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Generalizations of the Eneström–Kakeya Theorem Involving Weakened Hypotheses

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Abstract: The well-known Eneström–Kakeya Theorem states that, for $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$, a polynomial of degree n with real coefficients satisfying $0 \le a_0 \le a_1 \le \cdots \le a_n$, then all the zeros of P lie in $|z| \le 1$ in the complex plane. Motivated by recent results concerning an Eneström–Kakeya "type" condition on real coefficients, we give similar results with hypotheses concerning the real and imaginary parts of the coefficients and concerning the moduli of the coefficients. In this way, our results generalize the other recent results.

Keywords: polynomials; zeros; Eneström–Kakeya Theorem; coefficient monotonicity

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1. Introduction

The classical Eneström–Kakeya Theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [1] and Sōichi Kakeya in 1912 [2].

Theorem 1 (Eneström–Kakeya Theorem). *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \le a_0 \le a_1 \le \cdots \le a_n$, then all the zeros of P lie in $|z| \le 1$.*

A huge number of generalizations of the Eneström–Kakeya Theorem exist. Most of these involve weakening the condition on the coefficients. For a survey of such results up through 2014, see [3]. For example, Govil and Rahman ([4], Theorem 2) proved the following in 1968.

Theorem 2. *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n with complex coefficients such that $|arg(a_\ell) - \beta| \le \alpha \le \pi/2$ for $0 \le \ell \le n$ for some real β and $|a_0| \le |a_1| \le \cdots \le |a_n|$, then all the zeros of P lie in*

$$|z| \le \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{\ell=0}^{n-1}|a_\ell|.$$

With $\alpha = \beta = 0$, Theorem 2 reduces to the Eneström–Kakeya Theorem.

As a corollary to a more general result, Gardner and Govil ([5], Corollary 1) presented the following result concerning a monotonicity condition on the real and imaginary parts of the coefficients.

Theorem 3. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients, where $Re(a_\ell) = \alpha_\ell$ and $Im(a_\ell) = \beta_\ell$ for $\ell = 0, 1, \ldots, n$. Suppose that $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$ and $\beta_0 \le \beta_1 \le \cdots \le \beta_n$. Then, all the zeros of P lie in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \le |z| \le \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

With $\beta_\ell = 0$ for $\ell = 0, 1, \ldots, n$, Theorem 3 implies a result of Joyal, Labelle, and Rahman ([6], Theorem 3).

Aziz and Zargar [7] gave the following result in 2012, which involves a slight generalization of the Eneström–Kakeya monotonicity condition on the real coefficients of a polynomial.

Theorem 4. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with real coefficients such that, for some positive numbers k and ρ with $k \geq 1$ and $0 < \rho \leq 1$, the coefficients satisfy $0 \leq \rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq k a_n$. Then, all the zeros of P lie in the closed disk $|z + k - 1| \leq k + 2a_0(1-\rho)/a_n$.*

With $k = \rho = 1$, Theorem 4 implies the Eneström–Kakeya Theorem.

Recently, Shah et al. [8] proved the next result, which maintains the monotonicity condition on the "central" coefficients, but imposes no condition on the "tail" coefficients.

Theorem 5. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with real coefficients such that, for some positive numbers p and q with $0 \leq q \leq p \leq n$, the coefficients satisfy $a_q \leq a_{q+1} \leq \cdots \leq a_{p-1} \leq a_p$. Then, all the zeros of P lie in the closed annulus:*

$$\min\left\{1, \frac{|a_0|}{M_q - a_q + a_p + M_p + |a_n|}\right\} \leq |z| \leq \frac{|a_0| + M_q - a_q + a_p + M_p}{|a_n|},$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

With $p = n$ and $q = 0$, Theorem 5 implies a result of Joyal, Labelle, and Rahman [6]. Additionally with $a_0 \geq 0$, it then implies the Eneström–Kakeya Theorem.

The purpose of this paper is to combine the hypotheses of Theorems 4 and 5 and apply them to polynomials with complex coefficients. We apply the hypotheses to the real and imaginary parts of the coefficients and to the moduli of the coefficients.

2. Results

By introducing the parameters k and ρ of Aziz and Zargar, along with the parameters p and q of Shah et al. and imposing the hypothesis that results on the real and imaginary parts of the coefficients of a polynomial, we obtain the following.

Theorem 6. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients. Let $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ for $0 \leq \ell \leq n$. Suppose that, for some positive numbers k_R, k_I, ρ_R, ρ_I, p, and q with $k_R \geq 1$, $k_I \geq 1$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, and $0 \leq q \leq p \leq n$, the coefficients satisfy*

$$\rho_R \alpha_q \leq \alpha_{q+1} \leq \alpha_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_R \alpha_p$$

and

$$\rho_I \beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_I \beta_p.$$

Then, all the zeros of P lie in the closed annulus:

$$\min\left\{1, |a_0| \Big/ \Big(M_q + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_R \alpha_p + (k_R - 1)|\alpha_p|\right.$$

$$+k_I\beta_p+(k_I-1)|\beta_p|+M_p+|a_n|)\Big\} \le |z| \le \Big(|a_0|+M_q+(1-\rho_R)|\alpha_q|-\rho_R\alpha_q$$

$$+(1-\rho_I)|\beta_q|-\rho_I\beta_q+(k_R-1)|\alpha_p|+k_R\alpha_p+(k_I-1)|\beta_p|+k_I\beta_p+M_p\Big)\Big/|a_n|,$$

where $M_q=\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|$ *and* $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

Notice that, when $\beta_i=0$ for $0\le i\le n$ and $k_R=\rho_R=1$, then Theorem 6 reduces to Theorem 5. When $\beta_i=0$ for $0\le i\le n$, $k_R=\rho_R=1$, $q=0$, and $p=n$, then Theorem 6 reduces to a result of Joyal, Labelle, and Rahman [6]. If, in addition, $a_0\ge 0$, then it further reduces to the Eneström–Kakeya Theorem. With $k_R=k_I=\rho_R=\rho_I=1$, $q=0$, and $p=n$, then Theorem 6 reduces to Theorem 3.

By imposing a hypothesis similar to that of Theorem 6 on the moduli of the coefficients of a polynomial, we obtain the following.

Theorem 7. *Let* $P(z)=\sum_{\ell=0}^{n}a_\ell z^\ell$ *be a polynomial of degree n with complex coefficients. Suppose that, for some positive numbers* k, ρ, p, *and* q *with* $k\ge 1$, $0<\rho\le 1$, *and* $0\le q\le p\le n$, *the coefficients satisfy* $|arg(a_\ell)-\beta|\le\alpha\le\pi/2$ *for some real* β *and for* $q\le\ell\le p$, *and*

$$\rho|a_q|\le|a_{q+1}|\le|a_{q+2}|\le\cdots\le|a_{p-1}|\le k|a_p|.$$

Then, all the zeros of P lie in the closed annulus:

$$\min\Big\{1,|a_0|\Big/\Big(M_q+|a_q|+\rho|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha$$

$$+k|a_p|(\cos\alpha+\sin\alpha+1)-|a_p|+M_p+|a_n|\Big)\Big\}\le|z|\le\frac{1}{|a_n|}\Big(|a_0|+M_q+|a_q|$$

$$+\rho|a_q|(\sin\alpha-\cos\alpha-1)+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha+k|a_p|(\cos\alpha+\sin\alpha+1)-|a_p|+M_p\Big)$$

where $M_q=\sum_{\ell=1}^{q}|a_\ell-a_{\ell-1}|$ *and* $M_p=\sum_{\ell=p+1}^{n}|a_\ell-a_{\ell-1}|$.

With $q=0$, $p=n$ and $\rho=k=1$, Theorem 7 implies that P is nonzero in

$$|z|\le\cos\alpha+\sin\alpha+\frac{|a_0|(1-\sin\alpha-\cos\alpha)+2\sum_{\ell=0}^{n-1}|a_\ell|\sin\alpha}{|a_n|}$$

which is a slight improvement of Theorem 2. Additionally, with $\alpha=\beta=0$, it then reduces to the Eneström–Kakeya Theorem.

In connection with the Bernstein inequalities, Chan and Malik [9] (and, independently, Qazi [10]) considered the class of polynomials of the form $P(z)=a_0+\sum_{\ell=m}^{n}a_\ell z^\ell$. We can apply Theorems 6 and 7 to this class of polynomials by imposing the inequality hypotheses of those results. We obtain the following.

Corollary 1. *Let* $P(z)=a_0+\sum_{\ell=m}^{n}a_\ell z^\ell$ *be a polynomial of degree n with complex coefficients. Let* $\alpha_\ell=Re(a_\ell)$ *and* $\beta_\ell=Im(a_\ell)$ *for* $\ell=0$ *and* $m\le\ell\le n$. *Suppose that, for some positive numbers*

k_R, k_I, ρ_R, ρ_I, p, and q with $k_R \geq 1$, $k_I \geq 1$, $0 < \rho_R \leq 1$, $0 < \rho_I \leq 1$, and $1 \leq m \leq q \leq p \leq n$, the coefficients satisfy

$$\rho_R \alpha_q \leq \alpha_{q+1} \leq \alpha_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_R \alpha_p$$

and

$$\rho_I \beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_I \beta_p.$$

Then, all the zeros of P lie in the closed annulus given in Theorem 6, where we replace M_q of Theorem 6 with $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$.

Corollary 2. *Let $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients that satisfy $|\arg(a_\ell) - \beta| \leq \alpha \leq \pi/2$ for $q \leq \ell \leq p$ and for some real β. Suppose that, for some positive numbers k, ρ, p, and q with $k \geq 1$, $0 < \rho \leq 1$, and $1 \leq m \leq q \leq p \leq n$, the coefficients satisfy*

$$\rho|a_q| \leq |a_{q+1}| \leq |a_{q+2}| \leq \cdots \leq |a_{p-1}| \leq k|a_p|.$$

Then, all the zeros of P lie in the closed annulus given in Theorem 7, where we replace M_q of Theorem 7 with $M_q = \sum_{\ell=m}^{q} |a_\ell - a_{\ell-1}|$.

Theorems 6 and 7 also naturally apply to a polynomial of the form $P(z) = a_0 + \sum_{\ell=m}^{m'} a_\ell z^\ell + a_n z^n$, where $1 \leq m \leq m' \leq n-1$. With $1 \leq m \leq q \leq p \leq m' \leq n-1$, we obtain corollaries similar to Corollaries 1 and 2, where M_q is as given in the above corollaries, and $M_p = \sum_{\ell=p+1}^{m'} |a_\ell - a_{\ell-1}| + |a_n - a_{n-1}|$.

3. A Lemma

In proving Theorem 2, Govil and Rahman used the following ([4], Equation (6)).

Lemma 1. *Let $\{a_\ell\}_{\ell=1}^{n}$ be a set of complex numbers that satisfy $|\arg(a_\ell) - \beta| \leq \alpha \leq \pi/2$ for $0 \leq \ell \leq n$ and for some real β. Suppose $|a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_n|$. Then, for $\ell \in \{1, 2, \ldots, n\}$, we have*

$$|a_\ell - a_{\ell-1}| \leq (|a_\ell| - |a_{\ell-1}|)\cos\alpha + (|a_\ell| + |a_{\ell-1}|)\sin\alpha.$$

4. Proof of the Results

We now give a proof of Theorem 6.

Proof. Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_{n-1} z^{n-1} + a_n z^n$ be a polynomial of degree n satisfying the stated hypotheses. Define f by the equation:

$$P(z)(1-z) = a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + (a_{q+1} - a_q)z^{q+1} + \cdots$$

$$+(a_p - a_{p-1})z^p + (a_{p+1} - a_p)z^{p+1} + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} = f(z) - a_n z^{n+1}.$$

If $|z| = 1$, then

$$\begin{aligned} |f(z)| &= |a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p + \cdots \\ &\quad + (a_n - a_{n-1})z^n| \\ &\leq |a_0| + |a_1 - a_0| + \cdots + |a_{q-1} - a_{q-2}| + |a_q - a_{q-1}| + |a_{q+1} - a_q| + \cdots \\ &\quad + |a_p - a_{p-1}| + |a_{p+1} - a_p| + \cdots + |a_n + a_{n-1}|. \end{aligned} \tag{1}$$

Let $\alpha_\ell = \mathrm{Re}(a_\ell)$ and $\beta_\ell = \mathrm{Im}(a_\ell)$ for $q \leq \ell \leq p$. Thus, for $|z| = 1$,

$$|f(z)| \leq |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |\alpha_{q+1} + i\beta_{q+1} - \alpha_q - i\beta_q|$$

$$+|\alpha_{q+2} + i\beta_{q+2} - \alpha_{q+1} - i\beta_{q+1}| + \cdots + |\alpha_p + i\beta_p - \alpha_{p-1} - i\beta_{p-1}| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|.$$

Let $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Hence, for $|z| = 1$,

$$\begin{aligned} |f(z)| &\leq |a_0| + M_q + |\alpha_{q+1} - \alpha_q| + |\alpha_{q+2} - \alpha_{q+1}| + \cdots + |\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \beta_q| \\ &\quad + |\beta_{q+2} - \beta_{q+1}| + \cdots + |\beta_p - \beta_{p-1}| + M_p \\ &= |a_0| + M_q + |\alpha_{q+1} - \rho_R\alpha_q + \rho_R\alpha_q - \alpha_q| + |\alpha_{q+2} - \alpha_{q+1}| + \cdots \\ &\quad + |\alpha_{p-1} - \alpha_{p-2}| + |\alpha_p - k_R\alpha_p + k_R\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_I\beta_q + \rho_I\beta_q - \beta_q| \\ &\quad + |\beta_{q+2} - \beta_{q+1}| + \cdots + |\beta_{p-1} - \beta_{p-2}| + |\beta_p - k_I\beta_p + k_I\beta_p - \beta_{p-1}| + M_p. \end{aligned}$$

Since $\rho_R\alpha_q \leq \alpha_{q+1} \leq a_{q+2} \leq \cdots \leq \alpha_{p-1} \leq k_R\alpha_p$ and $\rho_I\beta_q \leq \beta_{q+1} \leq \beta_{q+2} \leq \cdots \leq \beta_{p-1} \leq k_I\beta_p$, where $p \geq q$, then for $|z| = 1$,

$$\begin{aligned} |f(z)| &\leq |a_0| + M_q + |\alpha_{q+1} - \rho_R\alpha_q + \rho_R\alpha_q - \alpha_q| - \alpha_{q+1} + \alpha_{p-1} \\ &\quad + |\alpha_p - k_R\alpha_p + k_R\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_I\beta_q + \rho_I\beta_q - \beta_q| - \beta_{q+1} + \beta_{p-1} \\ &\quad + |\beta_p - k_I\beta_p + k_I\beta_p - \beta_{p-1}| + M_p \\ &\leq |a_0| + M_q + |\alpha_{q+1} - \rho_R\alpha_q| + |\rho_R\alpha_q - \alpha_q| - \alpha_{q+1} + \alpha_{p-1} \\ &\quad + |\alpha_p - k_R\alpha_p| + |k_R\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_I\beta_q| + |\rho_I\beta_q - \beta_q| - \beta_{q+1} + \beta_{p-1} \\ &\quad + |\beta_p - k_I\beta_p| + |k_I\beta_p - \beta_{p-1}| + M_p \\ &= |a_0| + M_q + (\alpha_{q+1} - \rho_R\alpha_q) + (1 - \rho_R)|\alpha_q| - \alpha_{q+1} + \alpha_{p-1} \\ &\quad + (k_R - 1)|\alpha_p| + (k_R\alpha_p - \alpha_{p-1}) + (\beta_{q+1} - \rho_I\beta_q) + (1 - \rho_I)|\beta_q| - \beta_{q+1} \\ &\quad + \beta_{p-1} + (k_I - 1)|\beta_p| + (k_I\beta_p - \beta_{p-1}) + M_p. \end{aligned}$$

Notice that $z^n f\left(\frac{1}{z}\right) = \sum_{\ell=0}^{n} (a_\ell - a_{\ell-1}) z^{n-\ell}$, where $a_{-1} = 0$ has the same bound on $|z| = 1$ as $f(z)$. Since $z^n f\left(\frac{1}{z}\right)$ is analytic in $|z| \leq 1$, we have

$$\begin{aligned} \left| z^n f\left(\frac{1}{z}\right) \right| &\leq |a_0| + M_q - \rho_R\alpha_q + (1 - \rho_R)|\alpha_q| + (k_R - 1)|\alpha_p| \\ &\quad + k_R\alpha_p - \rho_I\beta_q + (1 - \rho_I)|\beta_q| + (k_I - 1)|\beta_p| + k_I\beta_p + M_p \end{aligned}$$

for $|z| \leq 1$, by the Maximum Modulus Theorem. Thus,

$$\begin{aligned}\left|f\left(\frac{1}{z}\right)\right| &= \frac{1}{|z|^n}\Big(|a_0| + M_q - \rho_R\alpha_q + (1-\rho_R)|\alpha_q| + (k_R-1)|\alpha_p| \\ &\quad + k_R\alpha_p - \rho_I\beta_q + (1-\rho_I)|\beta_q| + (k_I-1)|\beta_p| + k_I\beta_p + M_p\Big)\end{aligned}$$

for $|z| \leq 1$. Replacing z with $1/z$, we have

$$\begin{aligned}|f(z)| &\leq \Big(|a_0| + M_q - \rho_R\alpha_q + (1-\rho_R)|\alpha_q| + (k_R-1)|\alpha_p| \\ &\quad + k_R\alpha_p - \rho_I\beta_q + (1-\rho_I)|\beta_q| + (k_I-1)|\beta_p| + k_I\beta_p + M_p\Big)|z^n|\end{aligned}$$

for $|z| \geq 1$. We now have for $|z| \geq 1$

$$\begin{aligned}|(1-z)P(z)| &= |f(z) - a_n z^{n+1}| \geq |a_n||z^{n+1}| - |f(z)| \\ &\geq |z^n|\Big[|a_n||z| - \Big(|a_0| + M_q - \rho_R\alpha_q + (1-\rho_R)|\alpha_q| + (k_R-1)|\alpha_p| \\ &\quad + k_R\alpha_p - \rho_I\beta_q + (1-\rho_I)|\beta_q| + (k_I-1)|\beta_p| + k_I\beta_p + M_p\Big)\Big].\end{aligned}$$

Therefore if

$$\begin{aligned}|z| &> \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_R\alpha_q + (1-\rho_R)|\alpha_q| + (k_R-1)|\alpha_p| + k_R\alpha_p \\ &\quad - \rho_I\beta_q + (1-\rho_I)|\beta_q| + (k_I-1)|\beta_p| + k_I\beta_p + M_p\Big)\end{aligned}$$

then $(1-z)P(z) \neq 0$ and, hence, $P(z) \neq 0$. Therefore, all the zeros of P lie in

$$\begin{aligned}|z| &\leq \frac{1}{|a_n|}\Big(|a_0| + M_q - \rho_R\alpha_q + (1-\rho_R)|\alpha_q| - \rho_I\beta_q + (1-\rho_I)|\beta_q| \\ &\quad + (k_R-1)|\alpha_p| + k_R\alpha_p + (k_I-1)|\beta_p| + k_I\beta_p + M_p\Big),\end{aligned}$$

as claimed.

Consider the polynomial:

$$\begin{aligned}S(z) &= z^n P(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + a_{q+1} z^{n-q-1} + \cdots \\ &\quad + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \cdots + a_{n-1} z + a_n.\end{aligned}$$

Let

$$\begin{aligned}H(z) &= (1-z)S(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} + \cdots \\ &\quad + (a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n.\end{aligned}$$

With $\alpha_\ell = \mathrm{Re}(a_\ell)$ and $\beta_\ell = \mathrm{Im}(a_\ell)$ for $q \le \ell \le p$, we have

$$\begin{aligned}
|H(z)| \geq\; & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} + \cdots \\
& + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n| \Big) \\
=\; & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q + i\beta_q - \alpha_{q+1} - i\beta_{q+1}||z|^{n-q} + |\alpha_{q+1} + i\beta_{q+1} - \alpha_{q+2} - i\beta_{q+2}||z|^{n-q-1} + \\
& \cdots + |\alpha_{p-1} + i\beta_{p-1} - \alpha_p - i\beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots \\
& + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n| \Big) \\
\geq\; & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q - \rho_R\alpha_q + \rho_R\alpha_q - \alpha_{q+1}||z|^{n-q} + |\beta_q - \rho_I\beta_q + \rho_I\beta_q - \beta_{q+1}||z|^{n-q} \\
& + |\alpha_{q+1} - \alpha_{q+2}||z|^{n-q-1} + |\beta_{q+1} - \beta_{q+2}||z|^{n-q-1} + |\alpha_{q+2} - \alpha_{q+3}||z|^{n-q-2} \\
& + |\beta_{q+2} - \beta_{q+3}||z|^{n-q-2} + \cdots + |\alpha_{p-2} - \alpha_{p-1}|z|^{n-p+2} \\
& + |\beta_{p-2} - \beta_{p-1}||z|^{n-p+2} + |\alpha_{p-1} - k_R\alpha_p + k_R\alpha_p - \alpha_p||z|^{n-p+1} \\
& + |\beta_{p-1} - k_I\beta_p + k_I\beta_p - \beta_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \\
& \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n| \Big) \\
\geq\; & |a_0||z|^{n+1} - \Big(|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\
& + |\alpha_q|(1 - \rho_R)|z|^{n-q} + (\alpha_{q+1} - \rho_R\alpha_q)|z|^{n-q} + (\alpha_{q+2} - \alpha_{q+1})|z|^{n-q-1} \\
& + (\alpha_{q+3} - \alpha_{q+2})|z|^{n-q-2} + \cdots + (\alpha_{p-1} - \alpha_{p-2})|z|^{n-p+2} \\
& + (k_R\alpha_p - \alpha_{p-1})|z|^{n-p+1} + |\alpha_p|(k_R - 1)|z|^{n-p+1} + |\beta_q|(1 - \rho_I)|z|^{n-q} \\
& + (\beta_{q+1} - \rho_I\beta_q)|z|^{n-q} + (\beta_{q+2} - \beta_{q+1})|z|^{n-q-1} + (\beta_{q+3} - \beta_{q+2})|z|^{n-q-2} + \\
& \cdots + (\beta_{p-1} - \beta_{p-2})|z|^{n-p+2} + (k_I\beta_p - \beta_{p-1})|z|^{n-p+1} + |\beta_p|(k_I - 1)|z|^{n-p+1} \\
& + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n| \Big) \\
=\; & |z|^n \Bigg[|a_0||z| - \Bigg(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \cdots + \frac{|a_{q-1} - a_q|}{|z|^{q-1}} + \frac{|\alpha_q|(1 - \rho_R)}{|z|^q} \\
& + \frac{\alpha_{q+1} - \rho_R\alpha_q}{|z|^q} + \frac{\alpha_{q+2} - \alpha_{q+1}}{|z|^{q+1}} + \frac{\alpha_{q+3} - \alpha_{q+2}}{|z|^{q+2}} + \cdots \\
& + \frac{\alpha_{p-1} - \alpha_{p-2}}{|z|^{p-2}} + \frac{k_R\alpha_p - \alpha_{p-1}}{|z|^{p-1}} + \frac{|\alpha_p|(k_R - 1)}{|z|^{p-1}} + \frac{|\beta_q|(1 - \rho_I)}{|z|^q} \\
& + \frac{\beta_{q+1} - \rho_I\beta_q}{|z|^q} + \frac{\beta_{q+2} - \beta_{q+1}}{|z|^{q+1}} + \frac{\beta_{q+3} - \beta_{q+2}}{|z|^{q+2}} + \cdots \\
& + \frac{\beta_{p-1} - \beta_{p-2}}{|z|^{p-2}} + \frac{k_I\beta_p - \beta_{p-1}}{|z|^{p-1}} + \frac{|\beta_p|(k_I - 1)}{|z|^{p-1}} + \frac{|a_p - a_{p+1}|}{|z|^p} + \cdots \\
& + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \Bigg) \Bigg].
\end{aligned}$$

Now, with $|z| > 1$, so that $1/|z|^{n-j} < 1$ for $0 \le j < n$, we have

$$\begin{aligned}
|H(z)| &\ge |z|^n \Big[|a_0||z| - \Big(|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{q-1} - a_q| + |\alpha_q|(1-\rho_R) \\
&\quad + (\alpha_{q+1} - \rho_R \alpha_q) + (\alpha_{q+2} - \alpha_{q+1}) + (\alpha_{q+3} - \alpha_{q+2}) + \cdots + (\alpha_{p-1} - \alpha_{p-2}) \\
&\quad + (k_R \alpha_p - \alpha_{p-1}) + |\alpha_p|(k_R - 1) + |\beta_q|(1-\rho_I) + (\beta_{q+1} - \rho_I \beta_q) \\
&\quad + (\beta_{q+2} - \beta_{q+1}) + (\beta_{q+3} - \beta_{q+2}) + \cdots + (\beta_{p-1} - \beta_{p-2}) + (k_I \beta_p - \beta_{p-1}) \\
&\quad + |\beta_p|(k_I - 1) + |a_p - a_{p+1}| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n| \Big) \Big] \\
&= |z|^n \Big[|a_0||z| - \Big(\sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + k_R \alpha_p + (k_R - 1)|\alpha_p| \\
&\quad + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_I \beta_p + (k_I - 1)|\beta_p| + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n| \Big) \Big] \\
&> 0
\end{aligned}$$

if

$$\begin{aligned}
|z| &> \frac{1}{|a_0|} \Big(M_q + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_R \alpha_p + (k_R - 1)|\alpha_p| \\
&\quad + k_I \beta_p + (k_I - 1)|\beta_p| + M_p + |a_n| \Big).
\end{aligned}$$

Thus, all the zeros of $H(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
|z| &\le \frac{1}{|a_0|} \Big(M_q + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_R \alpha_p + (k_R - 1)|\alpha_p| \\
&\quad + k_I \beta_p + (k_I - 1)|\beta_p| + M_p + |a_n| \Big).
\end{aligned}$$

Hence, all the zeros of $H(z)$ and of $S(z)$ lie in

$$\begin{aligned}
|z| &\le \max \Big\{ 1, \big(M_q + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_R \alpha_p + (k_R - 1)|\alpha_p| \\
&\quad + k_I \beta_p + (k_I - 1)|\beta_p| + M_p + |a_n| \big) / |a_0| \Big\}.
\end{aligned}$$

Therefore, all the zeros of $P(z)$ lie in

$$\begin{aligned}
|z| &\ge \min \Big\{ 1, |a_0| / \big(M_q + (1-\rho_R)|\alpha_q| - \rho_R \alpha_q + (1-\rho_I)|\beta_q| - \rho_I \beta_q + k_R \alpha_p \\
&\quad + (k_R - 1)|\alpha_p| + k_I \beta_p + (k_I - 1)|\beta_p| + M_p + |a_n| \big) \Big\},
\end{aligned}$$

as claimed. □

We now give a proof of Theorem 7.

Proof. Let $P(z) = a_0 + a_1 z + \cdots + a_q z^q + \cdots + a_p z^p + \cdots + a_n z^n$ be a polynomial of degree n with complex coefficients such that $|\arg a_\ell - \beta| \le \alpha \le \pi/2$ for $\ell = q, q+1, \ldots, p$ and for some real β and $\rho|a_q| \le |a_{q+1}| \le \cdots \le k|a_p|$ for $0 < \rho \le 1$ and $k \ge 1$. Notice that we can assume without loss of generality that $\beta = 0$. Consider

$$P(z)(1-z) = a_0 + (a_1 - a_0)z + \cdots + (a_q - a_{q-1})z^q + \cdots + (a_p - a_{p-1})z^p + \cdots$$
$$+(a_n - a_{n-1})z^n - a_n z^{n+1} = f(z) - a_n z^{n+1}.$$

If $|z| = 1$, then

$$\begin{aligned}
|f(z)| &\le |a_0| + |a_1 - a_0| + \cdots + |a_{q-1} - a_{q-2}| + |a_q - a_{q-1}| + |a_{q+1} - a_q| + \cdots \\
&\quad +|a_p - a_{p-1}| + |a_{p+1} - a_p| + \cdots + |a_n + a_{n-1}| \text{ by } (1) \\
&= |a_0| + \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| \\
&\quad + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| \\
&= |a_0| + M_q + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + M_p
\end{aligned}$$

where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. Thus, for $|z| = 1$,

$$\begin{aligned}
|f(z)| &\le |a_0| + M_q + |a_{q+1} - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |a_p - a_{p-1}| + M_p \\
&= |a_0| + M_q + |a_{q+1} - \rho a_q + \rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad +|a_p - k a_p + k a_p - a_{p-1}| + M_p \\
&\le |a_0| + M_q + |a_{q+1} - \rho a_q| + |\rho a_q - a_q| + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| \\
&\quad +|a_p - k a_p| + |k a_p - a_{p-1}| + M_p \\
&\le |a_0| + M_q + (|a_{q+1}| - |\rho a_q|)\cos\alpha + (|a_{q+1}| + |\rho a_q|)\sin\alpha \\
&\quad +|a_q|(1-\rho) + \sum_{\ell=q+2}^{p-1} (|a_\ell| - |a_{\ell-1}|)\cos\alpha + \sum_{\ell=q+2}^{p-1} (|a_\ell| + |a_{\ell-1}|)\sin\alpha \\
&\quad +|a_p|(k-1) + (|k a_p| - |a_{p-1}|)\cos\alpha + (|k a_p| + |a_{p-1}|)\sin\alpha + M_p \\
&\quad \text{by Lemma } 1 \\
&= |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\
&\quad +2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p.
\end{aligned}$$

Hence, also,

$$\begin{aligned}
\left| z^n f\left(\frac{1}{z}\right) \right| &\le |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\
&\quad +2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p
\end{aligned}$$

for $|z| = 1$. By the Maximum Modulus Theorem,

$$\begin{aligned}\left|z^n f\left(\frac{1}{z}\right)\right| &\leq |a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\end{aligned}$$

holds inside the unit circle $|z| \leq 1$ as well. If $R > 1$, then $\frac{1}{R}e^{-i\theta}$ lies inside the unit circle for every real θ. Thus, it follows that

$$\begin{aligned}\left|f(Re^{i\theta})\right| &\leq \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg)R^n\end{aligned}$$

for every $R \geq 1$ and θ real. Hence, for every $|z| = R > 1$,

$$\begin{aligned}|P(z)(1-z)| &= |-a_n z^{n+1} + f(z)| \\ &\geq |a_n||R|^{n+1} - |f(z)| \\ &\geq |a_n|R^{n+1} - \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg)R^n \\ &= R^n\Bigg[|a_n|R - \Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg)\Bigg] \\ &> 0\end{aligned}$$

if

$$\begin{aligned}R &> \frac{1}{|a_n|}\Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg).\end{aligned}$$

Therefore all the zeros of $P(z)$ lie in

$$\begin{aligned}|z| &\leq \frac{1}{|a_n|}\Bigg(|a_0| + M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) \\ &+2\sum_{\ell=q+1}^{p-1}|a_\ell|\sin\alpha + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p\Bigg),\end{aligned}$$

as claimed.

Consider the polynomial:

$$S(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + \cdots + a_p z^{n-p} + \cdots + a_{n-1} z + a_n.$$

Let

$$\begin{aligned} H(z) &= (1-z)S(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_q - a_{q+1})z^{n-q} + \cdots \\ &\quad +(a_{p-1} - a_p)z^{n-p+1} + \cdots + (a_{n-2} - a_{n-1})z^2 + (a_{n-1} - a_n)z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} |H(z)| &\geq |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_q - a_{q+1}||z|^{n-q} + \cdots \\ &\quad +|a_{p-1} - a_p||z|^{n-p+1} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big] \\ &= |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\ &\quad +|a_q - \rho a_q + \rho a_q - a_{q+1}||z|^{n-q} + |a_{q+2} - a_{q+1}||z|^{n-q-1} + \cdots \\ &\quad +|a_{p-1} - a_{p-2}||z|^{n-p} + |a_{p-1} - k a_p + k a_p - a_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} \\ &\quad + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big] \\ &\geq |a_0||z|^{n+1} - \Big[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} \\ &\quad +|a_q - \rho a_q||z|^{n-q} + |\rho a_q - a_{q+1}||z|^{n-q} + |a_{q+2} - a_{q+1}||z|^{n-q-1} + \cdots \\ &\quad +|a_{p-1} - a_{p-2}||z|^{n-p} + |a_{p-1} - k a_p||z|^{n-p+1} + |k a_p - a_p||z|^{n-p+1} \\ &\quad +|a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-2} - a_{n-1}||z|^2 + |a_{n-1} - a_n||z| + |a_n|\Big]. \end{aligned}$$

Since $k \geq 1$ and $0 < \rho \leq 1$, then

$$\begin{aligned} |H(z)| &\geq |z|^n \Bigg[|a_0||z| - \Bigg(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \cdots + \frac{|a_{q-1} - a_q|}{z^{q-1}} + \frac{|a_q|(1-\rho)}{|z|^q} \\ &\quad + \frac{|\rho a_q - a_{q+1}|}{|z|^q} + \frac{|a_{q+2} - a_{q+1}|}{|z|^{q+1}} + \cdots + \frac{|a_{p-1} - a_{p-2}|}{|z|^{p-2}} + \frac{|a_{p-1} - k a_p|}{|z|^{p-1}} \\ &\quad + \frac{|a_p|(k-1)}{|z|^{p-1}} + \frac{|a_p + a_{p+1}|}{|z|^p} + \cdots + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\Bigg)\Bigg]. \end{aligned}$$

Now, with $|z| > 1$, so that $1/|z|^{n-j} < 1$ for $0 \le j < n$, we have

$$\begin{aligned}
|H(z)| &\geq |z|^n\Big[|a_0||z| - \Big(|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{q-1} - a_q| + |a_q|(1-\rho) \\
&\quad +|\rho a_q - a_{q+1}| + |a_{q+2} - a_{q+1}| + \cdots + |a_{p-1} - a_{p-2}| + |a_{p-1} - k a_p| \\
&\quad +|a_p|(k-1) + |a_p - a_{p+1}| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n|\Big)\Big] \\
&= |z|^n\Big[|a_0||z| - \Big(\sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}| + |a_q|(1-\rho) + |a_{q+1} - \rho a_q| \\
&\quad + \sum_{\ell=q+2}^{p-1} |a_\ell - a_{\ell-1}| + |k a_p - a_{p-1}| + |a_p|(k-1) + \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}| + |a_n|\Big)\Big] \\
&\geq |z|^n\Big[|a_0||z| - \Big(M_q + |a_q|(1-\rho) + (|a_{q+1}| - \rho|a_q|)\cos\alpha \\
&\quad +(|a_{q+1}| + \rho|a_q|)\sin\alpha - |a_{q+1}|(\cos\alpha + \sin\alpha) + 2\sum_{\ell=q+1}^{p-2} |a_\ell|\sin\alpha \\
&\quad +(k|a_p| - |a_{p-1}|)\cos\alpha + |a_{p-1}|(\cos\alpha + \sin\alpha) + (k|a_p| + |a_{p-1}|)\sin\alpha \\
&\quad +|a_p|(k-1) + M_p + |a_n|\Big)\Big] \text{ by Lemma 1} \\
&= |z|^n\Big[|a_0||z| - \Big(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
&\quad +k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p + |a_n|\Big)\Big] \\
&> 0
\end{aligned}$$

if

$$\begin{aligned}
|z| &> \frac{1}{|a_0|}\Big(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
&\quad +k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p + |a_n|\Big).
\end{aligned}$$

Thus, all the zeros of $H(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}
|z| &\leq \frac{1}{|a_0|}\Big(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
&\quad +k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p + |a_n|\Big).
\end{aligned}$$

Hence, all the zeros of $H(z)$ and of $S(z)$ lie in

$$\begin{aligned}
|z| &\leq \max\Big\{1, \Big(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell|\sin\alpha \\
&\quad +k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p + |a_n|\Big)\Big/|a_0|\Big\}.
\end{aligned}$$

Therefore, all the zeros of $P(z)$ lie in

$$|z| \geq \min\left\{1, |a_0| \Big/ \left(M_q + |a_q| + \rho|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{p-1} |a_\ell| \sin\alpha \right.\right.$$
$$\left.\left. + k|a_p|(\cos\alpha + \sin\alpha + 1) - |a_p| + M_p + |a_n|\right)\right\},$$

as claimed. □

5. Discussion

As explained in the Introduction, the hypotheses applied in this paper are consistent with several existing results of the Eneström–Kakeya type. In fact, the two main theorems proven in this paper are generalizations of some of these results. Future research could involve loosening the restrictions imposed on the coefficients of a polynomial given in Theorems 6 and 7. For example, a reversal in the monotonicity could be introduced, or additional parameters related to the monotonicity of the coefficients could be added to give generalizations of these theorems. In addition, the analytic theory of functions of a quaternionic variable could be applied to quaternionic polynomials (with the same or similar conditions imposed on the coefficients) to restrict the location of the zeros of such polynomials.

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