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Terracini Loci for Maps

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Abstract: Let X be a smooth projective variety and $f : X \to \mathbb{P}^r$ a morphism birational onto its image. We define the Terracini loci of the map f. Most results are only for the case $\dim X = 1$. With this new and more flexible definition, it is possible to prove strong nonemptiness results with the full classification of all exceptional cases. We also consider Terracini loci with restricted support (solutions not intersecting a closed set $B \subsetneq X$ or solutions containing a prescribed $p \in X$). Our definitions work both for the Zariski and the euclidean topology and we suggest extensions to the case of real varieties. We also define Terracini loci for joins of two or more subvarieties of the same projective space. The proofs use algebro-geometric tools.

Keywords: Terracini loci; curve; curves in projective spaces; joins of varieties; secant varieties

MSC: 14H55; 14N05

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1. Introduction

Zero-dimensional schemes, i.e., algebro-geometric generalizations of finite sets [1,2], entered the applied mathematic world at least from two paths:

(a) interpolation;
(b) describing the dimension of many varieties relevant to applications, e.g., the set of all tensors with fixed format and rank.

In the interpolation path, there are multivariate extensions of Hermite and Birkhoff interpolation in which, at certain nodes, one fixes the values of certain (or all up to a fixed order) partial derivatives [3,4]. Terracini loci correspond to taking all first-order derivatives. We recommend [5] and references therein if one wants to extend these interpolation problems (over the reals) with partially real solutions. We think it would be a rich topic of research and we explain the connection at the end of Section 4. These interpolation problems for multivariate polynomials are used in number theory and geometry and, thanks to the Terracini Lemma [6] (Cor. 1.11), enter the topic described in (b) [1,2,7–9], which is related to the present paper. For old and recent results on (b) related to partially symmetric tensors, see [10–12].

Let X be a smooth and connected curve of genus g defined over an algebraically closed field of characteristic 0. Fix a base point-free g^r_d, $r \geq 2$, and let $Y \subset \mathbb{P}^r$ be the image of X by the morphism v associated to the base point free g^r_d. We assume $\deg(Y) = d$, i.e., we assume that $v : X \to Y$ is the normalization map. For any zero-dimensional scheme $A \subset \mathbb{P}^r$, let $\langle A \rangle$ denote its linear span, i.e., the intersection of the hyperplanes of $\mathbb{P}^r$ containing A, with the convention $\langle A \rangle = \mathbb{P}^r$ if no hyperplane contains A. For any effective divisor $Z \subset X$, let $v(Z)$ be the scheme-theoretical image. For any zero-dimensional scheme $Z \subset X$, we say that $Z \in \tilde{\mathcal{T}}(g^r_d)$ or that $Z \in \tilde{\mathcal{T}}(g^r_d, \deg(Z)/2)$ if all connected components of Z have an even degree, $\dim\langle v(Z) \rangle \leq \deg(Z) - 2$ and $\langle v(Z) \rangle \neq \mathbb{P}^r$. Set $\tilde{\mathcal{T}}(g^r_d) := \cup_{x>0} \tilde{\mathcal{T}}(g^r_d, x)$.

We recall that, in the usual definitions of Terracini loci of Y, only smooth points of Y are considered (for very good reasons!) [13–15]. We believe that our definition is flexible and, perhaps, may be linked to the singularities of maps and their enumerative geometry [16,17].

The big restriction is that we are assuming that the connected components of Z have an even degree. In the original Terracini Lemma the points are general in Y and so of course they are smooth points of Y. But in the set-up of [6], tangent spaces at singular points may be used. In the case $\dim Y = 1$, there is a natural smooth variety associated to Y, its normalization, and a base point-free g^r_d, $d := \deg(Y)$, inducing the normalization map. However, even when v is the identity map with Y smooth, our definition here is different from the standard one (see Example 2). We give the following examples to help the reader.

Example 1. *Assume that the differential of the map $X \to \mathbb{P}^r$ obtained composing v with the inclusion $Y \subset \mathbb{P}^r$ is zero at some $p \in X$. Then $v(2p) = v(p)$ scheme-theoretically and, hence, $2p \in \tilde{\mathcal{T}}(g^r_d, 1)$. In all the quoted definitions of $\mathcal{T}(Y, x)$, we have $\mathcal{T}(Y, 1) = \emptyset$.*

Example 2. *Assume $r = 2$ and Y smooth, i.e., $X = Y$ and v the identity map. If $d \in \{2, 3\}$, then $\tilde{\mathcal{T}}(g^2_d, 2) = \emptyset$ by the Bezout theorem and the same is true for all quoted sets $\mathcal{T}(Y, 2)$. Now assume $d \geq 4$. We have $\tilde{\mathcal{T}}(g^2_d, 2) \neq \emptyset$ and $\mathcal{T}(Y, 2) \neq \emptyset$, but $\mathcal{T}(Y, 2)$ is the set of all points of contact of the bitangent lines of Y, while $\tilde{\mathcal{T}}(g^2_d, 2)$ is the union of $\mathcal{T}(Y, 2)$ and the non-ordinary inflectional points of Y, if any.*

Example 3. *Assume $r = 2$, $d \geq 3$ and Y singular. We saw that $\tilde{\mathcal{T}}(g^2_d, 1) \neq \emptyset$ if it has at least one cusp (or a singular point with a non-smooth branch). Fix $o \in \mathrm{Sing}(Y)$ and take $p \in X$ such that $v(p) = o$. Assume that Y at o has at least 2 branches that are tangent at o. The scheme $4p \subset X$ is an element of $\tilde{\mathcal{T}}(g^2_d, 2)$. Thus, if $d \geq 3$ and Y has at least one non-ordinary singularity (a point whose tangent cone is not formed by distinct lines with multiplicity one), then $\tilde{\mathcal{T}}(g^2_d) \neq \emptyset$ and these schemes $Z \subset X$ contributing to $\tilde{\mathcal{T}}(g^2_d)$ do not contribute to the usual Terracini locus $\mathcal{T}(X)$.*

See Remark 3 for the explanation of our assumption that our g^r_d is assumed to be base point free. See Remark 4 for the case $\deg(Y) < d$.

Proposition 1. *Assume $r \geq 3$ and that Y is singular. Then either $\tilde{\mathcal{T}}(g^r_d, 1) \neq \emptyset$ or $\tilde{\mathcal{T}}(g^r_d, 2) \neq \emptyset$.*

The following result extends [15] (Proposition 7) to the case in which Y is singular. It shows that, using our definition, we may extend and simplify previous results.

Theorem 1. *Assume $r \geq 3$ and odd. Then $\tilde{\mathcal{T}}(g^r_d) = \emptyset$ if and only if $d = r$, i.e., if and only if Y is the rational normal curve of $\mathbb{P}^r$.*

The proof of Theorem 1 is 2 lines: if Y is smooth, use [15] (Proposition 7), if Y is singular, use Proposition 1. However, our point here is that not only the new definition allows some proofs in new and more general cases, but that it works fine in the set up of [14,15], e.g., it would be easy to copy the proof of [15] (Proposition 7) in our language and then at a critical step, the case $r = 3$, we use our definition and Proposition 1.

If $r \geq 7$, the proof gives that if Y is not a rational normal curve, then at least one among $\tilde{\mathcal{T}}(g^r_d, 1)$, $\tilde{\mathcal{T}}(g^r_d, 2)$ and $\tilde{\mathcal{T}}(g^r_d, (r+1)/2)$ is not empty. However, the added generality is an illusion if Y is smooth (Remark 5).

The following result (the main result of the paper) shows that it is far easier to prove a nonemptiness theorem for $\tilde{\mathcal{T}}(g^r_d)$, even when g^r_d gives an embedding, than with the usual definition of Terracini loci. The drawback is that $\tilde{\mathcal{T}}(g^r_d) \neq \emptyset$ does not imply the corresponding result for the usual Terracini locus, even if we assume Y smooth.

Theorem 2. *Assume r even. We have $\tilde{\mathcal{T}}(g^r_d) = \emptyset$ if and only if Y is either a rational normal curve or a linearly normal degree $r+1$ smooth curve of genus 1 or $r = 2$ and Y is a nodal plane cubic.*

In the set up of Theorem 2 for $r \geq 4$, we have $d \in \{r, r+1\}$ and if $d = r+1$, we are also assuming that Y is smooth and linearly normal (equivalently, smooth and not rational).

For any smooth point o of a positive dimensional quasi-projective variety W, let $(2o, W)$ denote the closed subscheme of W with $(\mathcal{I}_{o,W})^2$ as its ideal sheaf. We have $\deg(2o, W) = \dim W + 1$ and $(2o, W)_{red} = \{o\}$. For any finite set S contained in the smooth locus of W, set $(2S, W) := \cup_{o \in S}(2o, W)$. We can extend in several different ways the definition to an arbitrary smooth and connected projective variety. The following definition differs from both [14,15] in that it allows points with image singular points of Y. However, it restricts the definition to 'first-order data,' specifically 'first-order zero-dimensional schemes.' For example, in the setup of Example 3, higher-order flexes at most points would not contribute according to the following definition.

Definition 1. *Let X be a smooth and connected projective variety and $f : X \to \mathbb{P}^r$ a morphism birational onto its image $Y := f(X)$ with Y not contained in a hyperplane of $\mathbb{P}^r$. Let $Z \subset X$ be a zero-dimensional scheme. We say that $Z \in \hat{\mathcal{T}}(f, x)$ if there is a finite set $S \subset X$ such that $\#S = x$, $Z \subsetneq 2S$, $\langle f(2S) \rangle \neq \mathbb{P}^r$ and $\dim\langle f(Z) \rangle \leq \deg(Z) - 2$. Let $\check{\mathcal{T}}(f, x)$ denote the set of all $Z \in \hat{\mathcal{T}}(f, x)$ such that $\#Z_{red} = x$. If we drop the assumption $Z \subsetneq 2S$, we get a set $\tilde{\mathcal{T}}(f, \deg(Z)/2)$. Set $\hat{\mathcal{T}}(f) := \cup_{x>0}\hat{\mathcal{T}}(f, x)$, $\check{\mathcal{T}}(f) := \cup_{x>0}\check{\mathcal{T}}(f, x)$ and $\tilde{\mathcal{T}}(f) := \cup_{x>0}\tilde{\mathcal{T}}(f, x)$.*

Remark 1. *Note that $\hat{\mathcal{T}}(f) = \check{\mathcal{T}}(f)$. A classical observation due to Chandler shows that, to see if $\hat{\mathcal{T}}(f, x) \neq \emptyset$, it is sufficient to check the zero-dimensional subschemes of X whose connected components have degree ≤ 2 [18,19], [13] (Lemma 2.8). Easy examples show that sometimes one needs to allow that some connected components have degree 1 [13] (Th. 4.2 4.2 for nd odd]). If $\dim X = 1$, we have $\tilde{\mathcal{T}}(g^r_d, x) = \tilde{\mathcal{T}}(f, x)$ and, hence, $\tilde{\mathcal{T}}(g^r_d) = \tilde{\mathcal{T}}(f)$, where f is the morphism associated to the base point-free g^r_d.*

Remark 2. *If $\dim X > 1$, Definition 1 often gives huge families of solutions (see Examples 10 and 11). When there are huge families, the point is to study the set of all solution Z, which is an algebraic set (Remark 6). Hence a good project is the study the geometry of $\hat{\mathcal{T}}(f, x)$.*

We devote a full section (Section 4) to the discussion of a few important related definitions:

- minimal Terracini loci;
- Terracini loci with restricted support;
- allowable points.

The minimality for Terracini loci is very important, as stressed with many examples in [13]. They are the building blocks for all Terracini loci.

The Terracini Lemma is true even for joins of finitely many varieties embedded in the same projective space [6] (Cor. 1.11]). We devote a full section (Section 6) to the definitions of joins for finitely many varieties. There are two options for the notions of minimality: minimality and weak minimality.

Structure of the Paper

Section 2 contains some remarks used in the proofs later. Some of them also clarify some elementary properties of our definitions.

Section 3 contains the proofs of the proposition and the 2 theorems stated in the introduction.

Section 4 is concerned with minimality (it also introduces *weak minimality*), Terracini loci with restricted support and allowable points. In the last part of the section, we discuss "restricted support" for the euclidean topology and show how it may be adapted to real solutions and partially complex solutions.

Section 5 gives examples (with X a surface) concerning the Terracini loci.

Section 6 contains the definition of Terracini loci for joins of different varieties, proves one result (Proposition 5) and shows that joins of different varieties are easier than the one of a single variety (Example 12 for joins, Example 11 in Section 5 for the secant variety of a variety).

2. Remarks and Preliminary Results

We first give the following 2 observations for the case of curves and then give 2 easy foundational statements for arbitrary $n := \dim X$ (Remarks 6 and 7)

Remark 3. *Assume that the g^r_d on X is not base point free and call B its base locus. Fix $p \in B$. Since $r \geq 1$, there is $D \in g^r_d(-B)$ such that $p \in D$. Thus, $B + D$ is an element of the g^r_d in which p occurs with multiplicity of at least 2. Thus, if we allow this g^r_d in the definition of $\tilde{\mathcal{T}}$, we get $2p \in \tilde{\mathcal{T}}(g^r_d, 1) \neq \emptyset$.*

Remark 4. *Assume $\deg(Y) < d$ and let $u : C \to Y$ be the normalization map. Since X is a smooth curve, v factors through u, i.e., there is a morphism $w : X \to C$ such that $v = u \circ w$. Moreover, $\deg(w)\deg(Y) = d$ with $\deg(w) \geq 2$. Assume for the moment that w is ramified, i.e., assume the existence of $o \in X$ such that the differential of w vanishes at o and, hence, $2o$ would give $\tilde{\mathcal{T}}(g^r_d, 1) \neq \emptyset$. Now assume that w is unramified. Thus, C has genus ≥ 1 and, hence, $\deg(Y) > r$. Assume for the moment $r > 1$. Our g^r_d is the composition of a $g^r_{\deg(Y)}$ on C and the unramified morphism w. Take a general $p \in Y$ and set $w^{-1}(p) = \{o_1, \dots, o_{\deg(w)}\}$. If $r > 1$, the scheme $2o_1 + 2o_2$ is an element of $\tilde{\mathcal{T}}(g^r_d, 2)$. Of course, if $r = 1$, then w ramifies and, hence, $\tilde{\mathcal{T}}(g^1_d, 1) \neq \emptyset$.*

Remark 5. *Take $Z \in \tilde{\mathcal{T}}(g^r_d, x)$ and set $a := \dim\langle v(Z)\rangle$. If $r \geq a+3$, then $Z \cup 2p \in \tilde{\mathcal{T}}(g^r_d, x+1)$ for all $p \in X \setminus Z_{red}$. The same holds for a variety X of dimension > 1 taking $Z \cup v$, where v is a degree 2 connected zero-dimensional scheme.*

Remark 6. *Let X be a smooth and connected projective variety. For any positive integers z the set $\mathrm{Hilb}^z(X)$ of all degree z zero-dimensional subschemes of X is a projective scheme. If $\dim X \leq 2$, this scheme is smooth, connected and of dimension $z \dim X$ and an open subset of it is formed by the set of all subsets of X with cardinality z. The other conditions in the definitions of $\tilde{\mathcal{T}}$ and $\hat{\mathcal{T}}$ are locally closed conditions and, hence, the set of the zero-dimensional schemes satisfying all these conditions is a finite union of irreducible quasi-projective varieties. If X is a smooth curve, the subset Γ_z of $\mathrm{Hilb}^z(X)$ formed by subschemes whose connected components have an even degree is either empty (case z odd) or an irreducible and smooth projective variety whose general element has $z/2$ connected components, each of them of degree 2. Note that Γ_z is projective, not only an open subset of a smooth projective variety.*

Remark 7. *In Definition 1, there is a key condition "$\langle f(2S)\rangle \neq \mathbb{P}^r$". Assume $n := \dim X > 1$. Fix a hyperplane $H \subset \mathbb{P}^r$ and set $X_1 := f^{-1}(f(X) \cap H)$. Set $f_1 := f_{|X_1} : X_1 \to H$. Assume that X_1 is smooth. Since f is birational onto its image, $n > 1$ and $\langle f(X)\rangle = \mathbb{P}^r$, X_1 is connected and $\dim X_1 = n - 1$. For any $p \in X_1$ let $(2p, X_1)$ denote the closed subscheme of X_1 with $(\mathcal{I}_{p,X_1})^2$ as its ideal sheaf. For any finite set $S \subset X_1$, set $(2S, X_1) := \cup_{p \in S}(2p, X_1)$. Take a finite set $S \subset X_1$ and assume $\langle f_1(S)\rangle \neq H$. Since $2S \cap X_1 = (2S, X_1)$, every zero-dimensional scheme $Z \subset (2S, X_1)$ contributing to $\tilde{\mathcal{T}}(f_1)$ contributes to $\tilde{\mathcal{T}}(f)$.*

Notation 1. *Take $f : X \to \mathbb{P}^r$ and $Z \subset X$ a zero-dimensional scheme. Set $\delta(Z) := \deg(Z) - 1 - \dim\langle f(Z)\rangle$.*

3. Proofs of the Results Stated in the Introduction

Proof of Proposition 1: If the differential of the map $X \to \mathbb{P}^r$ obtained composing v with the inclusion $Y \subset \mathbb{P}^r$ is zero at some $p \in X$, then $\tilde{\mathcal{T}}(g^r_d, 1) \neq \emptyset$ (Example 1). If the differential is non-zero everywhere, then there are $p_1, p_2 \in X$ such that $p_1 \neq p_2$ and $v(p_1) = v(p_2)$. Thus, $\dim\langle v(2p_1 + 2p_2)\rangle \leq 2$ and hence $2p_1 + 2p_2 \in \tilde{\mathcal{T}}(g^r_d, 2) \neq \emptyset$. □

Proof of Theorem 1: If Y is smooth, we use [15] (Proposition 7). If Y is singular, we use Proposition 1. □

Example 4. *Take $r = 2$ and $d \in \{2, 3\}$. Let $Y \subset \mathbb{P}^2$ be the image of the g^2_d. If $d = 2$, then obviously $\tilde{\mathcal{T}}(g^2_2) = \emptyset$. Now assume $d = 3$. By Bezout $\tilde{\mathcal{T}}(g^2_3, x) = 0$ for all $x > 1$. Thus, $\tilde{\mathcal{T}}(g^2_3) \neq \emptyset$ if and only if Y is a cuspidal cubic.*

Example 5. *Let $Y \subset \mathbb{P}^r$, $r \geq 4$, r even, be a smooth rational curve of degree $r + 1$. Thus, there is a rational normal curve $C \subset \mathbb{P}^{r+1}$ and $o \in \mathbb{P}^{r+1} \setminus C$ such that $Y = \ell_o(C)$, where $\ell_o : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r$ is the linear projection from o. Now assume $r = 4$. Call g^4_5 the linear series on Y induced by the inclusion $Y \subset \mathbb{P}^4$. Set $\ell := \ell_{o|C}$.*

Claim: *We have $\tilde{\mathcal{T}}(g^4_5, 1) = \emptyset$ and $\tilde{\mathcal{T}}(g^4_5, 2) \neq \emptyset$.*

Proof of Claim: Since C is a degree 5 rational normal curve, all its zero-dimensional schemes of degree ≤ 6 are linearly independent. Thus, for every zero-dimensional scheme $Z \subset Y$ with $\deg(Z) \in \{2, 4\}$, we have $\dim\langle Z\rangle \leq \deg(Z) - 2$ if and only if $o \in \langle \ell^{-1}(Z)\rangle$. Since Y is smooth, o is not contained in a tangent line of C. Hence, $\tilde{\mathcal{T}}(g^4_5, 1) = \emptyset$. Let $\mathcal{Z}$ be the family of all zero-dimensional schemes $Z \subset Y$ such that $\deg(Z) = 4$ and its connected components have an even degree (so either Z is connected or it has 2 connected components, each of them of degree 2). Note that $\tilde{\mathcal{T}}(g^4_5, 2) \neq \emptyset$ if and only if there is $Z \in \mathcal{Z}$ such that $o \in \ell^{-1}(Z)$. We have $\dim\langle \ell^{-1}(Z)\rangle = 3$ for all $Z \in \mathcal{Z}$. Note that $\mathcal{Z}$ is parametrized by the second symmetric power of C and in particular it is a 2-dimensional irreducible projective variety. Thus, $\cup_{Z\in\mathcal{Z}}\langle \ell^{-1}(Z)\rangle = \mathbb{P}^5$ (here we have equality and not just the density of the union; it is here where we use that we allow the case Z connected, i.e., we use our definition of Terracini locus). □

Proof of Theorem 2: If Y is a rational normal curve, i.e., if $d = r$, $\tilde{\mathcal{T}}(g^r_r) = \emptyset$ because every zero-dimensional scheme $Z \subset Y$ with $\deg(Z) \leq r + 1$ is linearly independent.

Now assume that Y is a linearly normal smooth curve of genus 1. Thus, $d = r + 1$. The integer $r + 1$ is a minimal degree of a linearly dependent zero-dimensional scheme $Z \subset Y$. Take one such scheme. Since r is even, $r + 1$ is odd and, hence, not all connected components of Z have an even degree. Thus, $\tilde{\mathcal{T}}(g^r_{r+1}) = \emptyset$.

Now assume $\tilde{\mathcal{T}}(g^r_d) = \emptyset$.

(a) Assume $r = 2$. The case $d \leq 3$ is described in Example 4. Thus, we may assume $d \geq 4$. Let g be the genus of Y. Let $Y^\vee$ denote the dual curve of Y. Call d^*, δ^* and k^* the invariant of $Y^\vee$ (the degree, the drop of genus and the cuspidal contribution). In the set up of the Plücker's formulas of [20] (p. 482), we have $r_1 = d^*$. By assumption, Y has no bitangent. By Examples 1–3, all singular points of Y are ordinary singular points, all flexes at smooth point of Y are ordinary flexes, i.e., the tangent line of Y at the point has order of contact 3 with the curve and $Y^\vee$ has only unibranch singularities. Since all flexes of Y are ordinary, $Y^\vee$ has only ordinary flexes [21] and, hence, $g = (d^* - 1)(d^* - 2)/2 - k^*$, where k^* is the number of cusps of $Y^\vee$. We have $k^* \leq d^*(d^* - 2)/3$ and, hence, $g \geq (d^* - 2)(d^* - 3)/6$. Since Y has no cuspidal locus, another of the Plücker formulas gives $d^* = 2d + 2g - 2$, contradicting the inequality $g \geq (d^* - 2)(d^* - 3)/6$.

(b) Assume $r = 4$. Proposition 1 implies that Y is smooth. By Example 5, we may assume $d \geq 6$. We take a general $p \in Y$. Since p is general, the tangent line T_pY has order of contact 2 with Y at p. By Remark 5, to get a contradiction, we reduce to prove that $\tilde{\mathcal{T}}(g^2_{d-2}) \neq \emptyset$ for a certain base point free and birational onto its image g^2_{d-2} on X. We apply step (a) to this g^2_{d-2}.

(c) Assume $r \geq 6$. We use induction on the even integer r. We project from a general tangent line of Y and reduce to the case $r - 2$ for a curve if degree $d - 2$, as we did in step (b) and use the inductive assumption. □

4. Minimal Terracini Loci and Allowed Support

For all the notions $\tilde{\mathcal{T}}$, $\hat{\mathcal{T}}$ and $\check{\mathcal{T}}$ of Terracini loci, there are at least 2 notions of minimality. We call $\mathbb{T}$ any $\tilde{\mathcal{T}}$, $\hat{\mathcal{T}}$ and $\check{\mathcal{T}}$ and, for instance, we write $\mathbb{T}(f, x)$ instead of $\tilde{\mathcal{T}}(g^r_n, x)$ or $\hat{\mathcal{T}}(f, x)$ or $\check{\mathcal{T}}(f, x)$. We recall that for every zero-dimensional scheme $Z \subset X$, X the smooth projective variety such that we are looking at the Terracini locus of $f : X \to \mathbb{P}^r$, $\delta(Z) := \deg(Z) - \dim\langle f(Z)\rangle - 1$. Fix $Z \in \mathbb{T}(f)$. We say that Z is *minimal* if $Z' \notin \mathbb{T}(f)$ for any $Z' \subsetneq Z$. We say that Z is *weakly minimal* if $\delta(Z') < \delta(Z)$ for all $Z' \subsetneq Z$ such that $Z' \in \mathbb{T}(f)$. Call $\mathbb{T}(f)'$ (resp. $\mathbb{T}(f)''$) the set of all minimal (resp. weakly minimal) elements of $\mathbb{T}(f)$.

Remark 8. *Assume $\mathbb{T}(f) \neq \emptyset$ and let x be the first integer such that $\mathbb{T}(f,x) \neq \emptyset$. Obviously, $\mathbb{T}(f,x) = \mathbb{T}(f,x)'$. Thus, in the set-up Theorems 1 and 2, we have $\tilde{\mathcal{T}}(g^r_d)' \neq \emptyset$, except in the listed cases with $\tilde{\mathcal{T}}(g^r_d) = \emptyset$. Moreover, (for any X) if there is $p \in X$ at which the differential of f is not injective, then $\mathbb{T}(f)' \neq \emptyset$.*

The following examples show that sometimes minimality is stronger than weak minimality (easy examples also exist for all $r > 2$).

Example 6. *Take a plane curve $Y \subset \mathbb{P}^2$ with a cuspidal point, a_1, whose tangent cone is formed by a line L (with multiplicity 2) tangent to Y at some $a_2 \in Y_{reg}$. Let $X \to Y$ denote the normalization map. Call g^2_d the linear series on X mapping X to Y. Take $p_1, p_2 \in X$ with images a_1 and a_2 in Y. Since $L \neq \mathbb{P}^2$, we have $2p_2 \notin \tilde{\mathcal{T}}(g^2_d)$. We have $\delta(2p_1 + 2p_2) > \delta(2p_1)$ and $\delta(2p_2) = 0$. Since $L \neq \mathbb{P}^2$, $2p_1 + 2p_2 \in \tilde{\mathcal{T}}(g^2_d)$. Hence, $2p_1 + 2p_2$ is weakly minimal, but not minimal.*

Example 7. *Take a plane curve Y_1 with 2 different cusps, b_1 and b_2, and let γ the associated linear system on the normalization $X_1 \to Y_1$. Take $o_1, o_2 \in X_1$ with images b_1 and b_2. Since $\{b_1, b_2\}$ spans a line, $2o_1 + 2o_2 \in \tilde{\mathcal{T}}(\gamma)$, $2o_1 \in \tilde{\mathcal{T}}(\gamma)$, $2o_2 \in \tilde{\mathcal{T}}(\gamma)$ and $\delta(2o_1 + 2o_2) > \delta(2o_i)$ for all i. Thus, $2o_1 + 2o_2$ is weakly minimal, but not minimal.*

Now we discuss two important refinements of the Terracini loci:

(1) closed subsets of X that we are forced to avoid;
(2) points of X or finite subsets of X that we may allow.

Obviously, (1) is important if we may take interpolation data only outside a closed subset B of X (see the end of the section for a discussion of more general B for the euclidean topology). This is important if we cannot access a small part, B, of the database. Nonemptiness outside B means that our problem has at least one solution without points in B.

Obviously, (2) can be used to shorten the computational task, if we computed in advance the data at the allowed point (or more that one point).

Take a smooth projective variety X and a morphism $f : X \to \mathbb{P}^r$, $r \geq 2$, birational onto its its image and with $\langle f(X)\rangle = \mathbb{P}^r$. We write $\mathbb{T}(f)$ and $\mathbb{T}(f,x)$ for any of $\tilde{\mathcal{T}}$, $\hat{\mathcal{T}}$ and $\check{\mathcal{T}}$, since the definitions in this section work for all definitions of Terracini loci. Take a closed subset $B \subsetneq X$ for the Zariski topology. Take any $Z \in \mathbb{T}(f,x)$. We say that $Z \in \mathbb{T}(f, \setminus B, x)$ if $Z_{red} \subset X \setminus B$. Set $\mathbb{T}(f, \setminus B) := \cup_{x>0}\mathbb{T}(f, \setminus B, x)$. Since $X \setminus B$ is an open subset of X, $\mathbb{T}(f, \setminus B, x)$ is an open subset of $\mathbb{T}(f,x)$ and $\mathbb{T}(f, \setminus B)$ (it may be empty). The same definition applies to the minimally Terracini and weakly minimal Terracini loci, $\mathbb{T}(f)'$ and $\mathbb{T}(f)''$. If $\dim X = 1$, then B is an arbitrary finite subset of X.

Fix $p \in X$. We say that p is *allowed* or that it is an *allowed point* for $\mathbb{T}(f)$ (or $\mathbb{T}(f)'$ or $\mathbb{T}(f)''$ or $\mathbb{T}(f,x)$ or $\mathbb{T}(f,x)'$ or $\mathbb{T}(f,x)''$) if there is $Z \in \mathbb{T}(f)$ (or $Z \in \mathbb{T}(f)'$ or $Z \in \mathbb{T}(f)''$ or $Z \in \mathbb{T}(f,x)$ or $Z \in \mathbb{T}(f,x)'$ or $Z \in \mathbb{T}(f,x)''$) such that $p \in Z_{red}$. Fix a finite set $A \subset X$, $A \neq \emptyset$. We say that A is *allowed* for $\mathbb{T}(f)$ (or $\mathbb{T}(f)'$ or $\mathbb{T}(f)''$ or for $\mathbb{T}(f,x)$ or $\mathbb{T}(f,x)'$ or $\mathbb{T}(f,x)''$) if there is $Z \in \mathbb{T}(f)$ (or $Z \in \mathbb{T}(f)'$ or $Z \in \mathbb{T}(f)''$ or $Z \in \mathbb{T}(f,x)$ or $Z \in \mathbb{T}(f,x)'$ or $Z \in \mathbb{T}(f,x)''$) such that $A \subseteq Z_{red}$.

The exceptional case in the next proposition is described in Example 8.

Proposition 2. *Let X be a smooth curve. Take a base point-free g^r_d, $d > r \geq 3$, on X birational onto its image. Assume r is odd. Fix $p \in X$. Let $f : X \to \mathbb{P}^r$ be morphism associated to the g^r_d. Set*

$Y := f(X)$. Then p is an allowed point for $\tilde{\mathcal{T}}(g^r_d)$, unless $d = r+1$, $X \cong \mathbb{P}^1$, f is an embedding and the tangent line of Y at $f(p)$ has order of contact 3 with Y at $f(p)$.

Proof. If the differential of f vanishes at p, then p is allowed for $\tilde{\mathcal{T}}(g^r_d, 1)'$. Thus, we may assume that the differential of f is non-zero at p. Now assume $f(p) \in \mathrm{Sing}(Y)$. Since the differential of f is non-zero at p, there is $q \in X \setminus \{p\}$ such that $f(p) = f(q)$. Thus, $\dim\langle f(2p+2q)\rangle \le 2$ and, hence, $2p+2q \in \tilde{\mathcal{T}}(g^r_d, 2)$. Thus, we may assume that Y is smooth at p. Let L be the tangent line of Y at $f(p)$. If L has a contact order of at least 4 with Y at $f(p)$, then $4p \in \tilde{\mathcal{T}}(g^r_d, 2)$. Now assume that L meets Y at some $a \ne p$ say $a = f(b)$. In this case $2p+2b \in \tilde{\mathcal{T}}(g^r_d, 2)$. Let $\ell : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ denote the linear projection from L. Since L meets only at $f(p)$ and Y is smooth at p, $\ell_{Y \setminus L \cap Y}$ extends to a morphism $\mu : Y \to \mathbb{P}^{r-2}$.

(a) Assume that L meets Y only at $f(p)$ and that L has order of contact 2 with X at $f(p)$. If $r = 3$, we just use that any morphism $X \to \mathbb{P}^1$ of degree $d-2 > 1$ is ramified, because $\mathbb{P}^1$ is algebraically simply connected. Thus, we may assume $r \ge 5$. We have $d-2 = \deg(\mu)\deg(\mu(Y))$.

(a1) Assume that μ is not birational onto its image. Call C the normalization of $\mu(Y)$. The morphism μ induces a degree $\deg(\mu)$ morphism $u : X \to C$. If u ramifies, there is a degree 2 subscheme $Z' \subset X$ such that $2p + Z' \in \tilde{\mathcal{T}}(g^r_d, 2)$. Now assume that u is unramified. Thus, C has genus ≥ 1. Thus, $z := \deg(\mu(Y)) > r-2$. Theorem 1 gives $\tilde{\mathcal{T}}(g^{r-2}_z) \ne \emptyset$. Lifting it to X by the map u and adding $2p$, we get an element of $\tilde{\mathcal{T}}(g^r_d)$.

(a2) Assume that μ is birational onto its image. Thus, the normalization map $X \to \mu(Y)$ is a g^{r-2}_{d-2}. Since $r-2 > d-2$, Theorem 1 gives the existence of $Z \in \tilde{\mathcal{T}}(g^{r-2}_{d-2})$. Since μ is induced by the linear projection from the tangent line of Y at $f(p)$, $2p + Z \in \tilde{\mathcal{T}}(g^r_d)$.

(b) Assume that L has order of contact 3 with Y at p. If $d \ge r+2$, then we conclude as in step (a) (note that in steps (a1) and (a2) we only used Theorem 1, not the statement of Proposition 2 for the integer $r-2$). Now assume $d = r+1$. Thus, either Y is smooth and rational, but not linearly normal, or $p_a(Y) = 1$. The first possibility is the exceptional case in the statement of the proposition. Now assume $p_a(Y) = 1$. In this case $d+1$ is the minimal degree of a linearly dependent zero-dimensional subscheme of Y and, hence, L cannot have order of contact > 2 with Y at the smooth point $f(p)$. $\square$

Example 8. *Take a smooth degree $r+1$ rational curve $X \subset \mathbb{P}^r$ spanning $\mathbb{P}^r$, $r \ge 3$. Fix $p \in X$ and let L be the tangent line of X at p. There are a degree $r+1$ rational normal curve $C \subset \mathbb{P}^{r+1}$ and $o \in \mathbb{P}^{r+1} \setminus C$ such that $X = \ell(C)$, where $\ell : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r$ denote the linear projection from o. Take $q \in C$ such that $\ell(q) = p$. Note that q is the unique point of C such that $\ell(q) = p$. Since every subscheme of degree $\le r+2$ of C is linearly independent, the tangent line of X at p has order of contact > 2 if and only if o is contained in the osculating plane of C at q. In particular, this is not the case if, for a fixed X, we take a general $p \in X$. Now assume that X is a general smooth degree $r+1$ rational curve of $\mathbb{P}^r$, i.e., assume that o is a general point of $\mathbb{P}^{r+1}$. Since o is general and $r+1 \ge 4$, o is contained in no osculating plane of C. Thus, every point of X is allowed for the g^r_{r+1} on X induced by the inclusion $X \subset \mathbb{P}^r$.*

Proposition 3. *Let X be a smooth curve. Take a base point free g^3_d, $d > 3$, on X inducing a morphism $f : X \to \mathbb{P}^3$ birational onto its image, with nowhere vanishing differential and with $\langle f(X)\rangle = \mathbb{P}^3$. Then a general $p \in X$ is allowable for $\tilde{\mathcal{T}}(g^3_d)'$.*

Proof. Set $Y := f(X)$. Since we chose p general after fixing the g^3_d, $f(p)$ is a smooth point of Y and the tangent line, L, of Y at $f(p)$ has order of contact 2 with Y at $f(p)$. Assume for the moment that L meets Y at a point $f(o)$ with $o \ne p$. Thus $2p + 2o \in \tilde{\mathcal{T}}(g^3_d, 2)$. Since the differential of f does not vanish at o, $2o \notin \tilde{\mathcal{T}}(g^3_d)$ and, hence, $2p + 2o \in \tilde{\mathcal{T}}(g^3_d, 2)$. Thus, we may assume $\deg(L \cap f(X)) = 2$. Thus, the linear projection from L induces a degree $d-2$ morphism $\mu : X \to \mathbb{P}^1$. Since $d-2 > 1$ and $\mathbb{P}^1$ is algebraically simply connected, there is $b \in X$ at which μ ramifies. Thus, $2p + 2b \in \tilde{\mathcal{T}}(g^3_d)$. Since f is a local embedding, $2p \notin \tilde{\mathcal{T}}(g^3_d)$ and $2b \notin \tilde{\mathcal{T}}(g^3_d)$. Thus, $2p + 2b \in \tilde{\mathcal{T}}(g^3_d)'$. $\square$

Remark 9. *In the set-up of Theorem 1, one expects that $\tilde{\mathcal{T}}(g^r_d)$ has a dimension of 1. Of course, sometimes it has a higher dimension, e.g., if the morphism associated to the g^r_d is ramified, say at the point p, and $r \geq 5$, then $2p + 2S \in \tilde{\mathcal{T}}(g^r_d)$ for all $S \subset X \setminus \{p\}$ such that $\#S = (r-1)/2$ and, hence, $\dim \tilde{\mathcal{T}}(g^r_d) \geq (r-1)/2$. We believe that, for "most" g^r_d's, $\tilde{\mathcal{T}}(g^r_d)$ has pure dimension 1. Proposition 2 and Example 8 show that $\tilde{\mathcal{T}}(g^r_d)$ has at least one component of positive dimension. There are g^r_d 's such that $\tilde{\mathcal{T}}(g^r_d)$ has isolated points. For instance, all points at which the g^r_d has a cusp, i.e., all points of $\tilde{\mathcal{T}}(g^r_d)$. Smooth plane curves of degree ≥ 4 give other examples.*

Proposition 4. *Fix an integer $d > 3$. Let X be a smooth projective curve and $B \subset X$ a finite set. Let g^3_d be a base point-free linear series such that the induced map $f : X \to \mathbb{P}^3$ is birational onto its image and its differential is everywhere non-zero. Then $\tilde{\mathcal{T}}(g^3_d, \setminus B) \neq \emptyset$ and a general $p \in X$ is allowable for $\tilde{\mathcal{T}}(g^3_d, \setminus B)$.*

Proof. Write $B = \{o_1, \dots, o_s\}$ and let $L_1, \dots, L_s$ be the tangent lines to $f(X)$ at $f(o_i)$. Since we are in characteristic zero, a general tangent line of $f(X)$ contains no point of $f(B)$. Since we are in characteristic zero, the differential of the rational map f_i from X to $\mathbb{P}^1$ induced by the linear projection from L_i has a non-zero differential at a general point of X. Fix a general $p \in X$ and let $L \subset \mathbb{P}^3$ the tangent line to $f(X)$ at $f(p)$. We just see that $L \cap L_i = \emptyset$ for all i. If there is $f(a) \in L \cap f(Y)$, $a \neq p$, then $a \notin B$ and $2p + 2a \in \tilde{\mathcal{T}}(g^3_d)$. Now assume that L meets $f(X)$ only at $f(p)$. Since p is general, the order of contact of L and $f(X)$ at $f(p)$ is 2. Thus, the linear projection from L induces a degree $d-2$ morphism $\mu : X \to \mathbb{P}^1$. Since $d - 2 > 1$, μ is ramified at some $b \in X$. Since $L \cap L_i = \emptyset$ for all i, $b \notin B$. Thus, $2p + 2b \in \tilde{\mathcal{T}}(g^3_d, \setminus B)$. $\square$

The following example shows that Proposition 4 is not always true in $\mathbb{P}^2$.

Example 9. *Take a smooth plane curve $X \subset \mathbb{P}^2$ of degree $d \geq 4$ and call g^2_d the linear series on X associated to $|\mathcal{O}_X(1)|$. Since X has bitangents , $\mathcal{T}(X, 2) \neq \emptyset$ and, hence, $\tilde{\mathcal{T}}(g^2_d, 2) \neq \emptyset$. Since X has only finitely many bitangents and flexes, there is a finite set B such that $\mathcal{T}(g^2_d, \setminus B) = \emptyset$.*

Remark 10. *In the definition of $\mathbb{T}(f, \setminus B)$, we must be alert to the logical quantifier: we first fix B and then take f hopefully to prove that $\mathbb{T}(f, \setminus B) \neq \emptyset$. If $\mathbb{T}(f) \neq \emptyset$, then finding B such that $\mathcal{T}(f, \setminus B) \neq \emptyset$ is trivial.*

Here we are using the Zariski topology. Now assume that the base field is the complex number field $\mathbb{C}$. We write $X(\mathbb{C})$ for the points of X with the euclidean topology. Thus, $X(\mathbb{C})$ is a compact and connected complex manifold of dimension $n := \dim X$ and, hence, a compact and connected orientable $2n$-dimensional topological manifold. In the definition of $\mathbb{T}(f, \setminus B)$, we may take as B any closed set $B \subsetneq X(\mathbb{C})$. Proposition 2 shows that often at least one point of some solution of $\mathbb{T}(f)$ may be found in $X(\mathbb{C}) \setminus B$. For any closed set $T \subseteq X$, we write $T(\mathbb{C})$ for the set $T(\mathbb{C})$ with the euclidean topology induced by the topology of $X(\mathbb{C})$. Even if T is very singular, $T(\mathbb{C})$ is not topologically very bad (it is a compact complex analytic space and in particular it is a compact CW complex and it has a triangulation). Fix any metric d on $X(\mathbb{C})$, inducing the euclidean topology, e.g., the one induced by an embedding of $X(\mathbb{C})$ in a big projective space equipped with the Fubini-Study metric. For all real numbers $\epsilon > 0$, let $T(\mathbb{C})_\epsilon$ denote the set of all $p \in X(\mathbb{C})$ whose distance from $T(\mathbb{C})$ is at most ϵ. Each $T(\mathbb{C})_\epsilon$ is a compact subset of $X(\mathbb{C})$. If $T \subsetneq X$, then there is a real number ϵ_0 such that $T(\mathbb{C})_\epsilon \subsetneq X(\mathbb{C})$ for all $0 < \epsilon \leq \epsilon_0$, because X is irreducible and, hence, $\dim T < \dim X$. Fix a Zariski closed set $B \subsetneq X$ and assume $\mathbb{T}(f, \setminus B) \neq \emptyset$. Take $Z \in \mathbb{T}(f, \setminus B)$. Since Z_{red} is a finite set $\epsilon_1 := d(Z_{red}, B(\mathbb{C}))$ is a positive real number. Note that $\mathbb{T}(f, \setminus B)_\epsilon \neq \emptyset$ for all $0 < \epsilon < \epsilon_1$.

Now assume that the algebraically closed base field is the complex number field $\mathbb{C}$ and that both X and f are defined over $\mathbb{R}$. Since X is smooth, the set $X(\mathbb{R})$ of all real $p \in X(\mathbb{C})$ is the union of finitely many connected components, each of them a compact differentiable manifold of dimension n. In this case, $\mathbb{T}(f)$ (or $\mathbb{T}(f, \backslash B)$ if B is defined over $\mathbb{R}$) has an involution $\sigma : \mathbb{T}(f) \to \mathbb{T}(f)$ induced by the complex conjugation of $\mathbb{C}$. Thus, it is natural to check if the involution σ has fixed points. These fix points, $\mathbb{T}(f)(\mathbb{R})$, are the "real solutions", but $Z \in \mathbb{T}(f)(\mathbb{R})$ may have $Z_{red} \nsubseteq X(\mathbb{R})$. For instance, take a smooth plane curve $X \subset \mathbb{P}^2$ with an ordinary bitangent tangent to X at 2 complex conjugate points of $X(\mathbb{C})$. Even worse, $X(\mathbb{R}) = \emptyset$ may occur (for any genus $g \geq 0$ there is a smooth genus g defined over $\mathbb{R}$ and with $X(\mathbb{R}) = \emptyset$ [22]. The reader is encouraged to look at "partially real" solutions in the sense of [5] and references therein.

5. Examples

Example 10. *Let $X \subset \mathbb{P}^4$ be a smooth and non-degenerate surface of degree d. Let $j : X \subset \mathbb{P}^4$ denote the inclusion. We have $\mathcal{T}(X,2) \neq \emptyset$ if and only if there is a hyperplane H tangent to X at 2 different points. If $d = 3$, then X is a ruled surface and the tangent plane at 2 different points of a line $L \subset X$ are contained in a hyperplane, because their intersection contains L. Now assume $d > 3$. Fix any hyperplane $H \subset \mathbb{P}^4$ such that $X \cap H$ is smooth and call g^3_d the linear series on $X \cap H$ we may apply Theorem 1. Thus, $\tilde{\mathcal{T}}(g^3_d) \neq \emptyset$. Hence, $\tilde{\mathcal{T}}(j)$ is huge.*

Example 11. *Let X be a smooth and connected projective surface and $f : X \to \mathbb{P}^5$ be an embedding such that $\langle f(X)\rangle = \mathbb{P}^5$. Set $Y := f(X)$. Y is a smooth surface. As in Example 10, using Theorem 2 instead of Theorem 1, we get that a general hyperplane H contains elements of $\tilde{\mathcal{T}}(f)$. We want to prove more. We have $\mathcal{T}(Y,2) \neq \emptyset$ if and only if at least 2 different tangent planes of Y meet. Assume that X is not isomorphic to $\mathbb{P}^2$. Fix $p \in Y$ and assume that T_pY meets Y only at p. We claim that there is $o \neq p$ such that $T_oY = T_pY$. Assume that this is not the case. Since T_pY meets Y only at p, the linear projection from T_pY induces a morphism $u : Y \setminus \{p\} \to \mathbb{P}^2$. By assumption, the differential of u is an isomorphism at all points of $Y \setminus \{o\}$. There is a birational morphism $v : Y_1 \to Y$ such that u extends to a morphism $w : Y_1 \to \mathbb{P}^2$. Assume for the moment $\deg(u) > 1$, i.e., $\deg(w) > 1$. Since $\mathbb{P}^2$ is algebraically simply connected the branch locus $B \subset \mathbb{P}^2$ is a non-empty effective divisor of $\mathbb{P}^2$ and in particular it is ample. Thus, its counterimage in Y_1 is not contained in the exceptional locus of w, i.e., there is $o \in Y \setminus \{p\}$ such that $T_oY \cap T_pY \neq \emptyset$. Now assume $\deg(u) = 1$, i.e., that u is generically injective. Since $Y \setminus \{o\}$ and $\mathbb{P}^2$ are smooth, Zariski Main Theorem (or a topological fact) gives that u is an open embedding. Since $Y \setminus (Y \setminus \{p\})$ is finite and u is an algebraic map, we get that $\mathbb{P}^2 \setminus u(Y \setminus \{p\})$ is finite. Thus, u extends to an isomorphism and, hence, $Y \cong \mathbb{P}^2$.*

6. Joins of Two or Finitely Many Embedded Varieties

Fix an integer $s \geq 2$ and s integral and non-degenerate varieties $Y_i \subset \mathbb{P}^r$, $1 \leq i \leq s$. The join $J(Y_1, \dots, Y_s)$ of $Y_1, \dots, Y_s$ is the closure in $\mathbb{P}^r$ of the union of all linear spans of sets $\{p_1, \dots, p_s\}$ with $p_i \in Y_i$ for all i. In the usual definition, one allows the case $Y_i = Y_j$ for some $i \neq j$, but we prefer to consider the case in which $Y_i \neq Y_j$ for all $i \neq j$ and we have positive integers $a_1, \dots, a_s$ such that each Y_i appears exactly a_i times in the join. Thus, we are looking at Terracini loci coming from the join of $\sigma_{a_1}(Y_1), \dots, \sigma_{a_s}(Y_s)$, where $\sigma_{a_i}(Y_i)$ denote the a_i-th secant variety of Y_i [2]. Set $x := a_1 + \cdots + a_s$. We are assuming $Y_i \neq Y_j$ for all $i \neq j$, but Y_i and Y_j are allowed to be projectively equivalent. Let X_i be a smooth projective variety and let $f_i : X_i \to Y_i$ be a morphism with $Y_i = f(X_i)$ and $f_i : X_i \to Y_i$ birational. We take $X_1, \dots, X_s$ as distinct abstract varieties (even if they are isomorphic) so that no point of X_i is a point of X_j for $i \neq j$. For all finite sets $S_i \subset X_i$, we write $(2S_i, X_i)$ as the union of the double points $2p$ of X_i, $p \in S$. We assume $\#S_i = a_i$ for all i and take zero-dimensional schemes $Z_i \subset (2S, X_i)$, $1 \leq i \leq s$. In the case $\dim X_i = 1$, we assume that each connected component of X_i has an even degree. We say that $(Z_1, \dots, Z_s)$ contributes to $\tilde{\mathcal{T}}(f_1, \dots, f_s; a_1, \dots, a_s)$ if $\langle \cup_{i=1}^s f(2S_i, X_i)\rangle \neq \mathbb{P}^r$ and $\dim\langle \cup_{i=1}^s f_i(Z_i)\rangle \leq \sum_{i=1}^s \deg(Z_i) - 2$. We get the same formula if $a_i = 0$ for some i taking $S_i = Z_i = \emptyset$, with the only restriction that $a_1 + \cdots + a_s > 0$. Set $\tilde{\mathcal{T}}(f_1,\dots,f_s) := \cup_{a_1\ge 0,\dots,a_s\ge 0, a_1+\cdots+a_s>0}\tilde{\mathcal{T}}(f_1,\dots,f_s;a_1,\dots,a_s)$ and $\tilde{\mathbb{T}}(f_1,\dots,f_s) := \cup_{a_1>0,\dots,a_s>0}\tilde{\mathcal{T}}(f_1,\dots,f_s;a_1,\dots,a_s)$. The notions of minimality and weak minimality for joins are different if we consider $\tilde{\mathcal{T}}(f_1,\dots,f_s)$ or $\tilde{\mathbb{T}}(f_1,\dots,f_s)$.

Note that, in the next proposition, we allow the case in which both Y_1 and Y_2 are rational normal curves, we only assume $Y_1 \ne Y_2$ as subsets of the same projective space.

Proposition 5. *Fix an odd integer $r \ge 3$ and irreducible and non-degenerate curves $Y_i \subset \mathbb{P}^r$, $i = 1,2$, such that $Y_1 \ne Y_2$. Let $f_i : X_i \to Y_i$ denote the normalization map. Fix an integer $0 < x < (r+1)/2$ and set $y := (r+1)/2 - x$. Then $\tilde{\mathcal{T}}(f_1,f_2;x,y) \ne \emptyset$. Moreover, $\tilde{\mathcal{T}}(f_1,f_2;x,y)$ has at least 2 irreducible families of elements of dimension $x+y-1$ and, as an allowable set for $\tilde{\mathcal{T}}(f_1,f_2;x,y)$, we may take the union of x general points of X_1 and $y-1$ general points of X_2.*

Proof. By Remark 5, it is sufficient to find integers $0 \le a \le x$ and $0 \le b \le y$, $a+b > 0$, such that $\tilde{\mathcal{T}}(f_1,f_2;a,b) \ne \emptyset$. Fix a general $S_1 \subset X_1$ such that $\#S_1 = x$ and set $Z_1 := 2S_1$. Since $Y_1 \ne Y_2$, $Y_1 \cap Y_2$ is finite. Since S_1 is general in X_1, $f_1(S_1) \cap Y_2 = \emptyset$. Set $V := \langle f(Z_1)\rangle$ If $\dim V \le 2x-2$, then $\tilde{\mathcal{T}}(f_1,f_2;x,0) \ne \emptyset$ and, hence, $\tilde{\mathcal{T}}(f_1,f_2;x,y) \ne \emptyset$. Now assume $\dim V = 2x-1$. If $Y_2 \cap V \ne \emptyset$, say $f_2(o) \in V$, the pair of schemes $(Z_1, 2o)$ gives $\tilde{\mathcal{T}}(f_1,f_2;x,y) \ne \emptyset$. Thus, we may assume $V \cap Y_2 = \emptyset$. Let $\mu : X_2 \to \mathbb{P}^{2r-2}$ denote the morphism induced by the composition of f_2 with the linear projection $\mathbb{P}^r \setminus V \to \mathbb{P}^{r-2x}$. Set $d := \deg(Y_2)$. Since Y_2 is non-degenerate, $d \ge r$ and, hence, $d > r-2x$. If $y = 1$, and, hence, $r - 2x = 1$, we use the ramification formula to say that μ is ramified at some $o \in X_2$ and, hence, we use the scheme $Z_1 \cup 2o$. Now assume $r - 2x \ge 3$. If $\deg(\mu) > 1$, one mimics step (a1) of the proof of Proposition 2. If $\deg(\mu) = 1$, we may use the statement of Theorem 1, because $d > r-2x$.

Now we prove the "Moreover" assertion. We may take as first x points on X_1 x general points of X_1. So these x points of X_1 are parametrized by a family of dimension x. If $y = 1$, then we proved the existence of our first irreducible family and even described it. Now assuming $y \ge 2$, instead of quoting Theorem 1, we take the proof of Proposition 2 and see that we may take $y-1$ general points of X_2 as part of our solution.

The other family is obtained first taking y general points of X_2 and then $x-1$ general points of X_1. □

The following example (the equivalent for joins of Example 11) shows how much easier it is to get results for joins of different varieties. We hope that the readers will give many more examples.

Example 12. *Take smooth and non-degenerate surfaces $X_i \subset \mathbb{P}^5$, $i = 1,2$, such that $X_1 \ne X_2$. Call f_1 and f_2 the inclusion of X_1 and X_2. Fix any $p \in X_1 \setminus X_1 \cap X_2$. We claim that p is allowable for $\tilde{\mathcal{T}}(f_1,f_2;1,1)$. If $T_pX_1 \cap X_2 \ne \emptyset$, then we may take any $o \in T_pX_1 \cap X_2$ and use that $T_pX \cap T_oX$ contains o ($o \ne p$, because $p \in X_1 \setminus X_1 \cap X_2$). Now assume $T_pX \cap X_2 = \emptyset$. In this case, the linear projection from T_pX induces a morphism $u : X_2 \to \mathbb{P}^2$ of degree $\deg(X_2)$. We have $\deg(X_2) > 1$, because X_2 spans $\mathbb{P}^5$. Since $\mathbb{P}^2$ is algebraically simply connected, the purity of the branch locus shows that u ramifies over a curve, i.e., there is a 1-dimensional family of $a \in X_2$ such that $T_pX \cap T_aX \ne \emptyset$.*

7. Methods

There are no experimental data and no part of a proof is completed numerically. All results are given with full proofs. Computers are not used for algebraic manipulations or computer graphics; however, a look at the references in [5] shows that they may be very useful for both branches discussed at the beginning of the introduction.

8. Conclusions

We gave a few related flexible definitions of Terracini loci and proved their power to prove that some Terracini loci are not empty. We consider stronger properties, which elements of Terracini may have: minimality, containing a prescribed point or omitting a prescribed bad set. Our main results are for the 1-dimensional case, but the definitions are general. We gave several examples in higher dimensions. We briefly mentioned at the end of Section 4 the case of real solutions and partially real solutions [5], which we believe may be greatly expanded by the readers.

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