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Some Properties of a Concircular Curvature Tensor on Generalized Sasakian-Space-Forms

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Abstract: The aim of the present paper is to study and investigate the geometrical properties of a concircular curvature tensor on generalized Sasakian-space-forms. In this manner, we obtained results for ϕ-concircularly flat, ϕ-semisymmetric, locally concircularly symmetric and locally concircularly ϕ-symmetric generalized Sasakian-space-forms. Finally, we construct examples of the generalized Sasakian-space-forms to verify some results.

Keywords: generalized Saskian-space-form; concircular curvature tensor; locally ϕ-symmetric

1. Introduction

In [1], the authors introduced and studied the notion of generalized Sasakian-space-forms with geometrical and physical significance. A generalized Sasakian-space-form is an almost contact metric manifold (M, ϕ, ξ, η, g), whose curvature tensor is defined as follows:

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

where f_1, f_2, f_3 are differentiable functions and X, Y, Z for vector fields on $M^{2n+1}(f_1, f_2, f_3)$. The Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-form, and cosymplectic and Kenmotsu space-forms are also considered particular types of generalized Sasakian-space-forms. Additionally, the generalized Sasakian-space-forms have been investigated in [2–21] and many others.

In Riemannian geometry, numerous researchers have studied curvature properties and how much they affected the manifold itself. Symmetry and flatness are two important curvature properties of the Riemannian manifold, which has many applications not only in mathematics but in many other sciences as well. In recent years, the notion of symmetric manifold has been studied by many authors by extending it to other sciences, such as the spacetime of general relativity [22] and locally symmetric f-associated standard static spacetimes [11]; for more detail, refer to [10,23].

In the context of generalized Sasakian-space-forms, Kim [13] investigated conformally flat and locally symmetric generalized Sasakian-space-forms. Generalized Sasakian-Space-Forms with Projective Curvature Tensor were explored by De and Sarkar [24] for some of their symmetric characteristics. Prakasha proved in [16] that every generalized Sasakian-space-form is Weyl-pseudo-symmetric. Further, Prakasha and Nagaraja [25] investigated quasi-conformally semisymmetric and quasi-conformally flat generalized Sasakian-space-forms. Sarkar and Akbar [26] constructed generalized Sasakian-space-forms that are conharmonically flat and conharmonically locally-symmetric. Hui and Prakasha [27] briefly looked at certain symmetric aspects of generalized Sasakian-space-forms using the C-Bochner curvature tensor. We investigated a generalized Sasakian-space-form that

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satisfies a specific curvature requirement on the concircular curvature tensor in the current research.

The paper is set up as follows: The definitions, basic formulas, and preliminary results of generalized Sasakian-space-forms are included in Section 1. In Section 2, we investigate ϕ-concircularly flat generalized Sasakian-space-forms and establish that a generalized Sasakian-space-form is ϕ-concircularly flat if and only if it is concircularly flat. In Section 3, we investigate ϕ-concircularly semisymmetric generalized Sasakian-space-forms and establish necessary and sufficient conditions for a generalized Sasakian-space-form to be ϕ-concircularly semisymmetric. The study of locally concircular symmetric and locally concircularly ϕ-symmetric generalized Sasakian-space-forms is covered in Section 4. Here it is shown that a generalized Sasakian-space-form is locally concircularly symmetric if and only if it is conformally flat. Additionally, we discover that a locally concircularly ϕ-symmetric generalized Sasakian-space-form is also conformally flat and hence, locally concircularly symmetric. Finally, some examples of generalized Sasakian-space-form with $f_1 = 1$ and $f_2 = f_3 = 0$ are given.

2. Generalized Sasakian-Space-Form

A $(2n+1)$-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold [28], if there exists on $M^{2n+1}(f_1, f_2, f_3)$ a $(1,1)$ tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{aligned} \phi^2(X) &= -X + \eta(X)\xi, \ \ \eta(\xi) = 1, && (2)\\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), && (3) \end{aligned}$$

for any vector fields X, Y on $M^{2n+1}(f_1, f_2, f_3)$.

For a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, the following relations exist in addition to relation (1) [1]:

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, && (4)\\ R(\xi,X)Y &= (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, && (5)\\ \eta(R(X,Y)Z) &= (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, && (6)\\ S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), && (7)\\ r &= 2n(2n+1)f_1 + 6nf_2 - 4nf_3. && (8) \end{aligned}$$

In [29], the authors compute the concircular curvature tensor $\tilde{\mathbf{Z}}$ for a $(2n+1)$-dimensional $(n>1)$ almost contact metric manifold:

$$\tilde{\mathbf{Z}}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \quad (9)$$

for any vector fields $X, Y, Z \in T(M)$.

By virtue of (1) in (9), we obtain

$$\begin{aligned} \tilde{\mathbf{Z}}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\}\\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\ &- \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \quad (10) \end{aligned}$$

Here, we recollect some results that will be required for further sections.

Theorem 1 ([30]). *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$.*

Theorem 2 ([24]). *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is Ricci semisymmetric if and only if $f_3=\frac{3f_2}{1-2n}$.*

Theorem 3 ([13]). *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is conformally flat if and only if $f_2=0$.*

3. ϕ-Concircularly Flat Generalized Sasakian-Space-Forms

Definition 1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is called ϕ-concircularly flat if it satisfies*

$$\phi^2\tilde{\mathbf{Z}}(\phi X,\phi Y)\phi Z=0, \tag{11}$$

for any vector fields $X, Y, Z\in T(M)$ [31].

As per the definition, it follows that every concircularly flat generalized Sasakian -space-form is ϕ-concircularly flat, but the converse never holds. Interestingly, in this section, we prove that the converse also holds good for a generalized Sasakian-space-form of dimension greater than three. The importance of studying ϕ-concircularly flat generalized Sasakian space forms can be seen in this section.

Let us consider that $M^{2n+1}(f_1,f_2,f_3)$ is concircularly flat. Then, by using (9) and Definition 1, we have

$$\phi^2[R(\phi X,\phi Y)\phi Z-\frac{r}{2n(2n+1)}\{g(\phi Y,\phi Z)\phi X-g(\phi X,\phi Z)\phi Y\}]=0. \tag{12}$$

Making use of (1) in (12), we obtain

$$\begin{aligned}&\phi^2[f_1(g(\phi Y,\phi Z)\phi X-g(\phi X,\phi Z)\phi Y)\\&+f_2(g(\phi X,\phi^2 Z)\phi^2 Y-g(\phi Y,\phi^2 Z)\phi^2 X+2g(\phi X,\phi^2 Y)\phi^2 Z)]\\=&\frac{r}{2n(2n+1)}\phi^2[g(Y,Z)\phi X-\eta(Y)\eta(Z)\phi X-g(X,Z)\phi Y+\eta(X)\eta(Z)\phi Y].\end{aligned} \tag{13}$$

Again, by virtue of (2) and (8) in (13), we have

$$\begin{aligned}&f_1\{g(Y,Z)\phi X-\eta(Y)\eta(Z)\phi X-g(X,Z)\phi Y+\eta(X)\eta(Z)\phi Y\}\\+\;&f_2\{g(X,\phi Z)\phi^2 Y-g(Y,\phi Z)\phi^2 X+2g(X,\phi Y)\phi^2 Z\}\\=\;&\frac{1}{(2n+1)}((2n+1)f_1+3f_2-2f_3)[g(Y,Z)\phi X-\eta(Y)\eta(Z)\phi X\\&-g(X,Z)\phi Y+\eta(X)\eta(Z)\phi Y].\end{aligned} \tag{14}$$

Taking the inner product with arbitrary vector field U in (14), we obtain

$$\begin{aligned}&f_1\{g(Y,Z)g(\phi X,U)-\eta(Y)\eta(Z)g(\phi X,U)-g(X,Z)g(\phi Y,U)+\eta(X)\eta(Z)g(\phi Y,U)\}\\+\;&f_2\{g(X,\phi Z)g(\phi^2 Y,U)-g(Y,\phi Z)g(\phi^2 X,U)+2g(X,\phi Y)g(\phi^2 Z,U)\}\\=\;&\frac{1}{(2n+1)}((2n+1)f_1+3f_2-2f_3)[g(Y,Z)g(\phi X,U)-\eta(Y)\eta(Z)g(\phi X,U)\\&-g(X,Z)g(\phi Y,U)+\eta(X)\eta(Z)g(\phi Y,U)].\end{aligned} \tag{15}$$

Setting $Y=Z=e_i$ in (15), where e_i is an orthogonal basis of the tangent space at each point of the manifold and taking summation over i, $i=1,2,\ldots,2n+1$, we have

$$3f_2g(\phi X,U)=\frac{3f_2-2f_3}{2n+1}(2n-1)g(\phi X,U), \tag{16}$$

which gives

$$3f_2 = \frac{3f_2 - 2f_3}{2n+1}(2n-1), \tag{17}$$

provided $g(\phi X, U) \neq 0$, which implies that

$$f_3 = \frac{3f_2}{1-2n}. \tag{18}$$

Hence, we can state the following:

Theorem 4. *For a $(2n+1)$-dimensional ϕ-concircularly flat $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, $f_3 = \frac{3f_2}{1-2n}$ holds.*

Suppose $f_3 = \frac{3f_2}{1-2n}$ holds. Then by virtue of Theorem 1, $M^{2n+1}(f_1, f_2, f_3)$ is concircularly flat. Therefore, $\tilde{\mathbf{Z}}(X,Y)Z = 0$, and hence $\phi^2(\tilde{\mathbf{Z}}(\phi X, \phi Y)\phi Z) = 0$. Thus, we state the following:

Theorem 5. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is ϕ-concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$.*

Using Theorems 1 and 5, we conclude the following Corollary:

Corollary 1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is ϕ-concircularly flat if and only if it is concircularly flat.*

Again, from (12), we have

$$\begin{aligned} &g(R(\phi X, \phi Y)\phi Z), \phi U) \\ &-\frac{r}{2n(2n+1)}\{g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)\} = 0. \end{aligned} \tag{19}$$

Using (1) and (2) in (19), we yield

$$\begin{aligned} &(f_1 - \frac{r}{2n(2n+1)})[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)] \\ + \quad &f_2[g(\phi X, Z)g(y, \phi U) - g(\phi Y, Z)g(X, \phi U) + 2g(\phi X, Y)g(Z, \phi W)] = 0. \end{aligned} \tag{20}$$

Setting $Y = Z = e_i$ in (20) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain

$$\begin{aligned} &(f_1 - \frac{r}{2n(2n+1)})[2ng(\phi X, \phi U) - g(\phi^2 X, \phi^2 U)] \\ + \quad &f_2[g(\phi X, \phi U) + 2g(\phi X, \phi U)] = 0. \end{aligned} \tag{21}$$

Using (2) in (21), we have

$$\begin{aligned} &(f_1 - \frac{r}{2n(2n+1)})[2ng(\phi X, \phi U) - g(X, U) + \eta(X)\eta(U)] \\ + \quad &f_2[g(\phi X, \phi U) + 2g(\phi X, \phi U)] = 0. \end{aligned} \tag{22}$$

Again using (3) and (8) in (22), we obtain

$$(2n-1)f_2 g(\phi X, \phi U) = 0. \tag{23}$$

which implies

$$f_2 = 0. \tag{24}$$

Provided $g(\phi X, \phi U) \neq 0$, we have the following:

Theorem 6. *If a* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form* $M^{2n+1}(f_1, f_2, f_3)$ *is* ϕ*-concircularly flat, then* $f_2 = 0$.

For a $(2n+1)$-dimensional generalized Sasakian-space-form, U.K. Kim [13] showed the following results:

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.
(ii) If $M^{2n+1}(f_1, f_2, f_3)$ is conformally flat and ξ is a Killing vector field, then $M^{2n+1}(f_1, f_2, f_3)$ is locally symmetric and has constant ϕ-sectional curvature.

In the perspective of the first part of the above hypothesis of Kim, we obtain the following:

Corollary 2. *For a* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form* $M^{2n+1}(f_1, f_2, f_3)$ ϕ*-concircularly flat and conformally flat are equivalent.*

In the perspective of the second part of the above hypothesis of Kim, we obtain the following:

Corollary 3. *A* ϕ*-concircularly flat* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form* $M^{2n+1}(f_1, f_2, f_3)$ *with* ξ *as a Killing vector field is locally symmetric and has constant* ϕ*-sectional curvature.*

Suppose $f_2 = 0$. Then from (18), we must have $f_3 = 0$; therefore, we obtain from (1) that

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}. \tag{25}$$

In [1], P. Alegre et al. demonstrated that a Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with specific form, $f_1 - 1 = f_2 = f_3$. In this case $f_2 = 0$ implies $f_3 = 0$ and $f_1 = 1$. Thus, from (25), we obtain $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$, indicating that the manifold has constant curvature 1. If a $(2n+1)$-dimensional $(n > 1)$ Riemannian manifold has constant curvature, it is known to be concircularly flat. Furthermore, concircular flatness implies ϕ-concircularly flat. Consequently, we can claim:

Corollary 4. *A* $(2n+1)$*-dimensional* $(n > 1)$ *Sasakian manifold is* ϕ*-concircularly flat if and only if the manifold is of constant curvature* 1.

4. ϕ-Concircularly Semisymmetric Generalized Sasakian-Space-Forms

A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is said to be ϕ-concircularly semisymmetric if it satisfies $\tilde{\mathbf{Z}}(X,Y) \cdot \phi = 0$, which implies ([7,32]),

$$(\tilde{\mathbf{Z}}(X,Y)\phi)Z = \tilde{\mathbf{Z}}(X,Y)\phi Z - \phi\tilde{\mathbf{Z}}(X,Y)Z = 0, \tag{26}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Now, from (9) it follows that

$$\tilde{\mathbf{Z}}(X,Y)\phi Z = R(X,Y)\phi Z - \frac{r}{2n(2n+1)}\{g(Y,\phi Z)X - g(X,\phi Z)Y\}. \tag{27}$$

By taking account of (1) in (27), we obtain

$$\begin{aligned}
&\tilde{\mathbf{Z}}(X,Y)\phi Z\\
=\;& f_1\{g(Y,\phi Z)X - g(X,\phi Z)Y\}\\
+\;& f_2\{g(X,\phi^2 Z)\phi Y - g(Y,\phi^2 Z)\phi X + 2g(X,\phi Y)\phi^2 Z\}\\
+\;& f_3\{\eta(X)\eta(\phi Z)Y - \eta(Y)\eta(\phi Z)X + g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\}\\
-\;& \frac{r}{2n(2n+1)}\{g(Y,\phi Z)X - g(X,\phi Z)Y\}. \qquad (28)
\end{aligned}$$

Again, by virtue of (2) and (8) in (28), we obtain

$$\begin{aligned}
&\tilde{\mathbf{Z}}(X,Y)\phi Z =\\
-\;& \frac{(3f_2-2f_3)}{2n+1}\{g(Y,\phi Z)X - g(X,\phi Z)Y\}\\
+\;& f_2\{g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\\
-\;& 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\} + f_3\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\}. \qquad (29)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\phi\tilde{\mathbf{Z}}(X,Y)Z =\\
-\;& \frac{(3f_2-2f_3)}{2n+1}\{g(Y,Z)\phi X - g(X,Z)\phi Y\}\\
+\;& f_2\{g(Y,\phi Z)X - g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\\
-\;& 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\} + f_3\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}. \qquad (30)
\end{aligned}$$

Substituting (29), (30) in (26)

$$\begin{aligned}
&\left[\frac{(3f_2-2f_3)}{2n+1} + f_2\right]\{g(Y,Z)\phi X - g(X,Z)\phi Y - g(Y,\phi Z)X + g(X,\phi Z)Y\}\\
+\;& (f_2-f_3)\{g(Y,\phi Z)\eta(X)\xi - g(X,\phi Z)\eta(Y)\xi - \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}\\
=\;& 0. \qquad (31)
\end{aligned}$$

Setting $Y = \xi$ in (31), we obtain

$$[3f_2 + (2n-1)f_3]\{g(X,\phi Z)\xi + \eta(Z)\phi X\} = 0. \qquad (32)$$

The above relation implies that

$$f_3 = \frac{3f_2}{1-2n}. \qquad (33)$$

Hence, we can state the following:

Theorem 7. *For a $(2n+1)$-dimensional $(n>1)$ ϕ-concircularly semisymmetric generalized Sasakian-space-form, $M^{2n+1}(f_1,f_2,f_3)$, $f_3 = \frac{3f_2}{1-2n}$ holds.*

Thus, in virtue of (33) and Theorem 1, we can state the following:

Theorem 8. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is ϕ-concircularly semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.*

In virtue of Theorems 2 and 8, we can state the following:

Corollary 5. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is ϕ-concircularly semisymmetric if and only if it is Ricci semisymmetric.*

By combining Theorems 1, 5, 8 and Corollary 5, we can state the following:

Corollary 6. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form. Then, the following statements are equivalent:*

(i) $M^{2n+1}(f_1, f_2, f_3)$ *is ϕ-concircularly semisymmtric,*
(ii) $M^{2n+1}(f_1, f_2, f_3)$ *is ϕ-concircularly flat,*
(iii) $M^{2n+1}(f_1, f_2, f_3)$ *is concircularly flat,*
(iv) $M^{2n+1}(f_1, f_2, f_3)$ *is Ricci semisymmetric,*
(v) $f_3 = \frac{3f_2}{1-2n}$ *holds on* $M^{2n+1}(f_1, f_2, f_3)$.

5. Locally Concircularly Symmetric and Locally Concircularly ϕ-Symmetric Generalized Sasakian-Space-Forms

If curvature tensor R of a Riemannian manifold $M^{2n+1}(f_1, f_2, f_3)$ is parallel, then the manifold is said to be locally symmetric, i.e., $\nabla R = 0$, where ∇ denotes the Levi–Civita connection. The concept of semisymmetric manifolds was introduced as an appropriate generalization of locally symmetric manifolds

$$(R(X,Y)\cdot R)(U,V)W = 0, \quad X, Y, U, V, W \in T(M)$$

and studied by many authors (e.g., [33–36]). A complete intrinsic classification of these spaces was given by Z.I. Szabo [37]. A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-forms $M^{2n+1}(f_1, f_2, f_3)$ is said to be locally concircularly symmetric if

$$(\nabla_W \tilde{\mathbf{Z}})(X,Y)Z = 0, \tag{34}$$

for all vector fields, X, Y, Z are orthogonal to ξ. It is called locally concircularly ϕ-symmetric if

$$\phi^2((\nabla_W \tilde{\mathbf{Z}})(X,Y)Z) = 0, \tag{35}$$

for all vector fields, X, Y, Z are orthogonal to ξ. First, by taking covariant differentiation on both sides of the Equation (10) with respect to the arbitrary vector field W, we obtain

$$\begin{aligned}
& (\nabla_W \tilde{\mathbf{Z}})(X,Y)Z \\
= \; & df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
+ \; & df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ \; & f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
- \; & g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
+ \; & 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
+ \; & df_3(W)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+ \; & g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\
+ \; & f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y \\
- \; & (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)(Z)X \\
+ \; & g(X,Y)(\nabla_W\eta)(Y)\xi + g(X,Y)\eta(Y)(\nabla_W\xi) \\
- \; & g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)(\nabla_W\xi)\} \\
- \; & \frac{dr(W)}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}.
\end{aligned} \tag{36}$$

If the vector fields X, Y, Z are orthogonal to ξ, then obtain from (36) that

$$\begin{aligned}
& (\nabla_W \tilde{\mathbf{Z}})(X,Y)Z \\
= \; & df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
+ \; & df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ \; & f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
- \; & g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
+ \; & 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
- \; & \frac{d(2n(2n+1)f_1 + 6nf_2 - 4nf_3)(W)}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \qquad (37)
\end{aligned}$$

Let $M^{2n+1}(f_1, f_2, f_3)$ be locally concircularly symmetric. Then from (34), we have

$$\begin{aligned}
& df_1(W)\{g(Y,Z)X - g(X,Z)Y\} + df_2(W)\{g(X,\phi Z)\phi Y \\
- \; & g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ \; & f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
- \; & g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
+ \; & 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
- \; & \frac{1}{2n+1}d((2n+1)f_1 + 3f_2 - 2f_3)(W))\{g(Y,Z)X - g(X,Z)Y\} = 0. \qquad (38)
\end{aligned}$$

Taking the inner product with respect to vector field U in (38), we obtain

$$\begin{aligned}
& df_1(W)\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\
+ \; & df_2(W)\{g(X,\phi Z)g(\phi Y,U) - g(Y,\phi Z)g(\phi X,U) + 2g(X,\phi Y)g(\phi Z,U)\} \\
+ \; & f_2\{g(X,\phi Z)g((\nabla_W\phi)Y,U) + g(X,(\nabla_W\phi)Z)g(\phi Y,U) \\
- \; & g(Y,\phi Z)g((\nabla_W\phi)X,U) - g(Y,(\nabla_W\phi)Z)g(\phi X,U) \\
+ \; & 2g(X,\phi Y)g((\nabla_W\phi)Z,U) + 2g(X,(\nabla_W\phi)Y)g(\phi Z,U)\} \\
- \; & \frac{1}{2n+1}d((2n+1)f_1 + 3f_2 - 2f_3)(W))\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\
= \; & 0. \qquad (39)
\end{aligned}$$

Let $\{e_i : i = 1, 2, \ldots 2n+1\}$ be an orthonormal basis of the tangent space at each point of the manifold. Setting $Z = U = e_i$ in (39) and taking summation over i, $1 \le i \le 2n+1$, we obtain

$$\begin{aligned}
& f_2\{-g(\phi X,(\nabla_W\phi)Y) + \sum_{1}^{2n+1} g(X,(\nabla_W\phi)e_i)g(\phi Y,e_i) \\
+ \; & g(\phi Y,(\nabla_W\phi)X) - \sum_{1}^{2n+1} g(Y,(\nabla_W\phi)e_i)g(\phi X,e_i) \\
+ \; & 2\sum_{1}^{2n+1} g(X,\phi Y)g((\nabla_W\phi)e_i,e_i)\} \\
= \; & 0. \qquad (40)
\end{aligned}$$

We now recall that $(\nabla_W g)(X,Y) = 0$, yielding

$$\nabla_W g(X,Y) - g(\nabla_W X,Y) - g(X,\nabla_W Y) = 0. \qquad (41)$$

Putting $X = e_i$ and $Y = \phi e_i$ in (41), we have $g(e_i, \phi\nabla_W e_i) - g(e_i, \nabla_W \phi e_i) = 0$. Thus, we have

$$g(e_i,(\nabla_W\phi)e_i) = 0. \qquad (42)$$

By virtue of (40) and (42), we obtain

$$f_2\{-g(\phi X,(\nabla_W\phi)Y)+\sum_1^{2n+1}g(X,(\nabla_W\phi)e_i)g(\phi Y,e_i)$$
$$+\quad g(\phi Y,(\nabla_W\phi)X)-\sum_1^{2n+1}g(Y,(\nabla_W\phi)e_i)g(\phi X,e_i)\}=0. \tag{43}$$

Equation (43) is true for any vector fields X, Y on the manifold. $X\neq Y$ gives

$$f_2=0.$$

Conversely, suppose that $f_2=0$. In addition, if we consider X, Y, Z orthogonal to ξ, then from (1), we find

$$R(X,Y)Z=f_1\{g(Y,Z)X-g(X,Z)Y\}. \tag{44}$$

From (44), we have

$$r=2n(2n+1)f_1. \tag{45}$$

By virtue of (8) and (45), we obtain $f_3=0$; hence, from (37), we obtain

$$(\nabla_W\tilde{\mathbf{Z}})(X,Y)Z=0. \tag{46}$$

Therefore, $M^{2n+1}(f_1,f_2,f_3)$ is locally concircularly symmetric. The above discussion helps us to state the following:

Theorem 9. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is locally concircularly symmetric if and only if $f_2=0$.*

Combining the results of Theorems 3 and 9, we obtain the following Corollary:

Corollary 7. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ is locally concircularly symmetric if and only if it is conformally flat.*

Next, suppose that $M^{2n+1}(f_1,f_2,f_3)$ is locally concircularly ϕ-symmetric. Then from the (2) and (35), we obtain

$$-(\nabla_W\tilde{\mathbf{Z}})(X,Y)Z+\eta((\nabla_W\tilde{\mathbf{Z}})(X,Y)Z)\xi=0. \tag{47}$$

Taking the inner product of Equation (47) with respect to an arbitrary vector field U, we have

$$-g((\nabla_W\tilde{\mathbf{Z}})(X,Y)Z,U)+\eta((\nabla_W\tilde{\mathbf{Z}})(X,Y)Z)\eta(U)=0. \tag{48}$$

By considering U is orthogonal to ξ, from (48), we obtain

$$g((\nabla_W\tilde{\mathbf{Z}})(X,Y)Z,U)=0. \tag{49}$$

The above equation is true for all vector fields U orthogonal to ξ. If we choose $U\neq 0$ and orthogonal to $(\nabla_W\tilde{\mathbf{Z}})(X,Y)Z$, then it follows that

$$(\nabla_W\tilde{\mathbf{Z}})(X,Y)Z=0.$$

Hence, $M^{2n+1}(f_1,f_2,f_3)$ is locally concircularly symmetric, and hence, by Theorem 7, it is conformally flat.

Conversely, let $M^{2n+1}(f_1, f_2, f_3)$ be conformally flat, and hence, $f_2 = 0$. Again, for X, Y, Z orthogonal to ξ, $f_2 = 0$ implies $f_3 = 0$. Therefore we obtain from (37), using (2) and considering X, Y, Z orthogonal to ξ,

$$\begin{aligned}
& \phi^2(\nabla_W \tilde{\mathbf{Z}})(X,Y)Z \\
= \;& df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
+ \;& df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ \;& f_2\{g(X,\phi Z)(\nabla_W\phi)Y + g(X,(\nabla_W\phi)Z)\phi Y \\
- \;& g(Y,\phi Z)(\nabla_W\phi)X - g(Y,(\nabla_W\phi)Z)\phi X \\
+ \;& 2g(X,\phi Y)(\nabla_W\phi)Z + 2g(X,(\nabla_W\phi)Y)\phi Z\} \\
- \;& \frac{1}{2n+1}d((2n+1)f_1 + 3f_2 - 2f_3)(W))\{g(Y,Z)X - g(X,Z)Y\}. \qquad (50)
\end{aligned}$$

Hence, for $f_2 = f_3 = 0$, from (50), we obtain

$$\phi^2(\nabla_W \tilde{\mathbf{Z}})(X,Y)Z = 0.$$

Therefore, $M^{2n+1}(f_1, f_2, f_3)$ is locally concircularly ϕ-symmetric.
This leads to the following Theorem:

Theorem 10. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is locally concircularly ϕ-symmetric if and only if it is conformally flat.*

Combining the results of Theorem 10 and Corollary 7, we find the following:

Theorem 11. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is locally concircularly ϕ-symmetric if and only if it is locally concircularly symmetric.*

Theorem 12. *In a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, the following conditions are equivalent:*

(i) $M^{2n+1}(f_1, f_2, f_3)$ *is locally concircularly symmetric,*
(ii) $M^{2n+1}(f_1, f_2, f_3)$ *is locally concircularly ϕ-symmetric,*
(iii) $M^{2n+1}(f_1, f_2, f_3)$ *is conformally flat,*
(iv) $f_2 = 0$ *holds on* $M^{2n+1}(f_1, f_2, f_3)$.

6. Examples

In this section, we give some examples on generalized Sasakian-space-form with $f_2 = f_3 = 0$ and $f_1 = 1$.

Example 1. *In [1], it is shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with*

$$f_1 = \frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{(f'')}{f}.$$

where $f = f(t)$, $t \in \mathbb{R}$. If we choose $f(t) = e^{-t}$, then $M^{2n+1}(f_1, f_2, f_3)$ is ϕ-concircularly flat generalized Sasakian-space-form, since $f_2 = 0$. Furthermore, we see that f_1 is constant. Therefore, Theorem 6 and Corollary 4 are verified.

Example 2. *The generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is a Sasakian manifold, then it has specific form, i.e., $f_1 - 1 = f_2 = f_3 = 0$, in this case, $f_1 = 1$, $f_2 = f_3 = 0$ since, by the result obtained in this paper, $M^{2n+1}(f_1, f_2, f_3)$ is ϕ-concircularly flat and its scalar curvature is constant.*

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