

Article

Verifying Measures of Quantum Entropy

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Abstract: This paper introduces a new measure of quantum entropy, called the effective quantum entropy (EQE). The EQE is an extension, to the quantum setting, of a recently derived classical generalized entropy. We present a thorough verification of its properties. As its predecessor, the EQE is a semi-strict quasi-concave function; it would be capable of generating many of the various measures of quantum entropy that are useful in practice. Thereafter, we construct a consistent estimator for our proposed measure and empirically test its estimation error, under different system dimensions and number of measurements. Overall, we build the grounds of the EQE, which will facilitate the analyses and verification of the next innovative quantum technologies.

Keywords: axioms; estimation; generalized entropy; quantum entropy; semi-strict quasi-concavity; schur-concavity

1. Introduction

Quantum communication systems bring higher performance possibilities in terms of storage, processing, transmission, and security [1]. Therefore, when designing the new tools to assess the complexity or efficiency of these new systems, the quantum nature of the produced quantities needs to be taken into account. For this purpose, the fundamental notions of classical information theory and communication theory should be appropriately extended to the quantum setting and thoroughly analyzed. One of these key notions is entropy.

Beyond its value in communications, quantum entropy has found usage in statistical physics (e.g., in the ergodic theorem or irreversible systems) [2], quantum computing [3], among many other new systems. For instance, in quantum cryptography, the quantum entropy (or an estimator of it) could help to verify the proper behavior of the quantum random number generators [4]. Similarly, given the complex sampling process, appropriate estimators of the quantum entropy often appear in the literature. In Ref. [5], the degree of unpredictability of the hyper-chaotic and high-dimensional temporal fluctuations of the output of a semiconductor laser is quantified using an estimator of the permutation entropy, which is an exceptionally robust measure in noisy environments. In Ref. [6], the chaotic signals generated by vertical-cavity surface-emitting lasers is numerically investigated using a similar estimator of the permutation entropy. In Ref. [7], the complexity of the chaotic intensity obtained from an external-cavity semiconductor laser is also investigated via the sample entropy (SampEn) algorithm.

Therefore, the literature on classical and quantum entropy measures and their properties and applications is now extensive. Carlen [8] showed an elementary introduction to the subject of trace inequalities and related topics in analysis, with a particular focus on results that are relevant to quantum statistical mechanics. Wilde's book [9], on the other hand, extensively surveyed the quantum information theory. Overall, numerous measures of classical entropy, and extensions to the quantum setting, have been introduced over the last few years:

Classical measures: Alfred Rényi proposed the Rényi entropy [10], which is a generalization of the Shannon entropy [11], by relaxing an axiom that characterizes it. That axiomatic approach has been beneficial and is summarized in a survey paper by Csiszar [12].



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On the other hand, Constantino Tsallis introduced the entropy in a nonextensive direction [13] as a generalization of the traditional Boltzmann–Gibbs entropy. Américo et al. recently proposed a new generalizing framework for conditional entropies given concavity, core-concavity, or quasi-concavity [14]. In general, a helpful tool to construct a new type of entropy is convexity/or concavity. We refer to Cambini’s book [15] for many valuable properties of the generalized convexity.

Quantum measures: As the most relevant extension in the quantum setting, the most natural extension of the entropy is the well-known von Neuman entropy $-\text{Tr}(\rho \log(\rho))$; for a density operator characterizing a quantum system, denoted ρ [16]. There also exist quantum Rényi entropy $(1-r)^{-1} \log(\text{Tr}(\rho^r))$ [17] by Muller et al. and quantum Tsallis entropy $(1-r)^{-1} \log(\text{Tr}(\rho^r) - 1)$ [18,19]; both with parameters $r > 0, r \neq 1$. Following then axiomatic approaches surveyed on Csiszar’s [12] in the classical setting, there exist axiomatic characterizations with a minimum of conditions for entropy as a function on the set of states. As such, and more recently, other generalized measures have also been introduced. Hu et al. [20] introduced a generalized family of quantum entropies with nonnegativity, continuity, and concavity. However, their generalization failed to satisfy subadditivity and additivity. Baumgartner [21] presented a new axiomatic characterization with a minimum of conditions for entropy as a function on the set of states in quantum mechanics. Bosyk et al. [22] presented another family of generalized quantum entropies. Their generalization is relatively unified, so they established basic properties satisfied by other well-known quantum entropies such as von Neumann and quantum versions of Rényi and Tsallis entropies. Furthermore, Fan et al. [23] studied the monotonicity of the unified quantum entropy following the work of Hu et al. [20]. Slomczynski et al. [24] studied two information-theoretical invariants for the projective unitary group acting on a finite-dimensional complex Hilbert space: PVM- and POVM-dynamical quantum entropies. They quantified the maximal randomness of the successive quantum measurement results in the case of the evolution of the system between every two consecutive measurements. Moreover, Ref. [25] proved essential properties of the quantum relative entropy through axiomatic approaches, i.e., continuity in the first argument, the validity of the data-processing inequality, additivity under tensor products, and super-additivity. Ref. [26] also studied super-additivity of quantum relative entropy and provided an extension of this inequality for arbitrary density operators. Ref. [27] followed Csiszar’s approach [12] in classical information theory, and showed the quantum α -relative entropies with parameter $\alpha \in (0, 1)$.

All in all, three distinctive approaches could be adopted to generate new classical measures [28]. Simply put, the new measures could be derived from axioms or principles, they could be extrapolated from physical experiments, or they could be qualified using a checklist of properties. This paper will adopt the latter approach to generate a new measure of quantum entropy. Our objective is to contribute to the growing tools for analyzing new quantum systems. Our main contributions are two-folds:

A new measure of quantum entropy of EQE: Generalizing frameworks on entropy has a long history in information theory. Similar to our lines of generalization in quantum entropy, Pastor et al. [28] constructed a sparsity measure to quantify the simplicity of the system. Our construction of the EQE is motivated by this sparsity measure as an inverse to quantify the randomness of the quantum system. We call this measure the effective quantum entropy (EQE). The EQE is derived from the classical entropy proposed by Ref. [28]; and the EQE will inherit atypical properties from it, e.g., the EQE will be semistrict quasi-concave, thus revealing original insights. However, the EQE will also satisfy several of the typical properties of quantum entropy, such as unitary invariance or Schur concavity. One of the practical advantages of the EQE is its broad sensitivity, which two parameters could control.

A partially consistent estimator for the EQE: We could compute the von Neumann or Rényi entropy if a quantum state is known. However, when characterizing an unknown system or when one seeks to verify that a system is behaving as desired experimentally, we need to use an estimator [29]. The estimation for quantum Rényi entropy has been extended to the quantum system [30]. On the other hand, Lopes [31] studied a ratio

form of an entropy in a classical setting. Ref. [31] introduced and studied the estimation of a family of entropy-based sparsity measures $(\|\mathbf{v}\|_q / \|\mathbf{v}\|_1)^{q/(1-q)}$ parameterized by $q \in [0, \infty]$, for a real vector \mathbf{v} . For an unknown state, we estimate the EQE given independent measurements of the state. Since sampling a state can be quite costly, it is desirable to minimize the number of measurements that are required to estimate the entropy to a desired precision and confidence. As explained by Keyl and Werner [32], a quantum system cannot be measured on a single system unlike a classical probability distribution. It can be estimated on an ensemble sequence of identically designed systems. Suppose we could fully determine the density operator on a single quantum system. Then, we could connect the measurement with a device, equipping multiple identical systems with the measured density, contradicting the no-cloning theorem [33]. We thus adopt this complexity as our figure-of-merit. Therefore, we numerically show the EQE’s parameters’ effect on the sampling requirements.

The paper is organized as follows. First, Section 2 introduces the preliminaries, including the main notation. The section will also present our proposed measure of quantum entropy, called the effective quantum entropy (EQE). We will then list and demonstrate a large set of properties of the EQE in Section 3. In Section 4, we will construct a consistent estimator of the EQE. Finally, Section 5 will present our conclusions and recommendations for future research. Our primary notation is summarized in Table 1.

Table 1. Summary of the main notation.

Symbol	Description
$\ \cdot\ _p$	Schatten p -(pseudo-)norm of matrices
$\cdot \oplus \cdot$	concatenation of vectors
$\cdot \succ \cdot$	majorization pre-order of density operators
α, β	parameters of the EQE, $\mathbf{H}_{\text{eq}}(\cdot)$
$\Theta(\cdot)$	increasing function
$\pi(\cdot)$	permutation (component-wise) of vectors
$\mathbf{H}_{\text{eq}}(\cdot)$	effective quantum entropy (EQE)
$h_{\text{eff}}(\cdot)$	effective (classical) entropy
$\text{sign}(\cdot)$	sign (component-wise) of vectors
\mathbf{v}, v_i, v_j	vector (in bold) and its components
ρ, σ, τ	density operators
μ, π	maximally mixing and pure state, respectively
U, V, W	unitary matrices
$(\cdot)^\dagger$	conjugate transpose (or Hermitian transpose)
$\text{Tr}(\cdot)$	trace of matrices
$\log(\cdot)$	natural logarithm of matrices
\mathbf{v}	probability mass function vector
$ \cdot $	component-wise absolute value
c_1, \dots, c_5	scalar (real numbers)
$\text{sup}(\cdot)$	supremum operator
$\cdot \otimes \cdot$	tensor product

2. Proposed Measure

In the following, $\|\rho\|_r = (\text{Tr} |\rho|^r)^{\frac{1}{r}}$ denotes the Schatten- r pseudo-norm of matrices [8], $\|\rho\|_r = (\text{Tr} |\rho|^r)^{\frac{1}{r}}$, for $r \in [0, \infty)$, where $\text{Tr}(\cdot)$ is the trace, and $|\cdot|$ is the component-wise absolute value.

2.1. Characterization of Classical Entropy

According to [28], a few distinctive approaches could be followed to generate new measures of classical entropy. In the quantum setting, that categorization holds. In brief, new measures of quantum entropy could be derived, proposed, or verified:

Axiomatic approach: New measures could be derived from the first principles. However, these principles could also impose strong requirements and produce fewer measures [34]. Examples include the von Neumann entropy, $-\text{Tr}(\rho \log(\rho))$ [3,16], and the quantum entropy of type r , $(r - 1)^{-1}(\|\rho\|_r - 1)$, with $r > 0, r \neq 1$ [35], which is an extension of the classical generalized entropy [36].

Operational approach: New measures could be designed to have a specific operational meaning in a physical or information-theoretic sense. However, the new measures could have a tight focus and serve only systems of similar nature as the one initially tested. Examples include the quantum min-entropy, $-\log \|\rho\|_\infty$, and quantum Hartley (max) entropy, $\log \|\rho\|_0$, which operational meaning was studied in the context of quantum systems in Ref. [37], and the quantum collision entropy, $\|\rho\|_2$ [17].

Verification approach: New measures could be proposed and then qualified as valid measures after verification of traditional properties. This approach could be innovative because the newly qualified measure could lead to atypical properties or original insights. Examples include the quantum Rényi entropy, $(1 - r)^{-1} \log(\|\rho\|_r^r)$ [17], and quantum Tsallis entropy $(1 - r)^{-1} \log(\|\rho\|_r^r - 1)$ [18], both with parameter $r > 0, r \neq 1$.

In this paper, we will follow the third approach. Our problem then will consist of three steps. First, to select a relevant measure of classical entropy. Second, to extend the selected measure into the quantum setting. Moreover, third, to verify that the extended measure satisfies the fundamental properties of quantum entropy. We will complete the first two steps in the rest of this Section.

2.2. The Effective Quantum Entropy (EQE)

We start by adopting the recently derived *effective entropy* measure in Ref. [28], which generates numerous of the standard measures of classical entropy by fixing one or two of its parameters or by applying an increasing transformation. This is a measure of generalized entropy, which follows a *quotient-of-functions* functional form, $h_{\text{eff}}(\mathbf{p}) = \frac{\|\mathbf{p}\|_\alpha}{\|\mathbf{p}\|_\beta}$, $0 \leq \alpha \leq 1 \leq \beta \leq \infty, \alpha \neq \beta$, where \mathbf{p} denotes a probability mass function. Intuitively, if the vector \mathbf{p} has k non-zero components, that is, if \mathbf{p} is associated to a process producing k different outcomes, then, $h_{\text{eff}}(\mathbf{p}) \leq k^{\frac{1}{\alpha} - \frac{1}{\beta}}$ by the Cauchy–Schwarz inequality, i.e., when k decreases or as the process becomes more certain, effective entropy decreases.

In quantum systems, a density operator, ρ , will play the role of a probability mass function, \mathbf{p} , in the classical setting. In particular, the maximally mixing state will resemble the uniform distribution (i.e., where the probability mass is equally allocated among all of the possible outcomes), and the pure state will mimic the distribution of a constant variable (i.e., where all the probability mass is concentrated into one single outcome). Then, in order to extend the effective entropy to the quantum setting, we first proceed to replace the vector r -norm, $\|\cdot\|_r$, by the Schatten- r pseudo-norm of matrices. We are now ready to propose a new measure.

Definition 1 (Effective Quantum Entropy). *The function,*

$$H_{\text{EQE}}(\rho) = \frac{\|\rho\|_\alpha}{\|\rho\|_\beta}, \quad 0 \leq \alpha \leq 1 \leq \beta \leq \infty, \alpha \neq \beta, \tag{1}$$

measures quantum entropy.

Similar to the effective entropy, the parameters α and β will tune the sensitiveness of the EQE. We will study the effect of both parameters later in Section 4. Before that, in the next Section 3, we will verify that the EQE satisfies the fundamental properties of quantum entropy. An essential tool in many of the proofs will be the notion of majorization. Let vectors, \mathbf{v} and \mathbf{w} , whose components, v_i and w_i , with $i = 1, \dots, n$, are set in decreasing order; it is said that “ \mathbf{v} is majorized by \mathbf{w} ”, denoted $\mathbf{v} \prec \mathbf{w}$, when $\sum_{i=1}^k v_i \leq \sum_{i=1}^k w_i$, for $k = 1, \dots, n - 1$, and $\sum_{i=1}^n v_i = \sum_{i=1}^n w_i$. Moreover, if the vectors do not share the same

length, then, by convention, the shorter one is considered complete by zero entries. We require to extend this notion to operate with density operators (instead of with vectors). To do so, we exploit that the quantum entropy of density operators is given by the set of its eigenvalues [21]. Then, we claim that the majorization relationship between density operators will be given by their set of eigenvalues (which could be represented by vectors). Our primary notation is summarized in Table 1.

3. Properties of Quantum Entropy

The previous Section 2 introduced the preliminaries and the effective quantum entropy. In this Section, we will verify that the EQE satisfies the fundamental properties of quantum entropy. Other practical properties will be verified as well. Unless otherwise noted, the primary notation from Table 1 holds.

3.1. Fundamental Properties

Property 1 (Unitary invariance). *Let $\sigma = V\rho V^\dagger$, with V unitary, e.g., a rotation. Then, $\mathbf{H}_{\text{EQE}}(\sigma) = \mathbf{H}_{\text{EQE}}(\rho)$. In particular, let $\rho = U\Lambda U^\dagger$ and $\sigma = V\Lambda V^\dagger$, with U, V unitary. Then, $\mathbf{H}_{\text{EQE}}(\sigma) = \mathbf{H}_{\text{EQE}}(\rho)$.*

Proof. The result follows from the unitary invariance of the Schatten pseudo-norm. \square

Property 2 (Schur concavity). *Let $\rho \prec \sigma$. Then, $\mathbf{H}_{\text{EQE}}(\rho) \geq \mathbf{H}_{\text{EQE}}(\sigma)$, with equality if and only if $\sigma = V\rho V^\dagger$, with V unitary.*

Proof. Note that $\|\cdot\|_r$ is a convex function when $r \geq 1$, and a concave one when $r \leq 1$. Then, if $\sigma \succ \rho$, we have that $\|\rho\|_\alpha \geq \|\sigma\|_\alpha$ and $\|\rho\|_\beta \leq \|\sigma\|_\beta$ by ([38], Proposition 3-C.1). So

$$\frac{\|\rho\|_\alpha}{\|\rho\|_\beta} \geq \frac{\|\sigma\|_\alpha}{\|\sigma\|_\beta}. \tag{2}$$

Therefore, the result follows. For the infinite-dimensional case, the result can be easily extended by following ([39], Property 2.2.7). \square

Property 3 (Semicontinuity). *It holds that $\mathbf{H}_{\text{EQE}}(\rho) = \sup_\sigma \{\mathbf{H}_{\text{EQE}}(\sigma) | \rho \prec \sigma\}$.*

Proof. Let $\rho \prec \sigma$. Then $\mathbf{H}_{\text{EQE}}(\rho) \geq \mathbf{H}_{\text{EQE}}(\sigma)$ by Property 2. Now, by taking the supremum with respect to σ , \sup_σ , in both sides, we have that

$$\mathbf{H}_{\text{EQE}}(\rho) \geq \sup_\sigma \{\mathbf{H}_{\text{EQE}}(\sigma) | \rho \prec \sigma\}. \tag{3}$$

The equality is satisfied since $\rho \prec \rho$. \square

Property 4 (Semistrict quasiconcavity). *Let ρ, σ such that $\mathbf{H}_{\text{EQE}}(\rho) \neq \mathbf{H}_{\text{EQE}}(\sigma)$, and $\tau = c_1\rho + (1 - c_1)\sigma$, with $c_1 \in (0, 1)$. Then, $\mathbf{H}_{\text{EQE}}(\tau) > \min\{\mathbf{H}_{\text{EQE}}(\rho), \mathbf{H}_{\text{EQE}}(\sigma)\}$.*

Proof. The result follows by the functional form of \mathbf{H}_{EQE} , and ([15], Theorem 2.3.8), which characterizes the quasi-concavity of the ratio of functions. Essentially, ([15], Theorem 2.3.8) says that a function $z(x) = f(x)/g(x)$ is semistrictly quasiconvex, if f is non-negative and convex, and g is positive and concave. \square

Property 5 (Expansibility). *Let ρ, σ with eigenvalues $\Lambda = [\lambda_1, \dots, \lambda_n]$ and $\Gamma = [\lambda_1, \dots, \lambda_n, 0]$, respectively. Then, $\mathbf{H}_{\text{EQE}}(\sigma) = \mathbf{H}_{\text{EQE}}(\rho)$.*

Proof. The result directly follows from the notion of majorization and by its convention on different dimensions. \square

These typical properties tell that the EQE is a semi-continuous and Schur concave function whose value depends on the eigenvalues of its argument. Besides this expected behavior, the EQE is semistrict quasi-concave, a new feature inherited from the classical effective entropy.

3.2. Other Properties

Property 6 (Concentration). Let ρ, σ with eigenvalues $\Lambda = [\lambda_1, \dots, \lambda_n]$, and $\Gamma = [\gamma_1, \dots, \gamma_{n-1}]$, respectively, with $\gamma_1 = \lambda_1 + \lambda_2$, and $\gamma_i = \lambda_{i+1}$, for $i = 2, \dots, n - 1$. Then, $\mathbf{H}_{\text{EQE}}(\rho) \geq \mathbf{H}_{\text{EQE}}(\sigma)$.

Proof. Let τ with eigenvalues $\Gamma = [\gamma_1, \dots, \gamma_n]$, with $\gamma_n = 0$. Then, $\mathbf{H}_{\text{EQE}}(\tau) = \mathbf{H}_{\text{EQE}}(\sigma)$ by Property 5. Note that

$$\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \gamma_i \text{ for } k = 1, \dots, n - 1, \text{ and } \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \gamma_i. \tag{4}$$

This implies that $\Lambda \prec \Gamma$. The result follows by Property 2. \square

Property 7 (Monotonicity). Let ρ be a density operator with eigenvalues $\Lambda = [\lambda_1, \dots, \lambda_n]$ where for $0 \leq \frac{c}{2} \leq a \leq b \leq c \leq 1$ and two indices $j < k$, $\lambda_j = a$ and $\lambda_k = c - a$, so $1 - c = \sum_{i \neq j, k} \lambda_i$. Let σ be a density operator with eigenvalues $\Gamma = [\gamma_1, \dots, \gamma_n]$ where $\gamma_i = \lambda_i$ for $i \neq j, k$, and $\gamma_j = b$ and $\gamma_k = c - b$. Then $\mathbf{H}_{\text{EQE}}(\rho) \geq \mathbf{H}_{\text{EQE}}(\sigma)$.

Proof. Since $\lambda_j = a \leq \gamma_j = b$, we have a majorization $\Lambda \prec \Gamma$. The result follows by Property 2. \square

Property 8 (Regularity). Let ρ be a density operator with largest eigenvalue, $\lambda_1 \geq \lambda_i$, for $i \neq 1$. And let $\sigma = (\rho + c_1|e_1\rangle\langle e_1|)/(1 + c_1)$ and $\tau = (\rho + d_1|e_1\rangle\langle e_1|)/(1 + d_1)$, where $|e_1\rangle$ is from an orthonormal basis of ρ and associated with the largest eigenvalue λ_1 , and $0 \leq c_1 < d_1$. Then, $\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\tau)$.

Proof. The eigenvalues of σ are $\Lambda := [\lambda_1 + c_1, \lambda_2, \dots, \lambda_n]/(1 + c_1)$, and the eigenvalues of τ are $\Gamma = [\lambda_1 + d_1, \lambda_2, \dots, \lambda_n]/(1 + d_1)$. Since $\frac{\lambda_1 + c_1}{1 + c_1} \leq \frac{\lambda_1 + d_1}{1 + d_1}$, $\Lambda \prec \Gamma$ thus $\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\tau)$, by Property 2. \square

Property 9 (Limits). Let μ, π be the maximally mixing and pure states. Then, $\mathbf{H}_{\text{EQE}}(\pi) \leq \mathbf{H}_{\text{EQE}}(\rho) \leq \mathbf{H}_{\text{EQE}}(\mu)$. Moreover, in a finite dimension, $|\mathbf{H}_{\text{EQE}}(\rho)| < \infty$.

Proof. For the first part of the inequality, note that $\pi \succ \rho \succ \mu$, for any ρ , by [40]. The result follows by Property 2.

For the second part of the finite EQE, note that $\|\rho\|_\beta \leq \|\rho\|_\alpha$, for $\alpha \leq \beta$, by ([41], Proposition 6.11), i.e., $1 \leq \mathbf{H}_{\text{EQE}}(\rho)$. For the upper bound, assume that $\text{rank}(\rho) < \infty$. Then,

$$\mathbf{H}_{\text{EQE}}(\rho) \leq \mathbf{H}_{\text{EQE}}(\mu) = (\text{rank } \rho)^{\frac{1}{\alpha} - \frac{1}{\beta}} < \infty \tag{5}$$

by applying the first part inequality. \square

Property 10 (Homogeneous growth). Let $\rho \neq I_n$, and $\sigma = \rho + c_1 I_n$, with $c_1 > 0$. Then, $\mathbf{H}_{\text{EQE}}(\sigma) > \mathbf{H}_{\text{EQE}}(\rho)$.

Proof. Let $\tau = c_2 \rho + (1 - c_2) I_n$, with $c_2 = \frac{1}{1 + c_1} < 1$, because $c_1 > 0$. First, $\mathbf{H}_{\text{EQE}}(\tau) = \mathbf{H}_{\text{EQE}}(\sigma)$, by Property 1. Then,

$$\mathbf{H}_{\text{EQE}}(\sigma) > \min\{\mathbf{H}_{\text{EQE}}(\rho), \mathbf{H}_{\text{EQE}}(I_n)\} \tag{6}$$

by Property 4. Note that I_n is a maximally mixing operator. Therefore, $\mathbf{H}_{\text{EQE}}(\sigma) > \mathbf{H}_{\text{EQE}}(\rho)$ by Property 9. \square

The next property is about acting a bistochastic operator. An operator \mathcal{E} between two finite dimensional densities is called bistochastic if given an orthonormal basis $\{|e_i\rangle\}_{i=1}^n$, \mathcal{E} can be represented as $\mathcal{E} = \sum_{i,j=1}^n \mathcal{E}_{i,j} |e_i\rangle\langle e_j|$ with $\sum_i \mathcal{E}_{i,j} = 1$ and $\sum_j \mathcal{E}_{i,j} = 1$.

Property 11 (Bistochastic operator). Let $\sigma = \mathcal{E}(\rho)$, with \mathcal{E} a bistochastic operator. Then, $\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\rho)$, with equality if and only if $\mathcal{E}(\rho) = V\rho V^\dagger$ for unitary V .

Proof. By the quantum Hardy–Littlewood–Pólya theorem, $\sigma \prec \rho$ ([42], Lemma 12.1). Then, the result follows by Property 2. \square

Property 12 (Mixture). Let ρ with eigenvalues $\Lambda = [\lambda_1, \dots, \lambda_n]$ with orthonormal basis $\{|\psi_i\rangle\}_{i=1}^n$, and $\sigma = \sum_{i=1}^n \lambda_i |\psi_i\rangle\langle\psi_i|$ be a mixture of arbitrary rank-one densities $\pi_i = |\psi_i\rangle\langle\psi_i|$. Then, $\mathbf{H}_{\text{EQE}}(\rho) \geq \mathbf{H}_{\text{EQE}}(\sigma)$.

Proof. By Schrödinger mixture theorem [42,43] (Theorem 8.2 in Ref. [42]), there exists a bistochastic map \mathcal{E} such that $\sigma = \mathcal{E}(\rho)$. Then, $\Lambda \prec \Gamma$ by Property 11. The result follows by Property 2. \square

The next property is about POVM (positive operator-valued measurement), which is a collection of $\{E_i\}$ of positive definite operators such that $\sum_{i=1}^m E_i = I_n$. Suppose that the measurement M_i from $E_i = M_i^\dagger M_i$ is performed upon a system in the state $|\psi_i\rangle$. For more details, see Section 2.2.6 in [3].

Property 13 (POVM). Let σ be an operator on an arbitrary orthonormal basis $\{|e_i\rangle\}_{i=1}^n$ with eigenvalues $\gamma_i^E(\rho) = \text{Tr}(E_i\rho)$ where $E = \{E_i\}_{i=1}^m$ is a rank-one POVM. Then, $\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\rho)$. Moreover, $\mathbf{H}_{\text{EQE}}(\rho) = \min_{E \in \mathbb{E}} \mathbf{H}_{\text{EQE}}(\sigma)$, $\forall E \in \mathbb{E}$, where \mathbb{E} is the set of all rank-one POVMs.

Proof. For any rank-one POVM $E = \{E_i\}_{i=1}^m$, there exist unitary operators U_i such that $M_i = U_i\sqrt{E_i}$, where E_i is the POVM to the measurement ([3] Exercise 2.63). Then $M_i^\dagger = M_i$. Let

$$\mathcal{E}_E(\rho) = \sum_{i=1}^m M_i\rho M_i = \sum_{i=1}^m \gamma_i^E(\rho) \frac{M_i\rho M_i}{\text{Tr}(E_i\rho)} = \sum_{i=1}^m \gamma_i^E(\rho) |\psi_i\rangle\langle\psi_i|, \tag{7}$$

where $|\psi_i\rangle$ is a normalized pure state from M_i since E_i is a rank-one operator. Then since $\sum_{i=1}^m E_i = I_n$, $\mathcal{E}_E(\rho)$ is a bistochastic operator. Therefore, the first inequality follows from Property 11.

$$\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\rho). \tag{8}$$

For the second property, since E can be arbitrary rank-on POVM,

$$\mathbf{H}_{\text{EQE}}(\rho) \leq \min_{E \in \mathbb{E}} \mathbf{H}_{\text{EQE}}(\sigma). \tag{9}$$

If we consider $E^* = \{|e_i\rangle\langle e_i|\}_{i=1}^n$ where $\{|e_i\rangle\}_{i=1}^n$ is an orthonormal basis that diagonalizes ρ . Then E^* is a rank-one POVM. Therefore the equality holds. c.f., see the proof of ([22], Proposition 8). \square

Property 14 (Reparametrization). Let σ be a density operator with eigenvalues $\gamma_i = \langle e_i|\rho|e_i\rangle$ by acting on an arbitrary orthonormal basis $\{|e_i\rangle\}_{i=1}^n$. Then, $\mathbf{H}_{\text{EQE}}(\sigma) \geq \mathbf{H}_{\text{EQE}}(\rho)$.

Proof. We can decompose ρ with the orthonormal basis $\{|e_i\rangle\}_{i=1}^n$ into the form $\rho = \sum_{i,j=1}^n \rho_{i,j} |e_i\rangle\langle e_j|$ where the diagonal terms are $\rho_{i,i} = \gamma_i$. Therefore, we can apply

the Schur–Horn theorem [44,45] implying that γ_i 's of diagonal terms of ρ and the eigenvalues of σ is majorized by the eigenvalues of ρ , i.e., $\sigma \prec \rho$. Then the proof follows by Property 2. c.f., see Proposition 5 in [22]. \square

Property 15 (Core-concavity). *Let ρ, σ such that, $\mathbf{H}_{\text{EQE}}(\sigma) < \mathbf{H}_{\text{EQE}}(\rho)$. Then, $\Theta \circ \mathbf{H}_{\text{EQE}}(\sigma) < \Theta \circ \mathbf{H}_{\text{EQE}}(\rho)$, for any increasing transformation Θ .*

Proof. The result follows from the monotonicity of increasing functions, c.f., proof of ([46], Property 9). \square

These *other* properties illustrate the response of the EQE when its argument is sparsified (i.e., the probability mass is concentrated in few, or even one possible outcome) or democratized (i.e., the probability mass is spread along any or all of the possible outcomes). Overall, these properties bring practical insights into the dynamics of the EQE.

3.3. Specific Properties of Composite Operators

Property 16 (Replication). *Let $\sigma = \rho \oplus \dots \oplus \rho$ (r -repeated states). Then, $\mathbf{H}_{\text{EQE}}(\sigma) > \mathbf{H}_{\text{EQE}}(\rho)$. In particular, let $\tilde{\mu}$ be the maximally mixing state of dimension \tilde{n} , with $\tilde{n} > n$. Then, $\mathbf{H}_{\text{EQE}}(\tilde{\mu}) > \mathbf{H}_{\text{EQE}}(\mu)$.*

Proof. Let τ with eigenvalues $\Xi = [\xi_1, \dots, \xi_{rn}]$, with $\xi_i = \lambda_i$, for $i = 1, \dots, n$ and $\xi_j = 0$, for $j = n + 1, \dots, rn$. Then, $\mathbf{H}_{\text{EQE}}(\tau) = \mathbf{H}_{\text{EQE}}(\rho)$ by Property 5.

Note that the eigenvalues of σ are $\Gamma = [\gamma_1, \dots, \gamma_{rn}]$, with $\gamma_{i+jn} = \frac{\lambda_i}{r}$ for $i = 1, \dots, n$, $j = 0, \dots, r - 1$. This implies that $\sum_{i=1}^{k-1} \gamma_i \leq \sum_{i=1}^{k-1} \xi_i$, for $k = 1, \dots, rn$ and $\sum_{i=1}^{rn} \gamma_i = \sum_{i=1}^{rn} \xi_i$, i.e., $\Gamma \prec \Xi$. The result follows by Property 2. \square

Property 17 (Decomposition). *Let π be the pure state, and $\sigma = \rho \otimes \pi$. Then, $\mathbf{H}_{\text{EQE}}(\sigma) = \mathbf{H}_{\text{EQE}}(\rho)$.*

Proof. Let $\rho = U\Lambda U^\dagger$, $\pi = V\Pi V^\dagger$, with U, V unitary, and $\Pi_{11} = 1$, $\Pi_{ij} = 0$ elsewhere. Then $\rho \otimes \pi = (U \otimes v \otimes e_1)\Lambda(U \otimes v \otimes e_1)^\dagger$, with $v_i = V_{i1}$. The result follows by Property 1. \square

Property 18 (Composite pure). *Let $\pi = \rho \otimes \sigma$ be a composite pure state. Then, $\mathbf{H}_{\text{EQE}}(\sigma) = \mathbf{H}_{\text{EQE}}(\rho)$.*

Proof. Applying the Schmidt decomposition ([42], Theorem 9.1), for any pure state $|\psi\rangle$ in $\rho \otimes \sigma$, we have

$$|\psi\rangle = \sum_{i=1}^n \sqrt{\lambda_i} |e_i\rangle \otimes |e'_i\rangle, \tag{10}$$

where $\{e_i\}_{i=1}^{n_\rho}$ and $\{e'_i\}_{i=1}^{n_\sigma}$ are orthonormal bases in ρ and σ , and note that $n = \min\{n_\rho, n_\sigma\}$ where n_ρ and n_σ are dimensions of operators ρ and σ . Therefore ρ and σ should have the forms:

$$\rho = \sum_{i=1}^n \lambda_i |e_i\rangle \langle e_i|, \text{ and } \sigma = \sum_{i=1}^n \lambda_i |e'_i\rangle \langle e'_i|. \tag{11}$$

Therefore, the eigenvalues of ρ and σ should be the same. Then the result follows by definition. c.f., see the proofs of ([20], Proposition 8) or ([22], Proposition 13). \square

We finish this Section by showing how the proposed fundamental properties of the EQE, and other derived properties, are also obeyed by the classical quantum entropy. Previous research works have proposed and extensively studied the inherent properties of classical quantum entropy [20–22], namely the unitary invariance, Schur concavity, semicontinuity and expansibility (Properties 1, 2, 3, and 5, respectively). Remarkably, most of the other properties proposed in this work (Properties 6–15), could be derived using the

Schur concavity of the EQE. Table 2 shows this mapping between our proposed properties to axioms, propositions, and properties by previous research. Granted, the originality and novelty of this work is based on the Property 4, which introduces the quasi-concavity nature of the quantum entropy. However, although the EQE verifies typical properties of composite states (Properties 16–18), the additivity properties for independent systems and subadditivity will not generally hold (except when one parameter is set equal to 1 and the other approaches 1). Nonetheless, numerical experiments suggested a tight bound that is worth further investigation.

Table 2. Mapping of the proposed properties to axioms, propositions, and properties by previous works.

Proposed Properties	Previously Adopted Properties
Fundamental properties:	
Property 1	([21], Axiom B), ([22], Property 6)
Property 1 (second part)	([21], Property 1)
Property 2	([20], Property 5), ([21], Axiom C'), ([22], Property 1)
Property 3	([21], Axiom F')
Property 4	
Property 5	[22]
Other properties:	
Property 6	[22]
Property 7	
Property 8	
Property 9	([20], Propositions 1 and 2), ([22], Proposition 2)
Property 10	
Property 11	([22], Property 7)
Property 12	([22], Property 4)
Property 13	([22], Proposition 8)
Property 14	([22], Proposition 5)
Property 15	
Specific to composite operators:	
Property 16	([21], Axiom C)
Property 17	([21], Axiom A)
Property 18	([20], Proposition 8), ([22], Proposition 13)

Conjecture 1 (Arithmetic Mean). *The proposed EQE would hold the following arithmetic mean property.*

$$H_{EQE}(\rho \oplus \sigma) \lesssim \frac{\|\rho\|_1}{\|\rho \oplus \sigma\|_1} H_{EQE}(\rho) + \frac{\|\sigma\|_1}{\|\rho \oplus \sigma\|_1} H_{EQE}(\sigma). \tag{12}$$

The next Section 4 will present an application example. We will discuss how to estimate the EQE of unknown systems in practice.

4. Numerical Experiments

In the estimation of EQE, the main challenge comes from the treatment of the ratio of two norms. Lopes [31] studied an estimation of the ratio of two norms in a sparsity setting, but it does not fit into the quantum setting because of different measurement procedures. For the quantum Rényi entropy with a parameter $\alpha > 0$, Acharya et al. [30] extensively studied the error bound of the Schur estimator for integral α 's and the natural Empirical Young Diagram (EYD) estimator [32,47,48] for non-integral $\alpha > 0$. For simplicity, we adopt the natural EYD estimator analyzed in Ref. [30] for any $\alpha, \beta > 0$. The analysis of the error bounds is beyond the scope of this paper. Instead, we illustrate the empirical performance of estimation errors by applying the EYD algorithm for any $\alpha, \beta > 0$.

EYD Algorithm

The EYD algorithm is a natural and the most straightforward way of estimating a spectrum of an operator ρ . As discussed in [30], it is a quantum analog of the classical empirical estimator. Let m be the number of observations, and γ_i be the number of observations of the state i . (1) For each symbol i , find the empirical distribution $\frac{\gamma_i}{m}$, (2) return an operator $\hat{\rho}$ of the mixed state with eigenvalues of $\frac{\gamma_i}{m}$. Then we estimate the α -norm of $\hat{\rho}$ as follows:

$$\|\hat{\rho}\|_r = \left[\sum \left(\frac{\gamma_i}{m} \right)^r \right]^{1/r}. \tag{13}$$

Therefore, our EQE estimator can be given by $\mathbf{H}_{\text{EQE}}(\hat{\rho}) = \frac{\|\hat{\rho}\|_\alpha}{\|\hat{\rho}\|_\beta}$. In general, we cannot guarantee the unbiasedness or the consistency of EYD estimator for our EQE.

Property 19. *The EYD estimator is not an unbiased estimator in general.*

$$\mathbb{E} \left(\frac{\gamma_i}{m} \right)^\alpha \leq \lambda_i^\alpha \quad \text{for } \alpha \leq 1, \quad \mathbb{E} \left(\frac{\gamma_i}{m} \right)^\beta \geq \lambda_i^\beta \quad \text{for } \beta \geq 1. \tag{14}$$

Proof. Jensen’s inequality implies both inequalities because $\mathbb{E} \left(\frac{\gamma_i}{m} \right) = \lambda_i$ for each $i = 1, \dots, n$. \square

However, if we fix $\beta = 1$, it is similar to the classical Rényi entropy, but we have an additional exponent $1/\alpha$ outside the summation. We can prove that it is still consistent.

Property 20. $\mathbf{H}_{\text{EQE}}(\hat{\rho})$ is consistent when $\beta = 1$.

Proof. For $\alpha = 1$, it is true because $\sum_i \left(\frac{\gamma_i}{m} \right) = \sum_i \lambda_i = 1$. From Lemma 20 in [30] (or applying the triangle inequality), we have when $\alpha < 1$,

$$\mathbb{E} \left| \sum_i \left(\frac{\gamma_i}{m} \right)^\alpha - \sum_i \lambda_i^\alpha \right| \leq O \left(\frac{1}{m^{\frac{\alpha}{2}}} \right). \tag{15}$$

Then for a sufficiently large constant $C(t) > 0$ not depending on m ,

$$\mathbb{P} \left[\left| \left(\sum_i \left(\frac{\gamma_i}{m} \right)^\alpha \right)^{1/\alpha} - \left(\sum_i \lambda_i^\alpha \right)^{1/\alpha} \right| > t \right] \leq \mathbb{P} \left[C(t) \left| \sum_i \left(\frac{\gamma_i}{m} \right)^\alpha - \sum_i \lambda_i^\alpha \right| > t^\alpha \right] \tag{16}$$

$$\leq \frac{C(t) \mathbb{E} \left| \sum_i \left(\frac{\gamma_i}{m} \right)^\alpha - \sum_i \lambda_i^\alpha \right|}{t^\alpha} \tag{17}$$

$$\leq O \left(\frac{1}{m^{\frac{\alpha}{2}}} \right), \tag{18}$$

where (17) is by applying Markov’s inequality, and (18) is from (15). Then the proof follows by taking $m \rightarrow \infty$.

For (16), let $\Gamma_m := \left(\sum_i \left(\frac{\gamma_i}{m} \right)^\alpha \right)^{1/\alpha}$ and $\lambda := \left(\sum_i \lambda_i^\alpha \right)^{1/\alpha}$. We know that $0 \leq \lambda \leq n^{1/\alpha}$. Then

$$\{ |\Gamma_m - \lambda| > t \} = \{ \Gamma_m > \lambda + t \} \cup \{ \Gamma_m < \lambda - t \}, \text{ and} \tag{19}$$

$$\{ C(t) |\Gamma_m^\alpha - \lambda^\alpha| > t^\alpha \} = \left\{ \Gamma_m > \left(\lambda^\alpha + \frac{t^\alpha}{C(t)} \right)^{1/\alpha} \right\} \cup \left\{ \Gamma_m < \left(\lambda^\alpha - \frac{t^\alpha}{C(t)} \right)^{1/\alpha} \right\}. \tag{20}$$

For a sufficiently large $C(t) > 0$ not depending on m , since λ lies in a compact domain, we can have $\lambda + t \geq \left(\lambda^\alpha + \frac{t^\alpha}{C(t)} \right)^{1/\alpha}$ and $\lambda - t \leq \left(\lambda^\alpha - \frac{t^\alpha}{C(t)} \right)^{1/\alpha}$. This means

$$\{\Gamma_m > \lambda + t\} \subseteq \left\{ \Gamma_m > \left(\lambda^\alpha + \frac{t^\alpha}{C(t)} \right)^{1/\alpha} \right\}, \text{ and} \tag{21}$$

$$\{\Gamma_m < \lambda - t\} \subseteq \left\{ \Gamma_m < \left(\lambda^\alpha - \frac{t^\alpha}{C(t)} \right)^{1/\alpha} \right\}, \tag{22}$$

implying that $\{|\Gamma_m - \lambda| > t\} \subseteq \{C(t)|\Gamma_m^\alpha - \lambda^\alpha| > t^\alpha\}$. Therefore (16) holds. \square

Then, Table 3 and Figure 1 show the behavior of the empirical L1-error of the EYD estimator of EQE. i.e., $L1\text{-error} = |\mathbf{H}_{EQE}(\hat{\rho}) - \mathbf{H}_{EQE}(\rho)|$. We report the number of required measurements to achieve a target L1-error given the dimension of ρ for various choices of α and β 's. For fixed $0 \leq \alpha \leq 1$ and $\beta \geq 1$, as the dimension increases, it requires more measurements to achieve the same L1-error, as expected. For fixed $\alpha \leq 1$ and the dimension, it tends to require more measurements to achieve the same L1-error as β increase. For fixed $\beta \geq 1$, it requires less measurements as α increases.

Table 3. The required number of measurement samples scaled by \log_{10} to achieve a target estimation L1-error for various α and β values. The value > 9.0 means that we are not able to find a proper number of measurements since it requires a large memory for simulation.

Case	Dimension	L1-Error	\log_{10} (# of Measurement Samples)								
			$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
$\beta = 2.0$	2	0.5	2.98	2.12	1.61	1.37	1.16	1.04	0.61	0.30	0.30
		0.75	2.79	1.92	1.44	1.16	0.87	0.61	0.30	0.30	0.30
		1.0	2.67	1.80	1.34	1.04	0.61	0.30	0.30	0.30	0.30
	4	0.5	4.16	3.06	2.38	2.05	1.83	1.57	1.32	1.04	0.91
		0.75	3.97	2.88	2.21	1.85	1.63	1.39	1.08	0.91	0.61
		1.0	3.85	2.74	2.07	1.71	1.49	1.22	0.91	0.61	0.30
	8	0.5	6.17	5.35	3.32	2.79	2.34	2.16	1.93	1.69	1.47
		0.75	5.92	5.51	3.10	2.62	2.17	1.93	1.73	1.45	1.20
		1.0	5.79	5.45	2.98	2.48	2.05	1.78	1.56	1.27	1.00
$\beta = 10.0$	2	0.5	3.99	3.05	2.43	2.12	1.88	1.75	1.59	1.35	1.15
		0.75	3.80	2.86	2.24	1.90	1.47	1.42	1.27	0.91	0.61
		1.0	3.68	2.70	2.07	1.69	1.25	1.13	0.91	0.30	0.30
	4	0.5	5.27	4.00	3.49	3.08	2.91	2.62	2.45	2.31	2.09
		0.75	5.08	3.80	3.23	2.89	2.65	2.38	2.19	2.02	1.83
		1.0	4.95	3.68	3.05	2.72	2.48	2.17	2.00	1.81	1.59
	8	0.5	7.13	6.35	5.89	4.09	3.66	3.34	3.20	3.03	2.93
		0.75	6.94	6.17	5.38	3.87	3.42	3.15	3.00	2.81	2.67
		1.0	6.53	5.73	4.55	3.71	3.27	3.03	2.84	2.64	2.45
$\beta = +\infty$	2	0.5	5.43	4.23	3.23	2.60	2.33	2.14	1.87	1.73	1.52
		0.75	5.25	3.87	2.88	2.23	1.88	1.63	1.34	1.10	1.06
		1.0	5.18	3.56	2.55	2.02	1.54	1.20	0.96	0.78	0.48
	4	0.5	8.34	6.84	4.96	4.35	3.83	3.44	3.20	3.08	2.91
		0.75	8.19	6.58	4.54	4.00	3.44	3.06	2.82	2.64	2.48
		1.0	7.80	6.15	4.28	3.66	3.17	2.82	2.55	2.31	2.14
	8	0.5	> 9.0	> 9.0	7.15	6.70	6.13	5.38	4.59	4.26	4.07
		0.75	> 9.0	8.71	6.92	6.41	5.39	4.52	4.12	3.94	3.63
		1.0	> 9.0	8.28	6.35	5.84	4.78	4.24	3.85	3.70	3.32

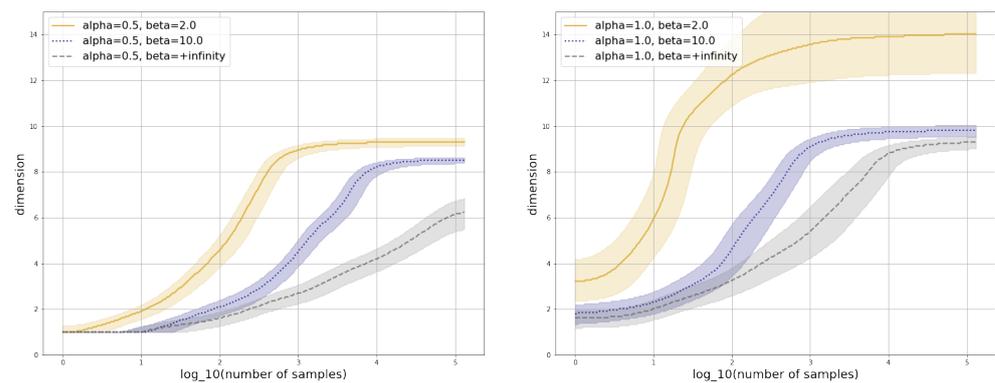


Figure 1. EQE estimation error contour plots for various α and β 's. We interpolate the curve on non-integer grid points over both dimension and number of measurements for a better visualization. We fix L1-error = 0.5 (lower band), L1-error = 0.75 (middle line), and L1-error = 1.0 (upper band) for illustrating the error-range.

5. Conclusions

In recent years, the measures to compute quantum entropy have been formulated from their counterparts in the classical setting. Similarly, a recently derived classical entropy has enabled a new formulation in this paper. We have introduced and demonstrated the effective quantum entropy (EQE). Like its predecessor, the EQE inherits a functional form, which conceptually unifies quotients of functions that have been repeatedly observed in the literature.

Lastly, quantum entropy is a central notion in modern (quantum) systems, aiming to exploit the large-dimension and correlation of underlying random processes. Granted, the possibilities for future research from this paper are extensive. An extension of EQE to relative and conditional quantum entropy would be desirable. It would also be interesting to explore how EQE can be efficiently optimized in democracy sensing algorithms. To that end, the work presented in this paper suggests new insights into the dynamics and properties of quantum entropy that may benefit the design of algorithms.

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