AppliedMath

Article

Radial Based Approximations for Arcsine, Arccosine, Arctangent and Applications

Roy M. Howard

School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, Perth 6845, Australia; r.howard@curtin.edu.au

Abstract: Based on the geometry of a radial function, a sequence of approximations for arcsine, arccosine and arctangent are detailed. The approximations for arcsine and arccosine are sharp at the points zero and one. Convergence of the approximations is proved and the convergence is significantly better than Taylor series approximations for arguments approaching one. The established approximations can be utilized as the basis for Newton-Raphson iteration and analytical approximations, of modest complexity, and with relative error bounds of the order of 10^{-16}, and lower, can be defined. Applications of the approximations include: first, upper and lower bounded functions, of arbitrary accuracy, for arcsine, arccosine and arctangent. Second, approximations with significantly higher accuracy based on the upper or lower bounded approximations. Third, approximations for the square of arcsine with better convergence than well established series for this function. Fourth, approximations to arccosine and arcsine, to even order powers, with relative errors that are significantly lower than published approximations. Fifth, approximations for the inverse tangent integral function and several unknown integrals.

Keywords: arcsine; arccosine; arctangent; two point spline approximation; upper and lower bounded functions; Newton-Raphson

MSC: 26A09; 26A18; 26D05; 41A15

Citation: Howard, R.M. Radial Based Approximations for Arcsine, Arccosine, Arctangent and Applications. *AppliedMath* **2023**, *3*, 343–394. https://doi.org/10.3390/appliedmath3020019

Received: 30 September 2022
Revised: 12 December 2022
Accepted: 15 December 2022
Published: 4 April 2023

1. Introduction

The elementary trigonometric functions are fundamental to many areas of mathematics with, for example, Fourier theory being widely used and finding widespread applications. The formulation of trigonometric results was pre-dated by interest in the geometry of triangles and this occurs well in antiquity, e.g., [1]. The fundamental functions of sine and cosine have a geometric basis and are naturally associated with an angle from the positive horizontal axis to a point on the unit circle. From angle addition and difference identities for sine and cosine, the derivatives of these functions can be defined and, subsequently, Taylor series approximations for sine and cosine can be established. Such approximations have reasonable convergence with a ninth order expansion having a relative error bound of 3.54×10^{-6} for the interval $[0, \pi/2]$. Naturally, many other approximations have been developed, e.g., [2–4].

The inverse trigonometric functions of arcsine, arccosine and arctangent are naturally of interest and find widespread use for both the general complex case and the real case. The arctangent function, for example, is found in the solution of the sine-Gordon partial differential equation for the case of soliton wave propagation, e.g., [5]. In statistical analysis the arcsine distribution is widely used and the arctangent function is the basis of a wide class of distributions, e.g., [6]. The graphs of sine, cosine, arcsine and arccosine are shown in Figure 1.

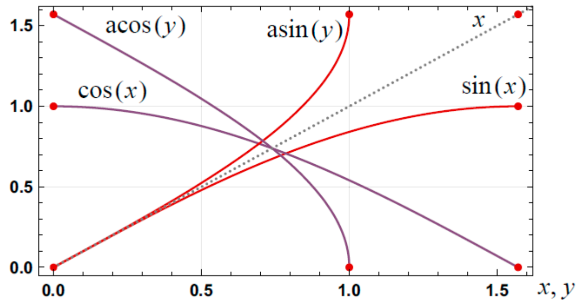

Figure 1. Graph of $y = f(x) = \sin(x)$, $x = f^{-1}(y) = \text{asin}(y)$, $y = g(x) = cos(x)$ and $x = g^{-1}(y) = \text{acos}(y)$ for $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 1$. Arcsine and arccosine are, respectively, written as asin and acos.

Taylor series expansions for arcsine and arccosine, unlike those for sine and cosine, have relatively poor convergence properties over the interval $[0,1]$ and a potential problem with respect to finding approximations is that both arcsine and arccosine have undefined derivatives at the point one. An overview of established approximations for arcsine and arctangent is provided in Section 2. In this paper, a geometric approach based on a radial function, whose derivatives are well defined at the point one, is used to establish new approximations for arccosine, arcsine and arctangent. The approximations for arccosine and arcsine are sharp (zero relative error) at the points zero and one and have a defined relative error bound over the interval $[0,1]$. Convergence of the approximations is proved and the convergence is significantly better, for arguments approaching one, than Taylor series approximations. The established approximations can be utilized as the basis for Newton-Raphson iteration and analytical approximations, of modest complexity, and with relative error bounds of the order of 10^{-16}, and lower, can be defined.

Applications for the established approximations are detailed and these include: First, approximations for arcsine, arccosine and arctangent to achieve a set relative error bound. Second, upper and lower bounded approximations, of arbitrary accuracy, for arcsine, arccosine and arctangent. Third, approximations to arccosine and arcsine, of even order powers, which have significantly lower relative error bounds than published approximations. Fourth, approximations for the inverse tangent integral function with significantly lower relative error bounds, over the interval $[0,\infty)$, than established Taylor series based approximations. Fifth, examples of approximations for unknown integrals.

1.1. Fundamental Relationships

For the real case the following relationships hold:

$$\begin{array}{ll} asin(-y) = -asin(y), & acos(-y) = \pi - acos(y), \;\; y \in [0,1] \\ atan(-y) = -atan(y), & y \in [0,\infty) \end{array} \tag{1}$$

Thus, it is sufficient to detail approximations over the interval $[0,1]$ for arcsine and arccosine and approximations over the positive real line for arctangent.

Fundamental relationships for arcsine, arccosine and arctangent, e.g., [7] (1.623, 1.624, p. 57) are:

$$\begin{aligned} asin(y) &= \frac{\pi}{2} - acos(y), \quad asin(y) = acos\left[\sqrt{1-y^2}\right], \\ acos(y) &= \frac{\pi}{2} - asin(y), \quad acos(y) = asin\left[\sqrt{1-y^2}\right], \quad 0 \leq y \leq 1, \end{aligned} \tag{2}$$

$$asin(y) = atan\left[\frac{y}{\sqrt{1-y^2}}\right], \; acos(y) = atan\left[\frac{\sqrt{1-y^2}}{y}\right], \; 0 \leq y \leq 1 \tag{3}$$

$$\begin{aligned} atan(y) &= asin\left[\frac{y}{\sqrt{1+y^2}}\right] = \frac{\pi}{2} - acos\left[\frac{y}{\sqrt{1+y^2}}\right] \\ atan(y) &= acos\left[\frac{1}{\sqrt{1+y^2}}\right] = \frac{\pi}{2} - asin\left[\frac{1}{\sqrt{1+y^2}}\right], \quad 0 \leq y \leq \infty. \end{aligned} \tag{4}$$

These relationships imply, for example, that approximations for arcsine and arctangent follow from an approximation to arccosine and approximations for arcsine and arccosine follow from an approximation to arctangent.

1.2. Notation

For an arbitrary function f, defined over the interval $[\alpha, \beta]$, an approximating function f_A has a relative error, at a point x_1, defined according to $re(x_1) = 1 - f_A(x_1)/f(x_1)$. The relative error bound for the approximating function, over the interval $[\alpha, \beta]$, is defined according to

$$\mathrm{re}_B = max\{|\mathrm{re}(x_1)| : x_1 \in [\alpha, \beta]\} \tag{5}$$

The notation $f^{(k)}$ is used for the kth derivative of a function. In equations, arcsine, arccosine and arctangent are abbreviated, respectively, as asin, acos and atan.

Mathematica has been used to facilitate analysis and to obtain numerical results. In general, the relative error results associated with approximations to arcsine, arccosine and arctangent have been obtained by sampling specified intervals, in either a linear or logarithmic manner, as appropriate, with 1000 points.

1.3. Paper Structure

A review of published approximations for arcsine and arctangent is provided in Section 2. In Section 3, the geometry, and analysis, of the radial function that underpins the proposed approximations for arccosine, arcsine and arctangent, is detailed. In Section 4, convergence of the approximations is detailed. In Section 5, the antisymmetric nature of the arctangent function is utilized to establish spline based approximations for this function. In Section 6, iteration, based on the proposed approximations, is utilized to detail approximations with quadratic convergence. Applications of the proposed approximations are detailed in Section 7 and conclusions are stated in Section 8.

2. Published Approximations for Arcsine and Arctangent

The Taylor series expansions for arcsine and arctangent, respectively, are, e.g., [8] (eqns. 4.24.1, 4.24.3, 4.24.4, p. 121)

$$\begin{aligned} \mathrm{asin}(y) &= y + \frac{y^3}{6} + \frac{3y^5}{40} + \frac{5y^7}{112} + \frac{35y^9}{1152} + \cdots = y + \sum_{k=1}^{\infty} \frac{\left[\prod_{i=0}^{k-1} 2i+1\right] y^{2k+1}}{(2k+1)\prod_{i=1}^{k} 2i} \\ &= \sum_{k=0}^{\infty} \frac{(2k)!y^{2k+1}}{2^{2k}(2k+1)(k!)^2}, \quad 0 \leq y < 1 \end{aligned} \tag{6}$$

$$\mathrm{atan}(y) = \begin{cases} y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} + \ldots + \frac{(-1)^k y^{2k+1}}{2k+1} + \ldots, & 0 \leq y < 1 \\ \frac{\pi}{2} - \frac{1}{y} + \frac{1}{3y^3} - \frac{1}{5y^5} + \frac{1}{7y^7} - \frac{1}{9y^9} + \ldots + \frac{(-1)^{k+1}}{(2k+1)y^{(2k+1)}} + \ldots, & y \geq 1 \end{cases} \tag{7}$$

For a set order, the relative error in a Taylor series approximation for arcsine increases sharply as $y \to 1$ (see Figure 2).

2.1. Approximation Form for Arcsine

The nature of arcsine is such that it has a rate of change of 1 at the origin and an infinite rate of change at the point one which complicates finding suitable approximations. An approximation form that has potential is $1-\sqrt{1-y}$, whose rate of change is $1/2\sqrt{1-y}$, with the rate of change being $1/2$ at the origin. As a starting point, consider the approximation form

$$s(y)=\alpha_0\left[1-\sqrt{1-y}\right]+\alpha_1 y+\alpha_2 y^2 \tag{8}$$

The three coefficients can be chosen to satisfy the constraints consistent with a sharp approximation at the points zero and one: $s(0)=0$, $s(1)=\pi/2$, $s^{(1)}(0)=1$ and $s^{(1)}(1)=\infty$. The constraints imply $\alpha_1=1-\alpha_0/2$, $\alpha_2=\pi/2-\alpha_0/2-1$, with α_0 being arbitrary. For the case of $\alpha_0=\pi/2$, the approximation is

$$s_1(y)=\frac{\pi}{2}\cdot\left[1-\sqrt{1-y}\right]+\left[1-\frac{\pi}{4}\right]y-\left[1-\frac{\pi}{4}\right]y^2 \tag{9}$$

which has a relative error bound, for the interval $[0,1]$, of 2.66×10^{-2}.

2.1.1. Optimized Coefficients

The coefficient α_0 can be optimized consistent with minimizing the relative error bound over the interval $[0,1]$. The optimum coefficient of $\alpha_0=\pi/2-1306/10,000$ leads to the approximation

$$\begin{aligned}&s_2(y)=\alpha_0\left[1-\sqrt{1-y}\right]+\alpha_1 y+\alpha_2 y^2,\\&\alpha_0=\frac{\pi}{2}-\frac{1306}{10,000},\ \alpha_1=\frac{10,653}{10,000}-\frac{\pi}{4},\ \alpha_2=\frac{\pi}{4}-\frac{9347}{10,000},\end{aligned} \tag{10}$$

which has a relative error bound, for the interval [0, 1], of 3.62×10^{-3}.

2.1.2. Padè Approximants

Given a suitable approximation form, Padè approximants can be utilized to find approximations with lower relative error bounds. For example, the form $\pi/2-\sqrt{1-y^2}\cdot p_{n,m}(y)$, where $p_{n,m}$ is an approximant of order n,m, can be utilized.

2.2. Published Approximations

The arcsine case is considered as related approximations for arccosine and arctangent follow from Equations (2) and (4). The following approximations are indicative of published approximations. First, the approximation

$$s_3(y)=\frac{\pi y}{2\left[y+\sqrt{1-y^2}\right]},\ y\in[0,1] \tag{11}$$

arises from the simple approximation for arctangent, e.g., [9] (eqn. 5), of

$$atan(y)\approx\frac{\pi y}{2(1+y)},\ y\in[0,\infty) \tag{12}$$

The maximum error in this approximation has a magnitude of 0.0711, but the relative error bound is 0.571, which occurs as y approaches zero.

Second, a Taylor series expansion for $\frac{asin(y)}{\sqrt{1-y^2}}$, e.g., [10] (eqn. 4) or $\frac{yasin(y)}{\sqrt{1-y^2}}$, e.g. [11], can be used. The latter yields the nth order approximation:

$$\begin{aligned} s_{4,n}(y) &= \sqrt{1-y^2}\sum_{k=1}^{n}\frac{2^{2k-1}(k!)^2 y^{2k-1}}{k(2k)!} \\ &= \sqrt{1-y^2}\left[y+\frac{2y^3}{3}+\frac{8y^5}{15}+\frac{16y^7}{35}+\frac{128y^9}{315}+\cdots\right] \end{aligned} \tag{13}$$

Consistent with a Taylor series, the relative error is low for $|y| \ll 1$ but, for a set order, becomes increasingly large as $y \to 1$.

Third, the following approximations are stated in [12] (eqns. 1.5 and 3.7):

$$s_5(y) = \frac{\pi y}{2+\sqrt{1-y^2}},\ s_6(y) = \frac{80y\left[1+\frac{5\sqrt{1-y^2}}{16}\right]}{57\left[1-\frac{2y^2}{19}+\frac{16\sqrt{1-y^2}}{19}\right]} \tag{14}$$

The first approximation is part of the Shafer-Fink inequality (e.g., [13]) is not sharp at the origin and has a relative error bound, for the interval $[0,1]$, of 4.72×10^{-2}. The second approximation is not sharp at $y = 1$ but has a relative error bound for the interval $[0,1]$, of 1.38×10^{-3}.

Fourth, the following approximation is detailed in [14] (eqn. 4.4.46, p. 81):

$$s_7(y) = \frac{\pi}{2} - \sqrt{1-y}\cdot\left[\alpha_0+\alpha_1 y+\alpha_2 y^2+\ldots+\alpha_7 y^7\right] \tag{15}$$

where

$$\begin{aligned} &\alpha_0 = \frac{\pi}{2}, && \alpha_1 = -0.2145988016, && \alpha_2 = 0.0889789874, \\ &\alpha_3 = -0.0501743046, && \alpha_4 = 0.0308918810, && \alpha_5 = -0.0170881256, \\ &\alpha_6 = 0.0066700901, && \alpha_7 = -0.0012624911. \end{aligned} \tag{16}$$

The relative error bound is 3.04×10^{-6} which occurs at the origin.

Fifth, [15] (Section 6.4), provides a basis for determining approximations for arcsine, arccosine and arctangent of arbitrary accuracy. Explicit formulas and results are detailed in Appendix A. For example, the following approximation for arcsine (as defined by $c_{2,2}\left[\sqrt{1-y^2}\right]$—see Equation (A13)) is

$$\begin{aligned} s_8(y) = &\frac{121\sqrt{2-\sqrt{2}\sqrt{1+\sqrt{1-y^2}}}}{120}\cdot\left[1-\frac{\sqrt{1+\sqrt{1-y^2}}}{121\sqrt{2}}\right] + \\ &\frac{\sqrt{2-\sqrt{2}\sqrt{1+\sqrt{1-y^2}}}}{\left[2+\sqrt{2}\sqrt{1+\sqrt{1-y^2}}\right]^{5/2}}\cdot\left[\frac{178}{15}+\frac{74\sqrt{1-y^2}}{15}+\frac{38\sqrt{2}\sqrt{1+\sqrt{1-y^2}}}{5}\right] \end{aligned} \tag{17}$$

and has a relative error bound of 1.71×10^{-5} that occurs at $y = 1$.

Comparison of Approximations

The graphs of the relative errors associated with the above approximations are shown in Figure 2.

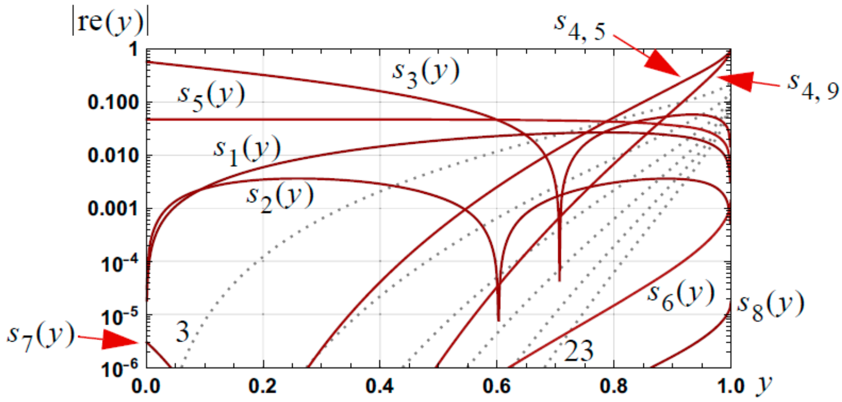

Figure 2. Graphs of the magnitude of the relative error in approximations to arcsine as defined in the text. Taylor series approximations, of orders 3, 7, 11, 15, 19, 23, are shown dotted.

3. Radial Based Two Point Spline Approximation for Arccosine Squared

Consider the geometry, as illustrated in Figure 3, associated with arcsine and arccosine and which underpins the four radial functions defined according to

$$r^2(y) = y^2 + \left[\frac{\pi}{2} - asin(y)\right]^2 = y^2 + acos(y)^2, \tag{18}$$

$$r_1^2(y) = (1-y)^2 + \left[\frac{\pi}{2} - asin(y)\right]^2 = (1-y)^2 + acos(y)^2, \tag{19}$$

$$r_2^2(y) = y^2 + asin(y)^2,\ r_3^2(y) = (1-y)^2 + asin(y)^2,\ y \in [0,1]. \tag{20}$$

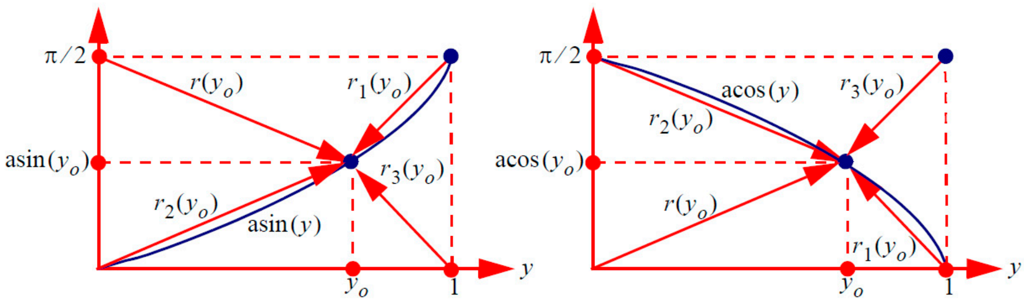

Figure 3. Illustration of four radial functions associated with arcsine and arccosine.

The graphs of these functions are shown in Figure 4. The functions r_2^2 and r_3^2 have undefined derivatives at the point $y = 1$, which does not facilitate function approximation. The function r^2 is smoother than r_1^2 and can be utilized as a basis for approximation. If there exists an nth order approximation, f_n, to r^2, then the relationships $acos(y) \approx \sqrt{f_n(y) - y^2}$, $asin(y) \approx \frac{\pi}{2} - acos(y)$ and $atan(y) = acos\left[1/\sqrt{1+y^2}\right]$ can be utilized to establish approximations for arccosine, arcsine and arctangent.

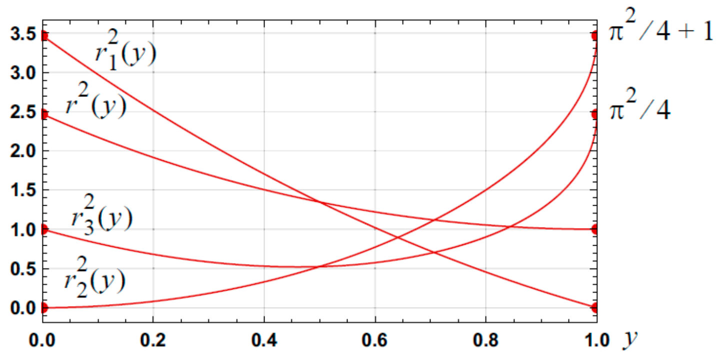

Figure 4. Graph of $r^2(y)$, $r_1^2(y)$, $r_2^2(y)$ and $r_3^2(y)$.

3.1. Approximations for Radial Function

The two point spline approximation detailed in [15] (eqn. 40), and the alternative form given in [16] (eqn. 70) can be utilized to establish convergent approximations to the radial function r^2 defined by Equation (18).

Theorem 1. *Two Point Spline Approximations for Radial Function.*

The nth order two point spline approximation to the radial function r^2, based on the points zero and one, is

$$f_n(y) = \sum_{k=0}^{2n+1} C_{n,k}y^k,\ n \in \{0,1,2,\ldots\} \tag{21}$$

where the coefficients $C_{n,k}$ are defined according to:

$$C_{n,k} = \begin{cases} \sum\limits_{r=0}^{k} \dfrac{(-1)^{k-r}(n+1)!}{(n+1+r-k)!(k-r)!}\cdot a_{n,r} = \dfrac{f^{(k)}(0)}{k!}, & 0 \le k \le n \\ \sum\limits_{r=k-n-1}^{n} \dfrac{(-1)^{k-r}(n+1)!}{(n+1+r-k)!(k-r)!}\cdot a_{n,r} + & \\ \sum\limits_{r=k-n-1}^{n} \dfrac{(-1)^{k-n-1}r!}{(r+n+1-k)!(k-n-1)!}\cdot b_{n,r}, & n+1 \le k \le 2n+1 \end{cases} \tag{22}$$

Here $f(y) = r^2(y)$ and

$$a_{n,r} = \sum_{u=0}^{r} \frac{f^{(r-u)}(0)}{(r-u)!}\cdot\frac{(n+u)!}{u!n!},\ b_{n,r} = \sum_{u=0}^{r} \frac{(-1)^{r-u}f^{(r-u)}(1)}{(r-u)!}\cdot\frac{(n+u)!}{u!n!} \tag{23}$$

$r \in \{0,1,\ldots,n\}$. The derivative values of f, at the points zero and one, are defined according to

$$\begin{aligned} &f(0) = \pi^2/4,\ f^{(1)}(0) = f^{(3)}(0) = -\pi,\ f^{(2)}(0) = 4,\ f^{(4)}(0) = 8, \\ &f^{(k)}(0) = (k-2)^2 f^{(k-2)}(0),\ k \in \{5,6,7,\ldots\} \end{aligned} \tag{24}$$

$$\begin{aligned} &f(1) = 1,\ f^{(1)}(1) = 0,\quad f^{(2)}(1) = 8/3,\quad f^{(3)}(1) = -8/15, \\ &f^{(k)}(1) = \frac{(-1)^k(k-1)^2}{2k-1}\left|f^{(k-1)}(1)\right|,\ k \in \{4,5,6,\ldots\}. \end{aligned} \tag{25}$$

Proof. The proofs for these results are detailed in Appendix F. □

3.1.1. Notes on Coefficients

Explicit expressions for the coefficients $C_{n,k}$, $n \in \{0,1,\ldots,6\}$, $k \in \{0,1,\ldots,2n+1\}$, are tabulated in Table A1 (Appendix B).

As $C_{n,k} = f^{(k)}(0)/k!$, $k \in \{0,1,\ldots,n\}$, and $f^{(k)}(0) = (k-2)^2 f^{(k-2)}(0)$, $k \in \{5,6,\ldots\}$, it follows that

$$C_{n,k} = \frac{f^{(k)}(0)}{k!} = \frac{(k-2)^2}{k(k-1)} \cdot \frac{f^{(k-2)}(0)}{(k-2)!} = \frac{(1-2/k)^2}{1-1/k} \cdot C_{n,k-2}, \quad \begin{cases} n \in \{5,6,\ldots\} \\ k \in \{5,6,\ldots,n\} \end{cases} \tag{26}$$

As $C_{n,4} = f^{(4)}(0)/4! = 1/3$, $C_{n,2} = f^{(2)}(0)/2 = 2$, $C_{n,0} = \pi^2/4$, $C_{n,3} = f^{(3)}(0)/3! = -\pi/6$, and $C_{n,1} = f^{(1)}(0) = -\pi$, it is the case that $|C_{n,k}| < |C_{n,k-2}|$ for $k \in \{2,3,\ldots,n\}$, $n \geq 2$. Hence, for n fixed, $n \geq 3$, the magnitudes of both even and odd order coefficients monotonically decrease as k increases and for $k \in \{3,4,\ldots,n\}$.

3.1.2. Explicit Approximations

Explicit approximations for r^2, of orders zero and one, are:

$$f_0(y) = \frac{\pi^2}{4} + y\left[1 - \frac{\pi^2}{4}\right] \tag{27}$$

$$f_1(y) = \frac{\pi^2}{4} - \pi y + \left[3 + 2\pi - \frac{3\pi^2}{4}\right]y^2 + \left[-2 - \pi + \frac{\pi^2}{2}\right]y^3 \tag{28}$$

Higher order approximations, up to order six, are detailed in Appendix B along with the relevant coefficients $C_{n,k}$, $k \in \{0,1,\ldots,2n+1\}$ (see Table A1).

3.1.3. Approximations for Arccosine, Arcsine and Arctangent

With the definition of

$$c_{n,k} = \begin{cases} -1, & n = 0, k = 2 \\ C_{n,k} - 1, & k = 2, n \in \{1,2,\ldots\} \\ C_{n,k}, & k \in \{0,1,3,\ldots,2n+1\},\ n \in \{0,1,\ldots\} \end{cases} \tag{29}$$

the approximations, as stated in Corollary 1, follow.

Corollary 1. *Approximations for Arccosine, Arcsine and Arctangent.*

The approximations for arccosine, arcsine and arctangent arising from the approximations specified in Theorem 1 are:

$$acos(y) \approx c_n(y) = \sqrt{\sum_{k=0}^{2n+1} c_{n,k} y^k},\ \mathrm{acos}(y) \approx c_n^A(y) = \frac{\pi}{2} - \sqrt{\sum_{k=0}^{2n+1} c_{n,k}(1-y^2)^{k/2}}, \tag{30}$$

$$asin(y) \approx s_n(y) = \frac{\pi}{2} - \sqrt{\sum_{k=0}^{2n+1} c_{n,k} y^k},\ \mathrm{asin}(y) \approx s_n^A(y) = \sqrt{\sum_{k=0}^{2n+1} c_{n,k}(1-y^2)^{k/2}}, \tag{31}$$

$$atan(y) \approx t_n(y) = \sqrt{\sum_{k=0}^{2n+1} \frac{c_{n,k}}{(1+y^2)^{k/2}}},\ \mathrm{atan}(y) \approx t_n^A(y) = \frac{\pi}{2} - \sqrt{\sum_{k=0}^{2n+1} \frac{c_{n,k} y^k}{(1+y^2)^{k/2}}}, \tag{32}$$

for $n \in \{1,2,3,\ldots\}$. The superscript A denotes alternative approximation forms. For the case of n= 0, the upper limit of the summations is 2 rather than 1.

Proof. These results follow directly from the definition $acos(y) = \sqrt{r^2(y) - y^2}$ (Equation (18)), and the approximations $f_n(y) = r^2(y)$ detailed in Theorem 1, leading to

$$acos(y) \approx \sqrt{f_n(y) - y^2} = \sqrt{\sum_{k=0}^{2n+1} C_{n,k} y^k - y^2} = \sqrt{\sum_{k=0}^{max\{2,2n+1\}} c_{n,k} y^k} \quad (33)$$

The approximations for the other results arise from the fundamental relationships detailed in Equations (2)–(4), and according to

$$\begin{aligned} s_n(y) &= \frac{\pi}{2} - c_n(y), \qquad t_n(y) = c_n\left[\frac{1}{\sqrt{1+y^2}}\right], \\ s_n^A(y) &= c_n\left[\sqrt{1-y^2}\right], c_n^A(y) = s_n\left[\sqrt{1-y^2}\right], t_n^A(y) = s_n\left[\frac{y}{\sqrt{1+y^2}}\right]. \end{aligned} \quad (34)$$

□

3.1.4. Explicit Approximations for Arccosine, Arcsine and Arctangent

Explicit approximations for arccosine, of orders zero, one and two, are:

$$c_0(y) = \sqrt{\frac{\pi^2}{4} + y\left[1 - \frac{\pi^2}{4}\right] - y^2} \quad (35)$$

$$c_1(y) = \sqrt{\frac{\pi^2}{4} - \pi y + c_{1,2}y^2 + c_{1,3}y^3} \quad (36)$$

$$c_2(y) = \sqrt{\frac{\pi^2}{4} - \pi y + y^2 + c_{2,3}y^3 + c_{2,4}y^4 + c_{2,5}y^5} \quad (37)$$

Approximations, of orders three to six, are detailed in Appendix C. Explicit approximations for arcsine, of orders zero to six, can then be specified by utilizing the relationships $s_i(y) = \pi/2 - c_i(y)$ and $s_i^A(y) = c_i\left[\sqrt{1-y^2}\right]$, $i \in \{0, 1, \ldots, 6\}$. Explicit approximations for arctangent follow from the relationships $t_i(y) = c_i\left[1/\sqrt{1+y^2}\right]$ and $t_n^A(y) = s_i\left[y/\sqrt{1+y^2}\right]$, $i \in \{0, 1, \ldots, 6\}$. For example, the second order approximation for arctangent is

$$t_2(y) = \sqrt{\frac{\pi^2}{4} - \frac{\pi}{\sqrt{1+y^2}} + \frac{1}{1+y^2} + \frac{c_{2,3}}{[1+y^2]^{3/2}} + \frac{c_{2,4}}{[1+y^2]^2} + \frac{c_{2,5}}{[1+y^2]^{5/2}}} \quad (38)$$

3.1.5. Relative Error Bounds for Arcsine, Arccosine and Arctangent

The relative error bounds for the approximations to r^2, arcsine, arccosine and arctangent, arising from the approximations stated in Theorem 1 and Corollary 1 are detailed in Table 1. The relative errors in the approximations, of orders one to five, for arcsine, arccosine and arctangent are shown in Figures 5 and 6. For example, the relative error bound associated with the fourth, $s_4(y)$, and sixth, $s_6(y)$, order approximations to arcsine, respectively, are 2.49×10^{-6} and 2.28×10^{-8}.

Table 1. Relative error bounds for approximations to r^2, arcsine, arccosine and arctangent. The interval [0, 1] is assumed for r^2, arcsine and arccosine whilst the interval [0, ∞) is assumed for arctangent.

Order of Approx.	Relative Error Bound: r^2	Relative Error Bound: $s_n(y), c_n^A(y), t_n^A(y)$	Relative Error Bound: $s_n^A(y), c_n(y), t_n(y)$
0	3.01×10^{-1}	5.33×10^{-1}	3.17×10^{-1}
1	4.22×10^{-3}	5.79×10^{-3}	2.92×10^{-3}
2	2.77×10^{-4}	3.64×10^{-4}	1.81×10^{-4}
3	2.20×10^{-5}	2.84×10^{-5}	1.42×10^{-5}
4	1.95×10^{-6}	2.49×10^{-6}	1.24×10^{-6}
5	1.84×10^{-7}	2.33×10^{-7}	1.16×10^{-7}
6	1.81×10^{-8}	2.28×10^{-8}	1.14×10^{-8}
8	1.92×10^{-10}	2.41×10^{-10}	1.20×10^{-10}
10	2.21×10^{-12}	2.76×10^{-12}	1.38×10^{-12}
12	2.68×10^{-14}	3.34×10^{-14}	1.66×10^{-14}
16	4.35×10^{-18}	5.41×10^{-18}	2.70×10^{-18}

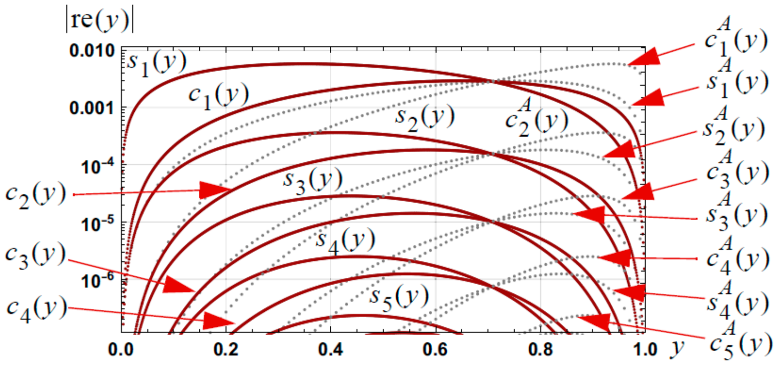

Figure 5. Graph of the relative error in approximations, of orders 1 to 5, for arcsine and arccosine. The dotted curves are for the approximations $s_n^A(y)$ and $c_n^A(y)$, $n \in \{1, 2, \ldots, 5\}$.

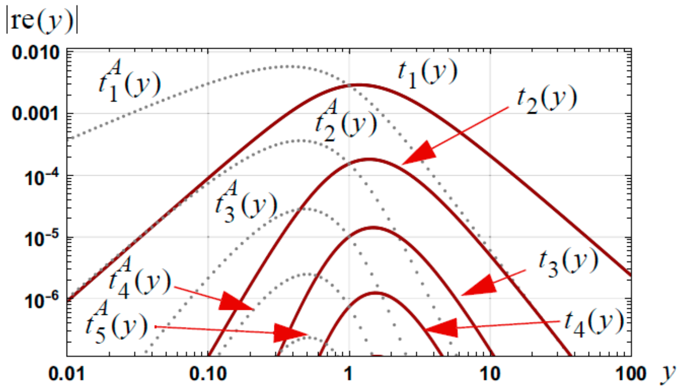

Figure 6. Graph of the relative error in approximations, of orders 1 to 5, for arctangent.

3.2. Alternative Approximations I: Differentiation of Arccosine Squared

Based on differentiation of the square of arccosine, alternative approximations for arccosine, arcsine and arctangent can be determined.

Theorem 2. *Alternative Approximations I: Differentiation of Arccosine Squared.*

Alternative approximations, of order n, $n \in \{1, 2, \ldots\}$, for arcsine and arccosine, over the interval [0, 1], and arctangent, over the interval [0, ∞), are:

$$\text{asin(y)} \approx s_n(y) = \frac{\pi}{2} - \sqrt{1-y^2}\sum_{k=0}^{2n} d_{n,k}y^k, \ \text{asin(y)} \approx s_n^A(y) = y\sum_{k=0}^{2n} d_{n,k}\left(1-y^2\right)^{k/2}, \tag{39}$$

$$\text{acos(y)} \approx c_n(y) = \sqrt{1-y^2}\sum_{k=0}^{2n} d_{n,k}y^k, \ \text{acos(y)} \approx c_n^A(y) = \frac{\pi}{2} - y\sum_{k=0}^{2n} d_{n,k}\left(1-y^2\right)^{k/2}, \tag{40}$$

$$\text{atan(y)} \approx t_n(y) = \frac{y}{\sqrt{1+y^2}}\sum_{k=0}^{2n} \frac{d_{n,k}}{(1+y^2)^{k/2}}, \ \text{atan(y)} \approx t_n^A(y) = \frac{\pi}{2} - \frac{1}{\sqrt{1+y^2}}\sum_{k=0}^{2n} \frac{d_{n,k}y^k}{(1+y^2)^{k/2}}, \tag{41}$$

where $d_{n,k} = \dfrac{-(k+1)c_{n,k+1}}{2}$, $k \in \{0,1,\ldots,2n\}$, *with* $c_{n,k}$ *being defined by Equation (29).*

Proof. Consider the nth order approximation for arccosine, as defined in Corollary 1: $acos(y) \approx \sqrt{p_n(y)}$, where $p_n(y) = \sum_{k=0}^{2n+1} c_{n,k}y^k$, $n \in \{1,2,\ldots\}$. Assuming convergence, it follows that $acos(y)^2 = p_\infty(y)$. Differentiation yields

$$\frac{-2acos(y)}{\sqrt{1-y^2}} = p_\infty^{(1)}(y), \ y \in [0,1), \tag{42}$$

which implies

$$\begin{aligned} acos(y) = \frac{-\sqrt{1-y^2}}{2} \cdot p_\infty^{(1)}(y) \approx c_n(y) \quad &= -\sqrt{1-y^2} \cdot \sum_{i=1}^{2n+1} \frac{ic_{n,i}}{2} \cdot y^{i-1} \\ &= \sqrt{1-y^2}\sum_{k=0}^{2n} d_{n,k}y^k \end{aligned} \tag{43}$$

after the index change of $k = i-1$ and where $d_{n,k} = -(k+1)c_{n,k+1}/2$. The approximation, defined by s_n, for arcsine follows from the relationship $asin(y) = \pi/2 - acos(y)$; the approximation for arctangent, defined by t_n, follows according to

$$atan(y) = acos\left[\frac{1}{\sqrt{1+y^2}}\right] \approx t_n(y) = \frac{y}{\sqrt{1+y^2}} \cdot \sum_{k=0}^{2n} \frac{d_{n,k}}{(1+y^2)^{k/2}} \tag{44}$$

The alternative approximations follow according to

$$s_n^A(y) = c_n\left[\sqrt{1-y^2}\right], \ c_n^A(y) = s_n\left[\sqrt{1-y^2}\right], \ t_n(y) = s_n\left[\frac{y}{\sqrt{1+y^2}}\right]. \tag{45}$$

□

3.2.1. Note

The same approximations can be derived by considering the relationship $\frac{d}{dy}atan(y)^2 = 2atan(y)/\left(1+y^2\right)$ which implies

$$atan(y) = \frac{1+y^2}{2} \cdot \frac{d}{dy}atan(y)^2 \tag{46}$$

Use of the arctangent approximation, $t_n(y)$, specified in Corollary 1 leads to the approximation

$$atan(y) \approx \frac{1+y^2}{2} \cdot \sum_{i=1}^{2n+1} \frac{-ic_{n,i}y}{(1+y^2)^{\frac{i}{2}+1}} = y \sum_{k=0}^{2\pi} \frac{d_{n,k}}{(1+y^2)^{(k+1)/2}} \qquad d_{n,k} = \frac{-(k+1)c_{n,k+1}}{2} \tag{47}$$

after the change of index $k = i - 1$. This result is consistent with $t_n(y)$ stated in Theorem 2.

3.2.2. Explicit Approximations for Arcsine and Arctangent

Approximations for arcsine, of orders one and two, are

$$\begin{aligned} s_1(y) &= \frac{\pi}{2} - \sqrt{1-y^2} \cdot \left[\frac{\pi}{2} + d_{1,1}y + d_{1,2}y^2\right] \\ s_1^A(y) &= y\left[\frac{\pi}{2} + d_{1,1}\sqrt{1-y^2} + d_{1,2}(1-y^2)\right] \\ & d_{1,1} = -2 - 2\pi + \frac{3\pi^2}{4},\ d_{1,2} = 3 + \frac{3\pi}{2} - \frac{3\pi^2}{4} \end{aligned} \tag{48}$$

$$\begin{aligned} s_2(y) &= \frac{\pi}{2} - \sqrt{1-y^2} \cdot \left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right] \\ s_2^A(y) &= y\left[\frac{\pi}{2} - \sqrt{1-y^2} + d_{2,2}(1-y^2) + d_{2,3}(1-y^2)^{\frac{3}{2}} + d_{2,4}(1-y^2)^2\right] \\ d_{2,2} &= -8 - 9\pi + \frac{15\pi^2}{4}, \quad d_{2,3} = \frac{70}{3} + 16\pi - \frac{15\pi^2}{2}, \quad d_{2,4} = \frac{-40}{3} - \frac{15\pi}{2} + \frac{15\pi^2}{4} \end{aligned} \tag{49}$$

Approximations, of orders three and four, are detailed in Appendix D. As an example, the approximations for arctangent, of order two, are:

$$\begin{aligned} t_2(y) &= \frac{y}{\sqrt{1+y^2}} \cdot \left[\frac{\pi}{2} - \frac{1}{\sqrt{1+y^2}} + \frac{d_{2,2}}{1+y^2} + \frac{d_{2,3}}{(1+y^2)^{3/2}} + \frac{d_{2,4}}{(1+y^2)^2}\right] \\ t_2^A(y) &= \frac{\pi}{2} - \frac{1}{\sqrt{1+y^2}} \cdot \left[\frac{\pi}{2} - \frac{y}{\sqrt{1+y^2}} + \frac{d_{2,2}y^2}{1+y^2} + \frac{d_{2,3}y^3}{(1+y^2)^{3/2}} + \frac{d_{2,4}y^4}{(1+y^2)^2}\right] \end{aligned} \tag{50}$$

3.2.3. Results

The relative error bounds associated with the approximations to arcsine, arccosine and arctangent, as specified by Theorem 2, are detailed in Table 2. The relative errors for arcsine, arccosine and arctangent are shown, respectively, in Figures 7–9.

Table 2. Relative error bounds, over the interval [0, 1] (arcsine and arccosine) and $[0, \infty)$ (arctangent), associated with the approximations to arcsine, arccosine and arctangent as defined in Theorem 2.

Order of Approx.	Relative Error Bound: s_n, c_n^A, t_n^A	Relative Error Bound: s_n^A, c_n, t_n
1	1.19×10^{-1}	7.51×10^{-3}
2	3.14×10^{-3}	5.54×10^{-4}
3	2.13×10^{-4}	4.89×10^{-5}
4	1.78×10^{-5}	4.72×10^{-6}
5	1.66×10^{-6}	4.80×10^{-7}
6	1.64×10^{-7}	5.05×10^{-8}
8	1.79×10^{-9}	5.99×10^{-10}
10	2.14×10^{-11}	7.54×10^{-12}
12	2.69×10^{-13}	9.85×10^{-14}
16	4.71×10^{-17}	1.81×10^{-17}

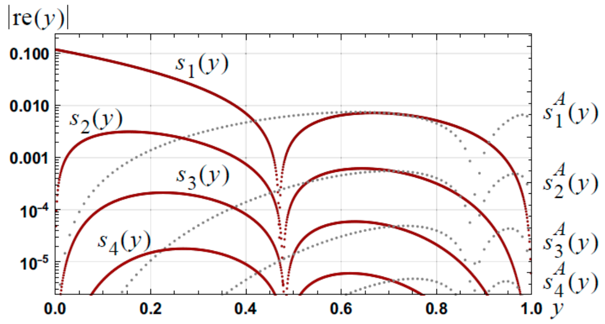

Figure 7. Graph of the relative errors in the approximations, as defined in Theorem 2, to arcsine.

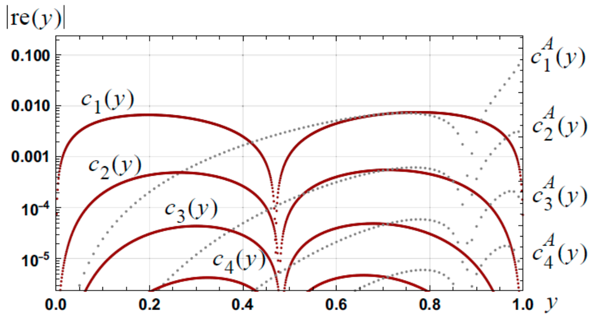

Figure 8. Graph of the relative errors in the approximations, as defined in Theorem 2, to arccosine.

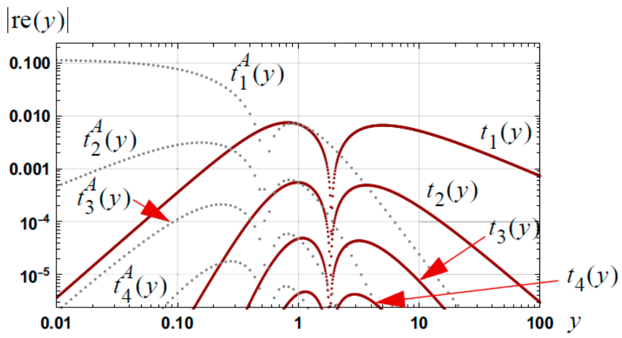

Figure 9. Graph of the relative errors in the approximations, as defined in Theorem 2, to arctangent.

3.2.4. Notes

The form of the approximation, as stated in Theorem 2, for arcsine:

$$s_n(y) = \frac{\pi}{2} - \sqrt{1-y^2} \sum_{k=0}^{2n} d_{n,k} y^k \tag{51}$$

is consistent with the optimum Padè approximant form specified by Abramowitz [14] and stated in Equation (15). The relative error bound for the Abramowitz approximation is 3.04×10^{-6}. The relative error bound for the 4th order approximation, s_4, as specified by Equation (A27), is 1.78×10^{-5} whilst a fifth order approximation, s_5, has a relative error bound of 1.66×10^{-6}.

A comparison of the results detailed in Tables 1 and 2 indicate that the approximations, as stated in Corollary 1, are more accurate than those specified in Theorem 2. For comparison, the fourth order approximations, s_4, for arcsine have the respective relative error bounds of 2.49×10^{-6} and 1.78×10^{-5}.

3.3. Alternative Approximations II: Integration of Arcsine

The integral of arcsine, e.g., [8] (4.26.14, p. 122), is:

$$\int_0^y \text{asin}(\lambda) d\lambda = y\text{asin}(y) + \sqrt{1-y^2} - 1, \quad |y| \le 1 \tag{52}$$

which implies

$$\text{asin}(y) = \frac{1}{y}\left[\int_0^y \text{asin}(\lambda) d\lambda + 1 - \sqrt{1-y^2}\right] \tag{53}$$

There is potential with this relationship, and based on approximations to arcsine that are integrable, to define new approximations to arcsine, with a lower relative error bound, than the approximations detailed in Corollary 1 and Theorem 2. The approximations to arcsine, as defined by s_n^A, in Theorem 2, are integrable and lead to the following approximations.

Theorem 3. *Alternative Approximations II—Integration of Arcsine.*

Alternative approximations, of order n, $n \in \{0, 1, 2, \cdots\}$, for arcsine, arccosine and arctangent, are:

$$\begin{aligned} \text{asin}(y) &\approx s_n(y) = \frac{1}{y}\left[1 - \sqrt{1-y^2} + \sum_{k=0}^{2n} d_{n,k} \cdot \frac{1 - \left(1-y^2\right)^{1+\frac{k}{2}}}{2+k}\right] \\ \text{asin}(y) &\approx s_n^A(y) = \frac{\pi}{2} - \frac{1}{\sqrt{1-y^2}}\left[1 - y + \sum_{k=0}^{2n} d_{n,k} \cdot \frac{1-y^{2+k}}{2+k}\right] \end{aligned} \tag{54}$$

$$\begin{aligned} \text{acos}(y) &\approx c_n(y) = \frac{\pi}{2} - \frac{1}{y}\left[1 - \sqrt{1-y^2} + \sum_{k=0}^{2n} d_{n,k} \cdot \frac{1 - \left(1-y^2\right)^{1+\frac{k}{2}}}{2+k}\right] \\ \text{acos}(y) &\approx c_n^A(y) = \frac{1}{\sqrt{1-y^2}}\left[1 - y + \sum_{k=0}^{2n} d_{n,k} \cdot \frac{1-y^{2+k}}{2+k}\right] \end{aligned} \tag{55}$$

$$\operatorname{atan}(y) \approx t_n(y) = \frac{\sqrt{1+y^2}}{y}\left[1 - \frac{1}{\sqrt{1+y^2}} + \sum_{k=0}^{2n} \frac{d_{n,k}}{2+k} \cdot \left[1 - \frac{1}{(1+y^2)^{1+\frac{k}{2}}}\right]\right]$$
$$\operatorname{atan}(y) \approx t_n^A(y) = \frac{\pi}{2} - \sqrt{1+y^2}\left[1 - \frac{y}{\sqrt{1+y^2}} + \sum_{k=0}^{2n} \frac{d_{n,k}}{2+k} \cdot \left[1 - \frac{y^{2+k}}{[1+y^2]^{1+k/2}}\right]\right] \tag{56}$$

where $d_{n,k} = \frac{-(k+1)c_{n,k+1}}{2}$ *with* $c_{n,k}$ *being defined by Equation (29).*

Proof. Consider the approximation for arcsine defined by s_n^A and stated in Theorem 2. Use of this approximation in Equation (53) leads to

$$\operatorname{asin}(y) \approx \frac{1}{y}[1 - \sqrt{1-y^2} + \sum_{k=0}^{2n} d_{n,k} \int_0^y \left[t\left(1-t^2\right)^{k/2}\right] dt \tag{57}$$

The result

$$\int_0^y t\left(1-t^2\right)^{k/2} dt = \frac{1 - \left(1-y^2\right)^{1+k/2}}{2+k} \tag{58}$$

leads to the approximation s_n defined in Equation (54). The alternative approximations follow according to $c_n(y) = \frac{\pi}{2} - s_n(y)$, $c_n^A(y) = s_n\left[\sqrt{1-y^2}\right]$, $t_n(y) = s_n\left[\frac{y}{\sqrt{1+y^2}}\right]$, $s_n^A(y) = c_n\left[\sqrt{1-y^2}\right]$ and $t_n^A(y) = c_n[1/\sqrt{1+y^2}$. □

3.3.1. Explicit Approximations for Arcsine

A second order approximations for arcsine is:

$$\begin{aligned} s_2(y) = \quad & \left[\frac{16}{3} + \frac{16\pi}{5} - \frac{3\pi^2}{2}\right] \cdot \frac{1}{y} \cdot \left[1 - \sqrt{1-y^2}\right] + \left[\frac{-32}{3} - \frac{8\pi}{5} + \frac{15\pi^2}{4}\right] y + \\ & \left[\frac{26}{3} + 6\pi - \frac{45\pi^2}{16}\right] y^3 + \left[\frac{-20}{9} - \frac{5\pi}{4} + \frac{5\pi^2}{8}\right] y^5 + \\ & \left[9 + \frac{32\pi}{5} - 3\pi^2\right] y\sqrt{1-y^2} + \left[\frac{-14}{3} - \frac{16\pi}{5} + \frac{3\pi^2}{2}\right] y^3\sqrt{1-y^2} \end{aligned} \tag{59}$$

and has a relative error bound of 1.56×10^{-4}. A fourth order approximation has a relative error bound of 1.00×10^{-6}.

3.3.2. Results

The relative error bounds associated with the approximations $s_n(y)$, $c_n^A(y)$ and $t_n(y)$ to arcsine, arccosine and arctangent, as specified by Theorem 3, are detailed in Table 3. The relative errors associated with $s_n^A(y)$, $c_n(y)$ and $t_n^A(y)$ become unbounded, respectively, at the points zero, one and zero. The graphs of the relative errors for $s_n(y)$ and $s_n^A(y)$ are shown in Figure 10.

Table 3. Relative error bounds associated with the approximations, specified in Theorem 3, for arcsine, arccosine (interval $[0, 1]$) and arctangent (interval $[0, \infty)$).

Order of Approx.	Relative Error Bound: $s_n(y)$, $c_n^A(y)$, $t_n(y)$
0	0.145
1	2.63×10^{-3}
2	1.56×10^{-4}
3	1.18×10^{-5}
4	1.00×10^{-6}
5	9.22×10^{-8}
6	8.91×10^{-9}
8	9.19×10^{-11}
10	1.03×10^{-12}
12	1.23×10^{-14}
16	1.95×10^{-18}

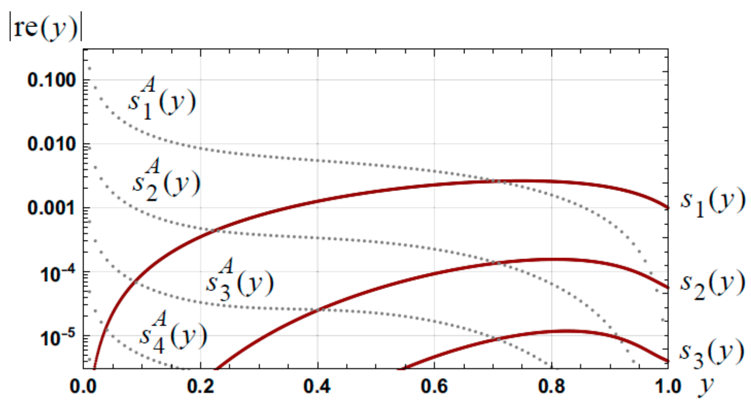

Figure 10. Graph of the relative errors in the approximations, as defined in Theorem 3, to arc-sin.

3.4. Alternative Approximations

Alternative approximations can be determined. For example, the relationship:

$$\int_0^y \operatorname{asin}(\lambda)^2 d\lambda = -2y + 2\sqrt{1-y^2}\operatorname{asin}(y) + y asin(y)^2 \tag{60}$$

leads to a quadratic equation for arcsine when an integrable approximation for $asin(y)^2$ is utilized. As a second example, the relationship

$$\int_0^y \sqrt{1-\lambda^2} d\lambda = \frac{y}{2}\sqrt{1-y^2} + \frac{\operatorname{asin}(y)}{2} \tag{61}$$

implies

$$\operatorname{asin}(y) = 2\int_0^y \sqrt{1-\lambda^2} d\lambda - y\sqrt{1-y^2} \tag{62}$$

and, thus, an approximation for arcsine can be determined when a suitable approximation for $\sqrt{1-y^2}$, which is integrable, is available.

4. Error and Convergence

Consider the definition of the square of the radial function r^2 as defined by Equation (18) and the error ε_n in the nth order approximation, f_n, to r^2, as defined in Theorem 1, i.e.,

$$r^2(y) = \operatorname{acos}(y)^2 + y^2 = f_n(y) + \varepsilon_n(y), \quad 0 \le y \le 1. \tag{63}$$

Consistent with the nature of a nth order two point spline approximation based on the points zero and one, it is the case that $\varepsilon_n^{(k)}(0) = \varepsilon_n^{(k)}(1) = 0,\ k \in \{0, 1, \ldots, n\}$.

From Equation (63) it follows that

$$\begin{aligned} \text{acos}(y) &= \sqrt{f_n(y) - y^2 + \varepsilon_n(y)} = \sqrt{f_n(y) - y^2} + \delta_{c,n}(y) = c_n(y) + \delta_{c,n}(y) \\ \delta_{c,n}(y) &= \sqrt{f_n(y) - y^2 + \varepsilon_n(y)} - \sqrt{f_n(y) - y^2} \end{aligned} \tag{64}$$

where $c_n(y) = \sqrt{f_n(y) - y^2}$ is the nth order approximation to arccosine defined in Corollary 1 and the error in this approximation is $\delta_{c,n}(y)$. For y fixed, and for the convergent case where $\underset{n\to\infty}{lim}\, \varepsilon_n(y) = 0$, it is the case that $\underset{n\to\infty}{lim}\, \delta_{c,n}(y) = 0$. Hence, for y fixed, convergence of $f_n(y)$ to $r^2(y)$ as n increases, is sufficient to guarantee the convergence of $c_n(y)$ to $\text{acos}(y)$.

Consider the nth order approximation to arcsine, $s_n(y) = \pi/2 - c_n(y)$, as given in Corollary 1. It then follows that

$$\text{asin}(y) = \frac{\pi}{2} - c_n(y) - \delta_{c,n}(y) = s_n(y) - \delta_{c,n}(y) \tag{65}$$

Again, for y fixed, a sufficient condition for convergence of $s_n(y)$ to $\text{asin}(y)$ is for $\underset{n\to\infty}{lim}\, \varepsilon_n(y) = 0$.

As $\text{atan}(y) = \text{acos}\left[1/\sqrt{1+y^2}\right]$, it follows that the error $\delta_{t,n}(y)$ in the approximation, t_n, to arctangent, as given by Corollary 1, yields the relationship

$$atan(y) = t_n(y) + \delta_{t,n}(y) = c_n\left[\frac{1}{\sqrt{1+y^2}}\right] + \delta_{c,n}\left[\frac{1}{\sqrt{1+y^2}}\right] \tag{66}$$

and, thus, $\delta_{t,n}(y) = \delta_{c,n}\left[1/\sqrt{1+y^2}\right]$. Again, for y fixed, convergence of $t_n(y)$ to atan(y) is guaranteed if $\underset{n\to\infty}{lim}\, \varepsilon_n(y) = 0$.

The goal, thus, is to establish convergence of the approximations specified by Theorem 1, i.e., to show that $\underset{n\to\infty}{lim}\, \varepsilon_n(y) = 0$. To achieve this goal, the approach is to determine a series for the error function ε_n and this can be achieved by first establishing a differential equation for ε_n.

4.1. Differential Equation for Error

Consider Equation (64): $\text{acos}(y) = \sqrt{f_n(y) + \varepsilon_n(y) - y^2}, y \in [0, 1]$. Differentiation yields

$$\frac{-1}{\sqrt{1-y^2}} = \frac{f_n^{(1)}(y) + \varepsilon_n^{(1)}(y) - 2y}{2\sqrt{f_n(y) + \varepsilon_n(y) - y^2}},\ y \in [0, 1) \tag{67}$$

and after squaring and simplification the equation becomes

$$4\left[f_n(y) + \varepsilon_n(y) - y^2\right] = \left(1 - y^2\right)\left[f_n^{(1)}(y) + \varepsilon_n^{(1)}(y) - 2y\right]^2 \tag{68}$$

Rearrangement leads to the differential equation for the error function:

$$\begin{gathered} (1-y^2)\left[\varepsilon_n^{(1)}(y)\right]^2 + 2(1-y^2)\left[f_n^{(1)}(y) - 2y\right]\varepsilon_n^{(1)}(y) - 4\varepsilon_n(y) + \\ (1-y^2)\left[f_n^{(1)}(y) - 2y\right]^2 - 4\left[f_n(y) - y^2\right] = 0,\ \ \varepsilon_n(0) = 0. \end{gathered} \tag{69}$$

A polynomial expansion can be used to solve for $\varepsilon_n(y)$.

Theorem 4. *Polynomial Form for Error Function.*

A polynomial form for the error function, ε_n, as defined by the differential equation specified in Equation (69), is

$$e_n(y) = \sum_{k=n+1}^{2n+1} [C_{k,k} - C_{n,k}]y^k + \sum_{k=2n+2}^{\infty} C_{k,k}y^k, \quad n \in \{3, 4, \ldots\} \tag{70}$$

where $C_{n,k}$ is the kth *coefficient defined in Theorem 1 and* $C_{k,k} = f^{(k)}(0)/k!$.

Proof. The proof is detailed in Appendix E. □

Explicit Approximations

Polynomial expansions for e_n, of orders three and four, are:

$$\begin{aligned} e_3(y) = \; & \left[\frac{-964}{45} - \frac{62\pi}{3} + \frac{35\pi^2}{4}\right]y^4 + \left[\frac{994}{15} + \frac{1837\pi}{40} - 21\pi^2\right]y^5 + \\ & \left[\frac{-2584}{45} - \frac{110\pi}{3} + \frac{35\pi^2}{2}\right]y^6 + \left[\frac{785}{45} + \frac{3401\pi}{336} - 5\pi^2\right]y^7 + \\ & \frac{4y^8}{35} - \frac{35\pi y^9}{1152} + \frac{128y^{10}}{1575} - \frac{63\pi y^{11}}{2816} + \ldots \end{aligned} \tag{71}$$

$$\begin{aligned} e_4(y) = \; & \left[\frac{-8704}{105} - \frac{2903\pi}{40} + \frac{63\pi^2}{2}\right]y^5 + \left[\frac{98,176}{315} + \frac{692\pi}{3} - 105\pi^2\right]y^6 + \\ & \left[\frac{-45,056}{105} - \frac{32,205\pi}{112} + 135\pi^2\right]y^7 + \left[\frac{5504}{21} + 164\pi - \frac{315\pi^2}{4}\right]y^8 + \\ & \left[\frac{-18,944}{315} - \frac{41,315\pi}{1152} + \frac{35\pi^2}{2}\right]y^9 + \frac{128y^{10}}{1575} - \frac{63\pi y^{11}}{2816} + \ldots \end{aligned} \tag{72}$$

4.2. Convergence

First, consistent with Equation (22), $C_{n,n} = f^{(n)}(0)/n!$. Second, consistent with Equation (26), it is the case that

$$C_{n,n} = \frac{(1 - 2/n)^2}{(1 - 1/n)} \cdot C_{n,n-2} \tag{73}$$

As discussed in Section 3.1.1, it is the case that $|C_{n,n}| < |C_{n,n-2}|$ and with $|C_{n,n}| < 1$ for $n > 2$. It then follows that $\lim_{n\to\infty} C_{n,n} = 0$ and the decrease in magnitude is monotonic as n increases for even and odd values. Third, from Equation (70) and the result $|C_{n,n}| < |C_{n,n-2}| = |C_{n-2,n-2}|$, it follows, for the case of $0 < y < 1$, y fixed, that

$$\begin{aligned} |e_n(y)| \; & \le \sum_{k=n+1}^{2n+1} |C_{k,k} - C_{n,k}|y^k + |C_{2n+2,2n+2}| \sum_{k=2n+2}^{\infty} y^k \\ & \le y^{n+1}\left[\sum_{k=n+1}^{2n+1} |C_{k,k} - C_{n,k}|\right] + |C_{2n+2,2n+2}| \cdot \frac{y^{2n+2}}{1-y} \end{aligned} \tag{74}$$

The graph of $S_n = \sum_{k=n+1}^{2n+1} |C_{k,k} - C_{n,k}|$ is shown in Figure 11. As this is bounded, and as $0 < y < 1$, it follows that $\lim_{n\to\infty} e_n(y) = 0$ for $0 < y < 1$.

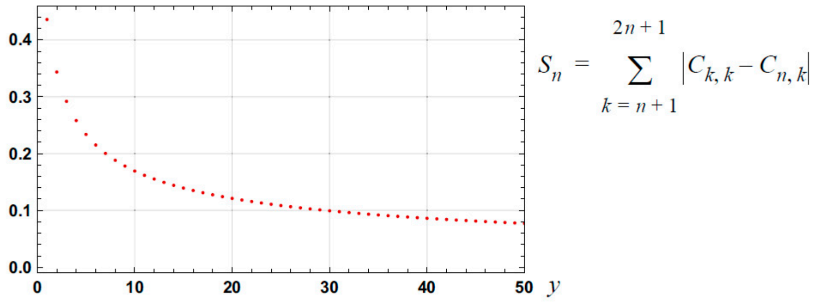

Figure 11. Graph of S_n for the case of $n \in \{1, 2, \ldots, 50\}$.

5. Direct Approximation for Arctangent

The approximations for arctangent detailed in Corollary 1, Theorem 2 and Theorem 3 are indirectly established. Direct approximations for arctangent can be established by utilizing the fundamental relationships $\text{atan}(y) + \text{acot}(y) = \pi/2$, $\text{acot}(y) = \text{atan}(1/y)$ which implies

$$\text{atan}(y) = \frac{\pi}{2} - \text{atan}(1/y), \qquad y > 0 \tag{75}$$

5.1. Approximations for Arctangent

The following theorem details a spline based approximation for arctangent.

Theorem 5. *Approximations for Arctangent.*

Given a nth order spline based approximation, $g_n(y)$, for $\text{atan}(y)$, $0 \le y \le 1$, based on the points zero and one, it is the case that

$$\text{atan}(y) \approx \begin{cases} g_n(y), & 0 \le y \le 1 \\ \dfrac{\pi}{2} - g_n\left[\dfrac{1}{y}\right], & y > 1 \end{cases} \tag{76}$$

The resulting nth order approximation, t_n, $n \in \{0, 1, 2, \ldots\}$, for arctangent is

$$\text{t}_n(y) = \begin{cases} \delta_{n,1}y + \delta_{n,2}y^2 + \ldots + \delta_{n,2n+1}y^{2n+1}, & 0 \le y \le 1 \\ \dfrac{\pi}{2} - \dfrac{\delta_{n,1}}{y} - \dfrac{\delta_{n,2}}{y^2} - \ldots - \dfrac{\delta_{n,2n+1}}{y^{2n+1}}, & 1 < y < \infty \end{cases} \tag{77}$$

where the coefficients $\delta_{n,i}$, $i \in \{1, \ldots, 2n+1\}$, are defined according to:

$$\delta_{n,i} = \begin{cases} \displaystyle\sum_{r=0}^{i} \frac{(-1)^{i-r}(n+1)!}{(r+n+1-i)!(i-r)!} \cdot a_{n,r}, & 1 \le i \le n \\ \displaystyle\sum_{r=i-n-1}^{n} \frac{(-1)^{i-r}(n+1)!}{(r+n+1-i)!(i-r)!} \cdot a_{n,r} + & \\ \displaystyle\sum_{r=i-n-1}^{n} \frac{(-1)^{i-n-1}r!}{(r+n+1-i)!(i-n-1)!} \cdot b_{n,r}, & n+1 \le i \le 2n+1 \end{cases} \tag{78}$$

Here:

$$a_{n,r} = \sum_{u=0}^{r} \frac{g^{(r-u)}(0)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!}, \quad b_{n,r} = \sum_{u=0}^{r} \frac{(-1)^{r-u}g^{(r-u)}(1)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \tag{79}$$

where $g(y) = \text{atan}(y)$ and

$$g^{(k)}(0) = \begin{cases} 0, & k \in \{0, 2, 4, \ldots\} \\ (-1)^{(k-1)/2}(k-1)!, & k \in \{1, 3, 5, \ldots\} \end{cases} \tag{80}$$

$$g^{(k)}(1) = \begin{cases} \frac{\pi}{4}, & k = 0 \\ \frac{(-1)^{k-1}}{2}, & k = 1, 2, 3 \\ 0, & k \in \{4, 8, \ldots\} \\ \frac{-(k-1)(k-2)}{2} \cdot g^{(k-2)}(1), & k \in \{5, 9, 13, \ldots\} \\ -(k-1)g^{(k-1)}(1), & k \in \{6, 10, 14, \ldots\} \\ \frac{-(k-1)}{2} \cdot g^{(k-1)}(1), & k \in \{7, 11, 15, \ldots\} \end{cases} \tag{81}$$

Proof. Consider the approximation $g_n(y)$ for $\text{atan}(y)$, $0 \leq y \leq 1$. The relationship $atan(y) = \pi/2 - atan(1/y)$ implies

$$atan(z) = \frac{\pi}{2} - atan\left[\frac{1}{z}\right] \approx \frac{\pi}{2} - g_n(y),\ z > 1, y = \frac{1}{z}, 0 \leq y < 1. \tag{82}$$

The formulas for $g^{(k)}(0)$, $g^{(k)}(1)$ and $\delta_{n,i}$ can be established in a manner consistent with the nature of the proof detailed in Appendix F. □

5.1.1. Analytical Approximations

Approximations for arctangent, of orders zero to two, are:

$$t_0(y) = \begin{cases} \frac{\pi y}{4}, & 0 \leq y \leq 1 \\ \frac{\pi}{2} - \frac{\pi}{4y}, & y > 1 \end{cases} \tag{83}$$

$$t_1(y) = \begin{cases} y - \left[\frac{5}{2} - \frac{3\pi}{4}\right]y^2 + \left[\frac{3}{2} - \frac{\pi}{2}\right]y^3, & 0 \leq y \leq 1 \\ \frac{\pi}{2} - \frac{1}{y} + \left[\frac{5}{2} - \frac{3\pi}{4}\right]\frac{1}{y^2} - \left[\frac{3}{2} - \frac{\pi}{2}\right]\frac{1}{y^3}, & y > 1 \end{cases} \tag{84}$$

$$t_2(y) = \begin{cases} y - \left[\frac{33}{4} - \frac{5\pi}{2}\right]y^3 + \left[12 - \frac{15\pi}{4}\right]y^4 - \left[\frac{19}{4} - \frac{3\pi}{2}\right]y^5, & 0 \leq y \leq 1 \\ \frac{\pi}{2} - \frac{1}{y} + \left[\frac{33}{4} - \frac{5\pi}{2}\right] \cdot \frac{1}{y^3} - \left[12 - \frac{15\pi}{4}\right] \cdot \frac{1}{y^4} + \left[\frac{19}{4} - \frac{3\pi}{2}\right] \cdot \frac{1}{y^5}, & y > 1 \end{cases} \tag{85}$$

Approximations, of orders three and four, are detailed in Appendix G.

5.1.2. Approximations for Arccosine and Arcsine

The relationships $\text{asin}(y) = \text{atan}\left[\frac{y}{\sqrt{1-y^2}}\right]$, $y \in [0, 1)$ and $\text{acos}(y) = \text{atan}\left[\sqrt{1-y^2}/y\right]$, $y \in (0, 1]$, imply the following approximations for arcsine and arccosine:

$$s_n(y) = \begin{cases} \frac{\delta_{n,1}y}{\sqrt{1-y^2}} + \frac{\delta_{n,2}y^2}{1-y^2} + \ldots + \frac{\delta_{n,2n+1}y^{2n+1}}{(1-y^2)^{n+1/2}}, & 0 \leq y \leq \frac{1}{\sqrt{2}} \\ \frac{\pi}{2} - \frac{\delta_{n,1}\sqrt{1-y^2}}{y} - \frac{\delta_{n,2}(1-y^2)}{y^2} - \ldots - \frac{\delta_{n,2n+1}\left[1-y^2\right]^{n+1/2}}{y^{2n+1}}, & \frac{1}{\sqrt{2}} < y \leq 1 \end{cases} \tag{86}$$

$$c_n(y) = \begin{cases} \dfrac{\pi}{2} - \dfrac{\delta_{n,1}y}{\sqrt{1-y^2}} - \dfrac{\delta_{n,2}y^2}{(1-y^2)} - \ldots - \dfrac{\delta_{n,2n+1}y^{2n+1}}{(1-y^2)^{n+1/2}}, & 0 \le y \le \dfrac{1}{\sqrt{2}} \\ \dfrac{\delta_{n,1}\sqrt{1-y^2}}{y} + \dfrac{\delta_{n,2}(1-y^2)}{y^2} + \ldots + \dfrac{\delta_{n,2n+1}(1-y^2)^{n+1/2}}{y^{2n+1}}, & \dfrac{1}{\sqrt{2}} \le y < 1 \end{cases} \tag{87}$$

Alternative approximations for arcsine and arccosine specified according to $s_n^A(y) = \pi/2 - \mathrm{acos}(y)$ and $c_n^A(y) = \pi/2 - \mathrm{asin}(y)$ lead to identical expressions, i.e., $s_n^A(y) = s_n(y)$ and $c_n^A(y) = c_n(y)$.

As an example, the third order approximation for arcsine is

$$s_3(y) = \begin{cases} \dfrac{y}{\sqrt{1-y^2}} - \dfrac{y^3}{3[1-y^2]^{3/2}} - \dfrac{\left[\dfrac{55}{2} - \dfrac{35\pi}{4}\right]y^4}{[1-y^2]^2} + \dfrac{\left[\dfrac{265}{4} - 21\pi\right]y^5}{[1-y^2]^{5/2}} - \\ \quad \dfrac{\left[\dfrac{331}{6} - \dfrac{35\pi}{2}\right]y^6}{[1-y^2]^3} + \dfrac{\left[\dfrac{63}{4} - 5\pi\right]y^7}{[1-y^2]^{7/2}}, \quad 0 \le y \le \dfrac{1}{\sqrt{2}} \\ \dfrac{\pi}{2} - \dfrac{\sqrt{1-y^2}}{y} + \dfrac{[1-y^2]^{3/2}}{3y^3} + \dfrac{\left[\dfrac{55}{2} - \dfrac{35\pi}{4}\right][1-y^2]^2}{y^4} - \dfrac{\left[\dfrac{265}{4} - 21\pi\right][1-y^2]^{5/2}}{y^5} \\ \quad + \dfrac{\left[\dfrac{331}{6} - \dfrac{35\pi}{2}\right][1-y^2]^3}{y^6} - \dfrac{\left[\dfrac{63}{4} - 5\pi\right][1-y^2]^{\frac{7}{2}}}{y^7}, \quad \dfrac{1}{\sqrt{2}} < y < 1 \end{cases} \tag{88}$$

5.1.3. Results

The relative errors associated with the approximations for arctangent, of orders one to six, are shown in Figure 12. The relative error bounds associated with the approximations to arctangent, arcsine and arccosine are detailed in Table 4. The relative error bound associate with the third order approximation for arcsine, as specified by Equation (88), is 3.73×10^{-5} which is comparable with the third order approximation specified in Corollary 1 whose relative error is 2.84×10^{-5}.

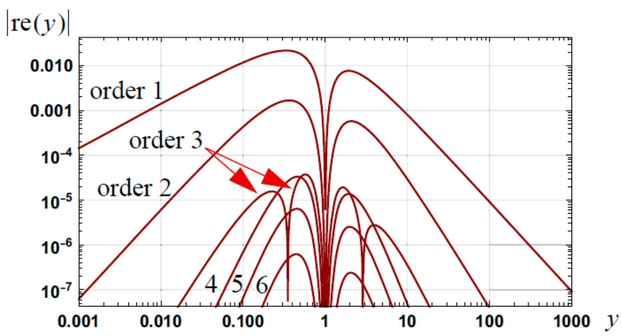

Figure 12. Graphs of the relative errors in approximations, of orders 1 to 6, for arctangent as defined in Theorem 5.

Table 4. Relative error bounds, associated with the approximations detailed in Theorem 5 and Theorem 6 for arcsine, arccosine and arctangent. The interval $[0, 1]$ is assumed for arcsine and arccosine; the interval $[0, \infty]$ for arctangent.

Order of Spline Approx.	Theorem 5—Relative Error Bounds : s_n,c_n,t_n	Theorem 6—Relative Error Bound for Arctangent. The Value Assumed for $\delta_{n,0}$ is the Second Value Stated in Equation (92).
0	2.15×10^{-1}	
1	2.18×10^{-2}	4.31×10^{-3}
2	1.68×10^{-3}	3.21×10^{-4}
3	3.73×10^{-5}	6.77×10^{-6}
4	3.34×10^{-5}	6.34×10^{-6}
5	6.39×10^{-6}	1.17×10^{-6}
6	6.22×10^{-7}	1.10×10^{-7}
8	1.82×10^{-8}	3.09×10^{-9}
10	3.74×10^{-10}	6.06×10^{-11}

5.2. Improved Approximation: Use of Integral for Arctangent

Consider the known integral

$$\int_0^y \lambda \mathrm{atan}(\lambda) d\lambda = \frac{y^2 \mathrm{atan}(y)}{2} + \frac{\mathrm{atan}(y)}{2} - \frac{y}{2} \tag{89}$$

which implies

$$atan(y) = \frac{2}{1+y^2} \cdot \left[\frac{y}{2} + \int_0^y \lambda \mathrm{atan}(\lambda) d\lambda \right]. \tag{90}$$

An integrable approximation for $y\mathrm{atan}(y)$, for $[0, 1]$, leads to an approximation for arctangent.

Theorem 6. *Improved Approximations for Arctangent.*

The nth order approximation for arctangent, based on Equation (90), is defined according to

$$t_n(y) = \frac{2}{1+y^2} \left[\frac{y}{2} + \begin{cases} \frac{\delta_{n,1} y^3}{3} + \frac{\delta_{n,2} y^4}{4} + \ldots + \frac{\delta_{n,2n+1} y^{2n+3}}{2n+3}, & 0 \le y \le 1 \\ \delta_{n,0} + \frac{\pi(y^2-1)}{4} - \delta_{n,1}(y-1) - \delta_{n,2} \ln \cdot (y) - \delta_{n,3} \left[1 - \frac{1}{y} \right] - & \\ \frac{\delta_{n,4}}{2} \left[1 - \frac{1}{y^2} \right] - \ldots - \frac{\delta_{n,2n+1}}{2n-1} \left[1 - \frac{1}{y^{2n-1}} \right], & 1 < y < \infty \end{cases} \right] \tag{91}$$

where the coefficients $\delta_{n,i}$ are defined in Equation (78) and

$$\delta_{n,0} = \frac{\pi}{4} - \frac{1}{2} \quad \text{or} \quad \delta_{n,0} = \frac{\delta_{n,1}}{3} + \frac{\delta_{n,2}}{4} + \ldots + \frac{\delta_{n,2n+1}}{2n+3} \tag{92}$$

Here $\delta_{n,0}$ is associated with $\int_0^1 \lambda \mathrm{atan}(\lambda) d\lambda$ and with the first value being exact. The second value yields a lower relative error bound for the interval $(1, \infty)$.

Proof. The approximations for arctangent, as defined in Theorem 5, when used in the integral in Equation (90), lead to the approximations specified by Equation (91). □

5.2.1. Explicit Expressions

Explicit approximations for arctangent, of orders one and two, are:

$$t_1(y) = \begin{cases} \frac{1}{1+y^2}\left[y + \frac{2y^2}{3} - \left[\frac{5}{4} - \frac{3\pi}{8}\right]y^4 + \left[\frac{3}{5} - \frac{\pi}{5}\right]y^5\right], & 0 \le y \le 1 \\ \frac{1}{1+y^2}\left[-\frac{59}{60} + \frac{27\pi}{40} + \frac{3-\pi}{y} - y + \left[5 - \frac{3\pi}{2}\right]\ln(y) + \frac{\pi y^2}{2}\right], & y > 1 \end{cases} \quad (93)$$

$$t_2(y) = \begin{cases} \frac{1}{1+y^2}\left[y + \frac{2y^2}{3} - \left[\frac{33}{10} - \pi\right]y^5 + \left[4 - \frac{5\pi}{4}\right]y^6 - \left[\frac{19}{14} - \frac{3\pi}{7}\right]y^7\right], & 0 \le y \le 1 \\ \frac{1}{1+y^2}\left[\begin{array}{c} \frac{1016}{105} - \frac{18\pi}{7} - \left[\frac{19}{6} - \pi\right]\frac{1}{y^3} + \left[12 - \frac{15\pi}{4}\right]\frac{1}{y^2} - \\ \left[\frac{33}{2} - 5\pi\right]\frac{1}{y} - y + \frac{\pi y^2}{2} \end{array}\right], & y > 1 \end{cases} \quad (94)$$

Third and fourth order approximations are detailed in Appendix H.

Explicit approximations for arcsine and arccosine can be defined by utilizing the relationships $\text{asin}(y) = \text{atan}\left[y/\sqrt{1-y^2}\right]$ and $\text{acos}(y) = \text{atan}\left[\sqrt{1-y^2}/y\right]$.

5.2.2. Results

The relative error bounds associated with the approximations to arctangent are detailed in Table 4 and the improvement over the original approximations is evident.

6. Improved Approximations via Iteration

Given an initial approximating function h_0 for the inverse, f^{-1}, of a function f, the ith iteration of the classical Newton-Raphson method of approximation leads to the ith order approximation

$$h_i(y) = h_{i-1}(y) - \frac{f[h_{i-1}(y)] - y}{f^{(1)}[h_{i-1}(y)]}, \quad h_0(y) \text{ known}, \ i \in \{1, 2, \ldots\}. \quad (95)$$

6.1. Newton-Raphson Iteration: Approximations and Results for Arcsine

The arcsine case is considered: An initial approximation to arcsine of $h_0(y) = s_n(y)$ $n \in \{0, 1, 2, \ldots\}$, as specified by Corollary 1, Theorem 2, Theorem 3 or Section 5.1.2, leads to the ith order iterative Newton-Raphson approximation:

$$\begin{aligned} h_i(y) &= h_{i-1}(y) - \frac{sin[h_{i-1}(y)] - y}{cos[h_{i-1}(y)]}, \ \ h_0(y) = s_n(y), \\ &= h_{i-1}(y) - tan[h_{i-1}(y)] + ysec[h_{i-1}(y)]. \end{aligned} \quad (96)$$

Iteration of orders one and two lead to the approximations:

$$h_1(y) = s_n(y) - \frac{sin[s_n(y)] - y}{cos[s_n(y)]} = s_n(y) - tan[s_n(y)] + ysec[s_n(y)] \quad (97)$$

$$\begin{aligned} h_2(y) &= s_n(y) - \frac{sin[s_n(y)] - y}{cos[s_n(y)]} - \frac{sin\left[s_n(y) - \frac{sin[s_n(y)] - y}{cos[s_n(y)]}\right] - y}{cos\left[s_n(y) - \frac{sin[s_n(y)] - y}{cos[s_n(y)]}\right]} \\ &= s_n(y) - tan[s_n(y)] + ysec[s_n(y)] - tan\left[\begin{array}{c} s_n(y) - tan[s_n(y)] + \\ ysec[s_n(y)] \end{array}\right] + \\ &\quad ysec[s_n(y) - tan[s_n(y)] + ysec[s_n(y)]] \end{aligned} \quad (98)$$

The approximation arising from a third order iteration is detailed in Appendix I.

Example and Results

As an example, consider the second order approximation for arcsine arising from Theorem 2 and defined by Equation (49):

$$h_0(y) = s_2(y) = \frac{\pi}{2} - \sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right], \quad y \in [0,1]. \tag{99}$$

The relative error bound associated with this approximation is 3.14×10^{-3}. The first order iteration of the Newton-Raphson method yields the approximation

$$\begin{aligned} f^{-1}(y) \approx h_1(y) = \; & \frac{\pi}{2} - \sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right] - \\ & \frac{cos\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right] - y}{sin\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right]} \end{aligned} \tag{100}$$

The relative error bound for this approximation, and associated with the interval [0, 1], is 2.13×10^{-7}. Second order iteration yields the approximation detailed in Equation (A62). The relative error bound associated with this approximation, for the interval [0, 1], is 5.68×10^{-15} The use of $h_0(y) = s_4(y)$, as specified by Equation (A27), rather than $h_0(y) = s_2(y)$, leads to a relative error bound of 3.05×10^{-23}.

Consider the fourth order approximation, s_4, defined by Equation (A27). A first order iteration of the Newton-Raphson method yields the approximation

$$\begin{aligned} f^{-1}(y) \approx g_1(y) = \; & \frac{\pi}{2} - \sqrt{1-y^2}\cdot\left[\begin{array}{c} \frac{\pi}{2} - y + \frac{\pi y^2}{4} - \frac{2y^3}{3} + d_{4,4}y^4 + d_{4,5}y^5 + d_{4,6}y^6 + \\ d_{4,7}y^7 + d_{4,8}y^8 \end{array}\right] - \\ & \frac{cos\left[\sqrt{1-y^2}\cdot\left[\begin{array}{c} \frac{\pi}{2} - y + \frac{\pi y^2}{4} - \frac{2y^3}{3} + d_{4,4}y^4 + d_{4,5}y^5 + d_{4,6}y^6 + \\ d_{4,7}y^7 + d_{4,8}y^8 \end{array}\right]\right] - y}{\sin\left[\sqrt{1-y^2}\cdot\left[\begin{array}{c} \frac{\pi}{2} - y + \frac{\pi y^2}{4} - \frac{2y^3}{3} + d_{4,4}y^4 + d_{4,5}y^5 + d_{4,6}y^6 + \\ d_{4,7}y^7 + d_{4,8}y^8 \end{array}\right]\right]} \end{aligned} \tag{101}$$

The relative error bound associated with this approximation is 1.44×10^{-11}.

The improvement that is possible with Newton-Raphson iteration is illustrated in Table 5 where the original approximations to arcsine and arctangent, based on $s_2(y)$, $s_2^A(y)$, $t_2(y)$ and $t_2^A(y)$ as defined in Theorem 2 and specified by Equations (49) and (50), are used. The quadratic convergence, with iteration, is evident. It is usual for the relative error improvement, with iteration, to be dependent on the relative error in the initial approximation. However, as the results in Table 5 indicate, the approximations of $s_2(y)$ and $t_2^A(y)$, with higher relative error bounds, lead to lower relative bounds with iteration than $s_2^A(y)$ and $t_2(y)$. This is due to the nature of the approximations.

Table 5. Relative error bounds for Newton-Raphson iterative approximations to arcsine and arctangent and based on $s_2(y)$, $s_2^A(y)$, $t_2(y)$ and $t_2^A(y)$ as defined in Theorem 2 and specified by Equations (49) and (50).

Order of Iteration	Relative Error Bound: $h_0(y)$=$s_2(y)$	Relative Error Bound: $h_0(y)$=$s_2^A(y)$	Relative Error Bound: $h_0(y)$=$t_2(y)$	Relative Error Bound: $h_0(y)$=$t_2^A(y)$
0	3.14×10^{-3}	5.54×10^{-4}	5.54×10^{-4}	3.14×10^{-3}
1	2.13×10^{-7}	6.52×10^{-7}	1.31×10^{-6}	4.26×10^{-7}
2	5.68×10^{-15}	1.43×10^{-12}	1.15×10^{-11}	4.55×10^{-14}
3	1.31×10^{-29}	7.98×10^{-24}	1.03×10^{-21}	1.68×10^{-27}
4	7.27×10^{-59}	2.68×10^{-46}	9.00×10^{-42}	2.39×10^{-54}
5	2.29×10^{-117}	3.13×10^{-91}	7.10×10^{-82}	4.95×10^{-108}

7. Applications

7.1. Approximations for a Set Relative Error Bounds: Arcsine

With the requirement of a set relative error bound in an approximation for arsine, arccosine or arctangent, an approximation form and a set order of approximation can be specified. The following details examples of approximations for arcsine and the interval [0, 1] is assumed.

For a relative error bound close to 10^{-4}, the approximation

$$s_2^A(y) = \sqrt{\frac{\pi^2}{4} - \pi\sqrt{1-y^2} + \left(1-y^2\right) + c_{2,3}\left(1-y^2\right)^{3/2} + c_{2,4}\left(1-y^2\right)^2 + c_{2,5}\left(1-y^2\right)^{5/2}} \tag{102}$$

as given by Corollary 1, yields a relative error bound of 1.81×10^{-4}. The approximation, s_2, defined by Equation (59) yields a relative error bound of 1.56×10^{-4}.

For a relative bound close to 10^{-6}, the approximation $s_4(y) = \frac{\pi}{2} - c_4(y)$, where c_4 is defined by Equation (A22), is

$$s_4(y) = \frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} + c_{4,5}y^5 + c_{4,6}y^6 + c_{4,7}y^7 + c_{4,8}y^8 + c_{4,9}y^9} \tag{103}$$

and has a relative error bound of 2.49×10^{-6}. The approximation defined by $s_{4,1}(y) = c_{4,1}\left[\sqrt{1-y^2}\right]$(see Equation (A14)) is

$$s_{4,1}(y) = 4\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}\sqrt{1 + \sqrt{1-y^2}}}}} \cdot \left[1 + \frac{10\left[1 + \frac{7}{10}\sqrt{2 + \sqrt{2 + \sqrt{2}\sqrt{1 + \sqrt{1-y^2}}}}\right]}{3\left[2 + \sqrt{2 + \sqrt{2 + \sqrt{2}\sqrt{1 + \sqrt{1-y^2}}}}\right]^{3/2}}\right] \tag{104}$$

and has a relative error bound of 1.19×10^{-6}. The approximation given by Abramowitz, as stated in Equation (15), has a relative error bound of 3.04×10^{-6}.

If a high accuracy approximation is required then two approaches can be used. First, higher order approximations as specified in Corollary 1, Theorem 2, Theorem 3 and Theorem 5 can be used. For example, the fifteenth order approximation, s_{15}, for arcsine detailed in Corollary 1 yields a relative error bound of 4.74×10^{-17}. Second, iterative approaches can be used. For example, the second order approximation, s_2, for arcsine arising from Theorem 2 and defined by Equation (49) and a second order iteration leading to Equation (A62) has a relative error bound of 5.68×10^{-15}. An alternative approximation can be defined by utilizing the zero order spline approximation, as specified by

Equation (117), and the sixth and seventh order approximations (the function $f_{0,6,7}$) which yields a relative error bound of 7.65×10^{-18} (see Table 6).

7.2. Upper and Lower Bounds for Arcsine, Arccosine and Arctangent

Lower, L, and upper, U, bounds for arcsine, i.e.,

$$L(y) < \mathrm{asin}(y) < U(y), \;\; 0 < y < 1, \tag{105}$$

lead to the following lower and upper bounds for arccosine and arctangent:

$$\frac{\pi}{2} - U(y) < \mathrm{acos}(y) < \frac{\pi}{2} - L(y), \;\; 0 < y < 1, \tag{106}$$

$$L\left[\frac{y}{\sqrt{1+y^2}}\right] < \mathrm{atan}(y) < U\left[\frac{y}{\sqrt{1+y^2}}\right], \;\; 0 < y < \infty. \tag{107}$$

7.2.1. Published Bounds for Arcsine

There is interest in upper and lower bounds for arcsine, e.g., [17–21]. The classic upper and lower bounded functions for arcsine are defined by the Shafer-Fink inequality [13]:

$$\frac{3y}{2+\sqrt{1-y^2}} \le asin(y) \le \frac{\pi y}{2+\sqrt{1-y^2}}, \; 0 \le y \le 1. \tag{108}$$

The relative error bound associated with the lower bounded function is 4.51×10^{-2}; the relative error bound associated with the upper bounded function is 4.72×10^{-2}.

Zhu [20] (eqn. 1.8), proposed the bounds:

$$\frac{6\left[\sqrt{1+y}-\sqrt{1-y}\right]}{4+\sqrt{1+y}+\sqrt{1-y}} \le asin(y) \le \frac{\frac{\pi\left(2-\sqrt{2}\right)}{\pi-2\sqrt{2}} \cdot \left[\sqrt{1+y}-\sqrt{1-y}\right]}{\frac{\sqrt{2}(4-\pi)}{\pi-2\sqrt{2}} + \sqrt{1+y}+\sqrt{1-y}}, \; 0 \le y \le 1 \tag{109}$$

where the lower relative error bound is 2.27×10^{-3} and the upper relative error bound is 5.61×10^{-4}.

Zhu [21] (Theorem 1), proposed the bounds

$$\begin{gathered} \frac{1}{2+\sqrt{1-y^2}}\left[a_n y^{2n+1} + \sum_{i=0}^{n-1} a_i y^{2i+1}\right] \le asin(y) \le \\ \frac{1}{2+\sqrt{1-y^2}}\left[b_n y^{2n+1} + \sum_{i=0}^{n-1} a_i y^{2i+1}\right], \; n \in \{2,3,\ldots\}, \quad 0 \le y \le 1, \end{gathered} \tag{110}$$

$$a_0 = 3, \; a_i = \frac{1}{2i+1}\left[\frac{(2i-1)!!}{2^{i-1} i!} - \frac{2^{i-1} i!}{i[(2i-1)!!]}\right], \; b_i = \pi - \sum_{k=0}^{i-1} a_k \tag{111}$$

The lower bound is equivalent to the bound proposed by Maleševí et al. [19] (eqn. 21). The relative errors in the bounds are low for $y \ll 1$ but increase as y increases. For the case of $n = 4$ the relative error bound for the lower bounded function is 0.0324; for the upper bounded function the relative error bound is 0.0159.

7.2.2. Proposed Bounds for Arcsine and Arccosine

Consider the approximations defined in Corollary 1 and whose relative errors are shown in Figure 5. As the graphs in this figure indicate, the approximations are either upper or lower bounds for arcsine and arccosine and this is confirmed by numerical analysis (for the orders considered) which shows that there are no roots, in the interval (0,1), for the error function associated with the approximations. The evidence is that the approximations, s_i, of orders 0, 2, 4, ... , are lower bounds for arcsine whilst the approximations of orders 1, 3, 5, ... are upper bounds. Thus, for example, second, s_2, and third, s_3, order approximations, as defined in Corollary 1, yield the inequalities

$$\begin{aligned} &\frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} - \pi y + y^2 + c_{2,3}y^3 + c_{2,4}y^4 + c_{2,5}y^5} \leq asin(y) \leq \\ &\frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + c_{3,4}y^4 + c_{3,5}y^5 + c_{3,6}y^6 + c_{3,7}y^7} \end{aligned} \tag{112}$$

for $y \in [0,1]$, where, as detailed in Table 1, the lower relative error bound is 3.64×10^{-4} and the upper relative error bound is 2.84×10^{-5}.

It then follows, from Equation (106), that

$$\begin{aligned} &\sqrt{\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + c_{3,4}y^4 + c_{3,5}y^5 + c_{3,6}y^6 + c_{3,7}y^7} \leq \mathrm{acos}(y) \leq \\ &\sqrt{\frac{\pi^2}{4} - \pi y + y^2 + c_{2,3}y^3 + c_{2,4}y^4 + c_{2,5}y^5} \end{aligned} \tag{113}$$

for $y \in [0,1]$. An analytical proof that the approximations for arcsine and arccosine, as detailed in Corollary 1, are upper/lower bounds is an unsolved problem.

7.2.3. Upper/Lower Bounds for Arctangent

As an example of upper and lower bounds that have been proposed for arctangent, consider the bounds proposed by Qiao and Chen [22] (Theorem 3.1 and Theorem 4.2) for $y > 0$:

$$\begin{aligned} &\frac{3\pi^2 y}{24 - \pi^2 + \sqrt{432 - 24\pi^2 + \pi^4 - 12\pi(12 - \pi^2)y + 36\pi^2 y^2}} < atan(y) < \\ &\frac{3\pi^2 y}{24 - \pi^2 + \sqrt{576 - 192\pi^2 + 16\pi^4 - 12\pi(12 - \pi^2)y + 36\pi^2 y^2}} \end{aligned} \tag{114}$$

$$\begin{aligned} &\frac{\pi}{2} - \frac{64 + 735y^2 + 945y^4}{15y[15 + 70y^2 + 63y^4]} + \frac{64}{43,659y^{11}} - \frac{1856}{464,373y^{13}} < atan(y) < \\ &\frac{\pi}{2} - \frac{64 + 735y^2 + 945y^4}{15y[15 + 70y^2 + 63y^4]} + \frac{64}{43,659y^{11}} \end{aligned} \tag{115}$$

The lower bounded function in Equation (114) has a relative error bound of 0.0520; the upper bounded function has a relative error bound of 0.0274. The error in the upper and lower bounded functions specified in Equation (115) diverges as $y \to 0$ but converges rapidly to zero for $y \gg 1$.

7.2.4. Proposed Bounds for Arctangent

As $atan(y) = acos\left[\frac{1}{\sqrt{1+y^2}}\right]$ it follows, from Equation (113), that the functions t_2 and t_3 defined in Corollary 1 are, respectively, upper and lower bounds for arctangent, i.e.,

$$\sqrt{\frac{\pi^2}{4} - \frac{\pi}{\sqrt{1+y^2}} + \frac{1}{1+y^2} - \frac{\pi}{6[1+y^2]^{3/2}} + \frac{c_{3,4}}{[1+y^2]^2} + \frac{c_{3,5}}{[1+y^2]^{5/2}} + \frac{c_{3,6}}{[1+y^2]^3} + \frac{c_{3,7}}{[1+y^2]^{7/2}}} \leq atan(y) \leq \sqrt{\frac{\pi^2}{4} - \frac{\pi}{\sqrt{1+y^2}} + \frac{1}{1+y^2} + \frac{c_{2,3}}{[1+y^2]^{3/2}} + \frac{c_{2,4}}{[1+y^2]^2} + \frac{c_{2,5}}{[1+y^2]^{5/2}}} \quad (116)$$

for $y \in [0, \infty)$. As detailed in Table 1, the relative error bound for the lower bounded function is 1.42×10^{-5} and 1.81×10^{-4} for the upper bounded function.

7.3. Spline Approximations Based on Upper/Lower Bounds

Consider upper, f_U, and lower, f_L, bounded functions for arcsine as illustrated in Figure 13. For y fixed at y_o, a spline approximation, based on the points $(\sin(u_o), u_o)$, $u_o = f_L(y_o)$ and $(\sin(v_o), v_o)$, $v_o = f_U(y_o)$, can readily be determined. From such an approximation, an approximation to $x_o = \text{asin}(y_o)$ can then be determined.

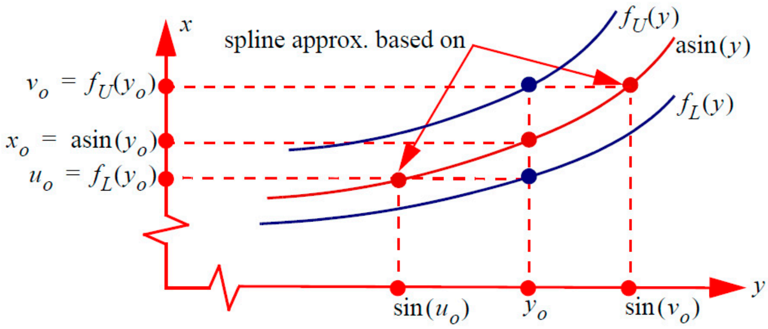

Figure 13. Illustration of upper and lower bounded approximations to arcsine and the two basis points $(\sin(u_o), u_o)$, $(\sin(v_o), v_o)$ for two point spline based approximations.

Theorem 7. *Spline Approximations Based on Upper/Lower Bounds.*

Consider lower, f_L, and upper, f_U, bounded approximations for arcsine. The zero order spline approximation for arcsine, based on the approximations f_L and f_U, is

$$f_0(y) = \frac{f_L(y)sin[f_U(y)] - f_U(y)sin[f_L(y)] + y[f_U(y) - f_L(y)]}{sin[f_U(y)] - sin[f_L(y)]}, \quad y \in (0, 1). \quad (117)$$

The nth order spline approximation for arcsine, based on the approximations f_L and f_U, is

$$f_n(y) = \frac{[\sin(v_o) - y]^{n+1}}{[\sin(v_o) - \sin(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [y - \sin(u_o)]^r \cdot \left[\begin{array}{l} \frac{(n+r)!u_o}{r!n![\sin(v_o) - \sin(u_o)]^r} + \\ \sum_{u=0}^{r-1} \frac{f^{(r-u)}[\sin(u_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[\sin(v_o) - \sin(u_o)]^u} \end{array} \right] + \frac{[y - \sin(u_o)]^{n+1}}{[\sin(v_o) - \sin(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [\sin(v_o) - y]^r \cdot \left[\begin{array}{l} \frac{(n+r)!v_o}{r!n![\sin(v_o) - \sin(u_o)]^r} + \\ \sum_{u=0}^{r-1} \frac{\left(-1\right)^{r-u} f^{(r-u)}[\sin(v_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[\sin(v_o) - \sin(u_o)]^u} \end{array} \right] \tag{118}$$

for $y \in (0,1)$, $u_0 = f_L(y)$, $v_0 = f_U(y)$, $f(y) = \text{asin}(y)$ *and*

$$f^{(k)}(y) = \sum_{i=0}^{k-1} \frac{d[k,i] \left[\frac{1 + (-1)^{k+i+1}}{2} \right] y^i}{(1 - y^2)^{\left\lfloor \frac{k+i+1}{2} \right\rfloor - \frac{1}{2}}}, \quad k \in \{1, 2, \ldots\} \tag{119}$$

where

$$\begin{aligned} &d[1,0] = 1, \\ &d[2,0] = 0, \quad d[2,1] = 1, \\ &d[3,0] = 1, \quad d[3,1] = 0, \quad d[3,2] = 3, \end{aligned} \tag{120}$$

and for $k > 3$:

$$d[k,i] = \begin{cases} d[k-1,1], i = 0 \\ (i+1)d[k-1,i+1] + 2\left[\left\lfloor \frac{k+i-1}{2} \right\rfloor - \frac{1}{2}\right] d[k-1,i-1], 1 \le i \le k-3 \\ 2\left[\left\lfloor \frac{k+i-1}{2} \right\rfloor - \frac{1}{2}\right] d[k-1,i-1], k-2 \le i \le k-1 \end{cases} \tag{121}$$

Proof. The proof is detailed in Appendix J. □

Results

Consider the approximations the approximation s_i, $i \in \{0, 1, 2, \ldots\}$, for arcsine as detailed in Corollary 1 where approximations, of order 0, 2, 4, ... , are lower bounds and the approximations, of orders 1, 3, 5, ... , are upper bounds. For example, with $f_L(y) = s_4(y)$ and $f_U(y) = s_5(y)$, $s_4(y) = \frac{\pi}{2} - c_4(y)$, $s_5(y) = \frac{\pi}{2} - c_5(y)$ with c_4 and c_5 defined by Equation (A22) and Equation (A23), the zero order spline approximation, as specified by Equation (117), is

$$f_{0,4,5}(y) = \frac{s_4(y)\sin[s_5(y)] - s_5(y)\sin[s_4(y)] + y[s_5(y) - s_4(y)]}{\sin[s_5(y)] - \sin[s_4(y)]} \tag{122}$$

The relative error bound for this approximation, over the interval $[0, 1]$, is 8.22×10^{-14}. Other results are detailed in Table 6 and clearly show the high accuracy of the approximations.

Table 6. Relative error bounds, over the interval $[0, 1]$, for spline approximations based on upper and lower bounded approximations to arcsine and as specified in Theorem 7.

Upper/Lower Bounded Functions: s_i Defined in Corollary 1	Spline Order	Notation for Approx.	Relative Error Bound
$f_L(y) = s_0(y),\ f_U(y) = s_1(y)$	0	$f_{0,0,1}$	2.43×10^{-4}
	1	$f_{1,0,1}$	1.45×10^{-7}
	2	$f_{2,0,1}$	1.31×10^{-10}
	3	$f_{3,0,1}$	1.44×10^{-13}
	4	$f_{4,\ 0,1}$	1.77×10^{-16}
$f_L(y) = s_2(y),\ f_U(y) = s_3(y)$	0	$f_{0,2,3}$	1.41×10^{-9}
	1	$f_{1,2,3}$	4.48×10^{-18}
	2	$f_{2,2,3}$	2.05×10^{-26}
	3	$f_{3,2,3}$	1.14×10^{-34}
	4	$f_{4,2,\ 3}$	6.98×10^{-43}
$f_L(y) = s_4(y),\ f_U(y) = s_5(y)$	0	$f_{0,4,5}$	8.22×10^{-14}
	1	$f_{1,4,5}$	1.48×10^{-26}
	2	$f_{2,4,5}$	3.78×10^{-39}
	3	$f_{3,4,5}$	1.16×10^{-51}
	4	$f_{4,4,5}$	3.95×10^{-64}
$f_L(y) = s_6(y),\ f_U(y) = s_7(y)$	0	$f_{0,6,7}$	7.56×10^{-18}
	1	$f_{1,6,7}$	1.27×10^{-34}
	2	$f_{2,6,7}$	2.97×10^{-51}
	3	$f_{3,6,7}$	8.30×10^{-68}
	4	$f_{4,6,7}$	2.57×10^{-84}

7.4. Approximations for Arcsine Squared and Higher Powers

There is interest in approximations for $\text{acos}(y)^k$, $\text{asin}(y)^k$, $\text{asin}(y)^k/y^k$, $k \in \{2, 3, \ldots\}$, e.g., [23–26]. The standard series for $\text{asin}(y)^2$, e.g., [7] (1.645.2), is

$$\text{asin}(y)^2 \approx S_{2,n}(y) = \sum_{k=0}^{2n+1} \frac{2^{2k}[k!]^2 y^{2k+2}}{(k+1)(2k+1)!} \tag{123}$$

The nth order approximation, s_n^A, specified in Corollary 1, leads to the approximations $s_{2,n}^A$ for $asin(y)^2$ defined according to

$$\begin{aligned} s_{2,n}^A(y) = \frac{\pi^2}{4} - \pi\sqrt{1-y^2} + (1-y^2) + c_{n,3}\left[1-y^2\right]^{3/2} + c_{n,4}\left[1-y^2\right]^2 + \ldots \\ + c_{n,2n+1}\left[1-y^2\right]^{n+1/2} \end{aligned} \tag{124}$$

for $n \geq 2$. The relative errors in $S_{2,n}$ and $s_{2,n}^A$ are shown in Figure 14. The approximations defined by $s_{2,n}^A$ have better overall relative error performance; in particular, they are sharp at the point one.

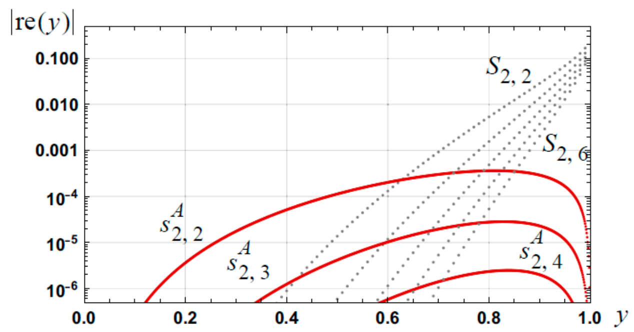

Figure 14. Graph of the relative errors in approximations to the square of arcsine as given by Equation (123) (orders 2 to 6) and Equation (124) (orders 2 to 4).

7.4.1. Approximations for Even Powers of Arcsine

Based on the approximation for the square of arcsine, as specified by Equation (124), the following result can be stated:

Theorem 8. *Approximation for Even Powers of Arcsine.*

Based on the nth order approximation, s_n^A, specified in Corollary 1, the even powers of arcsine can be approximated according to

$$asin(y)^{2m} \approx \left[s_n^A(y)\right]^{2m} = \sum_{k=0}^{m(2n+1)} \beta_k \left(1-y^2\right)^{k/2},\ m \in \{1,2,\ldots\} \tag{125}$$

where

$$\beta_k = \sum_{\substack{i_1+i_2+\ldots+i_m=k \\ i_1,i_2,\ldots,i_m \in \{0,1,2,\ldots,2n+1\}}} c_{n,i_1}c_{n,i_2}\ldots c_{n,i_m} \tag{126}$$

Here, $c_{n,i_1}c_{n,i_2}\ldots c_{n,i_m}$ are defined by Equation (29).

Proof. This result follows from expansion of s_n^A to the 2mth power, i.e.,

$$\left[s_n^A(y)\right]^{2m} = \sum_{i=1}^{2n+1}\cdots\sum_{i_m=1}^{2n+1} c_{n,i}c_{n,i_2}\cdots c_{n,i_m}\left(1-y^2\right)^{(i+i_2+\ldots+i_m)/2} \tag{127}$$

and collecting terms associated with $\left(1-y^2\right)^{k/2}$.□

7.4.2. Example

For example, the nth order approximation for $asin(y)^4$ is

$$\begin{aligned}\left[s_n^A(y)\right]^4 = \ & c_{n,0}^2 + 2c_{n,0}c_{n,1}\left(1-y^2\right)^{1/2} + \left[2c_{n,0}c_{n,2} + c_{n,1}^2\right]\left(1-y^2\right) + \\ & 2\left[c_{n,0}c_{n,3} + c_{n,1}c_{n,2}\right]\left(1-y^2\right)^{3/2} + \\ & \left[2c_{n,0}c_{n,4} + 2c_{n,1}c_{n,3} + c_{n,2}^2\right]\left(1-y^2\right)^2 + \ldots + \\ & 2c_{n,2n}c_{n,2n+1}\left(1-y^2\right)^{2n+1/2} + c_{n,2n+1}^2\left(1-y^2\right)^{2n+1}\end{aligned} \tag{128}$$

7.4.3. Roots of Arccosine: Approximations for Even Powers of Arccosine and Arcsine

The following theorem details a better approach for evaluating approximations for $asin(y)^{2k}$ and $acos(y)^{2k}$, $k \in \{1,2,\ldots\}$.

Theorem 9. *Root Based Approximation for Even Powers of Arccosine and Arcsine.*

Approximations of order n, for $\text{acos}(y)^{2k}$ *and* $\text{asin}(y)^{2k}$, $k \in \{1, 2, \ldots\}$, *respectively, are*

$$c_{2k,n}(y) = \frac{\pi^{2k}}{2^{2k}}(1-y)^k \left[1 - \frac{y}{r_1}\right]^k \left[1 - \frac{y}{r_1^*}\right]^k \left[1 - \frac{y}{r_2}\right]^k \left[1 - \frac{y}{r_2^*}\right]^k \cdots \left[1 - \frac{y}{r_n}\right]^k \left[1 - \frac{y}{r_n^*}\right]^k \quad (129)$$

$$\begin{aligned} s_{2k,n}(y) = \;& \frac{\pi^{2k}}{2^{2k}}\left(1 - \sqrt{1-y^2}\right)^k \left[1 - \frac{\sqrt{1-y^2}}{r_1}\right]^k \left[1 - \frac{\sqrt{1-y^2}}{r_1^*}\right]^k \cdot \\ & \left[1 - \frac{\sqrt{1-y^2}}{r_2}\right]^k \left[1 - \frac{\sqrt{1-y^2}}{r_2^*}\right]^k \cdots \left[1 - \frac{\sqrt{1-y^2}}{r_n}\right]^k \left[1 - \frac{\sqrt{1-y^2}}{r_n^*}\right]^k \end{aligned} \quad (130)$$

where r_i^* *is the conjugate of* r_i *and* $r_1, r_1^*, \ldots, r_n, r_n^*$ *are the roots of the nth order approximation* $c_n^2(y)$ *to* $\text{acos}(y)^2$ *defined in Corollary 1.*

Proof. Consider the nth order approximation $c_n^2(y)$ to $\text{acos}(y)^2$ defined in Corollary 1. This approximation is denoted $c_{2,n}$ and is of the form

$$c_{2,n}(y) = c_{n,0} + c_{n,1}y + c_{n,2}y^2 + \ldots + c_{n,2n+1}y^{2n+1} \quad (131)$$

This approximation can be written in the form

$$c_{2,n}(y) = \frac{\pi^2}{4}(1-y)\left[1 - \frac{y}{r_1}\right]\left[1 - \frac{y}{r_1^*}\right]\left[1 - \frac{y}{r_2}\right][1 - \frac{y}{r_2^*}] \cdots [1 - \frac{y}{r_n}]\left[1 - \frac{y}{r_n^*}\right] \quad (132)$$

It then follows that

$$\begin{aligned} \text{acos}(y)^{2k} \approx c_{2k,n}(y) = \;& \frac{\pi^{2k}}{2^{2k}}(1-y)^k \left[1 - \frac{y}{r_1}\right]^k \left[1 - \frac{y}{r_1^*}\right]^k \cdot \\ & \left[1 - \frac{y}{r_2}\right]^k \left[1 - \frac{y}{r_2^*}\right]^k \cdots \left[1 - \frac{y}{r_n}\right]^k \left[1 - \frac{y}{r_n^*}\right]^k \end{aligned} \quad (133)$$

The approximation, $s_{2k,n}(y)$, for $\text{asin}(y)^{2k}$ arises from the relationship $\text{asin}(y) = \text{acos}\left[\sqrt{1-y^2}\right]$. □

7.4.4. Approximations for Arccosine Squared

The second order approximation for $\text{acos}(y)^2$ is

$$\begin{aligned} c_{2,2}(y) = \;& \frac{\pi^2}{4} \cdot (1-y)\left[1 - \frac{y}{r_{21}}\right]\left[1 - \frac{y}{r_{21}^*}\right]\left[1 - \frac{y}{r_{22}}\right][1 - \frac{y}{r_{22}^*}] \\ & r_{21} = \frac{-1953}{2500} + \frac{j4507}{2000} \qquad r_{22} = \frac{12{,}833}{5000} + \frac{j8339}{5000} \end{aligned} \quad (134)$$

where $j = \sqrt{-1}$. The relative error bound for this approximation, over the interval [0, 1], is 3.66×10^{-4}. The fourth and sixth order approximations are detailed in Appendix K and have the respective relative error bounds of 2.48×10^{-6} and 2.25×10^{-8}. By using higher resolution in the approximations to the roots, slightly lower relative error bounds can be achieved. The stated root approximations represent a good compromise between accuracy and complexity.

7.4.5. Results

The relative error bounds associated with the nth order approximations for $\text{acos}(y)^{2k}$ and $\text{asin}(y)^{2k}$ are detailed in Table 7.

Table 7. Relative error bounds, over the interval [0, 1], for the approximations detailed in Theorem 9 for $\mathrm{acos}(y)^{2k}$ and $\mathrm{asin}(y)^{2k}$.

Order, n, of Approx.	Precision: Digits in Roots	Relative Error Bound: k = 1	Relative Error Bound: k = 2	Relative Error Bound: k = 3
2	5	3.66×10^{-4}	7.32×10^{-4}	1.10×10^{-3}
4	8	2.48×10^{-6}	4.96×10^{-6}	7.43×10^{-6}
6	9	2.25×10^{-8}	4.49×10^{-8}	6.74×10^{-8}
8	11	2.28×10^{-10}	4.55×10^{-10}	6.83×10^{-10}
10	13	2.93×10^{-12}	5.85×10^{-12}	8.78×10^{-12}

7.4.6. Comparison with Published Results

Borwein [23] details approximations for even powers of arcsine and approximations for powers of two, four, six, eight and ten are detailed in Appendix L. The approximation for arcsine to the sixth power is

$$asin(y)^6 \approx S_{6,n}(y) = \frac{45}{4}\sum_{k=1}^{n}\left[\sum_{m=1}^{k-1}\frac{1}{m^2}\cdot\sum_{p=1}^{m-1}\frac{1}{p^2}\right]\cdot\frac{2^{2k}[k!]^2 y^{2k}}{k^2(2k)!} \tag{135}$$

As an example, the relative error in approximations for $asin(y)^6$, as defined by $s_{6,n}(y)$ (Equation (130)) and the Borwein approximation $S_{6,n}(y)$, are shown in Figure 15. The clear advantage of the root based approach over the series defined by $S_{6,n}(y)$ is evident. In particular, the root based approximations are sharp at the point one.

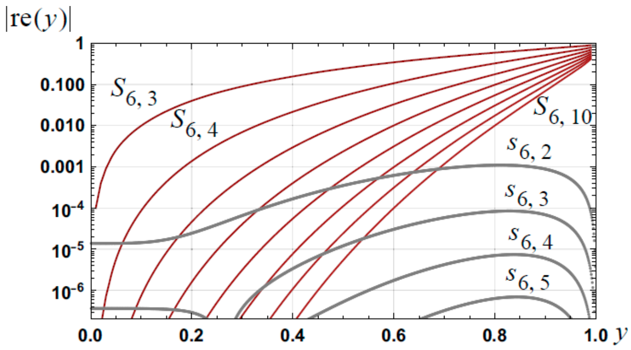

Figure 15. Graph of the relative error in approximations to $asin(y)^6$, as defined by $S_{6,n}(y)$ for $n \in \{3,4,5,6,7,8,9,10\}$, along with root based approximations $s_{6,n}(y)$ of orders 2, 3, 4, 5.

7.5. Approximations for the Inverse Tangent Integral Function

The inverse tangent integral function is defined according to

$$T(y) = \int_0^y \frac{atan(\lambda)}{\lambda} d\lambda \tag{136}$$

and an explicit series form (e.g., Mathematica) is

$$\begin{aligned} T(y) &= \frac{1}{2j}\cdot[\mathrm{Li}_2(jy) - \mathrm{Li}_2(-jy)],\ \ j = \sqrt{-1}, \\ \mathrm{Li}_n(z) &= \sum_{k=1}^{\infty}\frac{z^k}{k^n},\ \ |z| < 1,\ \ \text{analytical continuation for } |z| > 1. \end{aligned} \tag{137}$$

The Taylor series for arctangent, as given by Equation (7), leads to the *nth* order approximation, $T_{A,n}$, for T:

$$T_{A,n}(y) = [u(y) - u(y-1)\cdot \sum_{k=0}^{n} \frac{(-1)^k \cdot y^{2k+1}}{(2k+1)^2} + u(y-1)\left[\frac{\pi}{2}\cdot \ln(y) + \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2 y^{2k+1}}\right] \quad (138)$$

where u is the unit step function. The relative error in approximations, of orders one to ten, are shown in Figure 16.

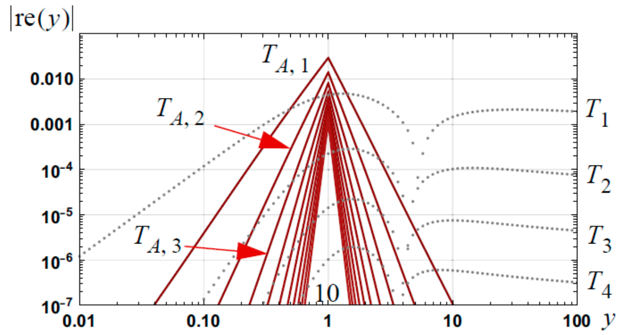

Figure 16. Graph of the relative errors in the Taylor series (orders one to ten) based approximations for the inverse tangent integral, as given by Equation (138), and the proposed approximations (orders one to four) as specified in Equation (139).

7.5.1. Inverse Tangent Integral Approximation

Based on the nth order approximation for arctangent, t_n, stated in Theorem 2, a nth order approximation to the inverse tangent integral is

$$T_n(y) = \sum_{k=0}^{2n} d_{n,k} \cdot \int_0^y \frac{1}{[1+\lambda^2]^{(k+1)/2}} d\lambda = \sum_{k=0}^{2n} d_{n,k} I_k(y), \quad y \geq 0, \quad (139)$$

where $d_{n,k}$ is defined in Theorem 2 and the integrals, $I_0,\ I_1, \cdots,\ I_5$ are defined according to

$$I_0(y) = \text{asinh}(y) = \ln\left[y + \sqrt{1+y^2}\right], \qquad I_1(y) = \text{atan}(y), \quad (140)$$

$$I_2(y) = \frac{y}{\sqrt{1+y^2}}, \qquad I_3(y) = \frac{y}{2(1+y^2)} + \frac{\text{atan}(y)}{2} \quad (141)$$

$$I_4(y) = \frac{y(1+2y^2/3)}{(1+y^2)^{3/2}}, \qquad I_5(y) = \frac{5y(1+3y^2/5)}{8(1+y^2)^2} + \frac{3\text{atan}(y)}{8} \quad (142)$$

The first order approximation, for the inverse arctangent integral, is

$$T_1(y) = \frac{\pi}{2}\cdot \ln\left[y + \sqrt{1+y^2}\right] + \left[-2 - 2\pi + \frac{3\pi^2}{4}\right]\text{atan}(y) + \left[3 + \frac{3\pi}{2} - \frac{3\pi^2}{4}\right]\cdot \frac{y}{\sqrt{1+y^2}} \quad (143)$$

Second and third order approximations are detailed in Appendix M.

7.5.2. Notes and Relative Error

The approximations, T_n, $n \in \{1, 2, \ldots\}$, are valid over the positive real line and the relative error in the approximations, of orders one to four, are shown in Figure 16. As is evident in this Figure, the approximations have a lower relative error bound than the disjointly defined Taylor series approximations defined by Equation (138). The relative error bounds associated with the approximations are detailed in Table 8.

Table 8. Relative error bounds, over the interval $[0, \infty)$, for Taylor series based approximation, and the approximations specified in Equation (139), for the inverse tangent integral function.

Order of Approx. n	Relative Error Bound: Taylor Series $T_{A,n}$	Relative Error Bound: T_n
1	2.96×10^{-2}	4.78×10^{-3}
2	1.41×10^{-2}	2.88×10^{-4}
3	8.17×10^{-3}	2.23×10^{-5}
4	5.31×10^{-3}	1.95×10^{-6}
5	3.72×10^{-3}	1.83×10^{-7}
6	2.74×10^{-3}	1.80×10^{-8}

7.5.3. Approximation of Catalan's Constant

As Catalan's constant can be defined according to

$$G = \int_0^1 \frac{\text{atan}(\lambda)}{\lambda} d\lambda \tag{144}$$

it follows that approximations for this constant, of orders two and four, can be defined according to

$$G_2 = \frac{\pi}{2} \cdot ln[1 + \sqrt{2}] + \frac{35}{6} - \frac{86\sqrt{2}}{9} + \pi\left[\frac{20}{3} - \frac{61}{4\sqrt{2}}\right] + \pi^2\left[\frac{1}{8} + \frac{55}{8\sqrt{2}}\right] - \frac{15\pi^3}{16} \tag{145}$$

$$\begin{aligned} G_4 = \frac{\pi}{2} \cdot ln[1 + \sqrt{2}] + \frac{298,369}{630} - \frac{2,609,456\sqrt{2}}{3675} + \pi\left[\frac{10,342}{21} - \frac{218,147}{224\sqrt{2}}\right] + \\ \pi^2\left[\frac{-557}{16} + \frac{14,529}{32\sqrt{2}}\right] - \frac{3465\pi^3}{64} \end{aligned} \tag{146}$$

The respective relative errors in these approximation are 2.25×10^{-4} and 1.03×10^{-6}.

7.6. Approximations for Unknown Integrals

The different forms for the approximations for arcsine, arccosine and arctangent, potentially, can lead to approximations for unknown integrals involving these functions. Four examples are detailed below.

7.6.1. Example 1

The function $4\text{acos}\left[e^{-t}\right]^2/\pi^2$ is an approximation to the unit step function for $y \geq 0$ after a transient rise time. Using the approximation form, c_n, detailed in Corollary 1 for arccosine, the approximation to the integral of this function (scaled by $\pi^2/4$) can be defined:

$$\int_0^y acos\left[e^{-t}\right]^2 dt \approx I_n(y) = \frac{\pi^2 y}{4} + \sum_{k=1}^{2n+1} \frac{c_{n,k}}{k}\left[1 - e^{-ky}\right],\ y > 0. \tag{147}$$

The third order approximation is

$$\begin{aligned} I_3(y) = \ & \left[\frac{139}{300} - \frac{271\pi}{630} - \frac{319\pi^2}{1680}\right] + \frac{\pi^2 y}{4} + \pi e^{-y} - \frac{e^{-2y}}{2} + \frac{\pi e^{-3y}}{18} + \\ & \left[\frac{-979}{180} - \frac{31\pi}{6} + \frac{35\pi^2}{16}\right] e^{-4y} + \left[\frac{944}{75} + \frac{46\pi}{5} - \frac{21\pi^2}{5}\right] e^{-5y} + \\ & \left[\frac{-48}{5} - \frac{55\pi}{9} + \frac{35\pi^2}{12}\right] e^{-6y} + \left[\frac{112}{45} + \frac{61\pi}{42} - \frac{5\pi^2}{7}\right] e^{-7y} \end{aligned} \tag{148}$$

and the relative error bound associated with this approximation, over the interval $[0, \infty)$, is 2.32×10^{-5}.

7.6.2. Example 2

Using the approximation form, t_n, detailed in Corollary 1 for arctangent, the following approximation can be defined

$$\int_0^y atan\left[\sqrt{e^t - 1}\right]^2 dt \approx I_n(y) = \frac{\pi^2 y}{4} + \sum_{k=1}^{2n+1} \frac{2c_{n,k}}{k} \cdot \left[1 - e^{-\frac{ky}{2}}\right], \ y > 0. \tag{149}$$

Mathematica, for example, specifies this integral in terms of the poly-logarithmic function. The third order approximation is

$$\begin{aligned} I_3(y) = \ & \left[\frac{139}{150} - \frac{271\pi}{315} - \frac{319\pi^2}{840}\right] + \frac{\pi^2 y}{4} + 2\pi e^{-y/2} - e^{-y} + \frac{\pi e^{-3y/2}}{9} + \\ & \left[\frac{-979}{90} - \frac{31\pi}{3} + \frac{35\pi^2}{8}\right] e^{-2y} + \left[\frac{1888}{75} + \frac{92\pi}{5} - \frac{42\pi^2}{5}\right] e^{-5y/2} + \\ & \left[\frac{-96}{5} - \frac{110\pi}{9} + \frac{35\pi^2}{6}\right] e^{-3y} + \left[\frac{224}{45} + \frac{61\pi}{21} - \frac{10\pi^2}{7}\right] e^{-7y/2} \end{aligned} \tag{150}$$

and the relative error bound associated with this approximation, over the interval $[0, \infty)$, is 2.32×10^{-5}.

7.6.3. Example 3

The following integral does not have an explicit analytical form but the approximations, t_n, detailed in Corollary 1, leads to

$$\int_0^y atan\left[\sqrt{\frac{e^{2t}}{(1+t)^2} - 1}\right]^2 dt \approx I_n(y) = \sum_{k=0}^{2n+1} c_{n,k} \int_0^y (1+t)^k e^{-kt} dt = \sum_{k=0}^{2n+1} p_k(y) e^{-ky}, \tag{151}$$

$y > 0$, where the polynomials $p_0, \ldots, p_{2n+1}$ can readily be established. For the case of $n = 2$, the relative error bound, associated the interval $[0, \infty)$, is 3.00×10^{-4}.

7.6.4. Example 4

Consider the definite integral defined by Sofo and Nimbran [27] (example 2.8, factor of $1/4$ missing):

$$I(1) = \int_0^1 t \ln(t)^2 \text{atan}(t)^2 dt \approx I_{S,n} = \frac{1}{4} \sum_{i=1}^{n} \left[\frac{(-1)^{i+1}}{i(i+1)^3} \cdot \sum_{k=1}^{i} \frac{1}{2k-1}\right] \tag{152}$$

The polynomial approximation, t_n, for arctangent detailed in Theorem 5 and for the interval $0 \leq y \leq 1$, yields

$$\begin{aligned} I(y) &= \int_0^y t\ln(t)^2 \text{atan}(t)^2\, dt \approx I_n(y) = \sum_{i=1}^{2n+1} \sum_{k=1}^{2n+1} \delta_{n,i}\delta_{n,k} \int_0^y t^{i+k+1} \ln(t)^2\, dt \\ &= \sum_{i=1}^{2n+1} \sum_{k=1}^{2n+1} \delta_{n,i}\delta_{n,k} \cdot \frac{2y^{i+k+2}}{(i+k+2)^3} \cdot \left[\begin{array}{l} 1 - [i+k+2]\ln(y) + \\ \left[2 + 2i + 2k + ik + \dfrac{i^2+k^2}{2} \right] \ln(y)^2 \end{array} \right] \end{aligned} \tag{153}$$

for $0 < y \leq 1$. For the case of $y = 1$ the approximation is

$$I_n(1) = \sum_{i=1}^{2n+1} \sum_{k=1}^{2n+1} \delta_{n,i}\delta_{n,k} \cdot \frac{2}{(i+k+2)^3} \tag{154}$$

The relative errors in the approximations $I_{S,n}$ and $I_n(1)$ are detailed in Table 9. The relative errors in the approximations $I_n(1)$, $n \in \{1, 2, \ldots, 6\}$ are shown in Figure 17. From the results shown in Table 9, it is clear that the approximations specified by Equation (154) converge significantly faster than the approximations detailed by Sofo and Nimbran [27] (Equation (152)). In addition, the approximation, t_n, for arctangent, underpins the more general approximation, as specified by Equation (153), for the integral $I(y)$, $0 < y \leq 1$.

Table 9. Table of the relative errors associated with the approximations $I_{S,n}$ and $I_n(y)$ as defined by Equations (152) and (154).

Order of Approx: n	**Relative Error in Approx: $I_{S,2n+1}$**	**Relative Error in Approx. $I_n(1)$**	**Relative Error Bound for $I_n(y)$, $0<y\leq 1$**
1	2.15×10^{-2}	3.16×10^{-2}	4.04×10^{-2}
2	5.44×10^{-3}	2.24×10^{-3}	2.96×10^{-3}
3	1.97×10^{-3}	3.18×10^{-5}	3.34×10^{-5}
4	8.85×10^{-4}	4.16×10^{-5}	5.01×10^{-5}
6	2.59×10^{-4}	6.82×10^{-7}	8.84×10^{-7}
8	1.02×10^{-4}	1.84×10^{-8}	2.30×10^{-8}
10	4.82×10^{-5}	3.48×10^{-10}	4.58×10^{-10}

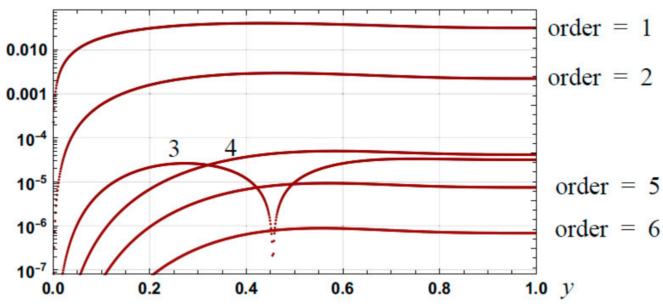

Figure 17. Graph of the relative errors in the approximations, of orders one to six, as defined by $I_n(y)$ (Equation (153)).

8. Summary and Conclusions

8.1. Summary of Results

The approximations detailed in the paper for arcsine and arctangent are tabulated, respectively, in Tables 10 and 11.

Table 10. Approximations for arcsine. The coefficients $c_{n.k}$, $d_{n.k}$ and $\delta_{n.k}$ are defined in the associated reference.

Reference	Approximation for Arcsine of Order n	Relative Error Bound for [0, 1], n = 4
Corollary 1	$\frac{\pi}{2} - \sqrt{\sum_{k=0}^{2n+1} c_{n,k} y^k}$, $\sqrt{\sum_{k=0}^{2n+1} c_{n,k} (1-y^2)^{k/2}}$	2.49×10^{-6} 1.24×10^{-6}
Theorem 2	$\frac{\pi}{2} - \sqrt{1-y^2} \sum_{k=0}^{2n} d_{n,k} y^k$, $y \sum_{k=0}^{2n} d_{n,k} (1-y^2)^{k/2}$	1.78×10^{-5} 4.72×10^{-6}
Theorem 3	$\frac{1}{y}\left[1 - \sqrt{1-y^2} + \sum_{k=0}^{2n} d_{n,k} \cdot \frac{1-(1-y^2)^{1+k/2}}{2+k}\right]$	1.00×10^{-6}
Theorem 5 (Equation (86))	$\frac{\delta_{n,1} y}{\sqrt{1-y^2}} + \frac{\delta_{n,2} y^2}{1-y^2} + \ldots + \frac{\delta_{n,2n+1} y^{2n+1}}{(1-y^2)^{n+1/2}},\ 0 \le y \le \frac{1}{\sqrt{2}}$ $\frac{\pi}{2} - \frac{\delta_{n,1}\sqrt{1-y^2}}{y} - \frac{\delta_{n,2}(1-y^2)}{y^2} - \ldots - \frac{\delta_{n,2n+1}[1-y^2]^{n+1/2}}{y^{2n+1}}$ $1/\sqrt{2} < y \le 1$	3.34×10^{-5}

Table 11. Approximations for arctangent. The coefficients $c_{n,k}$, $d_{n,k}$ and $\delta_{n,k}$ are defined in the associated reference.

Reference	Approximation for Arctangent of Order n	Relative Error Bound for [0,∞), n=4
Corollary 1	$\sqrt{\sum_{k=0}^{2n+1} \frac{c_{n,k}}{(1+y^2)^{k/2}}}$, $\frac{\pi}{2} - \sqrt{\sum_{k=0}^{2n+1} \frac{c_{n,k} y^k}{(1+y^2)^{k/2}}}$	1.24×10^{-6} 2.49×10^{-6}
Theorem 2	$y \sum_{k=0}^{2n} \frac{d_{n,k}}{(1+y^2)^{(k+1)/2}}$, $\frac{\pi}{2} - \sum_{k=0}^{2n} \frac{d_{n,k} y^k}{(1+y^2)^{(k+1)/2}}$	4.72×10^{-6} 1.78×10^{-5}
Theorem 3	$\frac{\sqrt{1+y^2}}{y}\left[1 - \frac{1}{\sqrt{1+y^2}} + \sum_{k=0}^{2n} \frac{d_{n,k}}{2+k} \cdot \left[1 - \frac{1}{(1+y^2)^{1+\frac{k}{2}}}\right]\right]$	1.00×10^{-6}
Theorem 5	$\delta_{n,1} y + \delta_{n,2} y^2 + \ldots + \delta_{n,2n+1} y^{2n+1},\ 0 \le y \le 1$ $\frac{\pi}{2} - \frac{\delta_{n,1}}{y} - \frac{\delta_{n,2}}{y^2} - \ldots - \frac{\delta_{n,2n+1}}{y^{2n+1}},\quad 1 < y < \infty$	3.34×10^{-5}
Theorem 6	$\frac{2}{1+y^2}\left[\frac{y}{2} + \begin{cases} \frac{\delta_{n,1} y^3}{3} + \frac{\delta_{n,2} y^4}{4} + \ldots + \frac{\delta_{n,2n+1} y^{2n+3}}{2n+3},\ 0 \le y \le 1 \\ \left[\frac{\delta_{n,1}}{3} + \ldots + \frac{\delta_{n,2n+1}}{2n+3}\right] + \frac{\pi(y^2-1)}{4} - \delta_{n,1}(y-1) - \\ \delta_{n,2}\ln(y) - \delta_{n,3}\left[1-\frac{1}{y}\right] - \ldots - \frac{\delta_{n,2n+1}}{2n-1}\left[1 - \frac{1}{y^{2n-1}}\right] \\ 1 < y < \infty \end{cases}\right.$	6.34×10^{-6}

For arcsine, the approximation form, s_n^A detailed in Theorem 2, can be written in the simple form

$$s_n^A(y) = y\left[p_1(y) + p_2(y)\sqrt{1-y^2}\right] \tag{155}$$

where p_1 and p_2 are polynomial functions. The approximation s_n, detailed in Theorem 3, has the lowest relative error bound for a set order (e.g., order four).

8.2. Conclusions

Based on the geometry of a radial function, and the use of a two point spline approximation, approximations of arbitrary accuracy, for arcsine, arccosine and arctangent, can be specified. Explicit expressions for the coefficients used in the approximations were detailed and convergence was proved. The approximations for arcsine and arccosine are sharp at the point zero and one and have a defined relative error bound for the interval $[0, 1]$. Alternative approximations were established based on a known integration result and a known differentiation result. The approximations have the forms detailed in Tables 10 and 11.

By utilizing the anti-symmetric relationship for arctangent around the point one, a two point spline approximation was used to establish approximations for this function as well as for arcsine and arccosine. Alternative approximations were established by using a known integral result.

Iteration utilizing the Newton-Raphson method, and based on any of the proposed approximations, yields results with significantly higher accuracy. The approximations exhibit quadratic convergence with iteration.

Applications of the approximations include: first, upper and lower bounded functions, of arbitrary accuracy, for arcsine, arccosine and arctangent. Second, it was shown how to use upper and lower bounded approximations to define approximations with significantly higher accuracy. Third, it was shown that the approximation s_n^A, detailed in Corollary 1, leads to a simple approximation form for the square of arcsine which has better convergence than established series for this function. By utilizing the roots of the square of the approximations to arccosine detailed in Corollary 1, it was shown how approximations to arccosine and arcsine, to even power orders, can be established. It was shown that the relative error bounds associated with such approximations are significantly lower that published approximations. Fourth, approximations for the inverse tangent integral function were proposed which have significantly lower relative error bounds over the interval $[0, \infty)$, than established Taylor series based approximations. Fifth, the approximation forms for arccosine and arctangent were utilized to establish approximations to several unknown integrals.

Funding: This research received no external funding.

Institutional Review Board Statement: Not relevant.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Approximations Based on Angle Subdivision

Given the coordinate (x, y) of a point on the first quadrant of the unit circle, and the corresponding angle θ, as defined by $\theta = acos(x)$ and $\theta = asin(y)$, the following definitions can be made:

$$s_i = \sin\left(\frac{\theta}{2^i}\right), \quad c_i = cos\left(\frac{\theta}{2^i}\right), \quad i \in \{0, 1, \ldots\}. \tag{A1}$$

Algorithms for determining s_i and c_i arise from half-angle formulas and are:

$$\begin{aligned} s_i &= \frac{1}{\sqrt{2}} \cdot \sqrt{1 - c_{i-1}}, \quad c_i = \frac{1}{\sqrt{2}} \cdot \sqrt{1 + c_{i-1}}, \qquad i \in \{1, 2, \ldots\} \\ s_0 &= y = \sqrt{1 - x^2} = \sin(\theta), \quad c_0 = x = \sqrt{1 - y^2} = cos(\theta) \end{aligned} \tag{A2}$$

The following result can be proved, following the approach detailed in [15] (Section 6.4 and Appendix I).

Theorem A1. *Approximation for Arcsine and Arccosine.*

Approximations for $asin(y)$ and $acos(x)$, of order n, are:

$$asin(y) \approx s_{i,n}(y) = 2^i \sum_{k=0}^{n} d_{n,k} s_i^{k+1}\left(\sqrt{1 - y^2}\right) \left[p[k, 0] + \frac{(-1)^k p\left[k, s_i\left(\sqrt{1 - y^2}\right)\right]}{\left[1 - s_i^2\left(\sqrt{1 - y^2}\right)\right]^{(2k+1)/2}} \right] \tag{A3}$$

$$\mathrm{acos}(x) \approx c_{i,n}(x) = 2^i \sum_{k=0}^{n} d_{n,k}\left[1 - c_i^2(x)\right]^{(k+1)/2} \left[p[k,0] + \frac{(-1)^k\, p\left[k,\ \sqrt{1 - c_i^2(x)}\right]}{c_i^{2k+1}(x)}\right] \quad \text{(A4)}$$

where

$$p(k,t) = \left(1 - t^2\right)\frac{d}{dt}p(k-1,\ t) + (2k-1)tp(k,t), \qquad p(0,t) = 1 \quad \text{(A5)}$$

$$d_{n,k} = \frac{n!}{(n-k)!(k+1)!} \cdot \frac{(2n+1-k)!}{2 \cdot (2n+1)!} \quad \text{(A6)}$$

Proof. The angle $\theta/2^i$ can be defined according to the standard path length formula along the unit circle from the point $(0,\ 1)$ to the point (s_i, c_i) (the point consistent with the angle $\pi/2 - \theta/2^i$):

$$\frac{\theta}{2^i} = \int_0^{s_i} \frac{1}{\sqrt{1-\lambda^2}}d\lambda = \int_0^{\sqrt{1-c_i^2}} \frac{1}{\sqrt{1-\lambda^2}}d\lambda, \qquad i \in \{1,2,\ \ldots\}. \quad \text{(A7)}$$

The integral can be approximated by using the general integral approximation [15] (eqn. 14):

$$\int_\alpha^t f(\lambda)d\lambda \ \approx\ \sum_{k=0}^{n} d_{n,\,k}(t-\alpha)^{k+1}\left[f^{(k)}(\alpha) + (-1)^k f^{(k)}(t)\right] \quad \text{(A8)}$$

where for the case being considered

$$f(t) = \frac{1}{\sqrt{1-t^2}}, \qquad f^{(k)}(t) = \frac{p(k,t)}{[1-t^2]^{(2k+1)/2}}, \quad k \in \{0,\ 1,\ \ldots\}. \quad \text{(A9)}$$

Here, $p(k,\ t)$ is specified by Equation (A5). For the case of $\alpha = 0$ and $t = s_i$ or $t = \sqrt{1-c_i^2}$, Equation (A8), respectively, leads to the required results:

$$\begin{aligned} \theta = \mathrm{asin}(y) \quad &\approx s_{i,n}(y) \\ &= 2^i \sum_{k=0}^{n} d_{n,k} s_i^{k+1}\left(\sqrt{1-y^2}\right)\left[p[k,0] + \frac{(-1)^k p\left[k, s_i\left(\sqrt{1-y^2}\right)\right]}{\left[1 - s_i^2\left(\sqrt{1-y^2}\right)\right]^{(2k+1)/2}}\right] \end{aligned} \quad \text{(A10)}$$

$$\begin{aligned} \theta = \mathrm{acos}(x) \quad &\approx c_{i,n}(x) \\ &= 2^i \sum_{k=0}^{n} d_{n,k}\left[1 - c_i^2(x)\right]^{(k+1)/2}\left[p[k,0] + \frac{(-1)^k\, p\left[k,\ \sqrt{1-c_i^2(x)}\right]}{c_i^{2k+1}(x)}\right] \end{aligned} \quad \text{(A11)}$$

□

Explicit Approximations for Arccosine

Some examples of the approximations for arccosine, as specified by Equation (A4), are detailed below: First, based on $\theta/2$, the second order spline approximation yields

$$c_{1,2}(x) = \frac{121\sqrt{1-x}}{120\sqrt{2}} \cdot \left[1 - \frac{x}{121}\right] + \frac{\sqrt{1-x}}{(1+x)^{5/2}} \cdot \left[\frac{13}{15} + \frac{19x}{10} + \frac{37x^2}{30}\right] \quad \text{(A12)}$$

which has a relative error bound, for the interval $[0,1]$ of 5.56×10^{-3}. Second, based on $\theta/4$, the second order spline approximation yields

$$\begin{aligned} c_{2,2}(x) = {} & \frac{121\sqrt{2-\sqrt{2}\sqrt{1+x}}}{120} \cdot \left[1 - \frac{\sqrt{1+x}}{121\sqrt{2}}\right] + \\ & \frac{\sqrt{2-\sqrt{2}\sqrt{1+x}}}{\left[2+\sqrt{2}\sqrt{1+x}\right]^{5/2}} \cdot \left[\frac{178}{15} + \frac{74x}{15} + \frac{38\sqrt{2}\sqrt{1+x}}{5}\right. \end{aligned} \tag{A13}$$

which has a relative error bound, for the interval $[0,1]$, of 1.71×10^{-5}. Third, based on $\theta/16$, the first order spline approximation yields

$$c_{4,1}(x) = 4\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}\sqrt{1+x}}}}\left[1 + \frac{10\left[1 + \frac{7}{10}\sqrt{2+\sqrt{2+\sqrt{2}\sqrt{1+x}}}\right]}{3\left[2+\sqrt{2+\sqrt{2+\sqrt{2}\sqrt{1+x}}}\right]^{3/2}}\right] \tag{A14}$$

which has a relative error bound, for the interval $[0,1]$, of 1.19×10^{-6}.

Appendix B. Explicit Approximations for Radial Function

Approximations for r^2, as specified by Theorem 1 and of orders one to six, are detailed below with the coefficients $C_{n,k}$, $k \in \{0,1,\ldots,2n+1\}$, being specified in Table A1:

$$f_1(y) = \frac{\pi^2}{4} - \pi y + C_{1,2}y^2 + C_{1,3}y^3 \tag{A15}$$

$$f_2(y) = \frac{\pi^2}{4} - \pi y + 2y^2 + C_{2,3}y^3 + C_{2,4}y^4 + C_{2,5}y^5 \tag{A16}$$

$$f_3(y) = \frac{\pi^2}{4} - \pi y + 2y^2 - \frac{\pi y^3}{6} + C_{3,4}y^4 + C_{3,5}y^5 + C_{3,6}y^6 + C_{3,7}y^7 \tag{A17}$$

$$f_4(y) = \frac{\pi^2}{4} - \pi y + 2y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} + C_{4,5}y^5 + C_{4,6}y^6 + C_{4,7}y^7 + C_{4,8}y^8 + C_{4,9}y^9 \tag{A18}$$

$$\begin{aligned} f_5(y) = {} & \frac{\pi^2}{4} - \pi y + 2y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} - \frac{3\pi y^5}{40} + C_{5,6}y^6 + C_{5,7}y^7 + C_{5,8}y^8 + \\ & C_{5,9}y^9 + C_{5,10}y^{10} + C_{5,11}y^{11} \end{aligned} \tag{A19}$$

$$\begin{aligned} f_6(y) = {} & \frac{\pi^2}{4} - \pi y + 2y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} - \frac{3\pi y^5}{40} + \frac{8y^6}{45} + C_{6,7}y^7 + C_{6,8}y^8 + C_{6,9}y^9 + \\ & C_{6,10}y^{10} + C_{6,11}y^{11} + C_{6,12}y^{12} + C_{6,13}y^{13} \end{aligned} \tag{A20}$$

Table A1. Table of coefficients. The lower order coefficients that are not listed are defined according to $C_{n,k} = C_{n-1,k}$, $k \in \{0, 1, \ldots, n-1\}$.

Order of Approx.	Coefficients
0	$C_{0,0} = \frac{\pi^2}{4}$, $C_{0,1} = 1 - \frac{\pi^2}{4}$
1	$C_{1,1} = -\pi$, $C_{1,2} = 3 + 2\pi - \frac{3\pi^2}{4}$, $C_{1,3} = -2 - \pi + \frac{\pi^2}{2}$
2	$C_{2,2} = 2$, $C_{2,3} = \frac{16}{3} + 6\pi - \frac{5\pi^2}{2}$ $C_{2,4} = \frac{-35}{3} - 8\pi + \frac{15\pi^2}{4}$, $C_{2,5} = \frac{16}{3} + 3\pi - \frac{3\pi^2}{2}$
3	$C_{3,3} = -\frac{\pi}{6}$, $C_{3,4} = \frac{979}{45} + \frac{62\pi}{3} - \frac{35\pi^2}{4}$, $C_{3,5} = \frac{-944}{15} - 46\pi + 21\pi^2$ $C_{3,6} = \frac{288}{5} + \frac{110\pi}{3} - \frac{35\pi^2}{2}$, $C_{3,7} = \frac{-784}{45} - \frac{61\pi}{6} + 5\pi^2$
4	$C_{4,4} = \frac{1}{3}$, $C_{4,5} = \frac{8704}{105} + \frac{145\pi}{2} - \frac{63\pi^2}{2}$ $C_{4,6} = \frac{-19,624}{63} - \frac{692\pi}{3} + 105\pi^2$, $C_{4,7} = \frac{45,056}{105} + \frac{575\pi}{2} - 135\pi^2$ $C_{4,8} = \frac{-27,508}{105} - 164\pi + \frac{315\pi^2}{4}$, $C_{4,9} = \frac{18,944}{315} + \frac{215\pi}{6} - \frac{35\pi^2}{2}$
5	$C_{5,5} = \frac{-3\pi}{40}$, $C_{5,6} = \frac{166,792}{525} + \frac{15,707\pi}{60} - \frac{231\pi^2}{2}$, $C_{5,7} = \frac{-66,304}{45} - \frac{8689\pi}{8} + 495\pi^2$ $C_{5,8} = \frac{854,948}{315} + \frac{3715\pi}{2} - \frac{3465\pi^2}{4}$, $C_{5,9} = \frac{-87,552}{35} - \frac{38,947\pi}{24} + 770\pi^2$ $C_{5,10} = \frac{364,288}{315} + \frac{14,409\pi}{20} - \frac{693\pi^2}{2}$, $C_{5,11} = \frac{-338,176}{1575} - \frac{5183\pi}{40} + 63\pi^2$
6	$C_{6,6} = \frac{8}{45}$, $C_{6,7} = \frac{63,125,504}{51,975} + \frac{9611\pi}{10} - 429\pi^2$ $C_{6,8} = \frac{-116,868,932}{17,325} - \frac{24,642\pi}{5} + \frac{9009\pi^2}{4}$, $C_{6,9} = \frac{6,002,688}{385} + \frac{43,043\pi}{4} - 5005\pi^2$ $C_{6,10} = \frac{-200,238,464}{10,395} - \frac{63,684\pi}{5} + 6006\pi^2$, $C_{6,11} = \frac{46,544,896}{3465} + 8589\pi - 4095\pi^2$ $C_{6,12} = \frac{-86,876,288}{17325} - \frac{46,814\pi}{15} + \frac{3003\pi^2}{2}$, $C_{6,13} = \frac{40,687,616}{51,975} + \frac{19,061\pi}{40} - 231\pi^2$

Appendix C. Explicit Approximations for Arccosine

Explicit approximations for arccosine, of orders three to six and arising from Corollary 1, are:

$$c_3(y) = \sqrt{\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + c_{3,4}y^4 + c_{3,5}y^5 + c_{3,6}y^6 + c_{3,7}y^7} \tag{A21}$$

$$c_4(y) = \sqrt{\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} + c_{4,5}y^5 + c_{4,6}y^6 + c_{4,7}y^7 + c_{4,8}y^8 + c_{4,9}y^9} \tag{A22}$$

$$c_5(y) = \sqrt{\begin{gathered}\frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} - \frac{3\pi y^5}{40} + c_{5,6}y^6 + c_{5,7}y^7 + c_{5,8}y^8 \\ + c_{5,9}y^9 + c_{5,10}y^{10} + c_{5,11}y^{11}\end{gathered}} \tag{A23}$$

$$c_6(y) = \sqrt{\begin{array}{c} \frac{\pi^2}{4} - \pi y + y^2 - \frac{\pi y^3}{6} + \frac{y^4}{3} - \frac{3\pi y^5}{40} + \frac{8y^6}{45} + c_{6,7}y^7 + c_{6,8}y^8 \\ +c_{6,9}y^9 + c_{6,10}y^{10} + c_{6,11}y^{11} + c_{6,12}y^{12} + c_{6,13}y^{13} \end{array}} \quad \text{(A24)}$$

Appendix D. Approximations for Arcsine of Orders Three to Four

Approximations for arcsine, of orders three and four and arising from Theorem 2, are:

$$s_3(y) = \frac{\pi}{2} - \sqrt{1-y^2} \cdot \left[\frac{\pi}{2} - y + \frac{\pi y^2}{4} + d_{3,3}y^3 + d_{3,4}y^4 + d_{3,5}y^5 + d_{3,6}y^6\right]$$
$$s_3^A(y) = y\left[\begin{array}{c} \frac{\pi}{2} - \sqrt{1-y^2} + \frac{\pi(1-y^2)}{4} + d_{3,3}\left(1-y^2\right)^{3/2} + d_{3,4}\left(1-y^2\right)^2 + \\ d_{3,5}\left(1-y^2\right)^{5/2} + d_{3,6}\left(1-y^2\right)^3 \end{array}\right] \quad \text{(A25)}$$

$$\begin{array}{ll} d_{3,3} = \frac{-1958}{45} - \frac{124\pi}{3} + \frac{35\pi^2}{2}, & d_{3,4} = \frac{472}{3} + 115\pi - \frac{105\pi^2}{2}, \\ d_{3,5} = \frac{-864}{5} - 110\pi + \frac{105\pi^2}{2}, & d_{3,6} = \frac{2744}{45} + \frac{427\pi}{12} - \frac{35\pi^2}{2}. \end{array} \quad \text{(A26)}$$

$$s_4(y) = \frac{\pi}{2} - \sqrt{1-y^2} \cdot \left[\begin{array}{c} \frac{\pi}{2} - y + \frac{\pi y^2}{4} - \frac{2y^3}{3} + d_{4,4}y^4 + d_{4,5}y^5 + d_{4,6}y^6 + \\ d_{4,7}y^7 + d_{4,8}y^8 \end{array}\right]$$
$$s_4^A(y) = y\left[\begin{array}{c} \frac{\pi}{2} - \sqrt{1-y^2} + \frac{\pi(1-y^2)}{4} - \frac{2\left(1-y^2\right)^{3/2}}{3} + d_{4,4}\left(1-y^2\right)^2 + \\ d_{4,5}\left(1-y^2\right)^{5/2} + d_{4,6}\left(1-y^2\right)^3 + d_{4,7}\left(1-y^2\right)^{7/2} + d_{4,8}\left(1-y^2\right)^4 \end{array}\right] \quad \text{(A27)}$$

$$\begin{array}{ll} d_{4,4} = \frac{-4352}{21} - \frac{725\pi}{4} + \frac{315\pi^2}{4}, & d_{4,5} = \frac{19,624}{21} + 692\pi - 315\pi^2, \\ d_{4,6} = \frac{-22,528}{15} - \frac{4025\pi}{4} + \frac{945\pi^2}{2}, & d_{4,7} = \frac{110,032}{105} + 656\pi - 315\pi^2, \\ d_{4,8} = \frac{-9472}{35} - \frac{645\pi}{4} + \frac{315\pi^2}{4}. & \end{array} \quad \text{(A28)}$$

Appendix E. Proof of Theorem 4

Consider the differential equation stated in Equation (68):

$$\left(1-y^2\right)\left[f_n^{(1)}(y) + \varepsilon_n^{(1)}(y) - 2y\right]^2 - 4\left[f_n(y) + \varepsilon_n(y) - y^2\right] = 0 \quad \text{(A29)}$$

and the nth order approximation, f_n, detailed in Theorem 1: $f_n(y) = C_{n,0} + C_{n,1}y + \ldots + C_{n,2n+1}y^{2n+1}$. As $\varepsilon_n(0) = 0$, the following form for the error function is assumed:

$$\begin{array}{c} \varepsilon_n(y) = [k_{n,1} - C_{n,1}]y + [k_{n,2} - C_{n,2} + 1]y^2 + [k_{n,3} - C_{n,3}]y^3 + \ldots + \\ [k_{n,2n+1} - C_{n,2n+1}]y^{2n+1} + k_{n,2n+2}y^{2n+2} + \ldots \end{array} \quad \text{(A30)}$$

with unknown coefficients $k_{n,1}, k_{n,2}, \cdots$. Use of this form in Equation (A29) leads to

$$\begin{array}{c} (1-y^2)\left[\begin{array}{c} k_{n,1} + 2k_{n,2}y + \ldots + (2n+1)k_{n,2n+1}y^{2n} + \\ (2n+2)k_{n,2n+2}y^{2n+1} + \ldots \end{array}\right]^2 - \\ 4\left[C_{n,0} + k_{n,1}y + k_{n,2}y^2 + \ldots + k_{n,2n+1}y^{2n+1} + k_{n,2n+2}y^{2n+2} + \ldots\right] = 0 \end{array} \quad \text{(A31)}$$

i.e.,

$$\left(1-y^2\right)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ijk_{n,i}k_{n,j}y^{i+j-2} - 4C_{n,0} - 4\sum_{i=1}^{\infty} k_{n,i}y^i = 0 \quad \text{(A32)}$$

As $C_{n,o} = \pi^2/4$, $n \in \{0,1,2,\cdots\}$, it follows that the coefficients $k_{n,i}$, $i \in \{1,2,\cdots\}$, are independent of n, leading to

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ijk_ik_jy^{i+j-2} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ijk_ik_jy^{i+j} - 4C_{n,0} - 4\sum_{i=1}^{\infty} k_iy^i = 0 \quad \text{(A33)}$$

By sequentially considering the coefficients of $y^0,\ y,\ y^2\cdots$, the constants k_i, $i \in \{1,2,\cdots\}$, can be determined. First, the coefficient of y^0 yields $k_1^2 = 4\,C_{n,0}$, leading to $k_1 = \pm 2\sqrt{C_{n,0}} = \pm\pi$. The negative solution is required as $\varepsilon_n(y) = [k_1 - C_{n,1}]y + \ldots$ and $C_{n,1} = -\pi$. Second, the coefficient of y yields $4k_1k_2 - 4k_1 = 0$, leading to $k_2 = 1$. Third, the coefficient of y^2 yields $6k_1k_3 + 4k_2^2 - k_1^2 - 4k_2 = 0$, leading to $k_3 = k_1/6 = -\pi/6$. For the general case, the coefficient of $y^{q-1},\ q \geq 3$, yields

$$\sum_{i,j\,\in\,\{1,2,\cdots\},\ i+j=q+1} ijk_ik_j - \sum_{i,j\,\in\,\{1,2,\cdots\},\ i+j=q-1} ijk_ik_j - 4k_{q-1} = 0 \quad \text{(A34)}$$

Thus:

$$\begin{aligned}&(1\cdot q)k_1k_q + 2(q-1)k_2k_{q-1} + \ldots + (q-1)(2)k_{q-1}k_2 + (q\cdot 1)k_qk_1 - \\ &\left[1(q-2)k_1k_{q-2} + 2(q-3)k_2k_{q-3} + \ldots + (q-2)(1)k_{q-2}k_1\right] - 4k_{q-1} = 0\end{aligned} \quad \text{(A35)}$$

i.e.,

$$2qk_1k_q + \sum_{u=2}^{q-1} u(q-u+1)k_uk_{q-u+1} - \sum_{u=1}^{q-2} u(q-u-1)k_uk_{q-u-1} - 4k_{q-1} = 0 \quad \text{(A36)}$$

leading to

$$k_q = \frac{4k_{q-1} - \sum_{u=2}^{q-1} u(q+1-u)k_uk_{q+1-u} + \sum_{u=1}^{q-2} u(q-u-1)k_uk_{q-u-1}}{2qk_1} \quad \text{(A37)}$$

for $q \in \{3,4,\ldots\}$.

Coefficient Values

Use of Equation (A37), for $q \geq 3$, leads to the following list of coefficient values:

$$\begin{aligned}&k_1 = -\pi, k_2 = 1, k_3 = \frac{-\pi}{6}, k_4 = \frac{1}{3}, \\ &k_5 = \frac{-3\pi}{40}, k_6 = \frac{8}{45}, k_7 = \frac{-5\pi}{112}, k_8 = \frac{4}{35}, \\ &k_9 = \frac{-35\pi}{1152}, k_{10} = \frac{128}{1575}, k_{11} = \frac{-63\pi}{2816}, k_{12} = \frac{128}{2079},\end{aligned} \quad \text{(A38)}$$

and the values are consistent with the result $k_i = C_{i,i}$, for $i \in \{1,3,4,\ldots\}$ (see Table A1 for $C_{1,1}, C_{3,3}, \ldots, C_{6,6}$). It is the case that $k_2 = C_{2,2} - 1$. These results are consistent, see Equation (A30), with the requirement that $f_n^{(i)}(0) = f^{(i)}(0)$, $i \in \{0,1,\ldots,n\}$ which implies $\varepsilon_n^{(i)}(0) = 0$, $i \in \{0,1,\ldots,n\}$.

From Equation (A30), the result $C_{n,i} = C_{i,i}$, $i \in \{1,2,\ldots,n\}$ then follows and, for $n \in \{3,4,\ldots\}$, it is the case that

$$\varepsilon_n(y) = \sum_{i=n+1}^{2n+1} [k_i - C_{n,i}]y^i + \sum_{i=2n+2}^{\infty} k_i y^i = \sum_{i=n+1}^{2n+1} [C_{i,i} - C_{n,i}]y^i + \sum_{i=2n+2}^{\infty} C_{i,i} y^i \quad \text{(A39)}$$

which is the required result.

Appendix F. Proof of Theorem 1

Consider the form for the nth order two point spline approximation, denoted f_n, to a function f as detailed in [15] (eqn. 40), and the alternative form given in [16] (eqn. 70). Based on the points zero and one, the nth order approximation is

$$f_n(y) = (1-y)^{n+1} \sum_{r=0}^{n} a_{n,r} y^r + y^{n+1} \sum_{r=0}^{n} b_{n,r} (1-y)^r, \quad \text{(A40)}$$

$$a_{n,r} = \sum_{u=0}^{r} \frac{f^{(r-u)}(0)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!}, \; b_{n,r} = \sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}(1)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!}, \quad \text{(A41)}$$

$r \in \{0,1,\ldots,n\}$, where $f(y) = r^2(y)$, $r^2(y) = acos(y)^2 + y^2$.

The sequence of numbers defined by $f^{(k)}(0)$ and $f^{(k)}(1)$, for $k \in \{0,1,2,\ldots\}$, respectively, are:

$$\frac{\pi^2}{4}, -\pi, 4, -\pi, 8, -9\pi, 128, -225\pi, 4608, -11,025\pi, 294,912, \ldots \quad \text{(A42)}$$

$$1, 0, \frac{8}{3}, \frac{-8}{15}, \frac{24}{35}, \frac{-128}{105}, \frac{640}{231}, \frac{-7680}{1001}, \frac{3584}{143}, \frac{-229,376}{2431}, \frac{18,579,456}{46,189}, \ldots \quad \text{(A43)}$$

For the first sequence, the ratios of the fifth to the third term, the seventh to the fifth term, ... are:

$$\frac{9}{1} = 3^2, \; \frac{225}{9} = 5^2, \; \frac{11,025}{225} = 49 = 7^2, \quad \cdots \quad \text{(A44)}$$

The ratios of the sixth to the fourth term, eight to the sixth term, . . . are:

$$\frac{128}{8} = 16 = 2^2{\cdot}2^2, \; \frac{4608}{128} = 36 = 2^2{\cdot}3^2, \; \frac{294,912}{4608} = 64 = 2^2{\cdot}4^2, \quad \cdots \quad \text{(A45)}$$

It then follows that the general iteration formula for $f^{(k)}(0)$ is:

$$\begin{gathered} f(0) = \frac{\pi^2}{4}, \; f^{(1)}(0) = f^{(3)}(0) = -\pi, \; f^{(2)}(0) = 4, \; f^{(4)}(0) = 8, \\ f^{(k)}(0) = (k-2)^2 f^{(k-2)}(0), \; k \in \{5,6,7,\ldots\}. \end{gathered} \quad \text{(A46)}$$

The general iteration form for $f^{(k)}(1)$ arises by considering the ratios $f^{(k)}(1)/f^{(k-1)}(1)$, for $k \in \{5,6,7,\ldots\}$, leading to:

$$\begin{gathered} f(1) = 1, \; f^{(1)}(1) = 0, \; f^{(2)}(1) = 8/3, \; f^{(3)}(1) = -8/15, \\ f^{(k)}(1) = \frac{(-1)^k (k-1)^2}{2k-1} \left| f^{(k-1)}(1) \right|, \; k \in \{4,5,6,\ldots\}. \end{gathered} \quad \text{(A47)}$$

Appendix F.1. Formula for Coefficients in Standard Polynomial Form

The goal is to write the approximation f_n, as defined by Equation (A40), in the form

$$f_n(y) = \sum_{k=0}^{2n+1} C_{n,k} y^k \tag{A48}$$

To this end, the binomial formula

$$(1-y)^i = \sum_{k=0}^{i} \frac{(-1)^k i!}{(i-k)!k!} \cdot y^k \tag{A49}$$

implies

$$\begin{aligned} f_n(y) = {} & [a_{n,0} + a_{n,1}y + \ldots + a_{n,n}y^n] \cdot \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^k + \\ & y^{n+1} \left[\begin{array}{c} b_{n,0} + b_{n,1}(1-y) + \ldots + b_{n,r} \sum_{k=0}^{r} \frac{(-1)^k r!}{(r-k)!k!} \cdot y^k + \ldots + \\ b_{n,n} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \cdot y^k \end{array} \right] \end{aligned} \tag{A50}$$

Thus:

$$\begin{aligned} f_n(y) = {} & a_{n,0} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^k + a_{n,1} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^{k+1} + \ldots + \\ & a_{n,r} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^{k+r} + \ldots + a_{n,n} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^{k+n} + \\ & b_{n,0}y^{n+1} + b_{n,1}\left(y^{n+1} - y^{n+2}\right) + \ldots + b_{n,r} \sum_{k=0}^{r} \frac{(-1)^k r!}{(r-k)!k!} \cdot y^{n+k+1} + \ldots + \\ & b_{n,n} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \cdot y^{n+k+1} \end{aligned} \tag{A51}$$

For $0 \le i \le n$, y^i is associated with the value of k in the summation $a_{n,r} \sum_{k=0}^{n+1} \frac{(-1)^k (n+1)!}{(n+1-k)!k!} \cdot y^{k+r}$ which is such that $k + r = i$, $k \ge 0$, i.e., $k = i - r$ and $0 \le r \le i$. Thus:

$$C_{n,i} = \sum_{r=0}^{i} \frac{(-1)^{i-r} (n+1)!}{(n+1+r-i)!(i-r)!} \cdot a_{n,r}, \qquad 0 \le i \le n. \tag{A52}$$

For $n + 1 \le i \le 2n + 1$, the lowest value of r, such that there is a term associated with y^i in $a_{n,r}$, satisfies the constraint $n + 1 + r = i$, i.e., $r = i - n - 1$. The term y^i is also associated with the index $n + k + 1 = i$, $k \ge 0$, in $b_{n,r}$, i.e., $k = i - n - 1$, and with the lowest value of r being consistent with $n + r + 1 = i$. Thus:

$$\begin{aligned} C_{n,i} = {} & \sum_{r=i-n-1}^{n} \frac{(-1)^{i-r} (n+1)!}{(n+1+r-i)!(i-r)!} \cdot a_{n,r} + \\ & \sum_{r=i-n-1}^{n} \frac{(-1)^{i-n-1} r!}{(r+n+1-i)!(i-n-1)!} \cdot b_{n,r}, \quad n + 1 \le i \le 2n + 1. \end{aligned} \tag{A53}$$

Appendix F.2. Nature of Coefficients

Consider $a_{n,r}$ and $C_{n,i}$ as defined by Equations (A41) and (A52), whereupon it follows that

$$a_{n,0} = f(0), \ \ a_{n,1} = (n+1)f(0) + f^{(1)}(0), \tag{A54}$$

$$C_{n,0} = a_{n,0} = f(0), \qquad C_{n,1} = -(n+1)a_{n,0} + a_{n,1} = f^{(1)}(0). \tag{A55}$$

It can readily be shown that

$$C_{n,i} = \frac{f^{(i)}(0)}{i!}, \ \ i \in \{0, 1, \ldots, n\}. \tag{A56}$$

This result is consistent with the requirement, $f_n^{(i)}(0) = f^{(i)}(0)$ for $i \in \{0, 1, \ldots, n\}$, associated with a two point spline approximation of order n.

Appendix G. Third and Fourth Order Approximations for Arctangent

Approximations for arctangent, of orders three and four and arising from Theorem 5, are:

$$t_3(y) = \begin{cases} y - \frac{y^3}{3} - \left[\frac{55}{2} - \frac{35\pi}{4}\right]y^4 + \left[\frac{265}{4} - 21\pi\right]y^5 - \left[\frac{331}{6} - \frac{35\pi}{2}\right]y^6 + \left[\frac{63}{4} - 5\pi\right]y^7, & \\ \qquad 0 \le y \le 1 & \\ \frac{\pi}{2} - \frac{1}{y} + \frac{1}{3y^3} + \left[\frac{55}{2} - \frac{35\pi}{4}\right] \cdot \frac{1}{y^4} - \left[\frac{265}{4} - 21\pi\right] \cdot \frac{1}{y^5} + \left[\frac{331}{6} - \frac{35\pi}{2}\right] \cdot \frac{1}{y^6} - & \\ \qquad \left[\frac{63}{4} - 5\pi\right] \cdot \frac{1}{y^7}, \qquad y > 1 & \end{cases} \tag{A57}$$

$$t_4(y) = \begin{cases} y - \frac{y^3}{3} - \left[\frac{395}{4} - \frac{63\pi}{2}\right]y^5 + \left[\frac{1979}{6} - 105\pi\right]y^6 - \left[\frac{1697}{4} - 135\pi\right]y^7 + & \\ \qquad \left[\frac{495}{2} - \frac{315\pi}{4}\right]y^8 - \left[55 - \frac{35\pi}{2}\right]y^9, \qquad 0 \le y \le 1 & \\ \frac{\pi}{2} - \frac{1}{y} + \frac{1}{3y^3} + \left[\frac{395}{4} - \frac{63\pi}{2}\right] \cdot \frac{1}{y^5} - \left[\frac{1979}{6} - 105\pi\right] \cdot \frac{1}{y^6} + & \\ \qquad \left[\frac{1697}{4} - 135\pi\right] \cdot \frac{1}{y^7} - \left[\frac{495}{2} - \frac{315\pi}{4}\right] \cdot \frac{1}{y^8} + \left[55 - \frac{35\pi}{2}\right] \cdot \frac{1}{y^9}, \quad y > 1 & \end{cases} \tag{A58}$$

Appendix H. Alternative Third and Fourth Order Approximations for Arctangent

Third and fourth order approximations for arctangent, and arising from Theorem 6, are:

$$t_3(y) = \begin{cases} \frac{1}{1+y^2} \left[\begin{array}{l} y + \frac{2y^3}{3} - \frac{2y^5}{15} - \left[\frac{55}{6} - \frac{35\pi}{12}\right]y^6 + \left[\frac{265}{14} - 6\pi\right]y^7 - \\ \qquad \left[\frac{331}{24} - \frac{35\pi}{8}\right]y^8 + \left[\frac{7}{2} - \frac{10\pi}{9}\right]y^9 \end{array} \right] & 0 \le y \le 1 \\ \frac{1}{1+y^2} \left[\begin{array}{l} \frac{6121}{840} - \frac{131\pi}{72} + \left[\frac{63}{10} - 2\pi\right] \cdot \frac{1}{y^5} - \left[\frac{331}{12} - \frac{35\pi}{4}\right] \cdot \frac{1}{y^4} + \\ \left[\frac{265}{6} - 14\pi\right] \cdot \frac{1}{y^3} - \left[\frac{55}{2} - \frac{35\pi}{4}\right] \cdot \frac{1}{y^2} - \frac{2}{3y} - y + \frac{\pi y^2}{2} \end{array} \right] & y > 1 \end{cases} \tag{A59}$$

$$t_4(y) = \begin{cases} \dfrac{1}{1+y^2}\left[\begin{array}{l} y + \dfrac{2y^3}{3} - \dfrac{2y^5}{15} - \left[\dfrac{395}{14} - 9\pi\right]y^7 + \left[\dfrac{1979}{24} - \dfrac{105\pi}{4}\right]y^8 - \\ \left[\dfrac{1697}{18} - 30\pi\right]y^9 + \left[\dfrac{99}{2} - \dfrac{63\pi}{4}\right]y^{10} - \left[10 - \dfrac{35\pi}{11}\right]y^{11} \end{array}\right] & 0 \le y \le 1 \\ \dfrac{1}{1+y^2}\left[\begin{array}{l} \dfrac{2339}{360} - \dfrac{69\pi}{44} - \left[\dfrac{110}{7} - 5\pi\right]\dfrac{1}{y^7} + \left[\dfrac{165}{2} - \dfrac{105\pi}{4}\right]\dfrac{1}{y^6} - \\ \left[\dfrac{1697}{10} - 54\pi\right]\dfrac{1}{y^5} + \left[\dfrac{1979}{12} - \dfrac{105\pi}{2}\right]\dfrac{1}{y^4} - \left[\dfrac{395}{6} - 21\pi\right]\dfrac{1}{y^3} - \dfrac{2}{3y} - y + \dfrac{\pi y^2}{2} \end{array}\right] & y > 1 \end{cases} \tag{A60}$$

Appendix I. Additional Approximations for Arcsine via Iteration

The third order iteration, arising from Equation (96), leads to the following approximation for arcsine:

$$\begin{aligned} h_3(y) = \; & s_n(y) - \frac{sin[s_n(y)] - y}{cos[s_n(y)]} - \frac{sin\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right] - y}{cos\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right]} - \\ & \frac{sin\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]} - \dfrac{sin\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right] - y}{cos\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right]}\right] - y}{cos\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]} - \dfrac{sin\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right] - y}{cos\left[s_n(y) - \dfrac{sin[s_n(y)] - y}{cos[s_n(y)]}\right]}\right]} \end{aligned} \tag{A61}$$

The second order iteration, based on Equation (99), leads to the following approximation for arcsine:

$$\begin{aligned} h_2(y) = \; & \frac{\pi}{2} - \sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right] - \\ & \frac{cos\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right] - y}{sin\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right]} - \\ & \frac{cos\left[\begin{array}{c} \sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right] + \\ \dfrac{cos\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right] - y}{sin\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right]} \end{array}\right] - y}{sin\left[\begin{array}{c} \sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right] + \\ \dfrac{cos\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right] - y}{sin\left[\sqrt{1-y^2}\left[\frac{\pi}{2} - y + d_{2,2}y^2 + d_{2,3}y^3 + d_{2,4}y^4\right]\right]} \end{array}\right]} \end{aligned} \tag{A62}$$

Appendix J. Proof of Theorem 7

A zero order spline approximation is simply an affine approximation between the two specified points. Consistent with the illustration of Figure 13, the zero order spline approximation, denoted f_0, to $\operatorname{asin}(y)$, is an affine approximation between the points $(\sin(u_0), u_0)$ and $(\sin(v_0), v_0)$ leading to

$$f_0(y) = u_o + [y - \sin(u_o)] \cdot \frac{v_o - u_o}{\sin(v_o) - \sin(u_o)}, \quad y \in [\sin(u_o), \sin(v_o)]. \tag{A63}$$

With the approximation $x_o = \operatorname{asin}(y_o) \approx f_o y_o$ it follows, after simplification, that

$$f_0(y_o) = \frac{u_o \sin(v_o) - v_o \sin(u_o) + y_o[v_o - u_o]}{\sin(v_o) - \sin(u_o)} \tag{A64}$$

Substitution of $u_0 = f_L(y_0)$ and $v_0 = f_U(y_0)$ yields the required result after the change in variable from y_0 to y.

General Result

Consider the general nth order spline approximation f_n to a function f over the interval $[\alpha, \beta]$, as given by [16] (eqn. 70):

$$f_n(x) = (\beta - x)^{n+1} \sum_{r=0}^{n} a_{n,r}(x - \alpha)^r + (x - \alpha)^{n+1} \sum_{r=0}^{n} b_{n,r}(\beta - x)^r \tag{A65}$$

where

$$\begin{aligned} a_{n,r} &= \frac{1}{(\beta - \alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{f^{(r-u)}(\alpha)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta - \alpha)^u}, \\ b_{n,r} &= \frac{1}{(\beta - \alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}(\beta)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta - \alpha)^u} \end{aligned} \tag{A66}$$

The general result stated in Theorem 7 arises with the definitions $f(y) = asin(y)$, and the interval $[\alpha, \beta]$ where $\alpha = sin(u_0)$, $\beta = sin(v_0)$ and $u_0 = f_L(y_0)$, $v_0 = f_U(y_0)$. The approximation is

$$\begin{aligned} f_n(y) = {} & \frac{[sin(v_o) - y]^{n+1}}{[sin(v_o) - sin(u_o)]^{n+1}} \cdot \\ & \sum_{r=0}^{n} [y - sin(u_o)]^r \cdot \left[\sum_{u=0}^{r} \frac{f^{(r-u)}[sin(u_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[sin(v_o) - sin(u_o)]^u} \right] + \\ & \frac{[y - sin(u_o)]^{n+1}}{[sin(v_o) - sin(u_o)]^{n+1}} \cdot \\ & \sum_{r=0}^{n} [sin(v_o) - y]^r \cdot \left[\sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}[sin(v_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[sin(v_o) - sin(u_o)]^u} \right] \end{aligned} \tag{A67}$$

for $y \in [sin(v_0), sin(v_0)]$ and where $f^{(k)}$ is the kth derivative of arcsine. An analytical expression for $f^{(k)}$ arises from noting that $f^{(1)}(y) = 1/\sqrt{1 - y^2}$ and that $f^{(k)}$ has the form

$$f^{(k)}(y) = \sum_{i=0}^{k-1} \frac{d[k,i] \left\lfloor \frac{1 + (-1)^{k+i+1}}{2} \right\rfloor y^i}{(1 - y^2)^{\left\lfloor \frac{k+i+1}{2} \right\rfloor - \frac{1}{2}}}, \ k \in \{1, 2, \ldots\} \tag{A68}$$

where the coefficients $d[k,i]$ are to be determined. By considering the forms for $f^{(k+1)}(y)$ and $f^{(k)}(y)$, the algorithm for the coefficients, as specified in Theorem 7, can be determined. Qi and Zheng [28] detail an alternative form for $f^{(k)}$. As $f^{(0)}[sin(u_o)] = u_o$ and $f^{(0)}[sin(v_o)] = v_o$, it then follows that

$$f_n(y) = \frac{[\sin(v_o) - y]^{n+1}}{[\sin(v_o) - \sin(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [y - \sin(u_o)]^r \cdot \left[\begin{array}{l} \frac{(n+r)!u_o}{r!n![\sin(v_o) - \sin(u_o)]^r} + \\ \sum_{u=0}^{r-1} \frac{f^{(r-u)}[\sin(u_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[\sin(v_o) - \sin(u_o)]^u} \end{array} \right] + \frac{[y - \sin(u_o)]^{n+1}}{[\sin(v_o) - \sin(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [\sin(v_o) - y]^r \cdot \left[\begin{array}{l} \frac{(n+r)!v_o}{r!n![\sin(v_o) - \sin(u_o)]^r} + \\ \sum_{u=0}^{r-1} \frac{(-1)^{r-u} f^{(r-u)}[\sin(v_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[\sin(v_o) - \sin(u_o)]^u} \end{array} \right] \quad \text{(A69)}$$

for $y \in [sin(u_o), sin(v_o)]$. The required result follows: the approximation for $asin(y_o)$ arises for the case of $y = y_o$.

Appendix K. Fourth and Sixth Order Approximations for Arccosine Squared

The fourth and sixth order approximations for arccosine squared, consistent with Theorem 9, are:

$$c_{2,4}(y) = \frac{\pi^2}{4}(1 - y) \prod_{i=1}^{4} \left[1 - \frac{y}{r_{4i}}\right] \left[1 - \frac{y}{r_{4i}^*}\right] \quad \text{(A70)}$$

$$\begin{array}{ll} r_{41} = \frac{-16,732,749}{12,500,000} + \frac{j6,808,161}{6,250,000} & r_{42} = \frac{-1,299,161}{12,500,000} + \frac{j25,525,407}{12,500,000} \\ r_{43} = \frac{1,168,741}{781,250} + \frac{j23,807,729}{12,500,000} & r_{44} = \frac{16,131,473}{6,250,000} + \frac{j9,610,843}{12,500,000} \end{array} \quad \text{(A71)}$$

$$c_{2,6}(y) = \frac{\pi^2}{4}(1 - y) \prod_{i=1}^{6} \left[1 - \frac{y}{r_{6i}}\right] \left[1 - \frac{y}{r_{6i}^*}\right] \quad \text{(A72)}$$

$$\begin{array}{ll} r_{61} = \frac{-333,602,739}{250,000,000} + \frac{j675,965,943}{10^9} & r_{62} = \frac{-788,537,601}{10^9} + \frac{j183,898,863}{125,000,000} \\ r_{63} = \frac{117,196,479}{10^9} + \frac{j117,896,643}{62,500,000} & r_{64} = \frac{1,129,571,433}{10^9} + \frac{j365,814,027}{200,000,000} \\ r_{65} = \frac{496,879,191}{250,000,000} + \frac{j82,357,137}{62,500,000} & r_{66} = \frac{1,238,163,489}{500,000,000} + \frac{j478,997,641}{10^9} \end{array} \quad \text{(A73)}$$

Appendix L. Approximations for Even Powers of Arcsine

Borwein [23] (eqn. 2.2 to 2.4) details approximations for even powers of arcsine and the approximations for powers of two, four, six, eight and ten are:

$$asin(y)^2 \approx S_{2,n}(y) = \frac{1}{2} \sum_{k=1}^{n} \frac{2^{2k}[k!]^2 y^{2k}}{k^2(2k)!} \quad \text{(A74)}$$

$$asin(y)^4 \approx S_{4,n}(y) = \frac{3}{2} \sum_{k=1}^{n} \left[\sum_{m=1}^{k-1} \frac{1}{m^2} \right] \cdot \frac{2^{2k}[k!]^2 y^{2k}}{k^2(2k)!} \quad \text{(A75)}$$

$$asin(y)^6 \approx S_{6,n}(y) = \frac{45}{4}\sum_{k=1}^{n}\left[\sum_{m=1}^{k-1}\frac{1}{m^2}\cdot\sum_{p=1}^{m-1}\frac{1}{p^2}\right]\cdot\frac{2^{2k}[k!]^2y^{2k}}{k^2(2k)!} \tag{A76}$$

$$asin(y)^8 \approx S_{8,n}(y) = \frac{315}{2}\sum_{k=1}^{n}\left[\sum_{m=1}^{k-1}\frac{1}{m^2}\cdot\sum_{p=1}^{m-1}\frac{1}{p^2}\cdot\sum_{q=1}^{p-1}\frac{1}{q^2}\right]\cdot\frac{2^{2k}[k!]^2y^{2k}}{k^2(2k)!} \tag{A77}$$

$$asin(y)^{10} \approx S_{10,n}(y) = \frac{10!}{4^5}\sum_{k=1}^{n}\left[\sum_{m=1}^{k-1}\frac{1}{m^2}\cdot\sum_{p=1}^{m-1}\frac{1}{p^2}\cdot\sum_{q=1}^{p-1}\frac{1}{q^2}\cdot\sum_{r=1}^{q-1}\frac{1}{r^2}\right]\cdot\frac{2^{2k}[k!]^2y^{2k}}{k^2(2k)!} \tag{A78}$$

Appendix M. Second and Third Order Approximations for Inverse Tangent Integral

Second and third order approximations for the inverse tangent integral are:

$$\begin{aligned} T_2(y) = \; & \frac{\pi}{2}\cdot\ln\left[y+\sqrt{1+y^2}\right] + \left[\frac{32}{3}+8\pi-\frac{15\pi^2}{4}\right]\cdot atan(y) + \\ & \left[-8-9\pi+\frac{15\pi^2}{4}\right]\cdot\frac{y}{\sqrt{1+y^2}} + \left[\frac{35}{3}+8\pi-\frac{15\pi^2}{4}\right]\cdot\frac{y}{1+y^2} + \\ & \left[\frac{-40}{3}-\frac{15\pi}{2}+\frac{15\pi^2}{4}\right]\cdot\frac{y}{(1+y^2)^{3/2}} + \left[\frac{-80}{9}-5\pi+\frac{5\pi^2}{2}\right]\cdot\frac{y^3}{(1+y^2)^{3/2}} \end{aligned} \tag{A79}$$

$$\begin{aligned} T_3(y) = \; & \frac{\pi}{2}\cdot ln\left[y+\sqrt{1+y^2}\right] + \left[\frac{-788}{9}-\frac{743\pi}{12}+\frac{455\pi^2}{16}\right]\cdot atan(y) + \frac{\pi}{4}\cdot\frac{y}{\sqrt{1+y^2}} + \\ & \left[\frac{-979}{45}-\frac{62\pi}{3}+\frac{35\pi^2}{4}\right]\cdot\frac{y}{1+y^2} + \left[\frac{472}{3}+115\pi-\frac{105\pi^2}{2}\right]\cdot\frac{y}{(1+y^2)^{3/2}} + \\ & \left[\frac{944}{9}+\frac{230\pi}{3}-35\pi^2\right]\cdot\frac{y^3}{(1+y^2)^{3/2}} + \left[-108-\frac{275\pi}{4}+\frac{525\pi^2}{16}\right]\cdot\frac{y}{(1+y^2)^2} \\ & \left[\frac{-324}{5}-\frac{165\pi}{4}+\frac{315\pi^2}{16}\right]\cdot\frac{y^3}{(1+y^2)^2} + \left[\frac{2744}{45}+\frac{427\pi}{12}-\frac{35\pi^2}{2}\right]\cdot\frac{y}{(1+y^2)^{5/2}} + \\ & \left[\frac{10,976}{135}+\frac{427\pi}{9}-\frac{70\pi^2}{3}\right]\cdot\frac{y^3}{(1+y^2)^{5/2}} + \left[\frac{21,952}{675}+\frac{854\pi}{45}-\frac{28\pi^2}{3}\right]\cdot\frac{y^5}{(1+y^2)^{5/2}} \end{aligned} \tag{A80}$$

References

1. Boyer, C.B. *A History of Mathematics*; John Wiley: New York, NY, USA, 1991.
2. Bercu, G. The natural approach of trigonometric inequalities—Padé approximant. *J. Math. Inequalities* **2017**, *11*, 181–191. [CrossRef]
3. Howard, R.M. Spline based series for sine and arbitrarily accurate bounds for sine, cosine and sine integral. *arXiv* **2020**. [CrossRef]
4. Stroethoff, K. Bhaskara's approximation for the sine. *Math. Enthus.* **2014**, *11*, 485–492. [CrossRef]
5. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. Method for solving the Sine-Gordon equation. *Phys. Rev. Lett.* **1973**, *30*, 1262–1264. [CrossRef]
6. Alkhairy, I.; Nagy, M.; Muse, A.H.; Hussam, E. The arctan-X family of distributions: Properties, simulation, and applications to actuarial sciences. *Complexity* **2021**, *2021*, 4689010. [CrossRef]
7. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 7th ed.; Jeffery, A., Zwillinger, D., Eds.; Academic Press: Washington, DC, USA, 2007.
8. Roy, R.; Olver, F.W.J. Elementary functions. In *NIST Handbook of Mathematical Functions*; Olver, F.W., Lozier, D.W., Boisvert, R.F., Clark, C.W., Eds.; National Institute of Standards and Technology: Gaithersburg, MD, USA; Cambridge University Press: Cambridge, UK, 2010; Chapter 4.
9. Gómez-Déniz, E.; Sarabia, J.M.; Calderín-Ojeda, E. The geometric arcTan distribution with applications to model demand for health services. *Commun. Stat. - Simul. Comput.* **2019**, *48*, 1101–1120. [CrossRef]
10. Scott, J.A. Another series for the inverse tangent. *Math. Gaz.* **2011**, *95*, 518–520. [CrossRef]
11. Bradley, D.M. A class of series acceleration formulae for Catalan's constant. *Ramanujan J.* **1999**, *3*, 159–173. [CrossRef]
12. Wu, S.; Bercu, G. Padé approximants for inverse trigonometric functions and their applications. *J. Inequalities Appl.* **2017**, *2017*, 31. [CrossRef]
13. Fink, A.M. Two inequalities. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat* **1995**, *6*, 49–50.
14. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*; Dover: Garden City, NY, USA, 1964.

15. Howard, R.M. Dual Taylor series, spline based function and integral approximation and applications. *Math. Comput. Appl.* **2019**, *24*, 35. [CrossRef]
16. Howard, R.M. Analytical approximations for the principal branch of the Lambert *W* function. *Eur. J. Math. Anal.* **2022**, *2*, 14. [CrossRef]
17. Bercu, G. Sharp refinements for the inverse sine function related to Shafer-Fink's inequality. *Math. Probl. Eng.* **2017**, *2017*, 9237932. [CrossRef]
18. Guo, B.N.; Luo, Q.M.; Qi, F. Sharpening and generalizations of Shafer-Fink's double inequality for the arc sine function. *Filomat* **2013**, *27*, 261–265. [CrossRef]
19. Maleševí, B.; Rašajski, M.; Lutovac, T. Refinements and generalizations of some inequalities of Shafer-Fink's type for the inverse sine function. *J. Inequalities Appl.* **2017**, *2017*, 275. [CrossRef]
20. Zhu, L. New inequalities of Shafer-Fink type for arc hyperbolic sine. *J. Inequalities Appl.* **2008**, *2008*, 368275. [CrossRef]
21. Zhu, L. The natural approaches of Shafer-Fink inequality for inverse sine function. *Mathematics* **2022**, *10*, 647. [CrossRef]
22. Qiao, Q.X.; Chen, C.P. Approximations to inverse tangent function. *J. Inequalities Appl.* **2018**, *141*. [CrossRef]
23. Borwein, J.M.; Chamberland, M. Integer Powers of Arcsin. *Int. J. Math. Math. Sci.* **2007**, *2007*, 19381. [CrossRef]
24. García-García, A.M.; Verbaarschot, J.J.M. Analytical spectral density of the Sachdev-Ye-Kitaev model at finite N. *Phys. Rev. D* **2017**, *96*, 066012. [CrossRef]
25. Kalmykov, M.Y.; Sheplyakov, A. Isjk—A C++ library for arbitrary-precision numeric evaluation of the generalized log-sine functions. *Comput. Phys. Commun.* **2005**, *172*, 45–59. [CrossRef]
26. Qi, F. Maclaurin's series expansions of real powers of inverse (hyperbolic) cosine and sine functions with applications. *Res. Sq.* **2021**. [CrossRef]
27. Sofo, A.; Nimbran, A.S. Euler-like sums via powers of log, arctan and arctanh functions. *Integral Transform. Spec. Funct.* **2020**, *31*, 966–981. [CrossRef]
28. Qi, F.; Zheng, M.M. Explicit expressions for a family of Bell polynomials and derivatives of some functions. *arXiv* **2014**. [CrossRef]