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Series Solution Method for Solving Sequential Caputo Fractional Differential Equations

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Abstract: Computing the solution of the Caputo fractional differential equation plays an important role in using the order of the fractional derivative as a parameter to enhance the model. In this work, we developed a power series solution method to solve a linear Caputo fractional differential equation of the order q, $0 < q < 1$, and this solution matches with the integer solution for $q = 1$. In addition, we also developed a series solution method for a linear sequential Caputo fractional differential equation with constant coefficients of order $2q$, which is sequential for order q with Caputo fractional initial conditions. The advantage of our method is that the fractional order q can be used as a parameter to enhance the mathematical model, compared with the integer model. The methods developed here, namely, the series solution method for solving Caputo fractional differential equations of constant coefficients, can be extended to Caputo sequential differential equation with variable coefficients, such as fractional Bessel's equation with fractional initial conditions.

Keywords: Mittag–Leffler function; Caputo fractional derivative; sequential; fractional power series

MSC: 34A08; 34A12

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1. Introduction

Although the notion and definition of fractional derivative was introduced in the 17th century, the analysis and applications of fractional dynamic equations with initial and boundary conditions have seen exponential growth in the past 30 decades. Among the many types of fractional derivatives introduced in the literature, the most used fractional derivatives are the Riemann–Liouville and the Caputo derivatives. The Caputo derivative was introduced in the beginning of the 20th century. The advantage of studying dynamic equations with the Caputo derivative is that the initial conditions and boundary conditions are the same for the corresponding integer dynamic equation closest to the fractional derivative involved. See [1–15] for some of the analysis and computational method of fractional dynamic and integral equations. Some of the monographs included in the list provide myriad applications of fractional dynamic equations in various branches of science and engineering. See [15–30] for some more applications of fractional dynamic equations. Some of the references provided above have the numerical methods of computing the solution. The Mittag–Leffler function plays an important role in the study of linear fractional dynamic equations. See [5,31] which are devoted entirely to the analysis of Mittag–Leffler functions. In [32], the authors demonstrated that fractional dynamic equation is a better mathematical model compared with the corresponding integer model for certain q values. They achieved this by comparing their solution with the realistic data available. In order to extend this to a variety of models, there is a need to compute the solution of a variety of Caputo fractional dynamic equations with the initial conditions. In general, it is rarely possible to compute the solution of a Caputo fractional differential equation of order q

with variable coefficients with initial conditions. However, it is very routine to compute the solution of the initial value problems when $q = 1$. This is one of the motivations for approaching the series solution method to solve Caputo fractional differential equations of order q with variable coefficients. Along similar lines, we used the method of series solution to solve Caputo sequential differential equations of order $2q$, which is sequential of order q with the initial conditions. This method will lay a foundation to solve Caputo sequential differential equations of order $2q$ which are sequential for order q with a variable coefficient and with initial conditions. This is still an open problem, which we plan to address in our future work. This is the motivation for our current work in this research article.

It is known that the Mittag–Leffler function is the generalization of the exponential function. The exponential function has been widely used to solve higher-order linear integer differential equations with constant coefficients and initial conditions. The Mittag–Leffler function has been used to solve a linear qth order Caputo fractional differential equation when $(n-1) < q < n$, with initial conditions. Although the Mittag–Leffler function has appeared in the solution by an iterative process and/or by using the Laplace transform method, the Laplace transform method is still very useful since the Caputo derivative is a convolution integral. However, the iterative method which uses the initial condition as the initial approximation cannot handle when we have linear terms involving a fractional derivative of order less than q. See the example from [12], which is ${}^cD^Q_{0+}u(t) + {}^cD^q_{0+}u(t) + u = 0$, $Q > q$. Since the integer derivative is sequential, in this work, we will consider only linear sequential Caputo fractional differential equations with the fractional type of initial conditions. See [23,29,33–39] for some of the analysis and computational methods developed for linear sequential differential equations with initial and boundary conditions. The Mittag–Leffler functions can also be used to solve the linear nq order Caputo fractional differential equations, which are sequential for order q with fractional initial conditions. However, it is not convenient to use the Mittag–Leffler function when the roots of the characteristic equation are coincident or complex. Similarly, for linear systems of the Caputo fractional differential system with initial conditions, the eigenvalue and eigenfunction method using the Mittag–Leffler function cannot be used. This is partly because the Mittag–Leffler function does not enjoy the nice property of the exponential function. It is also partly due to the fact that the product rule and the variation of the parameter method of the integer dynamic equations do not work for fractional dynamic equations. So, the better choice is the Laplace transform method. See [29,37–42] for results on Caputo sequential differential equations with constant coefficients and linear Caputo fractional differential systems with initial conditions. Also see [42,43] for some numerical results, which include some variable coefficient problems. In this work, we provided a series solution method to solve the Caputo fractional differential equation of order q, $0 < q < 1$ with variable coefficients. We also provided a series solution method to solve linear sequential differential equation with constant coefficients and with fractional initial conditions. The purpose is to use the value of q as a parameter to obtain a more physically realistic model. Just as an example, one can see that the fractional pendulum model exhibits damping without a damping term in the model, which is physically realistic. In our future work, we plan to extend the series solution method for the fractional Euler type of equation and also to fractional Bessel equations.

The layout of the article is as follows: In Section 2, we presented the definitions and known results, which are needed for our main results. In Section 3, we presented a series solution method for Caputo fractional differential equations of order q, with initial conditions and a specific variable coefficient. We also developed series solution for sequential Caputo fractional differential equations with fractional initial conditions.

2. Preliminary Results

In this section, we will recall some definitions and known results that play an important role in our main results.

Definition 1. *The Riemann–Liouville fractional integral of order q is defined by*

$$D_{0+}^{-q}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds, \tag{1}$$

where $0 < q \leq 1$ and $\Gamma(q)$ is the Gamma function.

Definition 2. *The Riemann–Liouville (left-sided) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined by*

$$D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}u(s)ds,\ t > 0. \tag{2}$$

The Caputo integral of order q for any function is the same as that of the Riemann–Liouville integral of order q.

Definition 3. *The Caputo (left-sided) fractional derivative of $u(t)$ of order nq, $n-1 \leq nq < n$ is defined by*

$${}^{c}D_{0+}^{nq}u(t) = \frac{1}{\Gamma(n-nq)}\int_0^t (t-s)^{n-nq-1}u^{(n)}(s)ds,\ t \in [0,\infty),\ t > 0, \tag{3}$$

where $u^{(n)}(t) = \frac{d^n(u)}{dt^n}$.

In particular, if q is an integer, then both the Caputo fractional derivative and integer derivative are one and the same.

See [5,6,12] for more details on the Caputo and Riemann–Liouville fractional derivatives.

Definition 4. *The Caputo (left) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined by*

$${}^{c}D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}u'(s)\mathrm{d}s. \tag{4}$$

We are just replacing n by 1 in the above definition of the Caputo derivative of order nq.

Next, we define the two-parameter Mittag–Leffler function, which will be useful on solving the systems of linear Caputo fractional differential equations using the Laplace transform. See [10,11,13] for more on fractional differential equations with applications.

Definition 5. *The two-parameter Mittag–Leffler function is defined as*

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)}, \tag{5}$$

where q, $r > 0$, and λ is a constant.

Furthermore, if $r = q$, then (5) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+q)}. \tag{6}$$

If $r = 1$ in (5), then

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+1)}. \tag{7}$$

If $q = r = 1$ in (5), then

$$E_{1,1}(\lambda t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(k+1)} = e^{\lambda t}, \tag{8}$$

where $e^{\lambda t}$ is the usual exponential function.

See [5,6,12] for more details on the Mittag–Leffler function.

Now, we define fractional trigonometric functions and generalized fractional trigonometric functions of order q, which will be required in our main results.

Definition 6. *The fractional trigonometric functions $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$, are given by*

$$\sin_{q,1}(\lambda t^q) = \frac{1}{2i}[E_{q,1}(i\lambda t^q) - E_{q,1}(-i\lambda t^q)], \tag{9}$$

and

$$\cos_{q,1}(\lambda t^q) = \frac{1}{2}[E_{q,1}(i\lambda t^q) + E_{q,1}(-i\lambda t^q)], \tag{10}$$

respectively.

We can also define $\sin_{q,q}(\lambda t^q)$ and $\cos_{q,q}(\lambda t^q)$ in a similar way using $E_{q,q}(\lambda t^q)$ in place of $E_{q,1}(\lambda t^q)$.

Definition 7. *The generalized fractional trigonometric functions $Gsin_{q,1}((\lambda + i\mu)t^q)$ and $Gcos_{q,1}((\lambda + i\mu)t^q)$, are given by*

$$Gsin_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2i}[E_{q,1}((\lambda + i\mu)t^q) - E_{q,1}((\lambda - i\mu)t^q)], \tag{11}$$

and

$$Gcos_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda + i\mu)t^q) + E_{q,1}((\lambda - i\mu)t^q)], \tag{12}$$

respectively.

We can also define $Gsin_{q,q}((\lambda + i\mu)t^q)$ and $Gcos_{q,q}((\lambda + i\mu)t^q)$ in a similar way.

Note that the generalized fractional trigonometric function cannot be expressed in simpler form as the integer trigonometric function since the Mittag–Leffler function does not enjoy the properties of an exponential function. When $q = 1$, then (9), (10), (11) and (12) will give $\sin \lambda t$, $\cos \lambda t$, $e^{\lambda t} \sin \mu t$ and $e^{\lambda t} \cos \mu t$, respectively.

Definition 8. *The Caputo fractional derivative of $u(t)$ of order nq for $n - 1 < nq < n$, is said to be a sequential Caputo fractional derivative of order q if the relation*

$${}^cD_{0+}^{nq} u(t) = {}^cD_{0+}^{q}({}^cD_{0+}^{(n-1)q})u(t), \tag{13}$$

holds for $n = 2, 3 \ldots$

Note that the Equation (13) can also be written as

$${}^cD_{0+}^{kq} u(t) = {}^cD_{0+}^{q}({}^cD_{0+}^{q})({}^cD_{0+}^{q}) \ldots k \text{ times} \ldots ({}^cD_{0+}^{q})u(t),$$

for $k = 2, 3, 4 \ldots$

See [33,37,42] for some more work on sequential fractional differential equations.

Note that we use the notation ${}^{sc}D_{0+}^{nq} u(t)$ in our main results, whose Caputo derivative ${}^cD_{0+}^{nq} u(t)$ of order nq is sequential for order q.

Next, we will consider the Caputo fractional homogeneous linear fractional differential equation of order q with initial conditions of the form

$$^{c}D_{0^+}^{q}u(t) - \lambda u(t) = 0, \;\; u(0) = u_0, \tag{14}$$

where $0 < q < 1$ for $t > 0$.

Then, the solution of (14) can be obtained as

$$u(t) = u_0 E_{q,1}(\lambda t^q),$$

where λ is a constant.

Remark 1. *The solution of* (14) *cannot be obtained if* λ *is a function of* t *by the method of integrating factor as in the integer order case when* $q = 1$ *or by the Laplace transform method for a fractional derivative of order* q*, when* $0 < q < 1$.

In the above equation, if q is replaced by nq such that $(n-1) < nq < n$, then in order to solve, we need the initial conditions

$$u^k(0) = b_k, \text{ for } k = 0, 1, 2, 3, \ldots (n-1).$$

See [6,12] for details of the solution when λ is a constant.

Replacing q by nq gives

$$^{c}D_{0^+}^{nq}u(t) - \lambda u(t) = 0, \; u^k(0) = b_k, \tag{15}$$

where $(n-1) < nq < n$, for $t > 0$.

Remark 2. *The Equation* (15) *cannot be solved if the Caputo fractional differential equation has lower-order terms of* $^{c}D_{0^+}^{kq}u(t)$ *for any integer* k*, and* $1 \leq k \leq (n-1)$*, with the following:*

(a) Constant coefficients;
(b) Variable coefficients.

This is one of the motivations for us to use the series solution method. The series solution method is also the most appropriate method to solve higher-order linear integer differential equations with variable coefficients.

3. Main Results

In this section, we develop series solution methods for linear sequential Caputo and linear sequential Riemann–Liouville fractional differential equations with initial conditions. It is to be noted that the initial conditions for the Caputo fractional initial value problem are those of the corresponding integer differential equation. In this work, the initial conditions are fractional initial conditions. Initially, we obtain the Caputo fractional linear differential equation with constant coefficients.

Consider the linear Caputo fractional differential equations with constant coefficients of order q, where $0 < q < 1$:

$$^{c}D_{0^+}^{q}u \mp \lambda u = 0, \;\; u(0) = u_0. \tag{16}$$

The solution $u(t)$ of (16) is given by

$$u(t) = u_0 E_{q,1}(\pm \lambda t^q).$$

The above solution can be obtained by using the Laplace transform methods as well as integral representation and iterative methods.

See [6,9,12] for the solution using different methods.

In the series solution method, we assume the solution of (16) to be

$$u(t) = \sum_{k=0}^{\infty} u_k t^{kq}.$$

It is easy to check that $u(0) = u_0$.
Finding the Caputo derivative of $u(t)$ and substituting it in (16), we obtain

$$\sum_{k=0}^{\infty} u_{k+1} \frac{\Gamma((k+1)q+1)}{\Gamma(kq+1)} t^{kq} \mp \lambda \sum_{k=0}^{\infty} u_k t^{kq} = 0.$$

From this, we obtain

$$u_n = u_0 \frac{(\pm\lambda)^n}{\Gamma(nq+1)}.$$

In particular, if $\lambda > 0$, then

$$u_n = u_0 \frac{\lambda^n}{\Gamma(nq+1)},$$

for $n = 1, 2, 3 \ldots$
If $-\lambda < 0$, then

$$u_n = u_0 \frac{(-\lambda)^n}{\Gamma(nq+1)},$$

for $n = 1, 2, 3 \ldots$
Substituting for u_n, the solution for (16) for $\lambda > 0$ is given by

$$u(t) = \sum_{k=0}^{\infty} \frac{(\lambda)^k}{\Gamma(nq+1)} t^{kq}, \tag{17}$$

and for $-\lambda < 0$, it is given by

$$u(t) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(nq+1)} t^{kq}. \tag{18}$$

Combining the above two equations, we can rewrite the solution of (16) as

$$u(t) = E_{q,1}(\pm\lambda t^q). \tag{19}$$

Remark 3. *Note that the solution* (19) *of* (16) *is still valid if* $\mp\lambda$ *is replaced by* $i \mp \lambda$ *for the complex roots case. Then, the corresponding solutions will be*

$$u = E_{(q,1)}(\pm i\lambda t^q).$$

Next, we consider the linear Caputo fractional differential equations with variable coefficients of order q, where $0 < q < 1$, with initial conditions of the form

$${}^cD_{0^+}^q u(t) \mp p(t)u = 0, \;\; u(0) = u_0. \tag{20}$$

It is known that there is no closed form of the formula to compute the solution of (20). In [43], we obtained a symbolic form of the solution. In order to obtain a nice series form of the solution, we will consider the special case of $p(t)$, namely, $p(t) = t^q$.

For that purpose, consider

$${}^cD_{0^+}^q u(t) - \lambda t^q u = 0, \;\; u(0) = u_0. \tag{21}$$

Assuming the solution

$$u(t) = \sum_{k=0}^{\infty} u_k t^{kq},$$

and substituting it into Equation (21), we obtain

$$\sum_{k=0}^{\infty} u_{k+1} t^{kq} \frac{\Gamma((k+1)q+1)}{\Gamma(kq+1)} + \lambda \sum_{0}^{\infty} u_k t^{kq} = 0.$$

This gives $u_1\Gamma(q+1) = 0$, which implies $u_1 = 0$.
Also, $u_2 \frac{\Gamma(2q+1)}{\Gamma(q+1)} = \lambda u_0$, gives $u_2 = \lambda u_0 \frac{\Gamma(q+1)}{\Gamma(2q+1)}$.
Similarly, we obtain

$$u_4 = \lambda u_2 \frac{\Gamma(q+1)}{\Gamma(2q+1)} = \lambda^2 u_0 \frac{\Gamma(q+1)\Gamma(3q+1)}{\Gamma(2q+1)\Gamma(4q+1)}.$$

Generalizing this for all n, we have

$$u_{2n+1} = 0,$$

for $n = 0, 1, 2, \ldots$ and

$$u_{2n} = u_0 \lambda^n \frac{\Gamma(q+1)\Gamma(3q+1)\ldots\Gamma((2n-1)q+1)}{\Gamma(2q+1)\Gamma(4q+1)\ldots\Gamma(2nq+1)},$$

for $n = 1, 2, \ldots$
Hence, the series solution of (21) is given by

$$u(t) = u_0\Big(1 + \sum_{k=1}^{\infty} \lambda^k \frac{\Gamma(q+1)\Gamma(3q+1)\ldots\Gamma((2k-1)q+1)}{\Gamma(2q+1)\Gamma(4q+1)\ldots\Gamma(2kq+1)} t^{2kq}\Big). \tag{22}$$

In particular, if $q = 1$, then

$$u(t) = u_0\Big(1 + \lambda\frac{t^2}{2} + \ldots\ldots + \lambda^n\Big(\frac{t^2}{2}\Big)^n + \ldots\Big) = u_0 e^{\frac{\lambda t^2}{2}}.$$

This is precisely the solution to

$$u' - \lambda t u = 0, \;\; u(0) = u_0.$$

Consider the linear sequential Caputo fractional differential equation of order $2q$, which is sequential of order q with constant coefficients of the form:

$${}^{sc}D_{0^+}^{2q} u + b\,{}^{sc}D_{0^+}^{q} u + cu = 0, \;\; u(0) = u_0, \;\; {}^{c}D_{0^+}^{q} u(t)|_{t=0} = u_1. \tag{23}$$

In this work, we will consider the following special case of (23),

- $b = 0$ and $c = \lambda^2$.
- Two real roots of the quadratic equation $r^2 + br + c = 0$, are λ_1 and λ_2 respectively such that $\lambda_1 \neq \lambda_2$.
- Two real roots of the quadratic equation $r^2 + br + c = 0$, are λ_1 and λ_2 respectively such that $\lambda_1 = \lambda_2$.

Case (i). When $b = 0$ and $c = \lambda^2$, we can rewrite (23) as,

$${}^{sc}D_{0^+}^{2q} u + \lambda^2 u = 0, \;\; u(0) = u_0, \;\; {}^{c}D_{0^+}^{q} u(t)|_{t=0} = u_1. \tag{24}$$

Using sequential derivative ${}^{sc}D_{0^+}^{2q} = {}^{sc}D_{0^+}^{q}({}^{sc}D_{0^+}^{q})$, (24) can be written as

$$({}^{sc}D_{0^+}^{q} + i\lambda)({}^{sc}D_{0^+}^{q} - i\lambda)u(t) = 0. \tag{25}$$

Using (25), we can see that $u(t) = E_{q,1}(i\lambda t^q)$ and $u(t) = E_{q,1}(-i\lambda t^q)$ are two linearly independent solutions of the linear Caputo homogeneous fractional differential Equation (24). Since (24) is a linear homogeneous Caputo fractional differential equation, $u(t) = \sin_{q,1}(\lambda t^q)$ and $u(t) = \cos_{q,1}(\lambda t^q)$ are also two linearly independent solutions of (24).

Remark 4. *Note that the two linearly independent solutions of* (24)*, namely* $u(t) = E_{q,1}(i\lambda t^q)$ *and* $u(t) = E_{q,1}(-i\lambda t^q)$*, can also be obtained by assuming*

$$u(t) = \sum_{k=0}^{\infty} u_k t^{kq}.$$

Case (ii) When two roots are λ_1 and λ_2 respectively such that $\lambda_1 \neq \lambda_2$, we can rewrite (23) as

$$({}^{sc}D_{0^+}^{q} - \lambda_1)({}^{sc}D_{0^+}^{q} - \lambda_2)u(t) = 0. \tag{26}$$

From (26), we can obtain two series solutions $u(t) = E_{q,1}(\lambda_1 t^q)$, and $u(t) = E_{q,1}(\lambda_2 t^q)$, which are linearly independent.

Remark 5. *Note that when we have complex roots* $\lambda_{1,2} = \lambda \pm i\mu$*, then the two linearly independent solutions are* $E_{q,1}((\lambda \pm i\mu)t^q)$*. Then, the appropriate linear combination of these two linearly independent solutions provides the two other linearly independent solutions, which are* $u(t) = G\sin_{q,1}(\lambda + i\mu)t^q$ *and* $u(t) = G\cos_{q,1}(\lambda + i\mu)t^q$*.*

Case (iii) When two roots are λ_1 and λ_2, respectively, such that $\lambda_1 = \lambda_2$, we consider a special case of the sequential Caputo fractional differential equation:

$${}^{sc}D_{0^+}^{2q}u - 2{}^{sc}D_{0^+}^{q}u + u = 0, \;\; u(0) = u_0, \;\; {}^{c}D_{0^+}^{q}u(t)|_{t=0} = u_1. \tag{27}$$

Let us start with the assumption that

$$u(t) = \sum_{k=0}^{\infty} u_k t^{kq},$$

be the solution of (27). Then finding ${}^{sc}D_{0^+}^{q}u$ and ${}^{sc}D_{0^+}^{2q}u$ and substituting into (27), we obtain

$$u(t) = u_0[1 + \sum_{k=2}^{\infty}(-1)^{k-1}(k-1)\frac{t^{kq}}{\Gamma(kq+1)}],$$

and

$$u(t) = u_1\sum_{k=1}^{\infty}(-1)^k k\frac{t^{nq}}{\Gamma(kq+1)}.$$

These solutions satisfy the initial condition as well.

Next, we consider the linear Riemann–Liouville differential equation with constant coefficients of order q where $0 < q < 1$ of the form

$$D_{0^+}^{q}u \mp \lambda u = 0, \;\; \Gamma(q)t^{1-q}u(t)|_{t=0} = u^0. \tag{28}$$

In this case, we cannot assume the solution to have a power series solution in powers of t^q, since there is a singularity near the initial condition. Hence, we assume that

$$u(t) = t^{q-1} \sum_{k=0}^{\infty} u_k \frac{(\lambda t^q)^k}{\Gamma((k+1)q)}.$$

From this, we can obtain

$$D_{0^+}^q u = \lambda t^{q-1} \sum_{k=0}^{\infty} u_{k+1} \frac{(\lambda t^q)^k}{\Gamma((k+1)q)}.$$

Substituting this into (28), we obtain $u_k = u_0$ for all $k = 1, 2, \ldots$ From this, we obtain the solution of (28) as

$$u(t) = u_0 t^{q-1} \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma((k+1)q)}. \tag{29}$$

Then,

$$t^{(1-q)}\Gamma(q)u(t)|_{t=0} = u_0 = u^0.$$

Thus, the solution of (28) satisfying the initial condition is given by

$$u(t) = u^0 t^{q-1} \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma((k+1)q)}. \tag{30}$$

Next, we consider the linear Caputo fractional of variable coefficients of Euler's type of the form

$$(t^{2q})^{sc}D_{0^+}^{2q} u + b(t^q)^{sc}D_{0^+}^{q} u + cu = 0 \ . \tag{31}$$

If $q = 1$, we can assume $u = t^r$ and obtain the solution based on the roots of the quadratic equation $r(r-1) + br + c = 0$. This method will not work for (31). However, it is easy to observe that

$$u = \frac{t^{2q}}{\Gamma(2q+1)},$$

is the solution of

$$(t^{2q})^{sc}D_{0^+}^{2q} u - \Gamma(2q+1)u = 0.$$

We plan to explore methods to solve Caputo fractional Euler's equation of the type (31).

4. Conclusions

The computation of the solution of Caputo and Riemann–Liouville fractional differential equations with initial conditions is very useful from a modeling point of view since the value of q can be used as a parameter to enhance the mathematical model. Unfortunately, the tools and methods easily available to solve integer differential equations are not available for fractional ones. For example, an integrating factor and variation of parameter methods, which are used to solve a linear first-order differential equation with variable coefficients, cannot be used to solve a Caputo fractional differential equation. In this work, we demonstrated that the series solution is the most appropriate method to solve the Caputo fractional differential equation with variable coefficients of order q when $0 < q < 1$.

Although the Mittag–Leffler function is an extension of the exponential function, we cannot readily use the Mittag–Leffler function to solve a linear Caputo fractional differential equation with constant coefficients of order nq and lower-order terms of order kq where $k < n$, and $(n-1) < nq < n$. We can use the Mittag–Leffler function just like the exponential function when the Caputo sequential derivative of order nq is sequential for order q. In addition, the initial conditions should be in terms of the fractional derivatives of order kq, where $k = 1, 2, \ldots (n-1)$. In this situation, we can also use the Laplace transform method. In this work, we provided the series solution approach for solving sequential Caputo differential equation with constant coefficients and with fractional initial conditions.

The reason to use the series solution method is to extend the series solution method to Caputo fractional, Caputo Legendre's equations and fractional Bessel's equation. Then, we can use the fractional order q as a parameter to improve the mathematical model that matches the data. In this work, we provided a series solution method for solving Caputo fractional differential equations of order q with a special variable coefficient, namely t^q. One can easily extend it to t^{nq} for any $n > 1$. However, the problem when the variable coefficient is any function of the form, say $p(t) = \sum_{k=0}^{n} a_k t^{kq}$, is yet to be solved.

Further, we provided a series solution method for solving sequential Caputo differential equations with constant coefficients and fractional initial conditions. However, it is still an open problem to solve a sequential Caputo fractional differential equation with a variable coefficient of Euler's type.

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