$$

$$\overline{x}_1 = \frac{-b - \sqrt{b^2 - 4a\overline{c}}}{2a}, \tag{165}$$

$$\underline{x}_2 = \frac{-b + \sqrt{b^2 - 4a\overline{c}}}{2a}, \tag{166}$$

$$\overline{x}_2 = \frac{-b + \sqrt{b^2 - 4a\underline{c}}}{2a}. \tag{167}$$

We now consider the application of the method of quantifier elimination [13,14], where the coefficient c is an interval parameter defined in Equation (163) while the coefficients a and b remain two deterministic parameters. This method has been applied to the present problem by Ioakimidis [15] (Section 3, pp. 4–10). Beyond a non-trivial interval c (that is, with $\underline{c} < \overline{c}$ in Equation (163)) Ioakimidis also assumed that $a > 0$, $b \neq 0$, and $b^2 - 4a\overline{c} > 0$ to make sure that the discriminant $\Delta := b^2 - 4ac$ of the interval quadratic Equation (1) is always positive ($\Delta > 0$) in the analysis. This happens since $a > 0$ and $\underline{c} < \overline{c}$. Under all these assumptions collectively denoted by the symbol $\mathcal{A}_{c,a>0}$, that is [15] (p. 4, Equation (4))

$$\mathcal{A}_{c,a>0} := a > 0 \wedge b \neq 0 \wedge \underline{c} \leq c \leq \overline{c} \wedge \underline{c} < \overline{c} \wedge b^2 - 4a\overline{c} > 0 \tag{168}$$

in a conjunctive logical form, the method of quantifier elimination yielded exactly the same Formulas (164)–(167) [15] (Section 3, pp. 4–10), which were previously obtained by Elishakoff and Daphnis [12] (Subsection 3.3, pp. 1028–1029, Equations (56)–(59)) by using the method of interval analysis.

On the other hand, in parallel with the method of quantifier elimination [13,14], Ioakimidis [15] (Section 3, pp. 4–10) also considered the direct method through the appropriate minimizations and maximizations with respect to the uncertain parameter c again with the help of the `Minimize` and `Maximize` commands of *Mathematica* [16]. The derived formulas for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the two roots $x_{1,2}$ (with $x_1 < x_2$ and again under the validity of the assumptions $\mathcal{A}_{c,a>0}$ defined in Equation (168)) again coincide with the above Formulas (164)–(167) having been derived by Elishakoff and Daphnis [12] (Subsection 3.3, pp. 1028–1029, Equations (56)–(59)) and also, subsequently, by Ioakimidis [15] (Section 3, pp. 4–10). This coincidence constitutes a direct verification of the validity of the formulas displayed in Equations (164)–(167) under the present assumptions $\mathcal{A}_{c,a>0}$.

Unfortunately, Ioakimidis [15] (Section 3, pp. 4–10) did not consider the case where $a < 0$, as is clear from the assumptions $\mathcal{A}_{c,a>0}$ defined in Equation (168), where it was explicitly assumed that $a > 0$ (first assumption $\mathcal{A}_{c,a>0}$). In this second but also important case, $a < 0$, and again assuming that we have two distinct real roots $x_{1,2}$, the above assumptions $\mathcal{A}_{c,a>0}$ take the slightly modified form

$$\mathcal{A}_{c,a<0} := a < 0 \wedge b \neq 0 \wedge \underline{c} \neq 0 \wedge \underline{c} \leq c \leq \overline{c} \wedge \underline{c} < \overline{c} \wedge b^2 - 4a\underline{c} > 0. \tag{169}$$

Now we simply have $a < 0$ instead of $a > 0$ in the original assumptions $\mathcal{A}_{c,a>0}$ defined in Equation (168) and, additionally, we have $\underline{c} \neq 0$ and a slight change in the expression of the discriminant Δ: now with $\underline{c}$ for $a < 0$ instead of $\overline{c}$ for $a > 0$ previously. The aim of this change in the expression of Δ is simply that the existence of two distinct real roots $x_{1,2}$ (with $x_1 < x_2$) of the interval quadratic Equation (1) should be assured for all values of the interval parameter $c \in C := [\underline{c}, \overline{c}]$, Equation (163).

In this second case with $a < 0$, working exactly as Ioakimidis did in the first case with $a > 0$ [15] (Section 3, pp. 4–10) and again employing the method of quantifier elimination [13,14], we obtain the four new bounds of the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1). Evidently, these new bounds are valid only under the validity of all the assumptions $\mathcal{A}_{c,a<0}$ defined in Equation (169) including the important assumption $a < 0$ concerning the coefficient a (first assumption) and have the following forms:

$$\underline{x}_1 = \frac{-b + \sqrt{b^2 - 4a\overline{c}}}{2a}, \tag{170}$$

$$\overline{x}_1 = \frac{-b + \sqrt{b^2 - 4a\underline{c}}}{2a}, \tag{171}$$

$$\underline{x}_2 = \frac{-b - \sqrt{b^2 - 4a\underline{c}}}{2a}, \tag{172}$$

$$\overline{x}_2 = \frac{-b - \sqrt{b^2 - 4a\overline{c}}}{2a}. \tag{173}$$

In these formulas, we observe that the bounds of the first root x_1 in the previous Formulas (164) and (165) now appear as the bounds of the second root x_2 in the new Formulas (172) and (173), respectively, and, conversely, the bounds of the second root x_2 in the previous Formulas (166) and (167) now appear as the bounds of the first root x_1 in the new Formulas (170) and (171), respectively. No other difference is observed between these sets of bounds: (i) the bounds (164)–(167) for $a > 0$ and (ii) the bounds (170)–(173) for $a < 0$, but continuously under the assumption $x_1 < x_2$.

Alternatively, in the same case, that is with $a < 0$, we also worked by using the direct method with the appropriate minimizations and maximizations (with the use of the `Minimize` and `Maximize` related commands of *Mathematica* [16]). Then we again obtained the bounds (170)–(173) of the roots $x_{1,2}$ of the interval quadratic Equation (1). Evidently, these two roots $x_{1,2}$ are now determined by the formulas in the second pair of Equation (2) because now $a < 0$. Clearly, this result by using the direct method constitutes a verification of the above Formulas (170)–(173) that are valid for $a < 0$.

Naturally, a second and perhaps simpler way of working in the present case, the case with $a < 0$, where the results of Ioakimidis [15] (Section 3, pp. 4–10) are inapplicable, is simply to transform the interval quadratic Equation (1) under consideration (here with $a < 0$) to an equivalent interval quadratic equation having the same distinct real roots $x_{1,2}$, but now with $a > 0$. This aim can easily be achieved through a multiplication of the initial interval quadratic Equation (1) by -1, that is

$$ax^2 + bx + c = 0 \ \text{ with } \ a < 0 \iff -ax^2 - bx - c = 0 \ \text{ now with } \ -a > 0 \tag{174}$$

as far as the coefficient of the quadratic term (x^2) is concerned, now $-a > 0$ instead of $a < 0$.

We take into consideration that in the present case (with $a < 0$) the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1) are given by the second pair of formulas in Equation (2), that is

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{175}$$

(provided that we wish that $x_1 < x_2$ as is here the case with $a < 0$) instead of the first pair of these formulas in Equation (2), that is

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{176}$$

which is valid in the case where $a > 0$, but again, we repeat, only if we wish that $x_1 < x_2$. Hence, for the determination of the bounds of the first root x_1 of the quadratic equation $ax^2 + bx + c = 0$ with $a < 0$ we are now obliged to use the bounds of the second root x_2 of the modified quadratic equation $-ax^2 - bx - c = 0$ with $-a > 0$. This is achieved by making the four simple substitutions

$$a \to -a, \quad b \to -b, \quad \underline{c} \to -\overline{c}, \quad \overline{c} \to -\underline{c} \quad \text{since} \quad -[\underline{c}, \overline{c}] = [-\overline{c}, -\underline{c}]. \tag{177}$$

Next, analogously, we can work with the second root x_2 of the quadratic equation $ax^2 + bx + c = 0$. Then we directly obtain again the Formulas (170)–(173). This result constitutes a second verification of these formulas (for $a < 0$) that are valid under the assumptions $\mathcal{A}_{c,a<0}$ defined in Equation (169).

Let us now consider the same problem, but by using the method of interval parametrization introduced by Elishakoff and Miglis [8] with the same interval $C := [\underline{c}, \overline{c}]$ for the uncertain coefficient (parameter) c that was defined in Equation (163) and again assuming the parameters a and b to be crisp.

Here, exactly as in the previous two sections concerning the uncertain coefficients a and b, we need to introduce two auxiliary quantities in order to parametrize this interval, $C := [\underline{c}, \overline{c}]$, of the interval coefficient c. These quantities are the average value (the midpoint) of the interval $C := [\underline{c}, \overline{c}]$ here denoted by the symbol c_{ave} and the deviation value (the radius) of the same interval C here denoted by the symbol c_{dev}. Now by using the interval $C := [\underline{c}, \overline{c}]$ in Equation (163) these two quantities, the quantities c_{ave} and c_{dev}, are defined as

$$c_{\text{ave}} \coloneqq \frac{\underline{c} + \overline{c}}{2}, \tag{178}$$

$$c_{\text{dev}} \coloneqq \frac{\overline{c} - \underline{c}}{2}. \tag{179}$$

Now, by using the above Equations (178) and (179), we can rewrite the uncertain coefficient c in the parametric form

$$c = c_{\text{ave}} + c_{\text{dev}}\, t \tag{180}$$

with parameter

$$t \in T := [-1, 1]. \tag{181}$$

As a consequence, by using the parametric Equation (180) for the uncertain coefficient c and assuming that $a > 0$ we can express the two roots $x_{1,2}$ of the present interval quadratic Equation (1) (with these roots here continuously assumed real and distinct with $x_1 < x_2$), which are displayed in the first pair of Equation (2), as

$$x_1 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}}\, t)}}{2a}, \tag{182}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}}\, t)}}{2a}. \tag{183}$$

Finally, we should again proceed to minimizations and maximizations with respect to the present parameter t of course under the continuous validity of the constraint (181) for this parameter t, that is $t \in T := [-1, 1]$ or, equivalently, $-1 \leq t \leq 1$. Hence, in this way, we can directly determine the intervals of both roots $x_{1,2}$. These computations can easily be performed by using the commands `Minimize` and `Maximize` of *Mathematica* [16] for minimizations and maximizations, respectively, or, alternatively, the analogous commands of another computer algebra system such as *Maple* either with concrete interval quadratic Equation (1), that is with known numerical values of a, b, $\underline{c}$, and $\overline{c}$, or even in the general case with these four quantities a, b, $\underline{c}$ and $\overline{c}$ simply being parameters.

Of course, as was already mentioned in Section 2 for the interval coefficient a, there is no need at all to use the present parametric form (180) for the derivation of the present bounds of the two roots $x_{1,2}$ of the interval quadratic Equation (1).

Now, in the special case where $a > 0$, under the four new assumptions (including that assuring that $\Delta > 0$)

$$\mathcal{A}_{c,a>0,t} := a > 0 \wedge c_{\text{dev}} > 0 \wedge b^2 - 4a(c_{\text{ave}} + c_{\text{dev}}) > 0 \wedge -1 \leq t \leq 1 \tag{184}$$

by using *Mathematica* [16] we obtain the following bounds of the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1):

$$\underline{x}_1 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} - c_{\text{dev}})}}{2a} \quad (\text{at } t = \underline{t} = -1), \tag{185}$$

$$\overline{x}_1 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}})}}{2a} \quad (\text{at } t = \overline{t} = 1), \tag{186}$$

$$\underline{x}_2 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}})}}{2a} \quad (\text{at } t = \overline{t} = 1), \tag{187}$$

$$\overline{x}_2 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} - c_{\text{dev}})}}{2a} \quad (\text{at } t = \underline{t} = -1). \tag{188}$$

Next, simply by using Equations (178) and (179) for the two quantities c_{ave} and c_{dev}, respectively, we directly transform the above Formulas (185)–(188) to the corresponding Formulas (164)–(167), respectively, already obtained (i) by the method of classic interval analysis [7], (ii) by the method of quantifier elimination [13,14], and (iii) by the direct method. The direct method is also based on minimizations and maximizations, but this time directly with respect to the uncertain coefficient (parameter) $c \in C := [\underline{c}, \overline{c}]$ instead of the parameter t (with $t \in T := [-1, 1]$) that is used in the present method of interval parametrization [8].

Of course, quite similar is the case where $a < 0$ together with $c \in C := [\underline{c}, \overline{c}]$, Equation (163), here again under the assumption that we have two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1) and, additionally, that $x_1 < x_2$. Here we will also study this case, $a < 0$, again by using the method of interval parametrization of Elishakoff and Miglis [8] exactly as we did previously, but now with $a < 0$. Evidently, in this case, it is the second pair of Formula (2) for the roots $x_{1,2}$ of the interval quadratic Equation (1) that will be used. Therefore, here the following formulas are valid:

$$x_1 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}}\,t)}}{2a}, \tag{189}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}}\,t)}}{2a} \tag{190}$$

instead of the similar Formulas (182) and (183) used previously in the case where $a > 0$.

In this case, $a < 0$, the previous assumptions $\mathcal{A}_{c,a>0,t}$ defined in Equation (184) take the slightly modified form

$$\mathcal{A}_{c,a<0,t} := a < 0 \wedge c_{\text{dev}} > 0 \wedge b^2 - 4a(c_{\text{ave}} - c_{\text{dev}}) > 0 \wedge -1 \leq t \leq 1. \tag{191}$$

Now by using again the `Minimize` and `Maximize` commands of *Mathematica* [16] we obtain the following four bounds of the two distinct real roots $x_{1,2}$ of the interval quadratic Equation (1):

$$\underline{x}_1 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}})}}{2a} \quad (\text{at } t = \bar{t} = 1), \tag{192}$$

$$\bar{x}_1 = \frac{-b + \sqrt{b^2 - 4a(c_{\text{ave}} - c_{\text{dev}})}}{2a} \quad (\text{at } t = \underline{t} = -1), \tag{193}$$

$$\underline{x}_2 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} - c_{\text{dev}})}}{2a} \quad (\text{at } t = \underline{t} = -1), \tag{194}$$

$$\bar{x}_2 = \frac{-b - \sqrt{b^2 - 4a(c_{\text{ave}} + c_{\text{dev}})}}{2a} \quad (\text{at } t = \bar{t} = 1). \tag{195}$$

These formulas essentially coincide with the Formulas (185)–(188), which resulted in the first case, that is the case where $a > 0$, but now the roles of the two roots $x_{1,2}$ of Equation (1) have been reversed.

Next, simply by using Equations (178) and (179) for the two quantities c_{ave} and c_{dev}, respectively, we directly transform the above Formulas (192)–(195) to the corresponding Formulas (170)–(173), which were obtained by the method of quantifier elimination [13,14] and also by the direct method.

Of course, as was already mentioned, there is no need to use the present parametric form (180) for the derivation of the present bounds of the two roots $x_{1,2}$ of the interval quadratic Equation (1). For example, in the case where $a < 0$, we also successfully used the modified parametric form

$$c = c_{\text{ave}} + c_{\text{dev}}\, t^5 \tag{196}$$

with t^5 instead of t and again with parameter $t \in T := [-1, 1]$. Having worked by using *Mathematica* exactly as previously, we again derived the same Formulas (192)–(195) and, next, the more convenient Formulas (170)–(173) for the bounds $\underline{x}_1$, $\bar{x}_1$, $\underline{x}_2$, and $\bar{x}_2$ of the roots $x_{1,2}$ that were also derived by using the initial parametric form (180). Of course, in practice, the use of the interval coefficient c as a parameter (as is the case in the direct method) without the use of a new parameter t (as is the case in Equations (180) and (196)) is sufficient for the computation of the aforementioned bounds.

Now that the formulas for the bounds of the roots $x_{1,2}$ have been established for the considered methods, we are able to proceed with a numerical example related, of course, to the present interval parameter c in Equation (163). This example was proposed and studied by Elishakoff and Daphnis [12] (Subsection 3.3, p. 1029). These authors assumed the following numerical values of the parameters a and b and a concrete interval c with numerical values of its endpoints [12] (p. 1029, Equation (60)):

$$a = 5, \quad b = 10, \quad c = [\underline{c}, \bar{c}] = [1, 2]. \tag{197}$$

Then by using the Formulas (164)–(167) derived by Elishakoff and Daphnis [12] (Subsection 3.3, pp. 1028–1029, Equations (56)–(59)) with classic interval analysis [7] and the above numerical values (197), Elishakoff and Daphnis computed the following bounds of the two roots $x_{1,2}$ [12] (p. 1029, Equations (61) and (62)):

$$\underline{x}_1 = -1.894427, \quad \bar{x}_1 = -1.774597, \tag{198}$$

$$\underline{x}_2 = -0.225403, \quad \bar{x}_2 = -0.105573. \tag{199}$$

Evidently, the same bounds also hold true when we work with the method of quantifier elimination.

On the other hand, by again using the numerical values (197) of a, b, $\underline{c}$, and $\overline{c}$ the direct approach yields the following results computed by Elishakoff and Daphnis [12] (p. 1029, Equations (63) and (64)):

$$\underline{x}_1 = -1.894427 \quad (\text{at } c = \underline{c} = 1), \quad \overline{x}_1 = -1.774597 \quad (\text{at } c = \overline{c} = 2), \tag{200}$$

$$\underline{x}_2 = -0.225403 \quad (\text{at } c = \overline{c} = 2), \quad \overline{x}_2 = -0.105573 \quad (\text{at } c = \underline{c} = 1). \tag{201}$$

Hence, the results (here the bounds of the roots $x_{1,2}$) derived by using (i) the method of classic interval analysis, (ii) the method of quantifier elimination and (iii) the direct method are identical.

Finally, by using the interval parametrization method [8] based on Equations (182) and (183), where we proceed to minimizations and maximizations with respect to the parameter $t \in T := [-1, 1]$, and again taking into consideration the numerical values (197), we obtain the following expected results:

$$\underline{x}_1 = -1.894427 \quad (\text{at } t = \underline{t} = -1), \quad \overline{x}_1 = -1.774597 \quad (\text{at } t = \overline{t} = 1), \tag{202}$$

$$\underline{x}_2 = -0.225403 \quad (\text{at } t = \overline{t} = 1), \quad \overline{x}_2 = -0.105573 \quad (\text{at } t = \underline{t} = -1). \tag{203}$$

Here we directly observe that the above values of the four bounds are identical to the corresponding previous values displayed in Equations (198)–(201) and are already found by using the other three methods.

Therefore, in the present numerical example, where the uncertain coefficient c in the interval quadratic Equation (1) is an interval parameter, $c \in C := [\underline{c}, \overline{c}] = [1, 2]$, we obtained the same results for the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$, and $\overline{x}_2$ of the roots $x_{1,2}$ of this quadratic equation by using all four methods under consideration. Here the formulas for these roots $x_{1,2}$ are displayed in the first pair of Equation (2) and this happens simply because $a = 5 > 0$, first Equation (197), in the present numerical example.

5. Some Generalized Babylonian Problems Involving Interval Quadratic Equations

As mentioned in the introduction of this paper, Babylonians have been handling problems that deal with quadratic equations. Let us consider some practical problems related to quadratic equations. The first one stems from tablet YBC 4663. According to Katz [20] (p. 23), Babylonians "applied this to various standard problems such as finding the length and width of a rectangle, given the semiperimeter and the area. For example, consider the problem $x + y = 6\frac{1}{2}$, $xy = 7\frac{1}{2}$ from tablet YBC 4663".

Let us solve this problem first in the crisp setting. We write the relevant equations again:

$$x + y = 6\tfrac{1}{2} = 6.5 \tag{204}$$

and

$$xy = 7\tfrac{1}{2} = 7.5. \tag{205}$$

So, using Equation (204), we have

$$y = 6\tfrac{1}{2} - x = 6.5 - x. \tag{206}$$

This means that, using Equations (205) and (206), we obtain

$$x(6.5 - x) = 7.5. \tag{207}$$

Equation (207) yields the quadratic equation

$$ax^2 + bx + c = -x^2 + 6.5x - 7.5 = 0. \tag{208}$$

We compute the discriminant Δ of this quadratic equation, Equation (208):

$$\Delta := b^2 - 4ac = 6.5^2 - 4(-1)(-7.5) = 42.25 - 30 = 12.25. \tag{209}$$

Using Equation (209) we compute the two roots $x_{1,2}$ (with $x_1 < x_2$) of the above quadratic Equation (208), which are given by the second pair of Formulas (2) because $a = -1 < 0$ in this quadratic equation. The values of these roots $x_{1,2}$ are

$$x_1 = x_{1,d} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-6.5 + \sqrt{12.25}}{-2} = 1.5, \tag{210}$$

$$x_2 = x_{2,d} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-6.5 - \sqrt{12.25}}{-2} = 5. \tag{211}$$

These roots $x_{1,d}$ and $x_{2,d}$ constitute the deterministic solution of the problem.

In reality, the measurements are associated with errors. So we will now consider the problem in a realistic interval setting with an interval semiperimeter $x + y$ and a deterministic (crisp) area xy, more explicitly,

$$x + y \in \left[6\tfrac{1}{4}, 6\tfrac{3}{4}\right] = [6.25, 6.75] \tag{212}$$

and

$$xy = 7\tfrac{1}{2} = 7.5. \tag{213}$$

Equations (212) and (213) yield the interval quadratic equation (here with interval coefficient b)

$$ax^2 + [\underline{b}, \overline{b}]\,x + c = -x^2 + [6.25, 6.75]\,x - 7.5 = 0 \tag{214}$$

evidently with

$$a = -1, \quad \underline{b} = 6.25, \quad \overline{b} = 6.75, \quad c = -7.5. \tag{215}$$

Since we know that by using classic interval analysis [7] we will obtain results that may not be the sharpest possible simply because the coefficient b appears twice in the Formula (2) for the two roots $x_{1,2}$, here we will employ the appropriate formulas obtained by using the method of quantifier elimination [13,14] together with the interval quadratic Equation (214) with interval coefficient b. Clearly, these are the formulas derived with $a < 0$, $c < 0$ and $b > 0$ (positive interval coefficient b).

These formulas, Equations (113)–(116), for the roots $x_{1,2}$ in the present case directly yield the bounds

$$\underline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} = \frac{-6.75 + \sqrt{6.75^2 - 4(-1)(-7.5)}}{-2} = 1.40253, \tag{216}$$

$$\overline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = \frac{-6.25 + \sqrt{6.25^2 - 4(-1)(-7.5)}}{-2} = 1.61980, \tag{217}$$

$$\underline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} = \frac{-6.25 - \sqrt{6.25^2 - 4(-1)(-7.5)}}{-2} = 4.63020, \tag{218}$$

$$\overline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} = \frac{-6.75 - \sqrt{6.75^2 - 4(-1)(-7.5)}}{-2} = 5.34747 \tag{219}$$

of course with $x_1 < x_2$ as is continuously assumed in this paper and, therefore, expected as well.

The alternative possibility is to multiply the interval quadratic Equation (214) by -1. Then we get the following modified, but equivalent with respect to the roots $x_{1,2}$, interval quadratic equation:

$$a^* x^2 + [\underline{b}^*, \overline{b}^*]\,x + c^* = x^2 + [-6.75, -6.25]\,x + 7.5 = 0 \tag{220}$$

evidently with exactly the same roots $x_{1,2}$ again with $x_1 < x_2$. In this equation, we obviously have

$$a^* = -a = 1 > 0, \quad c^* = -c = 7.5, \quad \underline{b}^* = -\overline{b} = -6.75, \quad \overline{b}^* = -\underline{b} = -6.25. \tag{221}$$

By using these modified numerical values (221) in the Formulas (78)–(81) concerning the present case, that is $a > 0$, $c > 0$ and $b < 0$, here with

$$a \to a^* = 1, \quad c \to c^* = 7.5, \quad \underline{b} \to \underline{b}^* = -6.75, \quad \overline{b} \to \overline{b}^* = -6.25 \tag{222}$$

we find the bounds

$$\underline{x}_1 = \frac{-\underline{b}^* - \sqrt{\underline{b}^{*2} - 4a^*c^*}}{2a^*} = \frac{-(-6.75) - \sqrt{(-6.75)^2 - 4 \cdot 1 \cdot 7.5}}{2} = 1.40253, \tag{223}$$

$$\overline{x}_1 = \frac{-\overline{b}^* - \sqrt{\overline{b}^{*2} - 4a^*c^*}}{2a^*} = \frac{-(-6.25) - \sqrt{(-6.25)^2 - 4 \cdot 1 \cdot 7.5}}{2} = 1.61980, \tag{224}$$

$$\underline{x}_2 = \frac{-\overline{b}^* + \sqrt{\overline{b}^{*2} - 4a^*c^*}}{2a^*} = \frac{-(-6.25) + \sqrt{(-6.25)^2 - 4 \cdot 1 \cdot 7.5}}{2} = 4.63020, \tag{225}$$

$$\overline{x}_2 = \frac{-\underline{b}^* + \sqrt{\underline{b}^{*2} - 4a^*c^*}}{2a^*} = \frac{-(-6.75) + \sqrt{(-6.75)^2 - 4 \cdot 1 \cdot 7.5}}{2} = 5.34747. \tag{226}$$

We directly observe that as was expected, the above four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of the roots $x_{1,2}$, here interval roots, coincide with the corresponding bounds already computed in Equations (216)–(219).

In the present first problem (with an interval semiperimeter $x + y$ and a deterministic area xy), we see that instead of obtaining the deterministic solutions $x_1 = x_{1,d} = 1.5$ and $x_2 = x_{2,d} = 5$ in Equations (210) and (211), respectively, we get the interval solutions

$$x_1 \in X_1 := [\underline{x}_1, \overline{x}_1] = [1.40253, 1.61980] \quad \text{and} \quad x_2 \in X_2 := [\underline{x}_2, \overline{x}_2] = [4.63020, 5.34747]. \tag{227}$$

These two interval solutions are clear from Equations (216)–(219) (with $a = -1 < 0$ there) as well as from Equations (223)–(226) (with $a^* = -a = 1 > 0$ there).

Moreover, we observe that the midpoints of the above intervals (227) are

$$x_{1,\mathrm{ave}} = \frac{1}{2}(\underline{x}_1 + \overline{x}_1) = 1.51117 \quad \text{and} \quad x_{2,\mathrm{ave}} = \frac{1}{2}(\underline{x}_2 + \overline{x}_2) = 4.98883 \tag{228}$$

for the first root x_1 and the second root x_2, respectively. Hence, these midpoints do not coincide with the deterministic solutions $x_1 = x_{1,d} = 1.5$ and $x_2 = x_{2,d} = 5$, respectively, although the differences

$$x_{1,\mathrm{ave}} - x_{1,d} = 1.51117 - 1.5 = 0.01117 \quad \text{and} \quad x_{2,\mathrm{ave}} - x_{2,d} = 4.98883 - 5 = -0.01117 \tag{229}$$

are, undoubtedly, sufficiently small. The aforementioned lack of coincidence having led to the non-zero differences (229) is due to the nonlinearity of the Formulas (210) and (211) for the roots $x_{1,2}$ of the quadratic Equation (208). This nonlinearity includes (i) the existence of a square root in both Formulas (210) and (211) for these roots $x_{1,2}$ and (ii) the appearance of the square b^2 of the interval coefficient b inside this square root.

This nonlinearity becomes even more clear if we write the two bounds $\underline{b}$ and $\overline{b}$ of the interval coefficient b in the forms

$$\underline{b} = b_{\mathrm{ave}} - b_{\mathrm{dev}} \quad \text{and} \quad \overline{b} = b_{\mathrm{ave}} + b_{\mathrm{dev}}, \tag{230}$$

where $b_{\mathrm{ave}} = (6.25 + 6.75)/2 = 13/2 = 6.5$. Next, we can use the expressions of the two bounds $\underline{x}_1$ and $\overline{x}_1$ of the first root x_1 in Equations (216) and (217), respectively, assuming that b_{dev} is a variable, and proceed to the derivation of the related Taylor–Maclaurin series of course again using *Mathematica* [16] for this task. Then we obtain the two Taylor–Maclaurin series (here with terms up to b_{dev}^{20}):

$$\begin{aligned}\{\underline{x}_1, \overline{x}_1\} \approx\ & 1.5 \mp 0.428571\, b_{\text{dev}} + 0.174927\, b_{\text{dev}}^2 \mp 0.0928185\, b_{\text{dev}}^3 + 0.0579933\, b_{\text{dev}}^4 \\ & \mp 0.0400500\, b_{\text{dev}}^5 + 0.0295094\, b_{\text{dev}}^6 \mp 0.0227373\, b_{\text{dev}}^7 + 0.0180995\, b_{\text{dev}}^8 \\ & \mp 0.0147690\, b_{\text{dev}}^9 + 0.0122880\, b_{\text{dev}}^{10} \mp 0.0103853\, b_{\text{dev}}^{11} + 0.00889112\, b_{\text{dev}}^{12} \\ & \mp 0.00769463\, b_{\text{dev}}^{13} + 0.00672056\, b_{\text{dev}}^{14} \mp 0.00591631\, b_{\text{dev}}^{15} + 0.00524417\, b_{\text{dev}}^{16} \\ & \mp 0.00467646\, b_{\text{dev}}^{17} + 0.00419246\, b_{\text{dev}}^{18} \mp 0.00377642\, b_{\text{dev}}^{19} + 0.00341615\, b_{\text{dev}}^{20}. \end{aligned} \tag{231}$$

In these equations and in the minus–plus ($\mp$) sign, the upper sign (the minus sign) corresponds to the lower bound $\underline{x}_1$ of the root x_1 whereas the lower sign (the plus sign) corresponds to the upper bound $\overline{x}_1$ of the same root x_1. Now, by adding the above two Taylor–Maclaurin series (231) and dividing the result by two we obtain the midpoint $x_{1,\text{ave}}$ of the root x_1. In this way, we easily find that

$$\begin{aligned} x_{1,\text{ave}} - x_{1,d} = \frac{1}{2}\left(\underline{x}_1 + \overline{x}_1\right) - x_{1,d} \approx\ & 0.174927\, b_{\text{dev}}^2 + 0.0579933\, b_{\text{dev}}^4 + 0.0295094\, b_{\text{dev}}^6 \\ & + 0.0180995\, b_{\text{dev}}^8 + 0.0122880\, b_{\text{dev}}^{10} + 0.00889112\, b_{\text{dev}}^{12} + 0.00672056\, b_{\text{dev}}^{14} \\ & + 0.00524417\, b_{\text{dev}}^{16} + 0.00419246\, b_{\text{dev}}^{18} + 0.00341615\, b_{\text{dev}}^{20} \end{aligned} \tag{232}$$

with $x_{1,d} = 1.5$, Equation (210). The nonlinear terms in the right-hand side of the above equation constitute the reason of the inequality $x_{1,\text{ave}} \neq x_{1,d}$, which is also clear from the first of Equation (229).

Now in the special case where $b_{\text{dev}} = 0.25$ (exactly as is the case in the present numerical example with $b_{\text{dev}} = (6.75 - 6.25)/2 = 0.50/2 = 0.25$) Equation (231) for the bounds $\underline{x}_1$ and $\overline{x}_1$ yield

$$\underline{x}_1 = 1.40253, \quad \overline{x}_1 = 1.61980, \quad \underline{x}_{1,\text{ave}} = 1.51117, \quad \underline{x}_{1,\text{ave}} - x_{1,d} = 0.01117. \tag{233}$$

Evidently, the above numerical results are in complete agreement with the corresponding results displayed in Equations (216) and (217) for the bounds $\underline{x}_1$ and $\overline{x}_1$, respectively, in the first of Equation (228) for the midpoint $x_{1,\text{ave}}$ and in the first of Equation (229) for the difference $x_{1,\text{ave}} - x_{1,d}$ as was expected.

In the same problem, completely analogous results were also derived for the second root x_2 of the interval quadratic Equation (214) with deterministic value of this root $x_{2,d} = 5$, Equation (211). Now Equation (232) for the difference $x_{1,\text{ave}} - x_{1,d}$, which concerns the first root x_1, takes the following completely analogous form, but now the difference $x_{2,\text{ave}} - x_{2,d}$, which concerns the second root x_2:

$$\begin{aligned} x_{2,\text{ave}} - x_{2,d} = \frac{1}{2}\left(\underline{x}_2 + \overline{x}_2\right) - x_{2,d} \approx\ & -0.174927\, b_{\text{dev}}^2 - 0.0579933\, b_{\text{dev}}^4 - 0.0295094\, b_{\text{dev}}^6 \\ & - 0.0180995\, b_{\text{dev}}^8 - 0.0122880\, b_{\text{dev}}^{10} - 0.00889112\, b_{\text{dev}}^{12} - 0.00672056\, b_{\text{dev}}^{14} \\ & - 0.00524417\, b_{\text{dev}}^{16} - 0.00419246\, b_{\text{dev}}^{18} - 0.00341615\, b_{\text{dev}}^{20} \end{aligned} \tag{234}$$

with $x_{2,d} = 5$, Equation (211). By comparing Equation (232) (for the first root x_1) and (234) (for the second root x_2) we directly observe that

$$x_{2,\text{ave}} - x_{2,d} = -(x_{1,\text{ave}} - x_{1,d}) \quad \text{or, equivalently,} \quad (x_{1,\text{ave}} - x_{1,d}) + (x_{2,\text{ave}} - x_{2,d}) = 0. \tag{235}$$

Now in the special case where $b_{\mathrm{dev}} = 0.25$ (exactly as is the case here), Equation (234) yields

$$\underline{x}_{2,\mathrm{ave}} - x_{2,d} = -0.01117 \tag{236}$$

which is in complete agreement with the second of Equations (229) and as a verification of this equation.

We could also consider the problem in an alternative interval setting, now with a deterministic (crisp) semiperimeter $x + y$ but with an interval area xy, as follows:

$$x + y = 6\tfrac{1}{2} = 6.5 \tag{237}$$

and

$$xy \in \left[7\tfrac{1}{4}, 7\tfrac{3}{4}\right] = [7.25, 7.75]. \tag{238}$$

These equations, Equations (237) and (238), directly lead to the interval quadratic equation (but now with interval coefficient c)

$$ax^2 + bx + [\underline{c}, \overline{c}] = -x^2 + 6.5x + [-7.75, -7.25] = 0. \tag{239}$$

Here we will again use the formulas obtained by the method of quantifier elimination [13,14], but now in the case where the coefficient a is negative ($a < 0$) and c is an interval coefficient. These formulas, Equations (170)–(173), and the present interval quadratic Equation (239) lead to the following bounds:

$$\underline{x}_1 = \frac{-b + \sqrt{b^2 - 4a\overline{c}}}{2a} = \frac{-6.5 + \sqrt{6.5^2 - 4(-1)(-7.25)}}{-2} = 1.42997, \tag{240}$$

$$\overline{x}_1 = \frac{-b + \sqrt{b^2 - 4a\underline{c}}}{2a} = \frac{-6.5 + \sqrt{6.5^2 - 4(-1)(-7.75)}}{-2} = 1.57295, \tag{241}$$

$$\underline{x}_2 = \frac{-b - \sqrt{b^2 - 4a\underline{c}}}{2a} = \frac{-6.5 - \sqrt{6.5^2 - 4(-1)(-7.75)}}{-2} = 4.92705, \tag{242}$$

$$\overline{x}_2 = \frac{-b - \sqrt{b^2 - 4a\overline{c}}}{2a} = \frac{-6.5 - \sqrt{6.5^2 - 4(-1)(-7.25)}}{-2} = 5.07003 \tag{243}$$

obviously again with $x_1 < x_2$ as is continuously assumed here and, therefore, completely expected.

On the other hand, exactly as in the previous generalized Babylonian problem, by multiplying the left-hand side of the interval quadratic Equation (239) under consideration by -1 we directly transform it to the modified, but equivalent with respect to the roots $x_{1,2}$, interval quadratic equation

$$a^*x^2 + b^*x + [\underline{c}^*, \overline{c}^*] = -ax^2 - bx + [-\overline{c}, -\underline{c}] = x^2 - 6.5x + [7.25, 7.75] = 0 \tag{244}$$

now with

$$a^* = -a = 1 > 0, \quad b^* = -b = -6.5, \quad \underline{c}^* = -\overline{c} = 7.25, \quad \overline{c}^* = -\underline{c} = 7.75. \tag{245}$$

Naturally, in this interval quadratic equation, Equation (244), we can use the initial Equations (164)–(167) for the sought four bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ of its two roots $x_{1,2}$ (now with $a^* = 1 > 0$ and again with the demand that $x_1 < x_2$) instead of Equations (170)–(173)

that were used previously (with $a = -1 < 0$, but again with $x_1 < x_2$). Then we easily find the following bounds for the roots $x_{1,2}$:

$$\underline{x}_1 = \frac{-b^* - \sqrt{b^{*2} - 4a^*\underline{c}^*}}{2a^*} = \frac{-(-6.5) - \sqrt{(-6.5)^2 - 4 \cdot 1 \cdot 7.25}}{2} = 1.42997, \tag{246}$$

$$\overline{x}_1 = \frac{-b^* - \sqrt{b^{*2} - 4a^*\overline{c}^*}}{2a^*} = \frac{-(-6.5) - \sqrt{(-6.5)^2 - 4 \cdot 1 \cdot 7.75}}{2} = 1.57295, \tag{247}$$

$$\underline{x}_2 = \frac{-b^* + \sqrt{b^{*2} - 4a^*\overline{c}^*}}{2a^*} = \frac{-(-6.5) + \sqrt{(-6.5)^2 - 4 \cdot 1 \cdot 7.75}}{2} = 4.92705, \tag{248}$$

$$\overline{x}_2 = \frac{-b^* + \sqrt{b^{*2} - 4a^*\underline{c}^*}}{2a^*} = \frac{-(-6.5) + \sqrt{(-6.5)^2 - 4 \cdot 1 \cdot 7.25}}{2} = 5.07003. \tag{249}$$

Of course, the obtained numerical values of the bounds $\underline{x}_1$, $\overline{x}_1$, $\underline{x}_2$ and $\overline{x}_2$ coincide with the numerical values in Equations (240)–(243) exactly as previously for the initial interval quadratic Equation (239).

In the present second problem (with a deterministic semiperimeter $x + y$ and an interval area xy), we observe that instead of obtaining the deterministic solutions $x_1 = x_{1,d} = 1.5$ and $x_2 = x_{2,d} = 5$ in Equations (210) and (211), respectively, here we get the interval solutions

$$x_1 \in X_1 := [\underline{x}_1, \overline{x}_1] = [1.42997, 1.57295] \quad \text{and} \quad x_2 \in X_2 := [\underline{x}_2, \overline{x}_2] = [4.92705, 5.07003]. \tag{250}$$

These two interval solutions are clear from Equations (240)–(243) (with $a = -1 < 0$) as well as from Equations (246)–(249) (with $a^* = -a = 1 > 0$).

Moreover, we observe that the midpoints of the above intervals (250) are

$$x_{1,\text{ave}} = \frac{1}{2}(\underline{x}_1 + \overline{x}_1) = 1.50146, \quad \text{and} \quad x_{2,\text{ave}} = \frac{1}{2}(\underline{x}_2 + \overline{x}_2) = 4.99854 \tag{251}$$

for the first root x_1 and the second root x_2, respectively. Hence, these midpoints do not coincide with the deterministic solutions $x_1 = x_{1,d} = 1.5$ and $x_2 = x_{2,d} = 5$, respectively, although the differences

$$x_{1,\text{ave}} - x_{1,d} = 1.50146 - 1.5 = 0.00146 \quad \text{and} \quad x_{2,\text{ave}} - x_{2,d} = 4.99854 - 5 = -0.00146 \tag{252}$$

are, undoubtedly, sufficiently small. The aforementioned lack of coincidence having led to the non-zero differences (252) is due to the nonlinearity of the Formulas (210) and (211) for the roots $x_{1,2}$ of the quadratic Equation (208). This nonlinearity is due only to the existence of a square root including the interval coefficient c in the Formulas (210) and (211) that hold true for these square roots.

This nonlinearity becomes even more clear if we write the two bounds $\underline{c}$ and $\overline{c}$ of the interval coefficient c in the forms

$$\underline{c} = c_{\text{ave}} - c_{\text{dev}} \quad \text{and} \quad \overline{c} = c_{\text{ave}} + c_{\text{dev}}, \tag{253}$$

where $c_{\text{ave}} = [(-7.75) + (-7.25)]/2 = -15/2 = -7.5$. Next, working analogously to the previous first generalized Babylonian problem, we can use the expressions (240) and (241) of the bounds $\underline{x}_1$ and $\overline{x}_1$, respectively, of the first root x_1 here also assuming that c_{dev} is a variable and proceed to the derivation of the two related Taylor–Maclaurin series of course again by using *Mathematica* [16] for this task. Then we obtain the following Taylor–Maclaurin series (here with terms up to c_{dev}^{20}):

$$\{\underline{x}_1, \overline{x}_1\} \approx 1.5 \mp 0.285714\, c_{\text{dev}} + 2.33236 \cdot 10^{-2}\, c_{\text{dev}}^2 \mp 3.80794 \cdot 10^{-3}\, c_{\text{dev}}^3 + 7.77130 \cdot 10^{-4}\, c_{\text{dev}}^4$$
$$\mp 1.77630 \cdot 10^{-4}\, c_{\text{dev}}^5 + 4.35012 \cdot 10^{-5}\, c_{\text{dev}}^6 \mp 1.11606 \cdot 10^{-5}\, c_{\text{dev}}^7 + 2.96099 \cdot 10^{-6}\, c_{\text{dev}}^8$$
$$\mp 8.05711 \cdot 10^{-7}\, c_{\text{dev}}^9 + 2.23626 \cdot 10^{-7}\, c_{\text{dev}}^{10} \mp 6.30633 \cdot 10^{-8}\, c_{\text{dev}}^{11} + 1.80181 \cdot 10^{-8}\, c_{\text{dev}}^{12}$$
$$\mp 5.20460 \cdot 10^{-9}\, c_{\text{dev}}^{13} + 1.51738 \cdot 10^{-9}\, c_{\text{dev}}^{14} \mp 4.45923 \cdot 10^{-10}\, c_{\text{dev}}^{15} + 1.31957 \cdot 10^{-10}\, c_{\text{dev}}^{16}$$
$$\mp 3.92860 \cdot 10^{-11}\, c_{\text{dev}}^{17} + 1.17591 \cdot 10^{-11}\, c_{\text{dev}}^{18} \mp 3.53657 \cdot 10^{-12}\, c_{\text{dev}}^{19} + 1.06819 \cdot 10^{-12}\, c_{\text{dev}}^{20}. \quad (254)$$

In these equations and in the minus–plus ($\mp$) sign, the upper sign (the minus sign) corresponds to the lower bound $\underline{x}_1$ of the root x_1 whereas the lower sign (the plus sign) corresponds to the upper bound $\overline{x}_1$ of the same root x_1—exactly as was the case in the first generalized Babylonian problem. Now, by adding the above two Taylor–Maclaurin series (254) and dividing the result by two we obtain the midpoint $x_{1,\text{ave}}$ of the root x_1. In this way, for the difference $x_{1,\text{ave}} - x_{1,d}$ we can easily find that

$$x_{1,\text{ave}} - x_{1,d} = \frac{1}{2}\left(\underline{x}_1 + \overline{x}_1\right) - x_{1,d} \approx 2.33236 \cdot 10^{-2}\, c_{\text{dev}}^2 + 7.77130 \cdot 10^{-4}\, c_{\text{dev}}^4$$
$$+ 4.35012 \cdot 10^{-5}\, c_{\text{dev}}^6 + 2.96099 \cdot 10^{-6}\, c_{\text{dev}}^8 + 2.23626 \cdot 10^{-7}\, c_{\text{dev}}^{10}$$
$$+ 1.80181 \cdot 10^{-8}\, c_{\text{dev}}^{12} + 1.51738 \cdot 10^{-9}\, c_{\text{dev}}^{14} + 1.31957 \cdot 10^{-10}\, c_{\text{dev}}^{16}$$
$$+ 1.17591 \cdot 10^{-11}\, c_{\text{dev}}^{18} + 1.06819 \cdot 10^{-12}\, c_{\text{dev}}^{20} \quad (255)$$

with $x_{1,d} = 1.5$, Equation (210). The nonlinear terms in the right-hand side of the above equation constitute the cause of the inequality $x_{1,\text{ave}} \neq x_{1,d}$. This inequality is also clear in the first of Equation (252).

Now in the special case where $c_{\text{dev}} = 0.25$ (exactly as is the case in the present numerical example with $c_{\text{dev}} = [(-7.25) - (-7.75)]/2 = (7.75 - 7.25)/2 = 0.50/2 = 0.25$), Equation (254) yield

$$\underline{x}_1 = 1.42997, \quad \overline{x}_1 = 1.57295, \quad x_{1,\text{ave}} = 1.50146, \quad x_{1,\text{ave}} - x_{1,d} = 0.00146. \quad (256)$$

Obviously, these numerical results (256) are in complete agreement with the corresponding results displayed in Equations (240), (241), the first of Equations (251) and the first of Equation (252) as was expected.

In the same generalized Babylonian problem, completely similar results were also derived for the second root x_2 of the interval quadratic Equation (239) with deterministic (crisp) value of this root $x_{2d} = 5$, Equation (211). Now Equation (255) (concerning the difference $x_{1,\text{ave}} - x_{1,d}$) takes the following analogous form (but now concerning the difference $x_{2,\text{ave}} - x_{2,d}$):

$$x_{2,\text{ave}} - x_{2,d} = \frac{1}{2}\left(\underline{x}_2 + \overline{x}_2\right) - x_{2,d} \approx -2.33236 \cdot 10^{-2}\, c_{\text{dev}}^2 - 7.77130 \cdot 10^{-4}\, c_{\text{dev}}^4$$
$$- 4.35012 \cdot 10^{-5}\, c_{\text{dev}}^6 - 2.96099 \cdot 10^{-6}\, c_{\text{dev}}^8 - 2.23626 \cdot 10^{-7}\, c_{\text{dev}}^{10}$$
$$- 1.80181 \cdot 10^{-8}\, c_{\text{dev}}^{12} - 1.51738 \cdot 10^{-9}\, c_{\text{dev}}^{14} - 1.31957 \cdot 10^{-10}\, c_{\text{dev}}^{16}$$
$$- 1.17591 \cdot 10^{-11}\, c_{\text{dev}}^{18} - 1.06819 \cdot 10^{-12}\, c_{\text{dev}}^{20} \quad (257)$$

with $x_{2,d} = 5$, Equation (211). By comparing Equations (255) (for the first root x_1) and (257) (for the second root x_2) we again observe that (exactly as in Equation (235) concerning the interval parameter b)

$$x_{2,\text{ave}} - x_{2,d} = -(x_{1,\text{ave}} - x_{1,d}) \quad \text{or, equivalently,} \quad (x_{1,\text{ave}} - x_{1,d}) + (x_{2,\text{ave}} - x_{2,d}) = 0. \quad (258)$$

Now in the special case where $c_{\text{dev}} = 0.25$ (exactly as is the case here), Equation (257) yields

$$\underline{x}_{2,\text{ave}} - x_{2,d} = -0.00146 \tag{259}$$

in complete agreement with the second of Equation (252) and also as a verification of this equation.

We note that because the coefficient b appears twice in Formula (2) for the two roots $x_{1,2}$, the resulting enclosures (intervals) X_1 and X_2 for these roots ($x_1 \in X_1$ and $x_2 \in X_2$) when the coefficient b is considered as an interval parameter are wider than the corresponding enclosures (intervals) that result when the coefficient c is considered as an interval parameter in the two uncertainty problems studied here. This becomes clear by comparing the corresponding intervals in Equation (227) (for an uncertainty in the coefficient b) and (250) (for an uncertainty in the coefficient c) and it happens mainly because c appears only once in Formula (2) contrary to the coefficient b, which appears twice. We also note that the two roots $x_{1,2}$ are almost (but, surely, not exactly) equally spread around the two deterministic values $x_1 = x_{1,d} = 1.5$ and $x_2 = x_{2,d} = 5$ computed in Equations (210) and (211), respectively, which correspond to the interval roots $x_1 \in X_1$ and $x_2 \in X_2$ in both generalized Babylonian problems under uncertainty conditions related to interval quadratic equations and studied here.

6. Kinematics Problem Using Interval Quadratic Equations

Let us consider another problem that is more modern. This is a classic Grade 12 Kinematics problem: "A pedestrian is running at his maximum speed of 6.0 m/s to catch a bus stopped by a traffic light. When he is 25 m from the bus, the light changes and the bus accelerates uniformly at 1.0 m/s^2. Find either (a) how far he has to run to catch the bus or (b) his frustration distance (closest approach)". This classic problem in kinematics was initially mentioned in the introductory physics textbook by Haber-Schaim, Dodge, and Walter [21] (Chapter 1, Problem 24); see also the note by Newburgh [22] and the technical report by Pisan and Bachmann [23] (p. 5, Example 2).

To solve this problem, we will write both motion equations of the bus and the pedestrian to see whether there is a chance that the pedestrian is running at a sufficient speed to catch this bus. We also assume that there is no initial speed for both the bus and the pedestrian and we will take the pedestrian's initial position as the origin of our referential in space. The time will also be $t = 0$ initially.

The bus's equation of motion is

$$x_{\text{bus}}(t) = \frac{1}{2}t^2 + 25 \tag{260}$$

while the pedestrian's one is

$$x_{\text{ped}}(t) = 6t \tag{261}$$

with the symbol t denoting the time in seconds in both Equations (260) and (261).

If the pedestrian catches the bus, this means that $x_{\text{ped}} = x_{\text{bus}}$. This yields

$$\frac{1}{2}t^2 + 25 = 6t \iff \frac{1}{2}t^2 - 6t + 25 = 0. \tag{262}$$

The discriminant Δ of the quadratic Equation (262) is

$$\Delta := b^2 - 4ac = (-6)^2 - 4 \times \frac{1}{2} \times 25 = 36 - 50 = -14 < 0. \tag{263}$$

This means that this equation has no real solutions. Thus, we conclude that the pedestrian is not going to catch the bus. Let us define the gap between the pedestrian and the bus as a function g:

$$g(t) := x_{\text{bus}}(t) - x_{\text{ped}}(t) = \left(\frac{1}{2}t^2 + 25\right) - 6t = \frac{1}{2}t^2 - 6t + 25. \tag{264}$$

Knowing that this function does not equal to zero for any value of time t (it is always positive), we need to find its minimum to find the closest approach of this pedestrian towards the bus. Let us consider the first and second derivatives of the function g defined in Equation (264)

$$g'(t) = t - 6 \quad \text{and} \quad g''(t) = 1. \tag{265}$$

Since $\forall t \in \mathbb{R}\ g''(t) = 1 > 0$, g is a convex function that admits a minimum when $g'(t) = 0$ hence for $t = 6$ s. For this value of time t using Equations (260) and (261) we obtain

$$x_{\text{bus}}(6) = \frac{1}{2}\,6^2 + 25 = 43 \text{ m}, \tag{266}$$

$$x_{\text{ped}}(6) = 6 \times 6 = 36 \text{ m}. \tag{267}$$

From these equations, Equations (266) and (267), it is directly concluded that the closest approach of the pedestrian to the bus (or his frustration distance) will be $x_{\text{bus}}(6) - x_{\text{ped}}(6) = 43 - 36 = 7$ m.

Now we will solve the same problem by considering interval parameters to make sure that this word problem becomes relevant to our study. Let us consider the same problem, but now the pedestrian, seeing that he will miss the bus if his running speed does not increase, gets some impetus as it were from the situation and now he runs at an estimated speed between 7.0 m/s and 7.5 m/s. Since the problem is the same, we will skip the beginning of the resolution and directly write that solving the present problem means solving the following interval quadratic equation:

$$\frac{1}{2}\,t^2 + [-7.5, -7]\,t + 25 = 0. \tag{268}$$

The discriminant Δ of this interval quadratic equation is

$$\Delta := b^2 - 4ac \in [-7.5, -7]^2 - 4 \times \frac{1}{2} \times 25 = [49, 56.25] - 50 = [-1, 6.25]. \tag{269}$$

This means that the pedestrian is catching the bus if the discriminant Δ is positive. This implies that

$$b \in B := [\underline{b}, \overline{b}] = [-7.5, -\sqrt{50}\,] \approx [-7.5, -7.07107]. \tag{270}$$

In order to make things simpler, we will consider the approximation $b \in B := [\underline{b}, \overline{b}] = [-7.5, -7.1]$. This means that it is sufficient that the pedestrian runs faster than 7.1 m/s in order to catch the bus.

Now, by using the appropriate expressions in Equations (72)–(75) (because here the uncertain coefficient b takes only negative values; therefore, $\underline{b} < 0$ and $\overline{b} < 0$) derived with the method of quantifier elimination in the present case, where $a > 0$, $c > 0$ and $\Delta > 0$, together with Equation (270) for the interval b, we get

$$\underline{t}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4ac}}{2a} = \frac{7.5 - \sqrt{7.5^2 - 50}}{1} = 5, \tag{271}$$

$$\overline{t}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4ac}}{2a} = \frac{7.1 - \sqrt{7.1^2 - 50}}{1} = \frac{71 - \sqrt{41}}{10} \approx 6.45969, \tag{272}$$

$$\underline{t}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4ac}}{2a} = \frac{7.1 + \sqrt{7.1^2 - 50}}{1} = \frac{71 + \sqrt{41}}{10} \approx 7.74031, \tag{273}$$

$$\overline{t}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4ac}}{2a} = \frac{7.5 + \sqrt{7.5^2 - 50}}{1} = 10. \tag{274}$$

Of course, it is clear that instead of appropriately using Equations (72)–(75) as we already did, it is also possible to use Equations (78)–(81) instead concerning the case $a > 0$, $c > 0$, $b < 0$, and $\Delta > 0$, which is exactly the case in the present interval quadratic Equation (268). The expressions in the latter equations are exactly those displayed in Equations (271)–(274) and already used for the computation of the sought bounds $\underline{t}_1$, $\overline{t}_1$, $\underline{t}_2$, and $\overline{t}_2$. Finally, the direct derivation of the same bounds with the method of quantifier elimination in the special case of Equation (268) was also seen to be possible and successful.

Because Equation (260) states that $x_{\text{bus}}(t) = \frac{1}{2}\,t^2 + 25$, we can calculate the distance that the pedestrian will need to run to catch the bus. The derived results imply that the pedestrian will have to run either between 37.5 m (for $t = \underline{t}_1$) and 45.8638 m (for $t = \overline{t}_1$) or between 54.9562 m (for $t = \underline{t}_2$) and 75 m (for $t = \overline{t}_2$) to catch the bus. The pedestrian will be then able to catch the bus with a time window that varies between 1.28062 s and 5 s (interval $[7.74031 - 6.45969, 10 - 5] = [1.28062, 5]$).

Of course, from a practical point of view in the present case of the interval B ($B = [-7.5, -7.1]$ or, equivalently, $-B = [7.1, 7.5]$), which (as was already mentioned) yields a positive discriminant $\Delta := b^2 - 4ac$ of the interval quadratic Equation (268) and, therefore, two distinct real roots $t_{1,2}$ of this equation with $t_1 < t_2$ as was assumed, it is only the smaller root t_1 of the interval quadratic Equation (268) that has a practical interest in the present kinematics problem. This happens because this root, the smaller root t_1, is the time that the pedestrian requires in order to catch the bus. On the contrary, as far as the second root t_2, the larger root of the same quadratic Equation (268), is concerned, it denotes the time that is required now for the bus to catch the pedestrian after the pedestrian had achieved to catch the bus at the previous time t_1 (with $t_1 < t_2$). This happens simply because the bus moves with a constant acceleration 1.0 m/s^2 whereas this is not the case for the pedestrian, who moves with a constant speed $-b \in [7.1, 7.5]$ m/s in the present numerical example.

7. A Simple Beam Problem

As a third application we consider a very simple beam problem in mechanics of materials. This is the problem of a beam of length L (with $L > 0$) simply supported at both its ends $x = 0$ and $x = L$ and loaded by a uniform distributed normal loading q (with $q > 0$) on its whole length $0 \le x \le L$. Then the following quadratic equation holds true, relating the position x on the present beam and the bending moment M on the same beam [24] (p. 193), which can also directly and quite easily be verified:

$$M = \frac{qL}{2}\,x - \frac{q}{2}\,x^2 \quad \text{or, equivalently,} \quad \frac{q}{2}\,x^2 - \frac{qL}{2}\,x + M = 0 \quad \text{with} \;\; 0 \le x \le L. \tag{275}$$

Because the present beam is simply supported at its ends $x = 0$ and $x = L$ and its distributed normal loading q is uniform and applied on its whole length $0 \le x \le L$, the problem is completely symmetrical with respect to the center (the midpoint) $x = L/2$ of the beam. Therefore, here we intend to restrict our attention only to the left half of the beam with $0 \le x < L/2$ or, equivalently, $x \in [0, L/2)$.

Here we are simply interested in the two roots $x_{1,2}$ of the quadratic Equation (275), that is in the points x of the beam where the bending moment has the value M provided that we know in advance both the intensity q of the uniform normal loading (with $q > 0$) and the length L (clearly with $L > 0$) of the present simply-supported beam. The discriminant Δ of the above quadratic Equation (275) is directly seen to be

$$\Delta = \Big(-\frac{qL}{2}\Big)^2 - 4\,\frac{q}{2}\,M = \frac{q^2L^2}{4} - 2qM = \frac{q}{4}\,(qL^2 - 8M). \tag{276}$$

Hence, under the present assumption of a positive intensity q of the distributed normal loading, that is $q > 0$, the above quadratic Equation (275) has two distinct real roots $x_{1,2}$ of course provided that

$$\Delta > 0, \quad \text{that is} \quad qL^2 - 8M > 0 \quad \text{or, equivalently,} \quad M < M_{\max} = \frac{qL^2}{8}. \tag{277}$$

In fact, the value $qL^2/8$ of the bending moment M is simply its maximum value $M_{\max}$ that appears at the center $x = L/2$ of the simply-supported beam under consideration as can easily be verified.

Here we are interested in the case where the bending moment M on the beam in the quadratic Equation (275) is an interval variable and it holds true that

$$M \in [\underline{M}, \overline{M}] \quad \text{with} \quad 0 < \underline{M} < \overline{M} < M_{\max} = \frac{qL^2}{8}. \tag{278}$$

It is clear that because of Equations (276) the last of the above inequalities, $\overline{M} < M_{\max}$, is equivalent to the assumption of a positive discriminant Δ, and, hence, of the existence of two points $x_{1,2}$ on the beam (one with $x \in [0, L/2)$ and one with $x \in (L/2, L]$) where the bending moment is equal to M.

Here our intention is simply to determine the interval $[\underline{x}, \overline{x}]$ of the position variable x on the left half of the beam (with $x \in [0, L/2)$ only) where the inequality constraint $\underline{M} \le M \le \overline{M}$ or, equivalently, $M \in [\underline{M}, \overline{M}]$ is satisfied by the bending moment M on the beam, which is the interval variable (parameter) in the present so the elementary beam problem related to the quadratic Equation (275).

To this end we will apply the same methodologies already reviewed in Section 2–4, but, clearly, our example essentially belongs to Section 4. For our symbolic computations we will again use the popular computer algebra system *Mathematica* [16]. Here our assumptions $\mathcal{A}_b$ have the following form (with the two position quantities x_{11} and x_{12} to be defined and used later):

$$\begin{aligned} \mathcal{A}_b \coloneqq\, & q > 0 \wedge L > 0 \wedge M > 0 \wedge \overline{M} < qL^2/8 \wedge \underline{M} \le M \le \overline{M} \wedge 0 < \underline{M} < \overline{M} \\ & \wedge 0 \le x < L/2 \wedge 0 < x_{11} < L/2 \wedge 0 < x_{12} < L/2. \end{aligned} \tag{279}$$

The root x_1 with $x_1 \in [0, L/2)$, which is the only root of Equation (275) in which we are interested in the present symmetrical beam problem, is given by the following simple formula directly derived by *Mathematica* or even by hand:

$$x_1 = \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8M}{q}}\right). \tag{280}$$

At first, we apply the direct method, which is simply based on the minimization and the maximization of the above root x_1 displayed in Equation (280) here with respect to the bending moment M on the beam and, naturally, under the validity of the above assumptions $\mathcal{A}_b$ defined in Equation (279). Here by using the `Minimize` and `Maximize` commands of *Mathematica* [16] for this minimization and maximization, respectively, we directly find the greatest lower bound (infimum) $\underline{x}_1 = \inf x_1$ and the least upper bound (supremum) $\overline{x}_1 = \sup x_1$ of this root x_1 of the interval quadratic Equation (275) (here with the bending moment M being an interval coefficient, $M \in [\underline{M}, \overline{M}]$), Equation (278), as follows:

$$\underline{x}_1 = \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\underline{M}}{q}}\right) \quad \text{for} \quad M = \underline{M}, \quad \overline{x}_1 = \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\overline{M}}{q}}\right) \quad \text{for} \quad M = \overline{M}. \tag{281}$$

Hence, the enclosure (the smallest interval) for the position variable x on the left part $x \in [0, L/2)$ of the beam that corresponds to the enclosure $[\underline{M}, \overline{M}]$ of the bending moment M on the beam is

$$x_1 \in X_1 := [\underline{x}_1, \overline{x}_1] = \left[\frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\underline{M}}{q}}\right), \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\overline{M}}{q}}\right)\right]. \tag{282}$$

It is clear that if the discriminant Δ of the interval quadratic Equation (275) determined in Equation (276) is equal to zero, $\Delta = 0$, or, alternatively but equivalently, $M = M_{\max} = qL^2/8$, then $x_1 = L/2$ and, therefore, there is no second real root $x_2 \in (L/2, L]$ of the present interval quadratic Equation (275).

Of course, the other three methods already reviewed previously are also applicable to the present problem of mechanics of materials. At first, as far as the method of classic interval analysis [7] is concerned, it is clear (under the assumptions $\Delta > 0$ assuring the existence of two real roots $x_{1,2}$ and also $q > 0$) that for the interval variable $M \in [\underline{M}, \overline{M}]$ by taking into account the Formula (280) for the first root x_1 of the interval quadratic Equation (275) being of interest here, we successively get the enclosures

$$\begin{aligned}
M \in [\underline{M}, \overline{M}] &\Rightarrow \frac{8M}{q} \in \left[\frac{8\underline{M}}{q}, \frac{8\overline{M}}{q}\right] \Rightarrow -\frac{8M}{q} \in \left[-\frac{8\overline{M}}{q}, -\frac{8\underline{M}}{q}\right] \\
&\Rightarrow L^2 - \frac{8M}{q} \in \left[L^2 - \frac{8\overline{M}}{q}, L^2 - \frac{8\underline{M}}{q}\right] \Rightarrow \sqrt{L^2 - \frac{8M}{q}} \in \left[\sqrt{L^2 - \frac{8\overline{M}}{q}}, \sqrt{L^2 - \frac{8\underline{M}}{q}}\,\right] \\
&\Rightarrow -\sqrt{L^2 - \frac{8M}{q}} \in \left[-\sqrt{L^2 - \frac{8\underline{M}}{q}}, -\sqrt{L^2 - \frac{8\overline{M}}{q}}\,\right] \\
&\Rightarrow \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8M}{q}}\right) \in \left[\frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\underline{M}}{q}}\right), \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\overline{M}}{q}}\right)\right] \\
&\Rightarrow x_1 \in X_1 := [\underline{x}_1, \overline{x}_1] = \left[\frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\underline{M}}{q}}\right), \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\overline{M}}{q}}\right)\right].
\end{aligned} \tag{283}$$

Hence, the same enclosure (282) of the first root x_1 of the interval quadratic Equation (275) was derived again now with elementary computations by using the method of classic interval analysis [7].

Now we are ready to proceed to the application of the method of quantifier elimination [13,14] to the present simple beam problem. Our quadratic Equation (275) can also be written in the form

$$p_M(x) = 0 \quad \text{with} \quad p_M(x) := \frac{q}{2}\,x^2 - \frac{qL}{2}\,x + M \quad \text{and} \quad 0 \le x \le L. \tag{284}$$

Evidently, our assumptions $\mathcal{A}_b$ defined in Equation (279) remain valid. Next, the universally quantified formula for the determination of a lower bound x_{11} of the position variable x ($x \ge x_{11}$) has the form

$$\forall M \in [\underline{M}, \overline{M}] \wedge \forall x < x_{11} \text{ it holds true that } p_M(x) \neq 0 \text{ under the assumptions } \mathcal{A}_b. \tag{285}$$

The above universally quantified Formula (285) excludes the existence of a root of the polynomial $p_M(x)$ defined in Equation (284) $\forall x < x_{11}$. Therefore, it is impossible that Equation (275) holds true for $x < x_{11}$. Hence, x_{11} is a lower bound of the sought root x_1 of Equation (275) although it is not necessarily the greatest lower bound (infimum) $\inf x_1$ of this root x_1. Next, by performing quantifier elimination to the above universally quantified Formula (285) (of course taking into consideration the assumptions $\mathcal{A}_b$) by using the implementation of quantifier elimination in *Mathematica* [16] we find the quantifier-free formula

$$x_{11} \le \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\underline{M}}{q}}\right). \tag{286}$$

Therefore, the greatest lower bound (infimum) $\underline{x}_1 = \inf x_1$ of the root x_1 is the greatest possible value of x_{11} and, clearly, because of the quantifier-free Formula (286), this bound $\underline{x}_1 = \inf x_1$ is given by the first of Equation (281) exactly as previously.

Quite similarly, we can determine the least upper bound (supremum) $\overline{x}_1$ of the same root x_1, that is, the smallest of its upper bounds x_{12}. The related universally quantified formula now has the form

$$\forall M \in [\underline{M}, \overline{M}] \wedge \forall x > x_{12} \text{ it holds true that } p_M(x) \neq 0 \text{ under the assumptions } \mathcal{A}_b. \tag{287}$$

Next, by performing quantifier elimination to the above universally quantified Formula (287) (of course taking again into account the assumptions $\mathcal{A}_b$ defined in Equation (279)) we find the quantifier-free formula

$$x_{12} \geq \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8\overline{M}}{q}}\right). \tag{288}$$

Therefore, the least upper bound (supremum) $\overline{x}_1 = \sup x_1$ of the root x_1 of Equation (275) of interest here is the least possible value of the upper bound x_{12} and, clearly, because of the above quantifier-free Formula (288), this upper bound $\overline{x}_1$ is given by the second of Equation (281), exactly as previously.

Finally, we consider the application of the fourth method, i.e., the method of interval parametrization. Here we selected to use the very simple parametrization of the bending moment M on the beam

$$M = \underline{M} + (\overline{M} - \underline{M})\,t \quad \text{with} \quad t \in [0, 1] \tag{289}$$

and the modified (but appropriate for the present method) assumptions

$$\mathcal{A}_b^* := q > 0 \wedge L > 0 \wedge \overline{M} < qL^2/8 \wedge 0 < \underline{M} < \overline{M} \wedge 0 \leq t \leq 1. \tag{290}$$

Then, as is directly verified, Equation (280) for the root x_1 takes the parametrized form

$$x_1 = \frac{1}{2}\left(L - \sqrt{L^2 - \frac{8[\underline{M} + (\overline{M} - \underline{M})t]}{q}}\right) \quad \text{with} \quad t \in [0, 1]. \tag{291}$$

By performing minimization and maximization to this root x_1 with respect to t we obtain the Formula (281) for the greatest lower bound $\underline{x}_1 = \inf x_1$ and the least upper bound $\overline{x}_1 = \sup x_1$ of this root.

Hence, our conclusion is that in the present elementary beam problem all four methods for the computation of enclosures for roots of interval quadratic equations reviewed in this paper lead to exactly the same and also sharp bounds $\underline{x}_1 = \inf x_1$ and $\overline{x}_1 = \sup x_1$ of the root x_1. Additionally, as far as the method of classic interval analysis [7] is concerned, here because of the appearance of the interval parameter M only once in the Formula (280) for the root x_1 under consideration, the obtained bounds $\underline{x}_1$ (lower bound) and $\overline{x}_1$ (upper bound) of this root x_1 are sharp. Similarly, here we were able to derive these bounds by hand when using the method of classic interval analysis [7].

On the other hand, the method of quantifier elimination requires the use of a computer algebra system (here *Mathematica*) as is essentially always the case. However, this method, in spite of the serious difficulty caused by its worst-case doubly-exponential computational complexity [25,26], has the advantages that it always leads to sharp bounds and, additionally, that its use does not require the analytical availability of the root under consideration, which here is the root x_1 displayed in Equation (280). This is in contrast to the other three methods where this analytical availability is necessary. In the present simple beam problem, this situation essentially does not cause a serious difficulty since the derivation of Equation (280) for the root $x_1 \in [0, L/2)$ is simple. However, in practice, there also exist more complex situations, where it is really difficult to determine the closed-form formulas for the roots of interest or these formulas even if they are available. Nevertheless, they are complicated.

Two such examples concerning the present problem of a simply-supported beam of length L under a uniform distributed normal loading q are the two cases where we are interested in the slope of the beam or, what is more important, in its deflection v. As far as the deflection v is concerned, in the present beam problem it is determined by the closed-form Formula [24] (p. 195, Equation (6)–(12))

$$v = \frac{qx}{24EI}\,(x^3 - 2L\,x^2 + L^3) \quad \text{or} \quad v = \frac{qx^4}{24EI} - \frac{qLx^3}{12EI} + \frac{qL^3x}{24EI}, \tag{292}$$

where the symbol EI denotes the flexural rigidity of the beam. Obviously, this equation is a quartic equation with respect to the position variable $x \in [0, L]$ on the beam. For computational convenience, the above Equation (292) can also be written in a simpler and now dimensionless form. This can easily be achieved by using the two dimensionless quantities $\xi := x/L$ and $Q := qL^4/(EI)$. Then the quartic Equation (292) takes its final and now dimensionless form

$$v = \frac{Q}{24}\,(\xi - 1)\xi(\xi^2 - \xi - 1) = \frac{Q}{24}\,\xi^4 - \frac{Q}{12}\,\xi^3 + \frac{Q}{24}\,\xi \tag{293}$$

or, better, by using a relevant quartic polynomial $p_v(\xi)$

$$p_v(\xi) = 0 \quad \text{with} \quad p_v(\xi) := \frac{Q}{24}\,(\xi - 1)\xi(\xi^2 - \xi - 1) - v = \frac{Q}{24}\,\xi^4 - \frac{Q}{12}\,\xi^3 + \frac{Q}{24}\,\xi - v. \tag{294}$$

The maximum value $v_{\max}$ of the deflection $v = v(\xi)$ of the beam (appearing at its center $\xi = 1/2$) is [24] (p. 195, Equation (6)–(13))

$$v_{\max} = \frac{5Q}{384} = \frac{5qL^4}{384EI}. \tag{295}$$

Assuming that v is an interval coefficient with $v \in [\underline{v}, \overline{v}]$ we can use the universally quantified formulas

$$\forall v \in [\underline{v}, \overline{v}] \wedge \forall \xi < \xi_{11} \text{ it holds true that } \quad p_v(\xi) \neq 0 \text{ under the assumptions } \mathcal{A}_v, \tag{296}$$

$$\forall v \in [\underline{v}, \overline{v}] \wedge \forall \xi > \xi_{12} \text{ it holds true that } \quad p_v(\xi) \neq 0 \text{ under the assumptions } \mathcal{A}_v, \tag{297}$$

which are analogous to the relevant Formulas (285) and (287) in the previous case for the bending moment M of the beam, but now for its deflection v. Here the assumptions $\mathcal{A}_v$ that we made have the form

$$\begin{aligned} \mathcal{A}_v \coloneqq Q > 0 \wedge 0 < v < \frac{5Q}{384} \wedge 0 < \underline{v} < \frac{5Q}{384} \wedge 0 < \overline{v} < \frac{5Q}{384} \\ \wedge\ \underline{v} \le v \le \overline{v} \wedge \underline{v} < \overline{v} \wedge 0 < \xi < \frac{1}{2} \wedge 0 < \xi_{11} < \frac{1}{2} \wedge 0 < \xi_{12} < \frac{1}{2}. \end{aligned} \tag{298}$$

Now we can perform quantifier elimination to the above universally quantified Formulas (296) and (297) again by using the computer algebra system *Mathematica* [16]. The resulting quantifier-free formulas can be written in their final forms as follows:

$$\xi_{11} \le \underline{r}_2 = \frac{1}{2}\left(1 - \sqrt{3 - 2\sqrt{1 + (96\underline{v}/Q)}}\right), \quad \xi_{12} \ge \overline{r}_2 = \frac{1}{2}\left(1 - \sqrt{3 - 2\sqrt{1 + (96\overline{v}/Q)}}\right), \tag{299}$$

where the new symbols $\underline{r}_2$ and $\overline{r}_2$ denote the second real roots of the following quartic polynomials:

$$\underline{p}_v^*(\xi) := Q\xi^4 - 2Q\xi^3 + Q\xi - 24\underline{v} \quad \text{and} \quad \overline{p}_v^*(\xi) := Q\xi^4 - 2Q\xi^3 + Q\xi - 24\overline{v} \tag{300}$$

respectively. The analytical expressions of these two roots, $\underline{r}_2$ and $\overline{r}_2$, have already been displayed in Equation (299) as they were explicitly computed with *Mathematica*. Therefore, finally, under the present assumption that $v \in [\underline{v}, \overline{v}]$ the corresponding enclosure for the position variable $x = L\xi$ on the left half $[0, L/2)$ of this simply-supported beam under a uniform distributed normal loading is

$$x \in [\underline{x}, \overline{x}] := [\underline{r}_2 L, \overline{r}_2 L]$$
$$= \left[\frac{L}{2} \left(1 - \sqrt{3 - 2\sqrt{1 + (96\underline{v}/Q)}}\right), \frac{L}{2} \left(1 - \sqrt{3 - 2\sqrt{1 + (96\overline{v}/Q)}}\right) \right] \text{ with } Q := \frac{qL^4}{EI}. \quad (301)$$

The above application of the method of quantifier elimination to the derivation of the enclosure for the position x on the left half $[0, L/2)$ of the present simply-supported beam when we have an interval deflection $v \in [\underline{v}, \overline{v}]$ has been an introductory example of a possible future generalization of this method to interval quartic equations. Similarly, these results can also be extended to interval cubic equations as is the case, e.g., with the slope $\theta \approx \mathrm{d}v/\mathrm{d}x$ of the present simply-supported beam. In principle, further generalizations of the same method, that is, to interval higher-degree equations (e.g., quintic equations) are also possible. However, in such a case, it is understood that the two bounds $\underline{x}_k$ and $\overline{x}_k$ of a root $x_k \in [\underline{x}_k, \overline{x}_k]$ will appear simply as the roots of appropriate polynomials and not in the form of analytical formulas in the derived quantifier-free formulas even in their final forms.

8. Conclusions

As seen throughout this article, the methods of quantifier elimination and of interval parametrization yield the exact (sharp) enclosures of both roots $x_{1,2}$ of the interval quadratic Equation (1) all the time, and they match the results obtained by the computer evaluation with the direct approach.

This study shows that classic interval analysis [7] is not always the most accurate method in the present task because sometimes it may lead to conservative bounds for the roots $x_{1,2}$, and equivalently to overestimated enclosures for the same roots. Here this may happen when the interval parameter is the coefficient b of the interval quadratic Equation (1) since b appears twice in the Formula (2) for its roots $x_{1,2}$, where the $\mp$ or $\pm$ sign appears. In the case of intervals, when we have the addition of two intervals there is no problem, but when we have the subtraction of two intervals the obtained interval may be conservative (not sharp) because of the dependency problem. Therefore, in the case of an interval parameter b, the interval of one of the roots (either x_1 or x_2) is conservative. This difficulty can be avoided by using the corresponding Fagnano formula for this root instead of the initial Sridhara formula, as was actually observed in the first two numerical examples of Section 3.

When it comes to the method of interval parametrization [8], it provides excellent results for every example (exactly as the direct method) and leads to formulas equivalent to those derived by the direct method, but including the auxiliary quantities (if any) used in the parametrization in their analytical forms, e.g., including the quantities a_{ave} and a_{dev} in Equations (15) and (16) instead of $\underline{a}$ and $\overline{a}$.

Reflecting upon the results, one could conclude that the method of quantifier elimination [13,14] is the perfect method as it yields analytical formulas that take every subcase into account. It is indeed the best way to deal with such problems as it yields analytical formulas that could then be evaluated by hand or with the help of a calculator. However, it also needs a computer algebra system such as *Mathematica* for establishing the formulas, but, most importantly, it does not work yet for more complex cases such as the case where two or even three of the coefficients a, b, and c of the quadratic Equation (1) are considered as intervals. Indeed, as stated by Ioakimidis is his paper [15] (p. 26): "... quantifier elimination cannot be performed in a reasonable CPU time. Therefore, all the present results were confined to only one interval parameter." This unfortunate situation with the method of quantifier elimination is due to the doubly-exponential computational complexity generally valid in quantifier elimination for real variables [25,26]. This leaves us with a problem: Is there a way to obtain analytical formulas for the roots of quadratic equations whose two or even all three coefficients are intervals? Our second paper in preparation aims to give an answer to this inquiry.

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