On New Types of Multivariate Trigonometric Copulas

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Abstract: Copulas are useful functions for modeling multivariate distributions through their univariate marginal distributions and dependence structures. They have a wide range of applications in all fields of science that deal with multivariate data. While there is a plethora of copulas, those based on trigonometric functions, especially in dimensions greater than two, have received much less attention. They are, however, of interest because of the properties of oscillation and periodicity of the trigonometric functions, which can appear in certain models of correlation of natural phenomena. In order to fill this gap, this paper introduces and investigates two new types of “multivariate trigonometric copulas”. Their main theoretical properties are studied, and some perspectives for applications are sketched for future work. In particular, we show that the proposed copulas are symmetric, not associative, with no orthant dependence, and with copula densities that have wide oscillations, which remains an uncommon property in the field. The expressions of their multivariate Spearman’s rho are also determined. Furthermore, the first type of the proposed copulas has the interesting feature of having a multivariate Spearman’s rho equal to 0 for all of the dimensions. Some graphic evidence supports the findings. Some mathematical formulas involving the product of n trigonometric functions may be of independent interest.

Keywords: multivariate copulas; trigonometric function; concordance measure; modeling

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1. Introduction

Copulas can be defined as multivariate functions that are used to extract and comprehend the dependence structures in multivariate distributions. On the other hand, we can build any multivariate distribution by describing the marginal distributions and copula separately. In recent years, copula modeling has become more relevant in a range of sectors, including economics, finance, engineering, medicine, biology, environmental sciences, and agronomy. Many data analysis scenarios rely on them. The mathematical definition of a copula is recalled below.

Definition 1. Let n be an integer greater or equal to 2. A n -dimensional (or multivariate with an implicit n) copula is a cumulative distribution function of a continuous distribution with n uniform marginals over the interval $[0, 1]$.

In the absolutely continuous case, a n -dimensional copula is a function $C : [0, 1]^n \rightarrow [0, 1]$ such that, for any $(x_1, \dots, x_n) \in [0, 1]^n$,

- (a) if $\min(x_1, \dots, x_n) = 0$, we have $C(x_1, \dots, x_n) = 0$,
- (b) for any $i = 1, \dots, n$, we have $C(1, \dots, 1, x_i, 1, \dots, 1) = x_i$,
- (c) we have

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} C(x_1, \dots, x_n) \geq 0.$$

This definition and all of the necessary information about the copulas can be found in [1–3]. Among the classical copulas (or family of copulas), we may cite the Frank copula (see [4]), the Gumbel–Hougaard copula (see [3]), the Clayton copula (see [3]), the Ali–Mikhail–Haq copula (see [3]), the Joe copula (see [5]), the Farlie–Gumbel–Morgenstern copula (see [3]), the Plackett copula (see [6]), the Raftery copula (see [7]), the Galambos copula (see [8]), the Hüsler–Reiss copula (see [9]), the elliptical copula (see [10]), the Fréchet copula (see [3]), and the Marshall–Olkin copula (see [11]). General constructions of

multivariate copulas can be found in [12–15]. Modern applications of the standard copulas can be found in [16–19]. When they are made explicit, the above mentioned copulas are often defined with exponential, logarithmic, power, and special integral functions.

On the other hand, recent years have been marked by a boom in the development of distributions defined by trigonometric functions, such as sin, cos, or tan. Numerous studies show that such trigonometric distributions benefit from flexible curvature properties, which allow them to extract a maximum amount of information about sophisticated phenomena. In particular, the corresponding estimated probability density function may be able to capture details of the shapes of a data histogram. This is particularly true for skewed, heavy-tailed, and bimodal data, whose fits are often underperformed by classical distributions. On this topic, we can consult [20–23], as well as the survey of [24]. However, the multivariate versions of trigonometric distributions remain underexplored, especially from the viewpoint of correlation. In particular, highlighted copulas that involve trigonometric functions are rare, despite a certain potential for applications. The most famous trigonometric copulas are the simple sine (SS) copulas, exemplified in ([25], Example 1 (point 4)) and ([26], Example 5), then further studied in [27], the sine-type copulas created by [28], the polynomial cosine copula introduced by [29], and the trigonometric archimedean copulas developed by [30,31]. The appeal of such copulas is that the trigonometric function's oscillating features can be used to describe various variable dependence structures that non-trigonometric copulas cannot. In the two dimensional case, we can think of the analysis of data as presenting scatter plots with oscillating patterns. The practical benefit is especially apparent when analyzing multivariate environmental data, such as those investigated by [30,31].

In this paper, we provide a contribution to the field by focusing on multivariate (absolutely continuous) copulas of the following special form:

$$C(x_1, \dots, x_n) = \prod_{i=1}^n x_i + \theta \Psi(x_1, \dots, x_n), \quad (1)$$

where θ is a tuning parameter and $\Psi : [0, 1]^n \rightarrow [0, 1]$ depends only on trigonometric functions (sin, cos, tan, ...). Surprisingly, such multivariate trigonometric copulas have not received much attention in the literature, despite the perspectives of modeling given by the oscillating properties offered by $\Psi(x_1, \dots, x_n)$. As an example, we may mention the SS copula defined by Equation (1) with the following multivariate separable sin-function:

$$\Psi(x_1, \dots, x_n) = \prod_{i=1}^n \sin(\pi x_i),$$

where $\theta \in [-1/\pi^n, 1/\pi^n]$. It is proved to have interesting shape and correlation properties, and can be more suitable to the famous FGM copula in some practical scenarios; see, for instance, [27]. As a new remark, the SS copula can be easily extended by considering Equation (1) with the following multivariate separable parametrized sin-function:

$$\Psi(x_1, \dots, x_n) = \prod_{i=1}^n [\sin(b_i \pi x_i)]^{a_i},$$

where $\theta \in [-1/(\pi^n \prod_{i=1}^n b_i a_i), 1/(\pi^n \prod_{i=1}^n b_i a_i)]$ and is under some restrictions on the involved parameters, which are, for any $i = 1, \dots, n$, b_i , a strictly positive integer, and,

- If $b_i = 1$, then $a_i \geq 1$;
- If $b_i \geq 2$, then a_i is an integer strictly greater than 1.

This copula can be asymmetric, orthant-dependent (positively or negatively), radially symmetric, and with no tail dependence. For practical reasons, the selection of these parameters remains somewhat subjective, but this generalization remains unexplored. However, because this generalization is based on the same separable structure as the SS copula definition, its mathematical novelty is limited.

In light of the foregoing, we propose SS copula extensions based on a mathematical approach aimed at breaking the separable structure of the related function $\Psi(x_1, \dots, x_n)$.

In particular, we demonstrate that valid copulas emerge from Equation (1) with a function $\Psi(x_1, \dots, x_n)$ of the following form:

$$\Psi(x_1, \dots, x_n) = \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \Phi \left(\sum_{i=1}^n x_i \right),$$

where $\Phi(x) \in \{1, \cos(\pi x), \sin(\pi x)\}$ and $\theta \in [-1/\pi^n, 1/\pi^n]$. The function $\Phi(x) = 1$ gives the SS copula, whereas $\Phi(x) \in \{\cos(\pi x), \sin(\pi x)\}$ opens up new avenues for statistical modeling. By adopting a theoretical viewpoint, we investigate the fundamental qualities of the corresponding copulas. We show that they are symmetric, not associative, without orthant dependence, and with copula densities that oscillate widely, which is a rare property among the existing copulas. Their multivariate Spearman's rho expressions are also determined. Surprisingly, one type of the considered multivariate trigonometric copula has the intriguing property of having a multivariate Spearman's rho equal to 0 for any n . It is thus a rare example of n -dimensional copula having this feature. The proofs are based on new trigonometric formulas involving multiple trigonometric functions. The bivariate cases are highlighted with illustrations by the means of graphics. We thus open a direction of research for an applied perspective in many multidimensional applied fields.

The rest of the paper is organized as follows. Section 2 presents and studies the first type of multivariate trigonometric copula. Properties and examples are described. The second type of multivariate trigonometric copula is described in Section 3, along with properties and examples. The proofs are postponed in Section 4. Section 5 concludes the paper.

2. First Type of Multivariate Trigonometric Copula

The first type of the proposed multivariate trigonometric copulas is presented in the next result.

Proposition 1. *The following function $C_1 : [0, 1]^n \rightarrow [0, 1]$ is a valid n -dimensional copula:*

$$C_1(x_1, \dots, x_n) = \prod_{i=1}^n x_i + \theta \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n},$$

where $u_n = 1$ if n is odd, and $u_n = 0$ if n is even, with $\theta \in [-1/\pi^n, 1/\pi^n]$.

For the purposes of this paper, the copula presented in Proposition 1 is called the type 1 extended multivariate sine (T1EMS) copula. Owing to a part of the proof of Proposition 1 (see Equation (5)), the corresponding copula density is

$$c_1(x_1, \dots, x_n) = 1 + \theta \pi^n \left[\cos \left(2\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \left[\sin \left(2\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n}.$$

Some properties of the T1EMS copula are described below. The parameter θ must be viewed as a dependence parameter; if $\theta = 0$, the T1EMS copula reduces to the independent copula. In addition, the T1EMS copula is symmetric: for any permuted vector of $(x_1, \dots, x_n)$, say $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$, we have $C_1(x_1, \dots, x_n) = C_1(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. The T1EMS copula is not radially symmetric, at least for all n ; for $n = 2$, we have $C_1(x_1, x_2) = x_1 x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi(x_1 + x_2))$ and the survival T1EMS copula is given by

$$\bar{C}_1(x_1, x_2) = x_1 + x_2 - 1 + C_1(1 - x_1, 1 - x_2) = x_1 x_2 - \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi(x_1 + x_2)),$$

so $\bar{C}_1(x_1, x_2) \neq C_1(x_1, x_2)$. Since $[\cos(\pi \sum_{i=1}^n x_i)]^{u_n} [\sin(\pi \sum_{i=1}^n x_i)]^{1-u_n} \in [-1, 1]$, there is no inequality of the form: $C_1(x_1, \dots, x_n) \geq \prod_{i=1}^n x_i$ or $C_1(x_1, \dots, x_n) \leq \prod_{i=1}^n x_i$ for any

$(x_1, \dots, x_n) \in [0, 1]^n$; there is no orthant dependence. The well-known Fréchet–Hoeffding theorem may be applied: we have

$$\max \left\{ 0, \sum_{i=1}^n x_i - n + 1 \right\} \leq C_1(x_1, \dots, x_n) \leq \min(x_1, \dots, x_n). \quad (2)$$

We can also note that the T1EMS copula is not associative: for instance, for $n = 2$ and $\theta = 1/(2\pi^2)$, we have $C_1(C_1(1/2, 1/3), 1/4) = 0.06678487$ and $C_1(1/2, C_1(1/3, 1/4)) = 0.07319695$, so $C_1(C_1(1/2, 1/3), 1/4) \neq C_1(1/2, C_1(1/3, 1/4))$. In particular, this implies that the T1EMS copula is not Archimedean.

An important concordance measure is now being examined. The multivariate Spearman's rho of a copula $C : [0, 1]^n \rightarrow [0, 1]$ is defined by

$$\rho_n(C) = \frac{n+1}{2^n - (n+1)} \left[2^n \int_0^1 \dots \int_0^1 C(x_1, \dots, x_n) dx_1 \dots dx_n - 1 \right]. \quad (3)$$

It can be thought of as a multivariate measure of average lower orthant dependence; see [32]. The following result investigates the multivariate Spearman's rho of the T1EMS copula.

Proposition 2. *The multivariate Spearman's rho of the T1EMS copula is equal to 0:*

$$\rho_n(C_1) = 0.$$

According to Proposition 2, the T1EMS copula is one of the few multivariate copulas with a multivariate Spearman's rho equal to 0 for all values of n . Since $\rho_n(C_1)$ is independent of θ , the independence role of θ is stochastic only. Explicit examples of the T1EMS copula for the first values of n are given in Table 1.

Table 1. The T1EMS copula for $n = 2, 3$, and 4.

n	$C_1(x_1, \dots, x_n)$
2	$x_1 x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi(x_1 + x_2))$ ($\theta \in [-1/\pi^2, 1/\pi^2]$)
3	$x_1 x_2 x_3 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \cos(\pi(x_1 + x_2 + x_3))$ ($\theta \in [-1/\pi^3, 1/\pi^3]$)
4	$x_1 x_2 x_3 x_4 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi x_4) \sin(\pi(x_1 + x_2 + x_3 + x_4))$ ($\theta \in [-1/\pi^4, 1/\pi^4]$)

Explicit examples of the T1EMS copula density and multivariate Spearman's rho are given in Table 2.

Table 2. The T1EMS copula density and multivariate Spearman's rho for $n = 2, 3$, and 4.

n	$c_1(x_1, \dots, x_n)$	$\rho_n(C_1)$
2	$1 + \theta \pi^2 \sin(2\pi(x_1 + x_2))$	0
3	$1 + \theta \pi^3 \cos(2\pi(x_1 + x_2 + x_3))$	0
4	$1 + \theta \pi^4 \sin(2\pi(x_1 + x_2 + x_3 + x_4))$	0

We recall that, for $n = 2$, the T1EMS copula is given as $C_1(x_1, x_2) = x_1x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi(x_1 + x_2))$ and the T1EMS copula density is given as $c_1(x_1, x_2) = 1 + \theta \pi^2 \sin(2\pi(x_1 + x_2))$. In order to understand the shape behavior of these two special functions, we perform a graphical analysis via simple three-dimensional (3D) and contour plots. Figures 1 and 2 show the 3D and contour plots of the copula, respectively.

Figure 1. Graphics of the 3D plot of the T1EMS copula for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

Figure 2. Graphics of the contour plot of the T1EMS copula for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

The typical triangle shapes of a bivariate copula can be seen in Figure 1. For $\theta = 1/\pi^2$, a kind of bump-shape appears. Such a bump seems typical of the presence of trigonometric functions in the copula definition (see [27,29]). Figure 2 illustrates the contour intensities of the triangles displayed in Figure 1. In it, the bump phenomenon is clearly visible.

Figures 3 and 4 show the 3D and contour plots of the considered T1EMS copula density, respectively.

From Figure 3, the copula density has “wide oscillating band shapes”, which are a consequence of the activated trigonometric term. Such kinds of shapes are uncommon for the densities of copulas defined by exponential, logarithmic, power, and special integral functions. The obtained contours in Figure 4 are parallel lines constituting homogeneous rainbows in terms of color intensity. Clearly, the T1EMS copula is an unusual copula that has such pronounced oscillating shapes, as far as we know. As a result, it is excellent for representing a type of specific oscillating-dependent structure that can be found in natural periodic processes, such as those involved in seasonality, for instance.

Figure 3. Graphics of the 3D plot of the T1EMS copula density for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

Figure 4. Graphics of the contour plot of the T1EMS copula density for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

3. Second Type of Multivariate Trigonometric Copula

The second type of multivariate trigonometric copula is twin to the T1EMS copula; depending on the parity of n , the sine term of the T1EMS copula is now replaced by a cosine term. Thus, the trigonometric feature is slightly changed, with some important consequences. This new type of copula is presented in the next result.

Proposition 3. The following function $C_2 : [0, 1]^n \rightarrow [0, 1]$ is a valid n -dimensional copula:

$$C_2(x_1, \dots, x_n) = \prod_{i=1}^n x_i + \theta \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n},$$

where $u_n = 1$ if n is odd, and $u_n = 0$ if n is even, with $\theta \in [-1/\pi^n, 1/\pi^n]$.

For the purposes of this paper, the copula presented in Proposition 3 is called the type 2 extended multivariate sine (T2EMS) copula. With reference to a part of the proof of Proposition 3 (see Equation (7)), the corresponding copula density is

$$c_2(x_1, \dots, x_n) = 1 + \theta \pi^n \left[\cos \left(2\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \left[\sin \left(2\pi \sum_{i=1}^n x_i \right) \right]^{u_n}.$$

The basic properties of the T2EMS copula are similar to those of the T1EMS copula. They are described below. Clearly, if $\theta = 0$, the T2EMS copula reduces to the independent copula. In addition, the T2EMS copula is symmetric: for any permuted vector of $(x_1, \dots, x_n)$, say $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$, we have $C_2(x_1, \dots, x_n) = C_2(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For $n = 2$, the T2EMS copula is radially symmetric: the survival T2EMS copula is given by

$$\bar{C}_2(x_1, x_2) = x_1 + x_2 - 1 + C_2(1 - x_1, 1 - x_2) = x_1 x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \cos(\pi(x_1 + x_2))$$

and we have $\bar{C}_2(x_1, x_2) = C_2(x_1, x_2)$. The case when n is greater than 3 asks for more investigation. Since $[\cos(\pi \sum_{i=1}^n x_i)]^{1-u_n} [\sin(\pi \sum_{i=1}^n x_i)]^{u_n} \in [-1, 1]$, there is no inequality of the form: $C_1(x_1, \dots, x_n) \geq \prod_{i=1}^n x_i$ or $C_1(x_1, \dots, x_n) \leq \prod_{i=1}^n x_i$ for any $(x_1, \dots, x_n) \in [0, 1]^n$; there is no orthant dependence. The Fréchet–Hoeffding theorem in Equation (2) can be applied with $C_2(x_1, \dots, x_n)$ instead of $C_1(x_1, \dots, x_n)$. We can also remark that the T2EMS copula is not associative; for instance, for $n = 2$ and $\theta = 1/\pi^2$, we have $C_2(C_2(1/2, 1/3), 1/4) = 0.03233124$ and $C_2(1/2, C_2(1/3, 1/4)) = 0.0291784$, so $C_2(C_2(1/2, 1/3), 1/4) \neq C_2(1/2, C_2(1/3, 1/4))$. Thus, the T2EMS copula is not Archimedean.

The following result investigates the multivariate Spearman's rho of the T2EMS copula, as defined in full generality by Equation (3).

Proposition 4. *The multivariate Spearman's rho of the T2EMS copula is*

$$\rho_n(C_2) = \frac{n+1}{2^n - (n+1)} \theta (-1)^{(n+u_n)/2+n}.$$

From Proposition 4, we see that, despite the relative complexity of the T2EMS copula, the expression of the multivariate Spearman's rho is quite simple and manageable, and we have

$$\rho_n(C_2) \in \left[-\frac{n+1}{\pi^n(2^n - (n+1))}, \frac{n+1}{\pi^n(2^n - (n+1))} \right].$$

Explicit examples of the T2EMS copula are given in Table 3.

Table 3. The T2EMS copula for $n = 2, 3$, and 4.

n	$C_2(x_1, \dots, x_n)$
2	$x_1 x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \cos(\pi(x_1 + x_2))$ ($\theta \in [-1/\pi^2, 1/\pi^2]$)
3	$x_1 x_2 x_3 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi(x_1 + x_2 + x_3))$ ($\theta \in [-1/\pi^3, 1/\pi^3]$)
4	$x_1 x_2 x_3 x_4 + \theta \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi x_4) \cos(\pi(x_1 + x_2 + x_3 + x_4))$ ($\theta \in [-1/\pi^4, 1/\pi^4]$)

Explicit examples of the T2EMS copula density and multivariate Spearman's rho are given in Table 4.

Table 4. The T2EMS copula density and multivariate Spearman's rho for $n = 2, 3$, and 4.

n	$c_2(x_1, \dots, x_n)$	$\rho_n(C_2)$
2	$1 + \theta \pi^2 \cos(2\pi(x_1 + x_2))$	-3θ
3	$1 + \theta \pi^3 \sin(2\pi(x_1 + x_2 + x_3))$	$-\theta$
4	$1 + \theta \pi^4 \cos(2\pi(x_1 + x_2 + x_3 + x_4))$	$\frac{5}{11}\theta$

We recall that, for $n = 2$, the T2EMS copula is given as $C_2(x_1, x_2) = x_1x_2 + \theta \sin(\pi x_1) \sin(\pi x_2) \cos(\pi(x_1 + x_2))$ and the T2EMS copula density is given as $c_2(x_1, x_2) = 1 + \theta \pi^2 \cos(2\pi(x_1 + x_2))$. In order to understand the shape behavior of these two special functions, we perform a graphical analysis via simple 3D and contour plots. Figures 5 and 6 show the 3D and contour plots of the copula, respectively.

Figure 5. Graphics of the 3D plot of the T2EMS copula for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

Figure 6. Graphics of the contour plot of the T2EMS copula for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

From Figure 5, we see the typical triangular shape of a bivariate copula. As for the T1EMS copula, it can be observed as a kind of bump into it for $\theta = -1/(2\pi^2)$. The contours of the obtained triangles are visualized in Figure 6.

Figures 7 and 8 show the 3D and contour plots of the copula density, respectively.

From Figures 7 and 8, we see intriguing oscillating shapes for the copula density, which are immediate consequences of the activated trigonometric term. To our knowledge, it is a rare copula that presents such pronounced oscillating shapes. Clearly, the T2EMS copula is ideal for modeling the kind of specific oscillating-dependent structure that can appear in some natural periodic phenomena (seasonality, ...). However, the practical aspects of the T1EMS and T2EMS copulas need more expertise and investigation, especially in the context of precision in data correlation analysis.

Figure 7. Graphics of the 3D plot of the T2EMS copula density for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

Figure 8. Graphics of the contour plot of the T2EMS copula density for $n = 2$ and (a) $\theta = -1/(2\pi^2)$, (b) $\theta = 1/(2\pi^2)$, and (c) $\theta = 1/\pi^2$.

4. Proofs

Proof of Proposition 1. The proof is based on Definition 1. Since $\sin(0) = \sin(\pi) = 0$, the points (a) and (b) are immediately satisfied by the T1EMS copula. Let us investigate the point (c), which requires more development. First, let us rewrite the T1EMS copula in terms of a finite sum of simple multivariate functions. By using the Euler formulas, $i^2 = -1$, and a suitable grouping decomposition, we get

$$\begin{aligned} & \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \\ &= \left\{ \prod_{i=1}^n \left(\frac{e^{i\pi x_i} - e^{-i\pi x_i}}{2i} \right) \right\} \left[\frac{e^{i\pi \sum_{i=1}^n x_i} + e^{-i\pi \sum_{i=1}^n x_i}}{2} \right]^{u_n} \left[\frac{e^{i\pi \sum_{i=1}^n x_i} - e^{-i\pi \sum_{i=1}^n x_i}}{2i} \right]^{1-u_n} \\ &= \frac{1}{2^n} \sum_{i=0}^{n-1} \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n-u_n)/2+i} \sin \left(2\pi \sum_{i=1}^n v_i x_i \right), \end{aligned} \quad (4)$$

where, for any $i = 1, \dots, n$,

$$W_{n,i} = \left\{ (v_1, \dots, v_n) \in \{0, 1\}^n \text{ such that } \sum_{j=1}^n v_j = n - i \right\}.$$

It is worth noting that $W_{n,0} = \{v_1 = 1, \dots, v_n = 1\}$. The above formula can be viewed as a generalization of the following classical formula: $\sin(\pi x) \cos(\pi x) = (1/2) \sin(2\pi x)$. As a more elaborated example, for $n = 3$, we get

$$\begin{aligned} & \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \cos(\pi(x_1 + x_2 + x_3)) \\ &= \frac{1}{8} (-\sin(2\pi(x_1 + x_2 + x_3)) + \sin(2\pi(x_1 + x_2)) + \sin(2\pi(x_1 + x_3)) + \sin(2\pi(x_2 + x_3)) \\ & \quad - \sin(2\pi x_1) - \sin(2\pi x_2) - \sin(2\pi x_3)). \end{aligned}$$

Therefore,

$$C_1(x_1, \dots, x_n) = \prod_{i=1}^n x_i + \frac{\theta}{2^n} \sum_{i=0}^{n-1} \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n-u_n)/2+i} \sin\left(2\pi \sum_{i=1}^n v_i x_i\right).$$

Since

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} \sin\left(2\pi \sum_{i=1}^n v_i x_i\right) \\ &= 2^n \pi^n (-1)^{(n-u_n)/2} \left[\prod_{i=1}^n v_i\right] \left[\cos\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{u_n} \left[\sin\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{1-u_n}, \end{aligned}$$

we have

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_1(x_1, \dots, x_n) \\ &= 1 + \frac{\theta}{2^n} \sum_{i=0}^{n-1} \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n-u_n)/2+i} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \sin\left(2\pi \sum_{i=1}^n v_i x_i\right) \\ &= 1 + \theta \pi^n \sum_{i=0}^{n-1} \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{n-u_n+i} \left[\prod_{i=1}^n v_i\right] \left[\cos\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{u_n} \left[\sin\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{1-u_n}. \end{aligned}$$

All of the terms in the sum are equal to 0 due to the product term $\prod_{i=1}^n v_i$, with the exception of $v_1 = \dots = v_n = 1$, which is reached for $i = 0$ by the definition of $W_{n,0}$. Moreover, since $n - u_n$ is always an even integer, we have $(-1)^{n-u_n} = 1$, and thus

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_1(x_1, \dots, x_n) \\ &= 1 + \theta \pi^n (-1)^{n-u_n} \left[\cos\left(2\pi \sum_{i=1}^n x_i\right)\right]^{u_n} \left[\sin\left(2\pi \sum_{i=1}^n x_i\right)\right]^{1-u_n} \\ &= 1 + \theta \pi^n \left[\cos\left(2\pi \sum_{i=1}^n x_i\right)\right]^{u_n} \left[\sin\left(2\pi \sum_{i=1}^n x_i\right)\right]^{1-u_n}. \end{aligned} \quad (5)$$

Since $\theta \in [-1/\pi^n, 1/\pi^n]$, and the product of the trigonometric terms belongs to $[-1, 1]$, we have

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_1(x_1, \dots, x_n) \\ &\geq 1 - |\theta| \pi^n \left| \left[\cos\left(2\pi \sum_{i=1}^n x_i\right)\right]^{u_n} \left[\sin\left(2\pi \sum_{i=1}^n x_i\right)\right]^{1-u_n} \right| \\ &\geq 1 - |\theta| \pi^n \geq 0. \end{aligned}$$

The point (c) is proved. This ends the proof of Proposition 1. $\square$

Proof of Proposition 2. We use the definition in Equation (3). To begin, let us calculate the integral term. Owing to Equation (4), we have

$$\begin{aligned} & \int_0^1 \dots \int_0^1 C_1(x_1, \dots, x_n) dx_1 \dots dx_n = \\ &= \int_0^1 \dots \int_0^1 \prod_{i=1}^n x_i dx_1 \dots dx_n \\ &+ \theta \int_0^1 \dots \int_0^1 \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} dx_1 \dots dx_n \\ &= \frac{1}{2^n} + \frac{\theta}{2^n} \sum_{i=0}^{n-1} \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n-u_n)/2+i} \int_0^1 \dots \int_0^1 \sin \left(2\pi \sum_{i=1}^n v_i x_i \right) dx_1 \dots dx_n. \end{aligned}$$

Now, set $m = \sum_{j=1}^n v_j$ and consider w_m such that $w_m = 1$ if m is odd, and $w_m = 0$ if m is even (the case $m = 0$ is excluded by the definition of $W_{n,i}$ with $i = 0, \dots, n-1$). By using the 2π -periodicity of the cosine and sine functions, we get

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \sin \left(2\pi \sum_{i=1}^n v_i x_i \right) dx_1 \dots dx_n \\ &= (-1)^{(m+w_m)/2} \frac{1}{2^m \pi^m} \left[\cos \left(2\pi \sum_{i=1}^n v_i x_i \right) \right]^{w_m} \left[\sin \left(2\pi \sum_{i=1}^n v_i x_i \right) \right]^{1-w_m} \Big|_{x_1=\dots=x_n=0}^{x_1=\dots=x_n=1} \\ &= (-1)^{(m+w_m)/2} \frac{1}{2^m \pi^m} [\cos(2m\pi) - 1]^{w_m} [\sin(2m\pi)]^{1-w_m} = 0. \end{aligned}$$

Therefore,

$$\int_0^1 \dots \int_0^1 C_1(x_1, \dots, x_n) dx_1 \dots dx_n = \frac{1}{2^n},$$

and

$$\rho_n(C_1) = \frac{n+1}{2^n - (n+1)} \left[2^n \int_0^1 \dots \int_0^1 C_1(x_1, \dots, x_n) dx_1 \dots dx_n - 1 \right] = 0.$$

The proof of Proposition 2 ends. $\square$

Proof of Proposition 3. The proof is similar to the proof of Proposition 1, but with other mathematical formulas; it is based on Definition 1. Since $\sin(0) = \sin(\pi) = 0$, the points (a) and (b) are immediately satisfied by the T2EMS copula. Let us investigate the point (c), which requires more development. First, as for the T1EMS copula, let us rewrite the T2EMS copula in terms of a finite sum of simple multivariate functions. By using the Euler formulas, $i^2 = -1$, and a suitable grouping decomposition, we get

$$\begin{aligned} & \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \\ &= \left\{ \prod_{i=1}^n \left(\frac{e^{i\pi x_i} - e^{-i\pi x_i}}{2i} \right) \right\} \left[\frac{e^{i\pi \sum_{i=1}^n x_i} + e^{-i\pi \sum_{i=1}^n x_i}}{2} \right]^{1-u_n} \left[\frac{e^{i\pi \sum_{i=1}^n x_i} - e^{-i\pi \sum_{i=1}^n x_i}}{2i} \right]^{u_n} \\ &= \frac{1}{2^n} \sum_{i=0}^n \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n+u_n)/2+i} \cos \left(2\pi \sum_{i=1}^n v_i x_i \right), \end{aligned} \quad (6)$$

where, for any $i = 1, \dots, n$,

$$W_{n,i} = \left\{ (v_1, \dots, v_n) \in \{0, 1\}^n \text{ such that } \sum_{j=1}^n v_j = n - i \right\}.$$

It is worth noting that $W_{n,0} = \{v_1 = 1, \dots, v_n = 1\}$ and $W_{n,n} = \{v_1 = 0, \dots, v_n = 0\}$. The above formula can be viewed as a generalization of the following classical formula: $[\sin(\pi x)]^2 = (1/2)[1 - \cos(2\pi x)]$. As a more elaborated example, for $n = 3$, we get

$$\begin{aligned} & \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi(x_1 + x_2 + x_3)) \\ &= \frac{1}{8} (\cos(2\pi(x_1 + x_2 + x_3)) - \cos(2\pi(x_1 + x_2)) - \cos(2\pi(x_1 + x_3)) - \cos(2\pi(x_2 + x_3)) \\ &+ \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi x_3) - 1). \end{aligned}$$

Therefore,

$$C_2(x_1, \dots, x_n) = \prod_{i=1}^n x_i + \frac{\theta}{2^n} \sum_{i=0}^n \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n+u_n)/2+i} \cos\left(2\pi \sum_{i=1}^n v_i x_i\right).$$

Since

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} \cos\left(2\pi \sum_{i=1}^n v_i x_i\right) \\ &= 2^n \pi^n (-1)^{(n+u_n)/2} \left[\prod_{i=1}^n v_i\right] \left[\cos\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{1-u_n} \left[\sin\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{u_n}, \end{aligned}$$

we have

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_2(x_1, \dots, x_n) \\ &= 1 + \frac{\theta}{2^n} \sum_{i=0}^n \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n+u_n)/2+i} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \cos\left(2\pi \sum_{i=1}^n v_i x_i\right) \\ &= 1 + \theta \pi^n \sum_{i=0}^n \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{n+u_n+i} \left[\prod_{i=1}^n v_i\right] \left[\cos\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{1-u_n} \left[\sin\left(2\pi \sum_{i=1}^n v_i x_i\right)\right]^{u_n}. \end{aligned}$$

Due to the product term $\prod_{i=1}^n v_i$, all of the terms in the sum are equal to 0, except for $v_1 = \dots = v_n = 1$, which is reached for $i = 0$ by the definition of $W_{n,0}$. Moreover, since $n + u_n$ is always an even integer, we have $(-1)^{n+u_n} = 1$, and thus

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_2(x_1, \dots, x_n) \\ &= 1 + \theta \pi^n (-1)^{n+u_n} \left[\cos\left(2\pi \sum_{i=1}^n x_i\right)\right]^{1-u_n} \left[\sin\left(2\pi \sum_{i=1}^n x_i\right)\right]^{u_n} \\ &= 1 + \theta \pi^n \left[\cos\left(2\pi \sum_{i=1}^n x_i\right)\right]^{1-u_n} \left[\sin\left(2\pi \sum_{i=1}^n x_i\right)\right]^{u_n}. \end{aligned} \quad (7)$$

Since $\theta \in [-1/\pi^n, 1/\pi^n]$, and the product of the trigonometric terms belongs to $[-1, 1]$, we have

$$\begin{aligned} & \frac{\partial^n}{\partial x_1 \dots \partial x_n} C_2(x_1, \dots, x_n) \\ & \geq 1 - |\theta| \pi^n \left| \left[\cos \left(2\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \right| \left| \left[\sin \left(2\pi \sum_{i=1}^n x_i \right) \right]^{u_n} \right| \\ & \geq 1 - |\theta| \pi^n \geq 0. \end{aligned}$$

The point (c) is established. This completes the proof of Proposition 3. $\square$

Proof of Proposition 4. We proceed in the same way as in the proof of Proposition 2. We use the definition in Equation (3), and begin by calculating the integral term by using Equation (6). We have

$$\begin{aligned} & \int_0^1 \dots \int_0^1 C_2(x_1, \dots, x_n) dx_1 \dots dx_n = \\ & = \int_0^1 \dots \int_0^1 \prod_{i=1}^n x_i dx_1 \dots dx_n \\ & + \theta \int_0^1 \dots \int_0^1 \left\{ \prod_{i=1}^n \sin(\pi x_i) \right\} \left[\cos \left(\pi \sum_{i=1}^n x_i \right) \right]^{1-u_n} \left[\sin \left(\pi \sum_{i=1}^n x_i \right) \right]^{u_n} dx_1 \dots dx_n \\ & = \frac{1}{2^n} + \frac{\theta}{2^n} \sum_{i=0}^n \sum_{(v_1, \dots, v_n) \in W_{n,i}} (-1)^{(n+u_n)/2+i} \int_0^1 \dots \int_0^1 \cos \left(2\pi \sum_{i=1}^n v_i x_i \right) dx_1 \dots dx_n. \end{aligned}$$

Now, set $m = \sum_{j=1}^n v_j$ and consider w_m such that $w_m = 1$ if m is odd, and $w_m = 0$ if m is even. Let us distinguish the case $m \neq 0$ and the case $m = 0$. For $m \neq 0$, the 2π -periodicity of the cosine and sine functions gives

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \cos \left(2\pi \sum_{i=1}^n v_i x_i \right) dx_1 \dots dx_n \\ & = (-1)^{(m-w_m)/2} \frac{1}{2^m \pi^m} \left[\cos \left(2\pi \sum_{i=1}^n v_i x_i \right) \right]^{1-w_m} \left[\sin \left(2\pi \sum_{i=1}^n v_i x_i \right) \right]^{w_m} \Big|_{x_1=\dots=x_n=0}^{x_1=\dots=x_n=1} \\ & = (-1)^{(m-w_m)/2} \frac{1}{2^m \pi^m} [\cos(2m\pi) - 1]^{1-w_m} [\sin(2m\pi)]^{w_m} = 0. \end{aligned}$$

For $m = 0$, it is clear that $v_1 = \dots = v_n = 0$, which implies that

$$\int_0^1 \dots \int_0^1 \cos \left(2\pi \sum_{i=1}^n v_i x_i \right) dx_1 \dots dx_n = \int_0^1 \dots \int_0^1 dx_1 \dots dx_n = 1.$$

Therefore, by considering only the non-nul term corresponding to $i = n$ by the definition of $W_{n,n}$, we have

$$\int_0^1 \dots \int_0^1 C_2(x_1, \dots, x_n) dx_1 \dots dx_n = \frac{1}{2^n} + \frac{\theta}{2^n} (-1)^{(n+u_n)/2+n}.$$

Hence

$$\begin{aligned} \rho_n(C_2) & = \frac{n+1}{2^n - (n+1)} \left[2^n \int_0^1 \dots \int_0^1 C_2(x_1, \dots, x_n) dx_1 \dots dx_n - 1 \right] \\ & = \frac{n+1}{2^n - (n+1)} \theta (-1)^{(n+u_n)/2+n}. \end{aligned}$$

The stated result is obtained, ending the proof. $\square$

5. Concluding Remarks and Perspectives

The article provides a theoretical contribution to the field of trigonometric copulas. Precisely, two new multivariate copulas were introduced and investigated. By shattering the separability of the "product sine function terms", they present an original alternative to the multivariate sine copula. We have established some of their characteristics that distinguish them from most classical copulas, particularly in terms of the fluctuating shapes of the copula densities and the possibility of a 0 value for the multivariate Spearman's rho (for one type of copula). Some possible directions for research on the suggested copulas are toward the practical aspects. The following works may be interesting applied perspectives.

- If we consider n cumulative distribution functions, say $F_1(x), \dots, F_n(x)$, then $H(x_1, \dots, x_n) = C_i(F_1(x_1), \dots, F_n(x_n))$ with $i = 1$ or 2 defines the cumulative distribution function of a new n -dimensional distribution that may be useful for multivariate data analysis, as well as the development of clustering methods or various regression models (for actual examples of such modeling approaches, see [33,34]);
- The elaboration of the procedure for estimating θ . The maximum likelihood estimation method can be employed in this regard (see [35]);
- In the spirit of [30,31], it needs to demonstrate the goodness of fit performance of the new copulas using real data analysis;
- Beyond probability and statistics, one can think of applying the proposed copulas in mathematical physics, as in the topics considered in [36–38];
- In view of the established theory, one can also think of constructing a new measure of correlation that fully takes into account the oscillating nature of the multivariate trigonometric copulas, following the ideas developed in [32].

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