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Equilibrium in a Failed Cartel: Set-Valued Reactions and Sources of Market Asymmetry

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Abstract

We study a three-firm market following the breakdown of an initially complete cartel, taking both the breakdown and the identity of the departing firm as exogenously given. In the baseline model, the former cartel members no longer coordinate: all three firms choose quantities simultaneously and non-cooperatively. Thus, the term *failed cartel* describes the institutional origin of the post-cartel market rather than continued partial coordination or an endogenously profitable deviation. Because firms' optimal quantities need not be unique, we represent their strategic behavior by set-valued best-response correspondences. We establish equilibrium existence for this simultaneous post-cartel Cournot model as a three-player specialization of the classical Debreu–Glicksberg–Fan framework, combining Berge's maximum theorem with the Kakutani–Fan–Glicksberg fixed-point theorem. The generalized concavity of inverse demand and convexity of costs provide sufficient conditions for existence without requiring unique optimal responses. We also examine the single-valued case, distinguishing the uniqueness of individual best responses from the uniqueness of equilibrium. To enrich the benchmark analysis following the cartel breakdown, we introduce a separate anticipatory extension belonging to the class of hierarchical multi-leader–follower games. Our extension is narrowly focused on separating strategic from technological sources of production asymmetry. In our specialization, Firm 3 has a uniquely determined continuous response, while Firms 1 and 2 choose their quantities simultaneously and anticipate that response, yielding a two-leader–one-follower game. Equilibrium existence is established separately under explicit continuity and quasi-concavity assumptions on the induced payoffs. The extension shows that, even under identical technologies, the ordering of equilibrium quantities depends on the sign of Firm 3's response to the leaders' production decisions: depending on this response, either Firms 1 and 2 or Firm 3 may individually produce the larger equilibrium quantity. Thus, firms' incentives to leave the cartel may differ according to the post-cartel market structure and their individual strategic positions within that structure.

Keywords: failed cartel; coupled optimization problems; set-valued reaction correspondence; best-response correspondence; equilibrium existence; equilibrium uniqueness; Berge's maximum theorem; Kakutani–Fan–Glicksberg fixed-point theorem

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1. Introduction

Many mathematical models arising in economics, game theory, operations research, engineering, and network sciences can be formulated as systems of coupled optimization problems. Rather than maximizing a single objective, several decision makers optimize individual objective functions whose values depend on the decisions of the remaining participants. Consequently, an equilibrium is characterized by the simultaneous compatibility of several interacting optimization problems. This viewpoint underlies a wide spectrum of mathematical theories, including competitive equilibrium, Nash games, generalized Nash equilibrium problems, variational inequalities, complementarity problems, and hierarchical optimization [1–4].

A central mathematical issue is that an optimization problem need not possess a unique solution. Given a continuous function on a compact feasible set, the Weierstrass theorem guarantees the existence of a maximizer, but it does not guarantee that this maximizer is unique. Consequently, the assignment that associates each parameter value with the set of all optimal solutions is generally a set-valued mapping, or correspondence. In particular, a parametric maximization problem of the form $\max_{y \in K(x)} f(x, y)$ generates the argmax correspondence $A(x) = \arg\max_{y \in K(x)} f(x, y)$.

Such correspondences arise naturally as reaction sets, best-response sets, optimal control sets, solution mappings of variational problems, and sets of admissible optimal decisions. Thus, the study of optimization problems with non-unique solutions leads naturally to set-valued analysis [4–6].

Parametric optimization also raises the question of how optimal solution sets change when the parameters of the problem vary. Berge's maximum theorem provides a classical answer: under appropriate continuity and compactness conditions, the optimal value depends continuously on the parameters, while the corresponding argmax mapping is upper semicontinuous and compact valued [5,6]. Upper semicontinuity expresses the stability of optimal solutions under perturbations: if a sequence of parameter values converges and a corresponding sequence of maximizers also converges, then its limit remains a maximizer of the limiting optimization problem. This property is essential when several parametric optimization problems are combined into a common equilibrium system.

Fixed-point theory provides the natural mathematical framework for such systems. When every optimization problem has a unique solution, the associated best-response operator is single valued, and the existence of an equilibrium may be studied through classical fixed-point results such as Brouwer's theorem. When optimal solutions are not unique, however, the best-response operator becomes set valued. In that case, Brouwer's theorem is replaced by fixed-point principles for correspondences, most notably the Kakutani–Fan–Glicksberg theorem [7–10]. These results guarantee a fixed point for an upper semicontinuous correspondence with nonempty, compact, and convex values defined on a nonempty compact convex set. The fixed-point formulation therefore converts a system of mutually dependent optimization problems into an equilibrium existence problem.

The distinction between single-valued and set-valued responses is important both mathematically and conceptually. A single-valued best response assigns one optimal decision to each configuration of the other participants' decisions. A set-valued best response retains all optimal decisions and therefore describes the optimization problem without imposing an artificial selection rule. Moreover, the single-valuedness of all reaction mappings does not by itself imply uniqueness of equilibrium: a continuous single-

valued best-response operator may have several fixed points. Additional conditions, such as contractivity, strict monotonicity, or diagonal strict concavity, are required to obtain uniqueness [3,4].

Cartels provide a natural setting in which these mathematical issues arise, as they embody the tension between cooperation and competition. By coordinating production, firms may restrict output and raise joint profits relative to non-cooperative competition, yet individual members retain incentives to deviate and expand output while benefiting from the restraint of others. The formation, stability, and breakdown of collusive agreements have therefore been central themes in oligopoly theory [11–15].

The standard benchmark for quantity competition is the Cournot model, in which firms choose quantities simultaneously and non-cooperatively [16–18]. A distinct strategic organization is the Stackelberg model, in which firms move sequentially and followers respond to the quantities chosen by leaders [17–19]. Between these benchmarks lies a broad literature on cartels, dominant firms and competitive fringes, and alternative forms of strategic asymmetry [20–22].

In the present paper, we consider the market structure that arises after one member leaves an initial cartel agreement. We use the term *failed cartel* to describe the origin of this post-cartel market structure. It does not imply that the two remaining firms continue to coordinate or maximize joint profits, nor does it impose a sequential order of moves. Following the cartel's breakdown, all three firms choose quantities simultaneously and non-cooperatively, as in a Cournot oligopoly. The cartel history nevertheless provides a natural setting in which to study the equilibrium consequences of the transition from coordination to non-cooperative competition. (We also consider a distinct anticipatory extension in triopoly settings, where Firms 1 and 2 choose their quantities simultaneously while anticipating Firm 3's optimal response to their joint production profile. Keeping this extension separate from the baseline failed-cartel model allows us to distinguish the consequences of cartel breakdown itself from those generated by an additional sequential, anticipatory structure).

Mathematically, the baseline post-cartel model is a standard three-firm simultaneous Cournot game. The previous cartel membership of the firms does not enter the best-response conditions and is relevant only for the institutional interpretation and the comparison with the full-cartel benchmark. Accordingly, the contribution of the baseline analysis does not lie in a new equilibrium existence principle. Rather, it lies in applying the classical set-valued existence framework to this benchmark and in separating three logically distinct questions: whether an equilibrium exists, whether each firm's best response is single valued, and whether the equilibrium itself is unique.

A fundamental difficulty is that the optimal production decisions of the firms need not be unique. In standard presentations of Cournot and Stackelberg competition, firms are frequently represented by single-valued reaction functions obtained from first-order optimality conditions. Such a representation is justified only when every firm's optimization problem has a unique maximizer. If strict concavity or differentiability is absent, several production levels may yield the same maximal profit. The follower's reaction is then a set

$$R_3(x_1, x_2) = \arg\max_{x_3 \in X_3} \Pi_3(x_1, x_2, x_3),$$

and the leaders' best responses are likewise described by argmax correspondences. The equilibrium problem must consequently be formulated as a fixed-point problem for a product of set-valued mappings.

This issue is not merely a technical possibility. The existence and uniqueness of Cournot equilibria have been the subject of extensive research. A large body of literature has established conditions under which equilibrium exists or is unique by imposing generalized

concavity assumptions on inverse demand, monotonicity properties of marginal revenue, supermodularity, or diagonal strict concavity [3,23–27]. These studies identify structural assumptions that guarantee the quasi-concavity of firms' profits, existence of maximizers, single-valued reactions, or uniqueness of equilibrium. They also demonstrate that these properties are logically distinct: the existence of an equilibrium does not require that every best response be unique, and the uniqueness of every individual maximizer does not automatically imply that the equilibrium itself is unique.

Our perspective is therefore to retain, rather than eliminate, the possible multiplicity of optimal responses. We represent each participant's behavior by the full set of maximizers of its payoff function. This formulation preserves all optimal decisions and avoids imposing an arbitrary selection from an argmax set. From the mathematical point of view, the model becomes a system of interacting set-valued optimization correspondences. From the economic point of view, it allows a firm to possess several equally profitable production responses to the same market configuration.

The analysis is deliberately developed in two stages. First, we formulate an abstract existence theorem solely in terms of three continuous payoff functions

$$\Pi_i : X_1 \times X_2 \times X_3 \to \mathbb{R}, \qquad i = 1, 2, 3,$$

and their associated argmax correspondences. Under compactness, convexity, continuity, and quasi-concavity assumptions, the three optimization problems generate nonempty, compact, convex-valued, and upper semicontinuous response correspondences. Their product therefore possesses a fixed point by the Kakutani–Fan–Glicksberg theorem. This fixed point simultaneously solves the three individual optimization problems.

Second, the abstract theorem is applied to the profit functions

$$\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i), \qquad i = 1, 2, 3.$$

The generalized concavity of the inverse demand function, together with the convexity and monotonicity of the cost functions, is used to verify the quasi-concavity conditions required by the abstract result. In this manner, the economic existence theorem is obtained as a direct consequence of a more general optimization and fixed-point principle. The separation between the abstract result and its Cournot application also makes the mathematical framework applicable to other systems of interdependent payoff functions.

The abstract formulation is not restricted to oligopoly theory. Similar structures arise whenever several agents solve coupled parametric optimization problems, and the solution of each problem may be non-unique. Examples include network and congestion games, resource-allocation models, hierarchical decision systems, engineering design problems, generalized Nash equilibrium problems, and control models with set-valued optimal feedback. Although the present paper focuses on a failed-cartel application, its main existence argument concerns fixed points of product correspondences generated by interacting optimization problems.

The use of set-valued responses also provides a transparent separation between existence, single-valuedness, and uniqueness. General continuity, compactness, convexity, and quasi-concavity assumptions are sufficient for existence. If each argmax set is a singleton, the response correspondences reduce to continuous reaction functions, and the equilibrium problem becomes a fixed-point equation for a single-valued operator. Nevertheless, the uniqueness of this fixed point requires further assumptions. In differentiable concave games, Rosen's diagonal strict concavity condition provides one important route to uniqueness [3,4]. Thus, uniqueness appears as an additional structural property rather than as a prerequisite for defining the model.

Beyond the abstract existence problem, we investigate the economic consequences of two distinct post-cartel market structures. In the simultaneous post-cartel Cournot model, differences in equilibrium quantities arise from technological asymmetry, including differences in production costs or feasible strategy sets. The identity of Firm 3 as the departing firm does not, by itself, generate an asymmetric strategic position. In the anticipatory extension, by contrast, Firms 1 and 2 incorporate the uniquely determined response of Firm 3 into their individual optimization problems. This introduces an additional strategic source of production asymmetry, which may persist even when all firms possess identical technologies. The technological and strategic effects may reinforce one another, offset one another, or imply opposite quantity orderings.

The contributions of the paper are both methodological and economic. First, we formulate the simultaneous three-firm equilibrium problem through possibly set-valued best-response correspondences and derive its existence as a specialization of the classical Debreu–Glicksberg–Fan framework. Second, we verify sufficient conditions for applying this framework to a post-cartel Cournot market by using generalized concavity of inverse demand and convexity of costs. Third, we distinguish the single-valuedness of the individual best responses from the uniqueness of equilibrium and provide examples with nondegenerate best-response sets and multiple equilibria. Fourth, we formulate a separate anticipatory single-valued extension, give sufficient conditions for its equilibrium existence, and investigate how technological and strategic effects jointly determine equilibrium quantity orderings. Finally, we compare profits and welfare across full-cartel, simultaneous Cournot–Nash, and anticipatory regimes in an illustrative linear benchmark.

Our analysis is related to the literature on partial cartels, cartel stability, and cartel–fringe market structures. One strand studies the formation and stability of partial cartels, asking whether a subset of firms can sustain cooperation when other firms remain independent and which coalition sizes are robust to individual incentives to enter or leave the cartel. Ref. [12] formalizes internal and external stability conditions for cartel membership, while refs. [13,14] examine stable cartel configurations when outsiders behave strategically in the product market. Allowing firms to differ in their productive capacities, ref. [28] endogenize the composition of a cartel in an infinitely repeated price game. They show that a stable cartel need not be all-inclusive: sufficiently large firms may find it optimal to participate, whereas firms with sufficiently small capacities may prefer to remain outside the cartel and benefit from the price umbrella created by the colluding firms. Their analysis demonstrates that firm heterogeneity affects not only the stability of collusion but also the identity of the firms that participate in a partial cartel. A central concern in this literature is therefore the endogenous determination of cartel membership and the conditions under which a particular cartel–outsider configuration can persist.

A related strand studies cartel–fringe market structures, focusing on the strategic interaction between a group of cooperating producers and firms that remain outside the cartel. Ref. [11] emphasizes the incentives of independent firms to expand production in response to cartel-induced output restrictions, while subsequent work considers how the nature of the fringe—competitive or strategically active—affects the profitability and behavior of the cartel. In particular, ref. [15] shows how strategic interaction with a competitive fringe can alter the output incentives of colluding firms. These models highlight that the consequences of cartelization depend not only on cooperation within the cartel but also on the responses of firms outside it.

Our focus differs from both strands. Rather than asking which coalition structures are stable or how an operating cartel interacts with an outside fringe, we take the breakdown of an initially complete cartel as given and study the non-cooperative market that follows. The term *failed cartel* refers to the origin of this market structure: after one firm leaves

the initial cartel agreement, the two remaining firms do not continue to maximize joint profits, nor does the defection itself impose a sequential order of moves. In the baseline model, all three firms instead compete simultaneously in quantities. Thus, the resulting market is a post-cartel Cournot game rather than a partial-cartel, cartel–fringe, or leader–follower structure. Unlike the endogenous cartel-formation framework of [28], we do not derive which firms join or leave the cartel from participation and incentive-compatibility conditions. Instead, we take the breakdown of the initial cartel and the identity of the departing firm as exogenously given.

The paper is also related to work emphasizing how strategic responses shape oligopoly outcomes. Ref. [29] provides the canonical distinction between strategic substitutes and strategic complements, and ref. [30] develops related ideas in models of strategic investment. Ref. [31] extends the strategic-substitutes and -complements framework to interdependent demands. In a cartel setting, ref. [15] shows that strategic interaction with a competitive fringe can substantially alter the output incentives of colluding firms. These contributions motivate our emphasis on best responses and on the way firms' optimization problems jointly determine the post-cartel equilibrium.

A separate connection arises through our anticipatory extension to include a benchmark from the class of hierarchical multi-leader–follower games. Early work on mixed Stackelberg–Cournot market structures was developed by [32], who studied a homogeneous-product oligopoly with one Stackelberg leader and several Cournot followers, analyzing the existence, uniqueness, and computation of the resulting Stackelberg–Nash–Cournot equilibrium. Ref. [33] extends this framework to multiple leaders who incorporate the aggregate follower response into their individual production decisions and derives conditions ensuring useful convexity and differentiability properties of that response. Ref. [34] further examines homogeneous-good oligopolies with multiple leaders and followers, showing that the allocation of firms across the two stages can affect aggregate output, market concentration, profits, and social welfare. In particular, sequential organization may generate greater concentration without necessarily reducing welfare, illustrating that the economic consequences of leader–follower structure cannot be inferred from conventional measures of market concentration alone.

A more general formulation of non-cooperative multi-leader–follower games was developed by [35]. They show that such games can be represented as generalized Nash equilibrium problems in which each leader solves a mathematical program with equilibrium constraints. This formulation also reveals a fundamental existence difficulty: the equilibrium constraints generally produce non-convex optimization problems, and consequently an equilibrium of the induced leader game need not exist. Pang and Fukushima therefore introduce remedial formulations based on convexified strategy sets and quasi-variational inequalities. The equilibrium problem with equilibrium constraints (EPEC) approach was further developed by [36] in the context of restructured electricity markets. Their analysis provides sufficient conditions for the existence of pure-strategy Nash equilibria and confirms that equilibrium existence in bi-level games depends on additional structural properties that are not required in an ordinary simultaneous Cournot model.

A stochastic extension of the multiple-leader–follower framework was studied by [37]. They consider a homogeneous-product oligopoly with several leaders and followers in which the leaders choose their supply levels before the realization of an uncertain demand function, whereas the followers make their decisions after observing both the leaders' quantities and the realized demand. They formulate the resulting solution as a stochastic multiple-leader Stackelberg–Nash–Cournot equilibrium and derive conditions for its existence and uniqueness.

Our anticipatory extension belongs to this broad class of hierarchical games, but is deliberately narrow in scope: it is designed to isolate production asymmetry arising from strategic anticipation from that generated by technological differences. When Firm 3's best response is singleton-valued, Firms 1 and 2 choose their quantities simultaneously while anticipating Firm 3's subsequent response, yielding a two-leader–one-follower game. Unlike the stochastic formulation of [37], demand is deterministic, as in [38]. Although the model has the hierarchical structure associated with an EPEC, we do not explicitly reformulate it as one. Instead, Firm 3's response is incorporated directly into the induced payoff functions of Firms 1 and 2. In the settings of our main analysis, equilibrium existence does not carry over to this hierarchical setting. Existence must therefore be established separately under explicit continuity and quasi-concavity assumptions on the induced payoffs.

The remainder of the paper is organized as follows. Section 2 introduces the market structure and the mathematical tools used in the analysis. Section 3 formulates the simultaneous post-cartel Cournot model and separates equilibrium existence, single-valuedness of individual best responses, and uniqueness of equilibrium. Section 4 develops the distinct anticipatory single-valued extension and studies the technological and strategic sources of production asymmetry. Section 5 presents the illustrative examples and equilibrium comparisons. Section 6 discusses the interpretation, empirical implications, and limitations of the results. Section 7 concludes.

2. Materials and Methods

The mathematical framework developed in this paper is based on the interaction of three coupled optimization problems. After the cartel agreement has broken down, each firm maximizes its own profit according to the Cournot principle, taking the quantities chosen by the other two firms as given. Since these optimization problems may admit more than one solution, the firms' optimal decisions are represented by set-valued best-response correspondences rather than by single-valued reaction functions. The post-cartel equilibrium can consequently be formulated as a fixed point of their product correspondence.

We first introduce the market structure, the profit functions, and the cooperative cartel benchmark. We then define the non-cooperative market that results after one firm leaves the agreement. The existence analysis is based on parametric optimization, generalized concavity, upper semicontinuous correspondences, and set-valued fixed-point theory. In particular, the main framework does not require differentiability, first-order optimality conditions, or uniqueness of the firms' optimal quantities.

Throughout the paper, we use the standard notation $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ for the set of non-negative real numbers and $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ for the set of strictly positive real numbers.

2.1. Market Structure and Profit Functions

Three firms compete in a homogeneous-good market, each producing quantities $x_1, x_2, x_3 \geq 0$. The aggregate market output is $Q = x_1 + x_2 + x_3$ and the market price is $P(Q)$, where $P : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function. The profit, or payoff, of firm i is given by $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$, for $i = 1, 2, 3$, where each firm i has a cost function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$.

This formulation includes the classical Cournot model, in which firms choose their quantities simultaneously and non-cooperatively [16–18,39]. In the present paper, it is applied to the market structure that arises after one member leaves an initial cartel agreement. The term *failed cartel* describes the origin of the post-cartel market structure; it does not impose a sequential order of moves and does not mean that the two remaining firms continue to maximize their joint profit.

Throughout the set-valued existence analysis, the inverse demand function is assumed to be positive and decreasing on the economically relevant output domain, while the cost functions are assumed to be continuous and convex. Additional assumptions will be stated explicitly whenever they are needed.

2.2. Best-Response Correspondences and Post-Cartel Cournot–Nash Equilibrium

Let $X_i \subset \mathbb{R}_+$, $i = 1,2,3$, be the admissible strategy set of firm i, and put $X = X_1 \times X_2 \times X_3$. For the existence results below, the sets X_i, $i = 1,2,3$, are assumed to be nonempty, compact, and convex. The boundedness of the admissible output sets has a natural economic interpretation. If inverse demand reaches the choke price at a finite aggregate quantity, or if sufficiently large output gives a firm a payoff below that obtained at zero output, quantities beyond an appropriate bound cannot be optimal. The feasible sets may therefore be restricted to compact intervals without excluding any profit-maximizing quantity.

In the Cournot framework, each firm chooses its own output while treating the output levels of the other two firms as fixed. Thus, the three firms solve $\max_{y_1 \in X_1} \Pi_1(y_1, x_2, x_3)$, $\max_{y_2 \in X_2} \Pi_2(x_1, y_2, x_3)$, and $\max_{y_3 \in X_3} \Pi_3(x_1, x_2, y_3)$, respectively.

This formulation follows the classical interpretation of quantity competition introduced by Cournot and subsequently developed in oligopoly theory [16–18,39].

The sets of optimal quantities of the firms are described by the following best-response correspondences:

$$R_1(x_2, x_3) = \arg\max_{y_1 \in X_1} \Pi_1(y_1, x_2, x_3),$$

$$R_2(x_1, x_3) = \arg\max_{y_2 \in X_2} \Pi_2(x_1, y_2, x_3),$$

and

$$R_3(x_1, x_2) = \arg\max_{y_3 \in X_3} \Pi_3(x_1, x_2, y_3).$$

In general, the sets $R_1(x_2, x_3)$, $R_2(x_1, x_3)$, and $R_3(x_1, x_2)$ need not be singletons. We therefore regard the best-response operators as set-valued correspondences rather than as single-valued reaction functions. This distinction is essential for the approach developed below, since the existence analysis is based on the existence of global maximizers and fixed points of set-valued correspondences, rather than on differentiability, first-order optimality conditions, or contractive properties.

A vector $(x_1^*, x_2^*, x_3^*) \in X$ is called a post-cartel Cournot–Nash equilibrium, or equivalently a failed-cartel Cournot equilibrium, if $x_1^* \in R_1(x_2^*, x_3^*)$, $x_2^* \in R_2(x_1^*, x_3^*)$, and $x_3^* \in R_3(x_1^*, x_2^*)$.

Equivalently, define the set-valued mapping $R : X \rightrightarrows X$ by

$$R(x_1, x_2, x_3) = R_1(x_2, x_3) \times R_2(x_1, x_3) \times R_3(x_1, x_2).$$

Thus, a post-cartel Cournot–Nash equilibrium is precisely a fixed point of the product correspondence R; that is, $(x_1^*, x_2^*, x_3^*) \in R(x_1^*, x_2^*, x_3^*)$.

The use of reaction correspondences is natural when optimal reactions are not necessarily unique. Under compactness of the strategy sets and continuity of the profit functions, the existence and regularity of the sets of maximizers can be studied by means of the maximum theorem [5]. If, in addition, the reaction correspondences have nonempty, compact, and convex values and satisfy the appropriate upper semicontinuity assumptions, the existence of an equilibrium can be obtained from fixed-point principles for set-valued mappings [7–9].

2.3. The Full-Cartel Benchmark

Before the breakdown of the agreement, the three firms coordinate their production decisions and maximize their aggregate profit [40–44]. The full-cartel problem is

$$\max_{(x_1,x_2,x_3)\in X}\left\{(x_1+x_2+x_3)P(x_1+x_2+x_3)-\sum_{i=1}^{3}c_i(x_i)\right\}.$$

Thus, the set of cartel-optimal production profiles is

$$C=\arg\max_{(x_1,x_2,x_3)\in X}\left\{(x_1+x_2+x_3)P(x_1+x_2+x_3)-\sum_{i=1}^{3}c_i(x_i)\right\}.$$

When it is convenient to formulate the cartel decision in terms of aggregate output, let t denote total production and let $c(t)$ denote the corresponding aggregate production cost. The cartel then solves $\max_{t\in X_C}\{tP(t)-c(t)\}$.

After an optimal aggregate quantity t^* has been selected, it may be allocated according to agreed shares $\alpha_i \geq 0$, $\sum_{i=1}^{3}\alpha_i = 1$, so that $x_i = \alpha_i t^*$. In the symmetric benchmark used below, the allocation is equal, with $\alpha_i = 1/3$.

This cooperative solution is used only as a benchmark for the subsequent non-cooperative outcomes. Once the cartel breaks down, its joint maximization problem no longer constrains the firms, and each firm maximizes its own profit independently.

2.4. The Post-Cartel Cournot Market

After Firm 3 leaves the agreement, all three firms choose their quantities independently and simultaneously. Firms 1 and 2 no longer coordinate their decisions, and Firm 3 does not acquire a follower position. Consequently, the resulting equilibrium concept is the standard three-firm Cournot–Nash equilibrium defined by the best-response correspondences introduced above.

The term *post-cartel* refers only to the institutional origin of the market. It performs no additional mathematical role in the baseline equilibrium conditions. Its relevance lies in providing the full-cartel benchmark and in motivating the comparison between coordinated and non-cooperative outcomes.

The anticipatory extension studied in Section 4 is a separate model. There, Firms 1 and 2 incorporate the uniquely determined response of Firm 3 into their induced payoffs. It therefore represents a two-leader–one-follower game and requires its own equilibrium definition and existence result.

2.5. Generalized Concavity

Generalized concavity provides a useful framework for extending classical concavity arguments to functions that need not be concave in the ordinary sense. Various forms of generalized concavity, including quasi-concavity, log-concavity, and α-concavity, have been widely used in optimization and economic analysis [45–47].

In oligopoly theory, generalized concavity assumptions have been employed to establish existence and uniqueness properties of Cournot equilibria under substantially weaker conditions than ordinary concavity of the inverse demand function. In particular, log-concavity, biconcavity, concave price flexibility, and α-concavity have been used to study the shape of revenue functions and the existence or uniqueness of equilibrium [23–27].

We recall the notion of α-concavity, which will be used below to formulate sufficient conditions for the existence and regularity of the maximizers arising in the post-cartel Cournot model.

Let $p:\Omega\to\mathbb{R}_{++}$ be a positive function defined on a convex set $\Omega\subseteq\mathbb{R}_{+}$.

Definition 1 ([45,48]). *A positive function p defined on a convex set $\Omega \subseteq \mathbb{R}_+$ is said to be α-concave, where $\alpha \in [-\infty, +\infty]$, if $p(\lambda x + (1-\lambda)y) \geq m_\alpha(p(x), p(y), \lambda)$ for all $x, y \in \Omega$ and all $\lambda \in [0,1]$, where*

$$m_\alpha(a,b,\lambda) = \begin{cases} a^\lambda b^{1-\lambda}, & \alpha = 0, \\ \max\{a,b\}, & \alpha = +\infty, \\ \min\{a,b\}, & \alpha = -\infty, \\ \left(\lambda a^\alpha + (1-\lambda)b^\alpha\right)^{1/\alpha}, & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, this notion reduces to ordinary concavity, whereas $\alpha = 0$ corresponds to log-concavity. Moreover, the generalized means m_α are monotone with respect to the parameter α. Consequently, α-concavity implies β-concavity whenever $\beta \leq \alpha$ [27,45].

Of particular importance for the present analysis is the case of negative values of α. For $n \geq 1$, a positive function p is $(-1)/n$-concave if

$$p(\lambda x + (1-\lambda)y) \geq \left(\lambda p(x)^{-1/n} + (1-\lambda)p(y)^{-1/n}\right)^{-n}.$$

Equivalently, p is $(-1)/n$-concave if and only if $\frac{1}{p^{1/n}}$ is convex.

This class contains, in particular, positive concave and positive log-concave functions and has proved useful in the analysis of revenue functions in Cournot models [27].

It is important to distinguish the roles of the different generalized concavity assumptions used below. For a single firm's optimization problem, (-1)-concavity of the inverse demand function is sufficient for the quasi-concavity of the firm's revenue, and hence for the convexity of its best-response set. This is the property needed in the existence argument. By contrast, $(-1/3)$-concavity is a stronger assumption because $-1 < -1/3$, and $(-1/3)$-concavity implies (-1)-concavity. In the three-firm model, the stronger condition is used only when an additional uniqueness result for the Cournot equilibrium is invoked. Thus, the $(-1/3)$-concavity assumption should not be regarded as necessary for the existence of a post-cartel Cournot–Nash equilibrium.

2.6. Fixed-Point Methods

The classical Banach contraction principle guarantees the existence and uniqueness of a fixed point for a contraction on a complete metric space [49]. Numerous extensions of this principle have subsequently been developed, including fixed-point results in partially ordered spaces and coupled, tripled, and more general n-tupled fixed-point structures [50–53].

Fixed-point methods have also proved useful in the analysis of market equilibria. Best-response functions and correspondences may be interpreted as operators whose fixed points represent production profiles that are consistent with the optimal decisions of all market participants. This viewpoint has been applied to nonlinear and non-differentiable response functions, as well as to a variety of oligopoly market structures [54–56].

The classical fixed-point problem concerns a single-valued mapping $T : X \to X$. A point $x^* \in X$ is called a fixed point of T if $T(x^*) = x^*$.

In equilibrium models, this equality expresses the consistency between a given production profile and the production profile generated by the corresponding best-response operator.

In many optimization and equilibrium problems, however, the optimal response to a given strategy profile need not be unique. The natural mathematical object is then a set-valued mapping, or correspondence, $F : X \rightrightarrows X$, where $F(x)$ is a nonempty subset of X. A point $x^* \in X$ is a fixed point of F if $x^* \in F(x^*)$. Thus, the fixed-point condition for a

correspondence requires the point to belong to the set of admissible responses generated by its own state.

Fixed-point theory for set-valued mappings has a long history. Kakutani's fixed-point theorem extends Brouwer's theorem to upper semicontinuous correspondences with nonempty compact convex values on compact convex subsets of finite-dimensional Euclidean spaces [7]. Fan and Glicksberg subsequently developed infinite-dimensional extensions for correspondences defined on compact convex subsets of locally convex topological vector spaces [8,9]. These results have become fundamental tools in mathematical economics and game theory, particularly in existence proofs for Nash equilibria and other equilibrium concepts [1,10,57].

The fixed-point argument used in the present paper is an application of this classical Debreu–Glicksberg–Fan framework to the three-firm post-cartel Cournot model. We do not claim a new set-valued fixed-point theorem. The role of the classical result is to establish the existence of an equilibrium while retaining all optimal quantities when the firms' maximization problems have non-unique solutions.

For the purposes of the present paper, the importance of set-valued fixed-point methods comes from the structure of the firms' best-response correspondences. If an optimization problem admits several maximizers, the corresponding reaction cannot, in general, be represented by a single-valued function. Instead, it is naturally described by an argmax correspondence. In the post-cartel Cournot model, each of the sets $R_1(x_2, x_3)$, $R_2(x_1, x_3)$, and $R_3(x_1, x_2)$ may contain more than one optimal quantity. For example, the best-response correspondence of the departing firm is $R_3(x_1, x_2) = \arg\max_{z \in X_3} \{zP(x_1 + x_2 + z) - c_3(z)\}$.

Even when the objective function is continuous and a maximum exists, the set $R_3(x_1, x_2)$ may contain more than one element.

This distinction is essential for the approach developed below. We do not impose differentiability assumptions in order to derive first-order conditions, nor do we assume that reaction mappings are contractions. Instead, the existence of maximizers is used to construct set-valued response correspondences. A post-cartel Cournot–Nash equilibrium is then formulated as a fixed point of the product of the three best-response correspondences.

Under appropriate compactness, convexity, and continuity assumptions, maximum theorems guarantee suitable properties of the argmax correspondences, while set-valued fixed-point principles provide the existence of a production profile consistent with the corresponding optimal responses [5,7–9].

Consequently, the fixed-point approach adopted in this paper differs from contraction-based methods used in some earlier oligopoly models. The central role is played not by metric contractivity or differentiability but by the existence of maximizers, the structural properties of the associated argmax correspondences, and the applicability of classical fixed-point principles for set-valued mappings.

2.7. Berge's Maximum Theorem

We recall the form of Berge's maximum theorem used below. This result is a standard tool for proving the existence and regularity of optimal solutions in parametric optimization problems. In the present paper, it is used to study the argmax correspondences generated by the profit-maximization problems of the firms [5,6].

Definition 2 ([6,58])**.** *Let X and Y be topological spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping. The mapping F is called upper semicontinuous at a point $x_0 \in X$ if, for every open set $V \subset Y$ satisfying $F(x_0) \subset V$, there exists a neighbourhood U of x_0 such that $F(x) \subset V$ for all $x \in U$.*

Definition 3 ([6,58]). *Let X and Y be topological spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping. The mapping F is called lower semicontinuous at a point $x_0 \in X$ if, for every open set $V \subset Y$ satisfying $F(x_0) \cap V \neq \varnothing$, there exists a neighbourhood U of x_0 such that $F(x) \cap V \neq \varnothing$ for all $x \in U$.*

Definition 4 ([6,58]). *A set-valued mapping $F : X \rightrightarrows Y$ is called continuous if it is both upper semicontinuous and lower semicontinuous.*

Proposition 1 ([58]). *Let X be a metric space, let Y be a compact metric space, and let $F : X \rightrightarrows Y$ have nonempty compact values. Then F is upper semicontinuous at x_0 if and only if, for every sequence $x_n \to x_0$ and every sequence $y_n \in F(x_n)$ such that $y_n \to y$, one has $y \in F(x_0)$.*

Theorem 1 (Berge's maximum theorem [5,6]). *Let X and Y be topological spaces, and let $K : X \rightrightarrows Y$ be a continuous set-valued mapping with nonempty compact values. Let $f : X \times Y \to \mathbb{R}$ be continuous. Define the value function $v(x) = \max_{y \in K(x)} f(x, y)$ and the solution correspondence $M(x) = \arg\max_{y \in K(x)} f(x, y)$.*

Then v is continuous, and M is nonempty-valued, compact-valued, and upper semicontinuous.

The following elementary consequence will be used when the three best-response correspondences are combined into a single product correspondence.

Proposition 2. *Let X be a metric space and, for $i = 1, \ldots, m$, let Y_i be a compact metric space. Suppose that $F_i : X \rightrightarrows Y_i$ is upper semicontinuous and has nonempty compact values. Then the product correspondence $F : X \rightrightarrows \prod_{i=1}^{m} Y_i$, defined by $F(x) = \prod_{i=1}^{m} F_i(x)$, is upper semicontinuous and has nonempty compact values.*

Proof. The nonemptiness and compactness of $F(x)$ follow from the corresponding properties of the finitely many sets $F_i(x)$. Let $x_n \to x$, let $y_n = (y_{1,n}, \ldots, y_{m,n}) \in F(x_n)$, and suppose that $y_n \to y = (y_1, \ldots, y_m)$. For every i, we have $y_{i,n} \in F_i(x_n)$ and $y_{i,n} \to y_i$. Since F_i is upper semicontinuous and compact-valued, Proposition 1 yields $y_i \in F_i(x)$. Therefore, $y \in \prod_{i=1}^{m} F_i(x) = F(x)$. Another application of Proposition 1 shows that F is upper semicontinuous. □

This proposition supplies the justification for the upper semicontinuity of the product of the firms' best-response correspondences used in the equilibrium existence proof below.

2.8. Kakutani–Fan–Glicksberg Fixed-Point Theorem

The Brouwer fixed-point theorem provides an existence result for continuous single-valued mappings defined on compact convex subsets of finite-dimensional Euclidean spaces. In optimization and equilibrium problems, however, the corresponding response operator is frequently set-valued, since an optimization problem may admit more than one maximizer. This motivates the use of fixed-point principles for correspondences.

Kakutani extended Brouwer's fixed-point theorem to upper semicontinuous set-valued mappings with nonempty, compact, and convex values [7]. Fan and Glicksberg subsequently obtained extensions to compact convex subsets of locally convex topological vector spaces [8,9]. These results form one of the standard fixed-point tools in mathematical economics and equilibrium theory; see also [6,10].

We recall the version, known as the Kakutani–Fan–Glicksberg fixed-point theorem, needed in the present paper.

Theorem 2 ([7–9]). *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space, and let $F : X \rightrightarrows X$ be an upper semicontinuous set-valued mapping such that, for every $x \in X$, the set $F(x)$ is nonempty, compact, and convex. Then F has a fixed point; that is, there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

In finite-dimensional Euclidean spaces, this result is usually referred to as Kakutani's fixed-point theorem. The Fan–Glicksberg extension allows the same fixed-point argument to be formulated in the more general setting of locally convex topological vector spaces. Since the strategy sets considered in the present paper are finite-dimensional, Kakutani's theorem would be sufficient for the main existence argument. We state the Kakutani–Fan–Glicksberg formulation because it provides a standard unified form of the set-valued fixed-point principle used below, not because a new fixed-point theorem is being proposed.

The theorem is particularly suitable for best-response correspondences. If the maximization problem defining a firm's optimal reaction has a nonempty set of maximizers, the resulting argmax operator is naturally set-valued. Under suitable continuity and compactness assumptions, Berge's maximum theorem provides the required regularity of the argmax correspondence, whereas the Kakutani–Fan–Glicksberg theorem yields the existence of a fixed point [5,7–9].

In the present model, Berge's maximum theorem is applied separately to the three firms' best-response correspondences. Their product is upper semicontinuous by Proposition 2, while the convexity of its values follows from the quasi-concavity of each firm's profit with respect to its own quantity. The Kakutani–Fan–Glicksberg theorem can then be applied to this product correspondence.

The following results of Kamburova and Marinov will be needed in the proofs of the main results below. We state them using explicit admissible domains and a precise description of the relevant intervals, thereby avoiding the potentially ambiguous wording of their original formulations.

Let $\Omega \subseteq \mathbb{R}_+$ be a convex interval, let $p : \Omega \to \mathbb{R}_{++}$, and, for a fixed $k \geq 0$, define $D_k = \{x \in \mathbb{R}_+ : x + k \in \Omega\}$. Thus, all the functions below are considered on the interval D_k.

Theorem 3 ([27]). *Let $p : \Omega \to \mathbb{R}_{++}$ be decreasing, continuous, and $(-1)/n$-concave, where $n \geq 1$. For $k \geq 0$, define $h_k : D_k \to \mathbb{R}_+$ by $h_k(x) = x\,p(x+k)^{1/n}$. Then h_k is strictly concave on every interval $I \subseteq D_k$ on which h_k is strictly increasing.*

Theorem 4 ([27]). *Let $p : \Omega \to \mathbb{R}_{++}$ be $(-1)/n$-concave, where $n \geq 1$, and let $k \geq 0$. Define $h_k(x) = x\,p(x+k)^{1/n}$ on D_k. The function h_k has no strict positive local minimum. Moreover, every positive local maximum of h_k is a global maximum on D_k.*

Lemma 1 (Single-peakedness of conditional revenue). *Let $\Omega \subseteq \mathbb{R}_+$ be a convex interval, let $p : \Omega \to \mathbb{R}_{++}$ be continuous, decreasing, and (-1)-concave, and fix $k \geq 0$. Define*

$$D_k = \{x \in \mathbb{R}_+ : x + k \in \Omega\} \qquad \text{and} \qquad h_k(x) = x\,p(x+k) \qquad \text{for} \qquad x \in D_k.$$

Then h_k is single-peaked in the following sense: it is either non-increasing on D_k, or there exists an interval $M_k \subseteq D_k$ consisting of its global maximizers such that h_k is strictly increasing and strictly concave before M_k and non-increasing from the left endpoint of M_k onward. In particular, h_k is quasi-concave on D_k.

Proof. Since p is positive and (-1)-concave, the function $g(q) = \frac{1}{p(q)}$ is convex on Ω. Moreover, since p is decreasing, g is non-decreasing.

We may write $h_k(x) = \frac{x}{g(x+k)}$.

For an interior point $x > 0$ of D_k, let $g'_+(x+k)$ denote the right derivative of the convex function g. The right derivative of h_k has the same sign as

$$A_k(x) = g(x+k) - xg'_+(x+k).$$

We show that A_k is non-increasing. Let $0 < x < y$ be interior points of D_k. By the convexity of g, we have

$$g(y+k) - g(x+k) \leq g'_+(y+k)(y-x),$$

while $g'_+(x+k) \leq g'_+(y+k)$. Consequently,

$$\begin{aligned} A_k(y) - A_k(x) &= g(y+k) - g(x+k) - yg'_+(y+k) + xg'_+(x+k) \\ &\leq x\left[g'_+(x+k) - g'_+(y+k)\right] \leq 0. \end{aligned}$$

Thus, A_k is non-increasing. It follows that the sign of the right derivative of h_k can change at most once, from positive to non-positive. In particular, h_k cannot be constant on an interval and then become increasing again.

If $A_k \leq 0$ throughout the interior of D_k, then h_k is non-increasing. Otherwise, h_k is strictly increasing until it reaches its set M_k of global maximizers and is non-increasing thereafter. By Theorem 3 with $n = 1$, h_k is strictly concave on every interval on which it is strictly increasing. Theorem 4 further ensures that every positive local maximum is global. Hence M_k is an interval, and the stated single-peaked structure follows.

Finally, the upper level sets of a function that is non-decreasing up to its set of maximizers and non-increasing thereafter are intervals.

Therefore, h_k is quasi-concave on D_k. □

For $n = 1$, the function $h_k(x) = xp(x+k)$ is the conditional revenue of a firm when the aggregate quantity k produced by the other firms is fixed. If $c : D_k \to \mathbb{R}$ is convex and increasing, then the profit function $\pi_k(x) = h_k(x) - c(x)$ is quasi-concave.

Indeed, if h_k is non-increasing, then π_k is non-increasing. Otherwise, h_k is strictly concave on its interval of increase, and hence $h_k - c$ is concave there, whereas both the non-increase of h_k and the increase of c imply that $h_k - c$ is non-increasing after that interval. Consequently, every upper level set of π_k is an interval.

Corollary 1 ([27]). *Let $N \geq 2$. Suppose that, for every $i = 1, \ldots, N$, the cost function c_i is convex and increasing. Assume that P is a positive, strictly decreasing, continuous, differentiable, and $(-1)/N$-concave function on $\mathbb{R}_+$. Then the corresponding Cournot game possesses a unique equilibrium.*

This corollary is a uniqueness result for the simultaneous Cournot–Nash game. In the homogeneous-good model considered in this paper, the market consists of three firms, $N = 3$; hence, the sufficient condition for uniqueness reduces to the $(-1)/3$-concavity of P. This stronger condition is not required for existence: (-1)-concavity is sufficient for the quasi-concavity of each firm's profit in its own quantity.

If compact strategy intervals $X_i = [0, M_i]$ are imposed, the uniqueness result applies provided that the bounds M_i do not create additional constrained equilibria; equivalently, every constrained equilibrium must lie in the region where the constrained and unconstrained best responses coincide.

2.9. Diagonal Strict Concavity

The existence results developed above guarantee that the best-response correspondences possess suitable properties for the application of set-valued fixed-point theorems, but they do not, in general, exclude the possibility of several equilibrium production profiles. A classical approach to uniqueness in concave games is based on the notion of diagonal strict concavity, introduced by Rosen [3].

Let $X = X_1 \times X_2 \times X_3$ and let $\Pi_i : X \to \mathbb{R}$, $i = 1, 2, 3$, denote the payoff functions of the three firms. For the purposes of this subsection only, assume additionally that each X_i is a nonempty compact convex set, that Π_i is continuous on X and concave with respect to the firm's own quantity x_i, and that the payoff functions are continuously differentiable with respect to the firms' strategy variables.

For a vector of positive weights $r = (r_1, r_2, r_3)$, where $r_i > 0$, the weighted pseudogradient of the game is defined by

$$g_r(x_1, x_2, x_3) = \left(r_1 \frac{\partial \Pi_1}{\partial x_1}(x_1, x_2, x_3), r_2 \frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, x_3), r_3 \frac{\partial \Pi_3}{\partial x_3}(x_1, x_2, x_3) \right).$$

Definition 5 ([3]). *The game is said to satisfy the diagonal strict concavity condition if there exists a weight vector $r \in \mathbb{R}^3_{++}$ such that, for every pair of distinct points $x, y \in X$, one has $\langle x - y, g_r(x) - g_r(y) \rangle < 0$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^3$.*

The diagonal strict concavity condition expresses a strict monotonicity property of the weighted pseudogradient of the game. It is stronger than the separate concavity of each payoff function in the firm's own strategy variable and controls the strategic interaction among the players.

The following classical consequence of Rosen's theorem records precisely the role of this condition in the present setting.

Proposition 3 ([3]). *Let X_i, $i = 1, 2, 3$, be nonempty compact convex sets. Suppose that every payoff function Π_i is continuous on X, concave in x_i, and continuously differentiable with respect to the strategy variables. If the game satisfies the diagonal strict concavity condition, then it has at most one Cournot–Nash equilibrium. Consequently, whenever existence is guaranteed, the equilibrium is unique.*

In the post-cartel model, this proposition applies to the simultaneous Cournot game played by the three firms after the cartel agreement has broken down. It does not concern a leader–follower game and does not involve a selection from the best-response correspondence of Firm 3.

In the present paper, diagonal strict concavity is not used to establish the existence of a post-cartel Cournot–Nash equilibrium. Existence is obtained from the properties of the maximization problems and the associated set-valued best-response correspondences. Rather, diagonal strict concavity is introduced as an additional condition that may be used to exclude multiple equilibrium production profiles when the corresponding payoff structure is differentiable and concave.

This distinction is important for the approach adopted below. The main existence result does not require differentiability, first-order conditions, or second-order conditions. When the additional smoothness and concavity assumptions stated above are satisfied, Rosen's diagonal strict concavity condition provides a classical uniqueness criterion complementary to the set-valued fixed-point existence argument. It is an optional sufficient condition for uniqueness and is not part of the assumptions of the main existence theorem.

3. Main Results

The post-cartel Cournot model considered in this paper leads naturally to optimization problems whose solutions need not be uniquely determined. Even when the corresponding profit functions are continuous and maximizers exist, the best-response set of a firm may consist of several production levels. Consequently, the best-response structure of the market is more naturally described by set-valued correspondences than by single-valued response functions.

The purpose of this section is to establish the existence of a post-cartel Cournot–Nash equilibrium and to identify additional conditions guaranteeing its uniqueness. The existence argument combines generalized concavity, Berge's maximum theorem, and the classical Kakutani–Fan–Glicksberg fixed-point theorem. Generalized concavity is used to establish the quasi-concavity of each firm's profit with respect to its own quantity and, consequently, the convexity of its best-response set. Berge's maximum theorem provides the nonemptiness, compactness, and upper semicontinuity of the individual argmax correspondences. These correspondences are then combined into a product correspondence, whose upper semicontinuity follows from Proposition 2. The Kakutani–Fan–Glicksberg theorem yields a fixed point of this product correspondence and hence a post-cartel Cournot–Nash equilibrium. When additional generalized concavity or differentiability assumptions are available, Corollary 1 or Rosen's diagonal strict concavity condition provides a complementary criterion for uniqueness.

Thus, the analysis does not rely on explicit reaction functions or on solving systems of first-order conditions. Instead, the existence of an equilibrium follows from the existence of global maximizers and a fixed point of the product of the three best-response correspondences.

3.1. Existence of Equilibrium

We begin with the simultaneous post-cartel Cournot model. Each firm maximizes its own profit while treating the quantities of the other two firms as given. Since the maximizer need not be unique, the firms' optimal decisions are represented by the best-response correspondences R_1, R_2, and R_3.

The existence proof is a three-player application of the classical Debreu–Glicksberg–Fan approach. Continuity and compactness ensure, by Berge's maximum theorem, that the best-response correspondences are nonempty, compact-valued, and upper semicontinuous, while quasi-concavity in each firm's own quantity ensures that their values are convex. Proposition 2 then permits the individual correspondences to be combined, and the Kakutani–Fan–Glicksberg fixed-point theorem yields an equilibrium [5,7–9]. No differentiability, first-order conditions, or selection from a best-response set is required.

Theorem 5 (Existence of a post-cartel Cournot–Nash equilibrium)**.** *Let $X_i \subset \mathbb{R}$, $i = 1, 2, 3$, be nonempty compact convex sets, and let $\Pi_i : X_1 \times X_2 \times X_3 \to \mathbb{R}$, $i = 1, 2, 3$, be continuous payoff functions.*

For every $(x_2, x_3) \in X_2 \times X_3$, define $R_1(x_2, x_3) = \arg\max_{x_1 \in X_1} \Pi_1(x_1, x_2, x_3)$. For every $(x_1, x_3) \in X_1 \times X_3$, define $R_2(x_1, x_3) = \arg\max_{x_2 \in X_2} \Pi_2(x_1, x_2, x_3)$. Finally, for every $(x_1, x_2) \in X_1 \times X_2$, define $R_3(x_1, x_2) = \arg\max_{x_3 \in X_3} \Pi_3(x_1, x_2, x_3)$.

Assume that $x_1 \mapsto \Pi_1(x_1, x_2, x_3)$ is quasi-concave on X_1 for every fixed (x_2, x_3), that $x_2 \mapsto \Pi_2(x_1, x_2, x_3)$ is quasi-concave on X_2 for every fixed (x_1, x_3), and that $x_3 \mapsto \Pi_3(x_1, x_2, x_3)$ is quasi-concave on X_3 for every fixed (x_1, x_2).

Then there exists a point $(x_1^*, x_2^*, x_3^*) \in X_1 \times X_2 \times X_3$ such that $x_1^* \in R_1(x_2^*, x_3^*)$, $x_2^* \in R_2(x_1^*, x_3^*)$, and $x_3^* \in R_3(x_1^*, x_2^*)$. Consequently, (x_1^*, x_2^*, x_3^*) is a post-cartel Cournot–Nash equilibrium.

Proof. Because each X_i is nonempty and compact and each Π_i is continuous, the Weierstrass theorem shows that every maximization problem defining R_i has a solution. Hence, the three best-response correspondences have nonempty values.

For fixed values of the quantities selected by the other two firms, the set R_i is the inverse image of the maximal payoff value under a continuous function. It is therefore closed in X_i and, since X_i is compact, it is compact.

We next verify convexity. Fix $i \in \{1,2,3\}$, fix the quantities of the other two firms, and let $u_i, v_i \in R_i$. If $\lambda \in [0,1]$, quasi-concavity of Π_i with respect to x_i gives

$$\Pi_i(\lambda u_i + (1-\lambda)v_i, x_{-i}) \geq \min\{\Pi_i(u_i, x_{-i}), \Pi_i(v_i, x_{-i})\}.$$

The two quantities on the right-hand side are maximal. Therefore, $\lambda u_i + (1-\lambda)v_i$ is also a maximizer, and R_i has convex values.

Since the feasible-set correspondence in each maximization problem is constant and compact-valued, Berge's maximum theorem implies that R_1, R_2, R_3 are upper semicontinuous and compact-valued.

The correspondences R_1, R_2, R_3 are naturally defined on different coordinate subproducts. To apply Proposition 2, we regard them as correspondences on the common domain $X_1 \times X_2 \times X_3$ by defining $\mathcal{R}_1(x_1, x_2, x_3) = R_1(x_2, x_3)$ and $\mathcal{R}_2(x_1, x_2, x_3) = R_2(x_1, x_3)$, and $\mathcal{R}_3(x_1, x_2, x_3) = R_3(x_1, x_2)$.

Each $\mathcal{R}_i$ is the composition of R_i with the corresponding continuous coordinate projection. Therefore, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are upper semicontinuous and have nonempty compact convex values.

Define $\mathcal{R} : X_1 \times X_2 \times X_3 \rightrightarrows X_1 \times X_2 \times X_3$ by

$$\mathcal{R}(x_1, x_2, x_3) = R_1(x_2, x_3) \times R_2(x_1, x_3) \times R_3(x_1, x_2).$$

Equivalently, $\mathcal{R}(x_1, x_2, x_3) = \mathcal{R}_1(x_1, x_2, x_3) \times \mathcal{R}_2(x_1, x_2, x_3) \times \mathcal{R}_3(x_1, x_2, x_3)$.

Thus, the three component correspondences have the common domain required in Proposition 2. By that proposition, $\mathcal{R}$ is upper semicontinuous and has nonempty compact values. Since each $\mathcal{R}_i$ has convex values, $\mathcal{R}$ also has convex values.

The set $X_1 \times X_2 \times X_3$ is nonempty, compact, and convex. Therefore, the Kakutani–Fan–Glicksberg fixed-point theorem yields a point $(x_1^*, x_2^*, x_3^*) \in X_1 \times X_2 \times X_3$ such that

$$(x_1^*, x_2^*, x_3^*) \in \mathcal{R}(x_1^*, x_2^*, x_3^*).$$

Equivalently, $x_1^* \in R_1(x_2^*, x_3^*)$, $x_2^* \in R_2(x_1^*, x_3^*)$, and $x_3^* \in R_3(x_1^*, x_2^*)$. Hence, (x_1^*, x_2^*, x_3^*) is a post-cartel Cournot–Nash equilibrium. □

The preceding theorem allows all three best-response correspondences to be set-valued. One may naturally ask whether this generality is necessary or whether the assumptions of continuity, compactness, and quasi-concavity typically imply unique optimal quantities. The following example shows that they do not: each best-response set may contain a nondegenerate interval.

Example 1. *Let* $X_1 = X_2 = X_3 = [0,2]$. *Consider the following payoff functions:*

$$\Pi_3(x_1, x_2, z) = -\operatorname{dist}(z, [a(x_1, x_2), b(x_1, x_2)])^2,$$

where $a(x_1, x_2) = \frac{x_1+x_2}{4}$ *and* $b(x_1, x_2) = 1 + \frac{x_1+x_2}{4}$, *and*

$$\Pi_1(x_1, x_2, z) = -\operatorname{dist}\left(x_1, \left[\frac{x_2+z}{4}, 1 + \frac{x_2+z}{4}\right]\right)^2,$$

$$\Pi_2(x_1, x_2, z) = -\operatorname{dist}\left(x_2, \left[\frac{x_1+z}{4}, 1+\frac{x_1+z}{4}\right]\right)^2.$$

The three payoff functions are continuous on $X_1 \times X_2 \times X_3$. Moreover, for every fixed value of the remaining variables, each payoff function is quasi-concave with respect to its own decision variable. We next determine the corresponding best-response correspondences.

Since $x_1, x_2 \in [0,2]$, we have $0 \le a(x_1, x_2) \le 1$ and $1 \le b(x_1, x_2) \le 2$.

For every fixed $(x_1, x_2) \in X_1 \times X_2$, the function $z \mapsto \Pi_3(x_1, x_2, z)$ attains its maximum value 0 at every point of the interval $[a(x_1, x_2), b(x_1, x_2)]$. Consequently, $R_3(x_1, x_2) = \arg\max_{z \in X_3} \Pi_3(x_1, x_2, z) = [a(x_1, x_2), b(x_1, x_2)]$.

Thus, $R_3(x_1, x_2)$ is nonempty, compact, and convex, but it is not single-valued.

For example, if $x_1 = x_2 = 1$, then $a(1,1) = \frac{1}{2}$, $b(1,1) = \frac{3}{2}$, and therefore $R_3(1,1) = [\frac{1}{2}, \frac{3}{2}]$. Hence, every quantity $z \in [\frac{1}{2}, \frac{3}{2}]$ is an optimal quantity of Firm 3.

For every fixed $(x_2, z) \in X_2 \times X_3$, the maximum value of $\Pi_1(\cdot, x_2, z)$ is 0, and it is attained at every point of the interval $\left[\frac{x_2+z}{4}, 1+\frac{x_2+z}{4}\right]$. Therefore, $R_1(x_2, z) = [\frac{x_2+z}{4}, 1+\frac{x_2+z}{4}]$.

Similarly, $R_2(x_1, z) = [\frac{x_1+z}{4}, 1+\frac{x_1+z}{4}]$.

Thus, all three firms may have set-valued best-response correspondences.

Consider, for instance, the point $(x_1, x_2, z) = (1,1,1)$. We have $R_3(1,1) = [\frac{1}{2}, \frac{3}{2}]$, while $R_1(1,1) = R_2(1,1) = [\frac{1}{2}, \frac{3}{2}]$. Consequently, $1 \in R_1(1,1)$, $1 \in R_2(1,1)$, and $1 \in R_3(1,1)$. Hence, $(1,1,1) \in R_1(1,1) \times R_2(1,1) \times R_3(1,1)$, and $(1,1,1)$ is a fixed point of the product best-response correspondence.

This example shows that continuity and quasi-concavity of the payoff functions do not imply uniqueness of the corresponding maximizers. The sets of maximizers may contain entire intervals. Nevertheless, these argmax sets remain nonempty, compact, and convex, which is precisely the structure required in Theorem 5.

This example illustrates the abstract existence theorem only. Its payoff functions are constructed to demonstrate nondegenerate best-response intervals and are not claimed to have the economic form $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$. The application of Theorem 5 to profit functions of that economic form is considered in the next subsection.

3.2. Application to the Price–Cost Functions Model

Theorem 5 establishes an existence result for a system of three interacting optimization problems formulated solely in terms of the payoff functions Π_i, $i = 1,2,3$. Its assumptions involve only continuity, quasi-concavity, and compactness, without referring to any particular economic model.

We now specialize this abstract framework to the post-cartel Cournot market introduced in Section 2.4. In this setting, the payoff functions are generated by the standard structure consisting of an inverse demand function and individual cost functions, $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$, $i = 1,2,3$.

The generalized concavity assumptions imposed on the inverse demand function, together with the convexity of the cost functions, guarantee the quasi-concavity of each firm's payoff with respect to its own decision variable. Consequently, all assumptions of Theorem 5 are satisfied, and the existence of a post-cartel Cournot–Nash equilibrium follows as a direct application of the abstract optimization theorem.

Corollary 2. *Let $X_i = [0, M_i] \subset \mathbb{R}_+$, $i = 1,2,3$, be nonempty compact intervals.*

Let $\Omega \subseteq \mathbb{R}_+$ be a convex interval containing $[0, M_1 + M_2 + M_3]$. Assume that the inverse demand function $P : \Omega \to \mathbb{R}_{++}$ is continuous, decreasing, and (-1)-concave.

Assume further that the cost functions $c_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,3$, are continuous, convex, and increasing.

For $i = 1,2,3$, define $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$.

Then the hypotheses of Theorem 5 are satisfied. Consequently, there exists a post-cartel Cournot–Nash equilibrium $(x_1^*, x_2^*, x_3^*) \in X_1 \times X_2 \times X_3$.

Proof. Since the inverse demand function and the cost functions are continuous, each payoff function $\Pi_i : X_1 \times X_2 \times X_3 \to \mathbb{R}$ is continuous.

Fix $i \in \{1,2,3\}$ and denote the aggregate output of the other two firms by $s \geq 0$. Since P is continuous, decreasing, and (-1)-concave, Lemma 1 shows that the conditional revenue function $r_s(x) = xP(x+s)$ is either non-increasing, or it is strictly increasing and strictly concave up to its set of global maximizers and non-increasing thereafter.

In the first case, $r_s - c_i$ is non-increasing because c_i is increasing. In the second case, $r_s - c_i$ is concave on the interval of increase because c_i is convex, and it is non-increasing thereafter. Hence, $r_s - c_i$ is quasi-concave in both cases. Consequently, the mapping $x_i \mapsto \Pi_i(x_1, x_2, x_3)$ is quasi-concave on X_i for every $i = 1,2,3$.

Therefore, all the assumptions of Theorem 5 are satisfied, and the conclusion follows. □

The (-1)-concavity assumption in Corollary 2 is used only to establish quasi-concavity of the individual profit functions and hence the existence of an equilibrium. It does not, by itself, imply uniqueness.

Suppose, in addition, that $P : \mathbb{R}_+ \to \mathbb{R}_{++}$ is strictly decreasing, differentiable, and $(-1/3)$-concave, and that the upper bounds M_i are nonbinding, so that the constrained and unconstrained best responses coincide at equilibrium. Since the market consists of three firms, the condition $(-1/3)$-concavity is precisely the $(-1)/N$-concavity condition with $N = 3$. Therefore, Corollary 1 implies that the post-cartel Cournot–Nash equilibrium is unique.

Thus, (-1)-concavity is sufficient for existence, whereas the stronger $(-1/3)$-concavity condition is invoked only for uniqueness.

The following example gives set-valued best responses and more than one fixed point in a standard price–cost model. The particular inverse demand used below, however, does not satisfy the (-1)-concavity assumption of Corollary 2.

Example 2. *Let* $X_1 = X_2 = X_3 = [0,3]$. *Assume that the inverse demand function is*

$$P(Q) = \begin{cases} 5 - Q, & 0 \leq Q \leq 3, \\ 1 + \dfrac{1}{Q-2}, & 3 \leq Q \leq 6, \\ 1 + \dfrac{1}{2(Q-4)}, & Q \geq 6, \end{cases}$$

and let $c_i(x) = x$ *for* $i = 1,2,3$.

The corresponding payoff functions are $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$, $i = 1,2,3$.

The inverse demand function is positive, continuous, and strictly decreasing. It is not (-1)-concave on the whole domain; indeed, on $3 < Q < 6$, one has $P(Q)^{-1} = (Q-2)/(Q-1)$, which is strictly concave rather than convex. Consequently, this example is not an application of Corollary 2; the equilibria below are verified directly.

We next show that the equilibrium is not unique.

Consider the production profile $(1,1,1)$. Direct computations show that $R_1(1,1) = R_2(1,1) = R_3(1,1) = [1,3]$.

Since $1 \in [1,3]$, we obtain $(1,1,1) \in R_1(1,1) \times R_2(1,1) \times R_3(1,1)$, and therefore $(1,1,1)$ is a post-cartel Cournot–Nash equilibrium.

Similarly, $R_1(2,2) = R_2(2,2) = R_3(2,2) = [2,3]$.

Hence, $(2,2,2)$ is also a post-cartel Cournot–Nash equilibrium.

The model therefore possesses at least two distinct equilibria, $(1,1,1)$ and $(2,2,2)$, both yielding strictly positive profits for every firm. Thus, the present example directly shows that its best-response correspondences and its equilibrium are not unique.

3.3. The Single-Valued Case

The previous results were formulated for general set-valued best-response correspondences. This level of generality is essential because, as illustrated by the preceding examples, multiple optimal strategies may occur even when the payoff functions satisfy all the assumptions of the existence theorem.

In many economic models, however, additional assumptions imply that every individual optimization problem admits a unique maximizer. In this case, the three best-response correspondences become single-valued functions. Consequently, the abstract set-valued equilibrium framework reduces to the classical formulation in terms of single-valued best-response functions. The single-valuedness of the individual best responses does not, by itself, imply that their common fixed point is unique.

Corollary 3. *Assume that all the hypotheses of Corollary 2 are satisfied. Suppose, in addition, that, for every point in their respective domains, the sets $R_1(x_2,x_3)$, $R_2(x_1,x_3)$, and $R_3(x_1,x_2)$ are singletons.*

Then there exist uniquely determined continuous best-response functions $b_1 : X_2 \times X_3 \to X_1$, $b_2 : X_1 \times X_3 \to X_2$, and $b_3 : X_1 \times X_2 \to X_3$ such that $R_1(x_2,x_3) = \{b_1(x_2,x_3)\}$, $R_2(x_1,x_3) = \{b_2(x_1,x_3)\}$, and $R_3(x_1,x_2) = \{b_3(x_1,x_2)\}$.

Define $T : X_1 \times X_2 \times X_3 \to X_1 \times X_2 \times X_3$ by

$$T(x_1,x_2,x_3) = (b_1(x_2,x_3), b_2(x_1,x_3), b_3(x_1,x_2)).$$

Then T is continuous and has at least one fixed point.

Every fixed point (x_1^*,x_2^*,x_3^*) of T satisfies $x_1^* = b_1(x_2^*,x_3^*)$, $x_2^* = b_2(x_1^*,x_3^*)$, and $x_3^* = b_3(x_1^*,x_2^*)$, and hence is a post-cartel Cournot–Nash equilibrium.

Proof. By Corollary 2, the correspondences $R_1 : X_2 \times X_3 \rightrightarrows X_1$, $R_2 : X_1 \times X_3 \rightrightarrows X_2$, and $R_3 : X_1 \times X_2 \rightrightarrows X_3$ are nonempty, compact-valued, and upper semicontinuous. By the additional assumption, each of them is singleton-valued and therefore determines a uniquely defined function b_i.

We verify continuity for b_3; the arguments for b_1 and b_2 are identical. Let $(x_1^n,x_2^n) \to (x_1,x_2)$ and put $z_n = b_3(x_1^n,x_2^n)$. Since X_3 is compact, every subsequence of $\{z_n\}$ has a convergent subsequence. If $z_{n_k} \to z$, then $z_{n_k} \in R_3(x_1^{n_k},x_2^{n_k})$. Upper semicontinuity and Proposition 1 imply that $z \in R_3(x_1,x_2) = \{b_3(x_1,x_2)\}$. Hence, $z = b_3(x_1,x_2)$. Thus, every convergent subsequence has the same limit, and compactness of X_3 gives $b_3(x_1^n,x_2^n) \to b_3(x_1,x_2)$. Therefore, b_3 is continuous.

The same argument proves the continuity of b_1 and b_2. Consequently, T is continuous. Since $X_1 \times X_2 \times X_3$ is nonempty, compact, and convex, Brouwer's fixed-point theorem yields a point (x_1^*,x_2^*,x_3^*) satisfying $T(x_1^*,x_2^*,x_3^*) = (x_1^*,x_2^*,x_3^*)$.

By the definition of T, this equality is equivalent to $x_1^* \in R_1(x_2^*,x_3^*)$, $x_2^* \in R_2(x_1^*,x_3^*)$, and $x_3^* \in R_3(x_1^*,x_2^*)$. Therefore, (x_1^*,x_2^*,x_3^*) is a post-cartel Cournot–Nash equilibrium. □

The following example illustrates Corollary 3. We consider a nonlinear Cournot market in which all individual optimization problems admit unique maximizers. Consequently, the best-response correspondences reduce to continuous single-valued functions, and the market equilibrium is characterized by a common fixed point of these functions. Its uniqueness will be established separately by applying Corollary 1.

Example 3. *Let* $X_1 = X_2 = X_3 = [0,2]$, *and consider the inverse demand function* $P(Q) = \frac{1}{(1+Q)^5}$. *Let the cost functions be* $c_1(x) = c_2(x) = 0.0004201x$ *and* $c_3(x) = 0.0017854x + \frac{0.001}{2}x^2$.

The numerical cost coefficients are used to construct a normalized illustrative market with an interior, asymmetric, and unique equilibrium. They are not obtained from an empirical calibration and are not intended to represent a particular industry. Their role is to provide a transparent example in which Firms 1 and 2 have identical technologies, Firm 3 has a different marginal-cost structure, and all equilibrium quantities lie strictly inside the prescribed strategy intervals.

The corresponding profit functions are $\Pi_i(x_1, x_2, x_3) = x_i P(x_1 + x_2 + x_3) - c_i(x_i)$, $i = 1, 2, 3$.

The inverse demand function P is positive, continuous, and strictly decreasing. Moreover, $P(Q)^{-1/3} = (1+Q)^{5/3}$ is convex on $\mathbb{R}_+$, and hence P is $(-1/3)$-concave. Since $(-1/3)$-concavity implies (-1)-concavity, the inverse demand function satisfies the generalized concavity condition of Corollary 2. The cost functions are continuous, convex, and increasing. Therefore, all the hypotheses of Corollary 2 are satisfied.

We next verify that each individual optimization problem has a unique solution. Fix $i \in \{1,2,3\}$, and denote the aggregate production of the other two firms by $s = \sum_{j \neq i} x_j$. The profit of Firm i, considered as a function of its own production x, is $\varphi_{i,s}(x) = \frac{x}{(1+s+x)^5} - c_i(x)$ for $x \in [0,2]$.

Let $A = 1 + s > 0$. For Firms 1 and 2, we have $\varphi'_{i,s}(x) = \frac{A-4x}{(A+x)^6} - 0.0004201$, $i = 1, 2$, whereas, for Firm 3, $\varphi'_{3,s}(x) = \frac{A-4x}{(A+x)^6} - 0.0017854 - 0.001x$.

The function $h_A(x) = \frac{A-4x}{(A+x)^6}$ satisfies $h'_A(x) = \frac{20x-10A}{(A+x)^7}$. In particular, h_A is strictly decreasing on $[0, A/4]$, which is precisely the interval on which h_A may be positive. For $x \geq A/4$, we have $h_A(x) \leq 0$.

It follows that each derivative $\varphi'_{i,s}$ can vanish at most once. Consequently, $\varphi_{i,s}$ is either strictly decreasing on $[0,2]$, or it first increases and then strictly decreases. Thus, $\varphi_{i,s}$ is strictly quasi-concave and possesses a unique maximizer on $[0,2]$.

Therefore, for every admissible production profile of the other firms, the sets $R_1(x_2, x_3)$, $R_2(x_1, x_3)$, and $R_3(x_1, x_2)$ are singletons. Hence, they determine the single-valued best-response functions $b_1 : X_2 \times X_3 \to X_1$, $b_2 : X_1 \times X_3 \to X_2$, and $b_3 : X_1 \times X_2 \to X_3$.

Numerical solution of the three simultaneous best-response equations gives the approximate equilibrium quantities $(x_1^*, x_2^*, x_3^*) \approx (0.445044, 0.445044, 0.394932)$.

All the hypotheses of Corollary 2 are satisfied, and P is differentiable. Moreover, since $s \in [0,4]$, every positive maximizer satisfies $x < (1+s)/4 \leq 5/4 < 2$, so the upper bound 2 is nonbinding. It therefore follows from Corollary 1 that this equilibrium is unique.

This example illustrates two logically distinct properties. First, every individual best-response correspondence is singleton-valued, so the set-valued formulation reduces to the continuous mapping T. Second, the stronger $(-1/3)$-concavity assumption guarantees that T has only one fixed point. The latter conclusion does not follow from single-valuedness alone.

3.4. Technological Asymmetry and Equilibrium Quantities

The general existence theorem does not require differentiability and allows the firms' best-response correspondences to be set-valued. Consequently, no first-order conditions are used in the existence analysis. In the present subsection, we impose additional smoothness and interiority assumptions only for the purpose of comparing the quantities produced by different firms at a given post-cartel Cournot–Nash equilibrium.

The comparison below concerns technological asymmetry, represented by differences between the firms' marginal costs. Since all three firms choose their quantities simultane-

ously and take the quantities of the other firms as given, no derivative of one firm's best response with respect to another firm's quantity enters the equilibrium conditions.

Proposition 4. *Let P and c_i, $i = 1,2,3$ be continuously differentiable on the relevant domains, and let (x_1^*, x_2^*, x_3^*) be an interior post-cartel Cournot–Nash equilibrium. Put $Q^* = x_1^* + x_2^* + x_3^*$ and assume that $P'(Q^*) < 0$.*

Then, for every pair of distinct firms $i, j \in \{1,2,3\}$, $P'(Q^*)(x_i^* - x_j^*) = c_i'(x_i^*) - c_j'(x_j^*)$.

Consequently, $c_i'(x_i^*) < c_j'(x_j^*)$ if and only if $x_i^* > x_j^*$, and $c_i'(x_i^*) > c_j'(x_j^*)$ if and only if $x_i^* < x_j^*$.

Proof. Since the equilibrium is interior and the payoff functions are differentiable, the individual optimality condition for Firm i is

$$P(Q^*) + x_i^* P'(Q^*) - c_i'(x_i^*) = 0. \tag{1}$$

Similarly, the optimality condition for Firm j is

$$P(Q^*) + x_j^* P'(Q^*) - c_j'(x_j^*) = 0. \tag{2}$$

Subtracting (2) from (1) gives $P'(Q^*)(x_i^* - x_j^*) = c_i'(x_i^*) - c_j'(x_j^*)$.

Since $P'(Q^*) < 0$, the two sides show that the difference between the equilibrium quantities has the opposite sign to the difference between the corresponding marginal costs. This proves the asserted equivalences. □

The preceding proposition does not use the first-order conditions as sufficient conditions for equilibrium. The existence of the equilibrium and the global optimality of each quantity are assumed from the preceding set-valued analysis. The first-order conditions are used here only as necessary conditions at an already established interior equilibrium.

Corollary 4. *Under the hypotheses of Proposition 4, suppose that all three firms have the same continuously differentiable convex cost function c. Then $x_1^* = x_2^* = x_3^*$.*

Proof. Suppose, for example, that $x_i^* > x_j^*$. Since c is convex, its derivative is nondecreasing, and hence $c'(x_i^*) \geq c'(x_j^*)$. On the other hand, Proposition 4 and $P'(Q^*) < 0$ imply $c'(x_i^*) < c'(x_j^*)$, which is a contradiction. The case $x_i^* < x_j^*$ leads to the same contradiction. Therefore, $x_i^* = x_j^*$ for every $i, j \in \{1,2,3\}$. □

Corollary 5. *Suppose that the hypotheses of Proposition 4 hold and that $c_i(x) = a_i x$, where $a_i \geq 0$, for $i = 1,2,3$. Then, for every pair of firms $i \neq j$, one has $a_i < a_j$ if and only if $x_i^* > x_j^*$.*

Proof. Since $c_i'(x) = a_i$, the conclusion follows immediately from Proposition 4. □

These results show that, within the simultaneous post-cartel Cournot model, differences between equilibrium quantities are generated by technological asymmetry. A firm with a lower marginal cost at equilibrium produces a strictly larger quantity. If the firms have identical convex technologies, their equilibrium quantities are equal.

The fact that Firm 3 is the firm that left the cartel does not, by itself, create an asymmetry in equilibrium production. Such an asymmetry would require differences in costs, feasible production sets, or other primitives of the firms' payoff functions. A comparison based on the derivative of Firm 3's response with respect to the quantities of Firms 1 and 2 would instead belong to a sequential leader–follower model and is therefore not part of the post-cartel Cournot framework studied here.

4. An Anticipatory Single-Valued Post-Cartel Extension

The post-cartel Cournot model studied in the preceding sections is a simultaneous game: each firm maximizes its individual profit while treating the quantities of the other firms as given. In particular, Firms 1 and 2 do not take into account how the optimal quantity of Firm 3 may change when they alter their own production.

We now consider a distinct anticipatory extension of that model. Assume that the best-response correspondence of Firm 3 is singleton-valued so that $R_3(x_1, x_2) = \{b_3(x_1, x_2)\}$. Firms 1 and 2 are assumed to anticipate this response when selecting their quantities. Accordingly, they maximize the induced payoff functions $\Psi_i(x_1, x_2) = \Pi_i(x_1, x_2, b_3(x_1, x_2))$, $i = 1, 2$.

This extension is a two-leader–one-follower game: Firms 1 and 2 choose their quantities simultaneously, while Firm 3 responds to their joint production profile. Such hierarchical non-cooperative structures are closely related to equilibrium problems with equilibrium constraints, in which several leaders optimize subject to the equilibrium response of a lower-level problem [35,36]. It is therefore different from the simultaneous three-firm Cournot game considered in Theorem 5. In particular, the existence conclusion of Theorem 5 cannot be invoked directly for the anticipatory game.

4.1. Equilibrium Formulation and Existence

Definition 6. *A triple $(x_1^*, x_2^*, x_3^*) \in X_1 \times X_2 \times X_3$ is called an anticipatory post-cartel equilibrium if $x_1^* \in \arg\max_{x_1 \in X_1} \Psi_1(x_1, x_2^*)$, $x_2^* \in \arg\max_{x_2 \in X_2} \Psi_2(x_1^*, x_2)$, and $x_3^* = b_3(x_1^*, x_2^*)$.*

The following result gives sufficient conditions for the existence of an anticipatory post-cartel equilibrium. Unlike the simultaneous model, the anticipatory model is formulated in terms of the induced leader payoffs. Their quasi-concavity does not follow automatically from the quasi-concavity of the original profit functions and must therefore be established separately.

The continuity of the follower response, however, need not be imposed as an independent assumption. As shown in Corollary 3, an upper semicontinuous singleton-valued best-response correspondence with values in a compact strategy set determines a continuous response function. We incorporate this observation directly into the following theorem.

Theorem 6. *Let $X_i \subset \mathbb{R}$, $i = 1, 2, 3$, be nonempty compact convex sets, and let $\Pi_i : X_1 \times X_2 \times X_3 \to \mathbb{R}$, $i = 1, 2, 3$, be continuous.*

Suppose that the best-response correspondence $R_3 : X_1 \times X_2 \rightrightarrows X_3$ is upper semicontinuous and singleton-valued. Write $R_3(x_1, x_2) = \{b_3(x_1, x_2)\}$ and define

$$\Psi_i(x_1, x_2) = \Pi_i\big(x_1, x_2, b_3(x_1, x_2)\big), \qquad i = 1, 2.$$

Assume that, for every fixed $x_2 \in X_2$, the function $x_1 \mapsto \Psi_1(x_1, x_2)$ is quasi-concave on X_1, and that, for every fixed $x_1 \in X_1$, the function $x_2 \mapsto \Psi_2(x_1, x_2)$ is quasi-concave on X_2.

Then the anticipatory post-cartel game possesses at least one equilibrium.

Proof. Since R_3 is upper semicontinuous and singleton-valued and X_3 is compact, the argument used in Corollary 3 shows that the uniquely determined response function $b_3 : X_1 \times X_2 \to X_3$ is continuous. Consequently, the continuity of the original profit functions implies that the induced payoff functions Ψ_1 and Ψ_2 are continuous.

Define the induced best-response correspondences $\widehat{R}_1 : X_2 \rightrightarrows X_1$ and $\widehat{R}_2 : X_1 \rightrightarrows X_2$ by $\widehat{R}_1(x_2) = \arg\max_{x_1 \in X_1} \Psi_1(x_1, x_2)$ and $\widehat{R}_2(x_1) = \arg\max_{x_2 \in X_2} \Psi_2(x_1, x_2)$, respectively.

By Berge's maximum theorem, these correspondences are nonempty, compact-valued, and upper semicontinuous. Their values are convex because the induced payoffs are quasi-concave in the corresponding own quantity.

Therefore, the product correspondence $\widehat{R}(x_1, x_2) = \widehat{R}_1(x_2) \times \widehat{R}_2(x_1)$ is upper semicontinuous and has nonempty compact convex values. The Kakutani–Fan–Glicksberg fixed-point theorem yields a point $(x_1^*, x_2^*) \in X_1 \times X_2$ such that $x_1^* \in \widehat{R}_1(x_2^*)$ and $x_2^* \in \widehat{R}_2(x_1^*)$.

Set $x_3^* = b_3(x_1^*, x_2^*)$. Then (x_1^*, x_2^*, x_3^*) is an anticipatory post-cartel equilibrium. □

The quasi-concavity assumption in Theorem 6 is substantive because it concerns the payoffs obtained after the follower response has been substituted into the leaders' optimization problems. The following result gives a sufficient condition directly in terms of the demand and cost primitives. In particular, it covers the linear-quadratic class used in several of the examples below.

Lemma 2. *Let the inverse demand function be $P(Q) = a - bQ$ for $b > 0$, on the aggregate-output interval generated by the strategy sets, and let the cost functions be*

$$c_i(x_i) = d_i x_i + \frac{\gamma_i}{2} x_i^2, \qquad \gamma_i \geq 0, \qquad i = 1, 2, 3.$$

Suppose that, for every $(x_1, x_2) \in X_1 \times X_2$, the unique best response of Firm 3 is interior to X_3.

Then b_3 is affine and continuous. Moreover, for each $i \in \{1, 2\}$ and every fixed quantity of the other leader, the induced payoff Ψ_i is strictly concave in x_i. Consequently, the anticipatory post-cartel game possesses at least one equilibrium.

Proof. For fixed (x_1, x_2), the profit of Firm 3 is

$$\Pi_3(x_1, x_2, x_3) = x_3\big[a - b(x_1 + x_2 + x_3)\big] - d_3 x_3 - \frac{\gamma_3}{2} x_3^2.$$

Its second derivative with respect to x_3 is

$$\frac{\partial^2 \Pi_3}{\partial x_3^2} = -(2b + \gamma_3) < 0.$$

Hence the follower's optimization problem has a unique maximizer. Under the assumed interiority condition, its first-order condition gives

$$b_3(x_1, x_2) = \frac{a - d_3 - b(x_1 + x_2)}{2b + \gamma_3}.$$

Thus, b_3 is affine and continuous, with $\frac{\partial b_3}{\partial x_i} = -\frac{b}{2b+\gamma_3}$ for $i = 1, 2$.

Fix $i \in \{1, 2\}$, and denote the other leader by j. After substituting the follower response, the aggregate output changes with x_i at the constant rate

$$1 + \frac{\partial b_3}{\partial x_i} = 1 - \frac{b}{2b + \gamma_3} = \frac{b + \gamma_3}{2b + \gamma_3} > 0.$$

Therefore, for fixed x_j, the second derivative of the induced payoff of Firm i is

$$\frac{\partial^2 \Psi_i}{\partial x_i^2} = -2b \frac{b + \gamma_3}{2b + \gamma_3} - \gamma_i < 0.$$

Hence $\Psi_i(\cdot, x_j)$ is strictly concave, and therefore quasi-concave, on X_i.

All the hypotheses of Theorem 6 are consequently satisfied, which proves the existence of an anticipatory post-cartel equilibrium. □

More generally, when b_3 and the original profit functions are twice continuously differentiable, a directly verifiable sufficient condition for the quasi-concavity required in Theorem 6 is $\frac{\partial^2 \Psi_i}{\partial x_i^2}(x_1, x_2) \leq 0$ throughout $X_1 \times X_2$, for $i = 1, 2$. If the inequality is strict, each induced optimization problem has a unique maximizer. This criterion is useful when the follower response is defined implicitly rather than available in closed form, as illustrated in Example 8.

4.2. Technological and Strategic Sources of Asymmetry

We now examine an interior equilibrium of the anticipatory game under additional differentiability assumptions. These assumptions are used only for the comparison result below and are not part of the general set-valued framework.

Proposition 5. *Assume that P, c_1, c_2, c_3, and b_3 are continuously differentiable on the relevant domains. Let (x_1^*, x_2^*, x_3^*) be an interior anticipatory post-cartel equilibrium and put $Q^* = x_1^* + x_2^* + x_3^*$. Assume that $P'(Q^*) < 0$.*

For $i = 1, 2$, put $r_i^* = \frac{\partial b_3}{\partial x_i}(x_1^*, x_2^*)$. Then

$$P'(Q^*)\left[x_i^*(1 + r_i^*) - x_3^*\right] = c_i'(x_i^*) - c_3'(x_3^*).$$

Consequently, $c_i'(x_i^*) < c_3'(x_3^*)$ if and only if $x_i^*(1 + r_i^*) > x_3^*$, whereas $c_i'(x_i^*) > c_3'(x_3^*)$ if and only if $x_i^*(1 + r_i^*) < x_3^*$.

Proof. Since x_i^* is an interior maximizer of the induced payoff Ψ_i, its necessary first-order condition is

$$P(Q^*) + x_i^* P'(Q^*)(1 + r_i^*) - c_i'(x_i^*) = 0. \tag{3}$$

Since $x_3^* = b_3(x_1^*, x_2^*)$ is an interior maximizer of the profit of Firm 3, its necessary first-order condition is

$$P(Q^*) + x_3^* P'(Q^*) - c_3'(x_3^*) = 0. \tag{4}$$

Subtracting (4) from (3) gives the stated identity. The two equivalences follow from $P'(Q^*) < 0$. □

The identity in Proposition 5 separates two different sources of equilibrium asymmetry. The difference $c_i'(x_i^*) - c_3'(x_3^*)$ represents the technological effect, whereas r_i^* represents the strategic effect generated by the anticipated response of Firm 3. Neither effect can generally be ignored.

In particular, marginal costs alone do not determine the ordering of the actual quantities. A sufficiently strong strategic response may reinforce, offset, or reverse the production advantage associated with differences in marginal costs.

Corollary 6. *Assume that the hypotheses of Proposition 5 are satisfied and that all three firms have the same continuously differentiable convex cost function c. Assume also that $x_i^* > 0$ for $i = 1, 2$.*

Then, for $i = 1, 2$, the following conclusions hold:

1. *If $\frac{\partial b_3}{\partial x_i}(x_1^*, x_2^*) < 0$, then $x_i^* > x_3^*$.*
2. *If $\frac{\partial b_3}{\partial x_i}(x_1^*, x_2^*) > 0$, then $x_i^* < x_3^*$.*

If, in addition, $x_1^* = x_2^*$, then a decreasing response of Firm 3 to both leaders gives $x_1^* = x_2^* > x_3^*$, whereas an increasing response gives $x_1^* = x_2^* < x_3^*$.

Proof. Fix $i \in \{1, 2\}$. First, suppose that $r_i^* < 0$ and, contrary to the assertion, that $x_i^* \leq x_3^*$. Convexity of c implies $c'(x_i^*) \leq c'(x_3^*)$.

On the other hand, $x_i^*(1+r_i^*) < x_i^* \leq x_3^*$. Since $P'(Q^*) < 0$, Proposition 5 implies $c'(x_i^*) > c'(x_3^*)$, which is a contradiction. Therefore, $x_i^* > x_3^*$.

Next, suppose that $r_i^* > 0$ and, contrary to the assertion, that $x_i^* \geq x_3^*$. Convexity of c gives $c'(x_i^*) \geq c'(x_3^*)$, whereas $x_i^*(1+r_i^*) > x_i^* \geq x_3^*$. Since $P'(Q^*) < 0$, Proposition 5 yields $c'(x_i^*) < c'(x_3^*)$, again a contradiction. Hence, $x_i^* < x_3^*$. □

The sign-based interpretation in the preceding corollary is closely related to the classical distinction between strategic substitutes and strategic complements and to the associated taxonomy of strategic postures [29–31]. The connection should not, however, be interpreted as an exact identification of the two concepts. In the classical terminology, strategic substitutability or complementarity describes how a rival's action affects a firm's marginal incentive to choose its own action. In the present anticipatory model, $\partial b_3/\partial x_i$ instead measures the endogenous response of the follower that Firm i incorporates directly into its induced first-order condition. A negative response relaxes the competitive effect of an expansion by Firm i, whereas a positive response reinforces it.

Comparisons between leader and follower quantities also have precedents in the literature on multiple-leader–follower oligopolies [33,34,38]. Accordingly, Corollary 6 is not intended as a new general principle concerning strategic substitutes and complements. Its role is to specialize this established strategic logic to a two-leader–one-follower setting with two simultaneously optimizing leaders. The more specific contribution of the present analysis is the exact identity in Proposition 5, which separates the technological component $c_i'(x_i^*) - c_3'(x_3^*)$ from the strategic component generated by $\partial b_3/\partial x_i$. This decomposition shows explicitly how the two components may reinforce, offset, or reverse one another in determining the ordering of equilibrium quantities.

4.3. Differentiability of the Response of Firm 3

The differentiability of b_3 used above is an additional assumption. It may be verified locally by applying the implicit function theorem to the first-order condition of Firm 3.

Suppose that P and c_3 are twice continuously differentiable and that the response $x_3 = b_3(x_1, x_2)$ is interior. Its first-order condition is $P(Q) + x_3P'(Q) - c_3'(x_3) = 0$, where $Q = x_1 + x_2 + x_3$.

The derivative of the left-hand side with respect to x_3 is $2P'(Q) + x_3P''(Q) - c_3''(x_3)$. Therefore, if $2P'(Q) + x_3P''(Q) - c_3''(x_3) \neq 0$, the implicit function theorem yields a locally unique continuously differentiable solution branch of the first-order condition through $x_3 = b_3(x_1, x_2)$. Since b_3 is already assumed to be the single-valued continuous interior best response, this local branch coincides with b_3.

Moreover, for $i = 1, 2$, there hold

$$\frac{\partial b_3}{\partial x_i} = -\frac{P'(Q) + x_3P''(Q)}{2P'(Q) + x_3P''(Q) - c_3''(x_3)}.$$

These conditions clarify that the differentiable response analysis belongs to the smooth, locally single-valued extension of the model. It does not replace the set-valued formulation used for the general existence result.

4.4. Equilibrium Comparison from an Enclosing Region

The comparison results obtained above do not require the equilibrium to be known in closed form. Suppose that analytical estimates, interval methods, or validated numerical computations provide a compact set $U \subseteq X_1 \times X_2$ containing an interior anticipatory post-cartel equilibrium (x_1^*, x_2^*). The ordering of the equilibrium quantities can then be determined by verifying the relevant inequalities uniformly on U.

For $(x_1, x_2) \in U$, let $x_3 = b_3(x_1, x_2)$ and $Q = x_1 + x_2 + b_3(x_1, x_2)$. For $i \in \{1, 2\}$, define

$$\mathcal{H}_i(x_1, x_2) = P'(Q)\left[x_i\left(1 + \frac{\partial b_3}{\partial x_i}(x_1, x_2)\right) - b_3(x_1, x_2)\right] - \left[c_i'(x_i) - c_3'\big(b_3(x_1, x_2)\big)\right].$$

By Proposition 5, every interior anticipatory post-cartel equilibrium satisfies $\mathcal{H}_i(x_1^*, x_2^*) = 0$.

Proposition 6. *Assume that $P'(Q) < 0$ on the output region generated by U and that the differentiability assumptions of Proposition 5 are satisfied. Let $(x_1^*, x_2^*) \in U$ be an interior anticipatory post-cartel equilibrium and set $x_3^* = b_3(x_1^*, x_2^*)$. For $i \in \{1, 2\}$, the following statements hold:*

1. *If $0 \notin \mathcal{H}_i\big(U \cap \{(x_1, x_2) : x_i \leq b_3(x_1, x_2)\}\big)$, then $x_i^* > x_3^*$.*
2. *If $0 \notin \mathcal{H}_i\big(U \cap \{(x_1, x_2) : x_i \geq b_3(x_1, x_2)\}\big)$, then $x_i^* < x_3^*$.*

If, in addition, $c_1 = c_2 = c_3 = c$, where c is continuously differentiable and convex, then the following simpler criteria apply:

3. *If $\frac{\partial b_3}{\partial x_i}(x_1, x_2) < 0$ throughout U, then every interior anticipatory post-cartel equilibrium contained in U satisfies $x_i^* > x_3^*$.*
4. *If $\frac{\partial b_3}{\partial x_i}(x_1, x_2) > 0$ throughout U, then every interior anticipatory post-cartel equilibrium contained in U satisfies $x_i^* < x_3^*$.*

Proof. If the hypothesis in part (i) holds but $x_i^* \leq x_3^*$, then (x_1^*, x_2^*) belongs to $U \cap \{(x_1, x_2) : x_i \leq b_3(x_1, x_2)\}$. This contradicts $\mathcal{H}_i(x_1^*, x_2^*) = 0$. Hence $x_i^* > x_3^*$. Part (ii) follows analogously.

Suppose now that the firms have identical technologies. If $\partial b_3/\partial x_i < 0$ and $x_i \leq b_3(x_1, x_2)$, then convexity of c gives $c'(x_i) - c'(b_3(x_1, x_2)) \leq 0$, whereas

$$x_i\left(1 + \frac{\partial b_3}{\partial x_i}(x_1, x_2)\right) - b_3(x_1, x_2) = x_i - b_3(x_1, x_2) + x_i\frac{\partial b_3}{\partial x_i}(x_1, x_2) < 0.$$

Since $P'(Q) < 0$, it follows that $\mathcal{H}_i(x_1, x_2) > 0$. Part (i) therefore yields $x_i^* > x_3^*$, proving part (iii).

Similarly, if $\partial b_3/\partial x_i > 0$ and $x_i \geq b_3(x_1, x_2)$, then $c'(x_i) - c'(b_3(x_1, x_2)) \geq 0$ and

$$x_i\left(1 + \frac{\partial b_3}{\partial x_i}(x_1, x_2)\right) - b_3(x_1, x_2) = x_i - b_3(x_1, x_2) + x_i\frac{\partial b_3}{\partial x_i}(x_1, x_2) > 0.$$

Consequently, $\mathcal{H}_i(x_1, x_2) < 0$, and part (ii) gives $x_i^* < x_3^*$. This proves part (iv). □

The proposition can be applied without solving the induced maximization problem exactly. It is sufficient to construct a verified enclosure U and establish the required sign condition throughout the relevant part of that enclosure. The resulting comparison is valid for every interior anticipatory post-cartel equilibrium contained in U, even when the equilibrium is not unique.

5. Illustrative Examples and Equilibrium Configurations

The examples below illustrate the simultaneous post-cartel Cournot model and the anticipatory extension separately. In the simultaneous model, quantity differences arise from technological heterogeneity, whereas the anticipatory model introduces the additional strategic effect generated by the response of Firm 3.

The anticipatory examples show how technological and strategic effects may reinforce, offset, or reverse one another. Each example therefore identifies explicitly the equilibrium concept being used, since an equilibrium of the simultaneous model need not be an equilibrium of the anticipatory extension, even when the market primitives are identical.

5.1. Reinforcing Technological and Strategic Effects

The following example concerns the anticipatory single-valued extension introduced in Section 4. It illustrates a case in which technological and strategic asymmetry work in the same direction. Firms 1 and 2 have lower marginal costs than Firm 3, while an increase in the quantity of either of them induces Firm 3 to reduce its optimal production. Both effects contribute to larger equilibrium quantities for Firms 1 and 2.

Example 4. *Let* $X_1 = X_2 = [0, 35]$ *and* $X_3 = [0, 25]$. *Consider the inverse demand function* $P(Q) = 100 - Q$ *and the cost functions* $c_1(x) = c_2(x) = 0.25x^2$ *and* $c_3(x) = x^2$.

The inverse demand function is positive and strictly decreasing on the relevant aggregate-output domain, with $P'(Q) = -1$. The cost functions are increasing and strictly convex, with $c_1'(x) = c_2'(x) = 0.5x$ and $c_3'(x) = 2x$. Thus, at a common positive quantity, Firm 3 has a higher marginal cost than Firms 1 and 2.

The numerical coefficients define a normalized illustrative market with linear inverse demand and quadratic costs. They are not obtained from an empirical calibration. The larger quadratic-cost coefficient of Firm 3 is chosen to represent a technological disadvantage, while the linear inverse demand gives an explicitly computable decreasing response.

This model belongs to the linear-quadratic class covered by Lemma 2.

For fixed $(x_1, x_2) \in X_1 \times X_2$, Firm 3 maximizes $x_3(100 - x_1 - x_2 - x_3) - x_3^2$. Since this objective function is strictly concave in x_3, its unique maximizer is $b_3(x_1, x_2) = (100 - x_1 - x_2)/4$. The prescribed strategy intervals imply $7.5 \leq b_3(x_1, x_2) \leq 25$, so the response is interior, continuous, and single-valued throughout $X_1 \times X_2$.

Moreover, $\partial b_3/\partial x_1 = \partial b_3/\partial x_2 = -1/4 < 0$. Thus, an expansion by either Firm 1 or Firm 2 induces Firm 3 to contract its production.

The induced payoff functions of Firms 1 and 2 are $\Psi_i(x_1, x_2) = \Pi_i(x_1, x_2, b_3(x_1, x_2))$, $i = 1, 2$. Direct substitution gives $\Psi_1(x_1, x_2) = 75x_1 - x_1^2 - \frac{3}{4}x_1x_2$ and $\Psi_2(x_1, x_2) = 75x_2 - x_2^2 - \frac{3}{4}x_1x_2$. Each induced payoff is continuous and strictly concave with respect to the corresponding own quantity. Therefore, the hypotheses of Theorem 6 are satisfied, and the anticipatory game possesses an equilibrium.

Strict concavity implies that the best responses of the two leaders are

$$B_1(x_2) = \mathrm{proj}_{[0,35]}\left(\frac{75 - \frac{3}{4}x_2}{2}\right), \qquad B_2(x_1) = \mathrm{proj}_{[0,35]}\left(\frac{75 - \frac{3}{4}x_1}{2}\right),$$

where proj_I denotes the metric projection onto the interval I. Since projection onto a closed interval is non-expansive, the joint best-response mapping $B(x_1, x_2) = (B_1(x_2), B_2(x_1))$ is a contraction in the maximum norm with contraction constant $3/8$. Hence, it has a unique fixed point. Equivalently, the coefficient matrix of the interior first-order system is strictly diagonally dominant.

The unique fixed point is interior and therefore satisfies

$$75 - 2x_1 - \frac{3}{4}x_2 = 0, \qquad 75 - \frac{3}{4}x_1 - 2x_2 = 0.$$

The unique solution is $x_1^* = x_2^* = 300/11$. The corresponding response of Firm 3 is $x_3^* = b_3(x_1^*, x_2^*) = 125/11$.

Hence, the unique anticipatory equilibrium is $(x_1^*, x_2^*, x_3^*) = (300/11, 300/11, 125/11)$, and $x_1^* = x_2^* > x_3^*$.

At the equilibrium, $c_1'(x_1^*) = c_2'(x_2^*) = 150/11$, whereas $c_3'(x_3^*) = 250/11$. Therefore, $c_i'(x_i^*) < c_3'(x_3^*)$ for $i = 1, 2$. Furthermore,

$$x_i^*\left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^* = \frac{300}{11}\left(1 - \frac{1}{4}\right) - \frac{125}{11} = \frac{100}{11} > 0.$$

Since $P'(Q^*) = -1$, Proposition 5 gives

$$P'(Q^*)\left[x_i^*\left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^*\right] = -\frac{100}{11} = c_i'(x_i^*) - c_3'(x_3^*).$$

Thus, the technological and strategic effects reinforce one another. Firms 1 and 2 have lower marginal costs, and their production expansions induce a contraction in the output of Firm 3. The resulting equilibrium advantage cannot be attributed exclusively to strategic aggressiveness, because the technologies are not identical. Instead, the example illustrates the combined effect described by Proposition 5.

5.2. A Technological Advantage Dominating an Opposing Strategic Effect

The following example concerns the anticipatory extension. It uses the same inverse demand and cost functions as Example 3, but the equilibrium concept is different. In Example 3, all three firms choose their quantities simultaneously and treat the quantities of the other firms as given. In the present example, Firms 1 and 2 anticipate the single-valued response of Firm 3.

This distinction leads to different equilibrium quantities even though the market primitives are identical. The example therefore directly illustrates why the simultaneous Cournot model and the anticipatory extension must be treated separately.

Example 5. *Let* $X_1 = X_2 = X_3 = [0, 2]$, *let* $P(Q) = 1/(1 + Q)^5$, *and consider the cost functions* $c_1(x) = c_2(x) = 0.0004201x$ *and* $c_3(x) = 0.0017854x + \frac{0.001}{2}x^2$.

As explained in Example 3, the numerical coefficients are chosen to provide a normalized illustrative market with an interior asymmetric equilibrium. They are not obtained from an empirical calibration. Firms 1 and 2 have constant and identical marginal costs, whereas Firm 3 has a higher and increasing marginal cost.

The inverse demand function is positive and strictly decreasing. Moreover, it is $(-1/3)$-concave because $P(Q)^{-1/3} = (1 + Q)^{5/3}$ is convex.

We first verify that the response of Firm 3 is uniquely determined. For fixed $(x_1, x_2) \in [0, 2]^2$, put $A = 1 + x_1 + x_2$. At an interior maximizer x_3, the first-order condition of Firm 3 is

$$F(A, x_3) = \frac{A - 4x_3}{(A + x_3)^6} - 0.0017854 - 0.001x_3 = 0.$$

The function $h_A(x_3) = (A - 4x_3)/(A + x_3)^6$ satisfies $h_A'(x_3) = (20x_3 - 10A)/(A + x_3)^7$. Hence, h_A is strictly decreasing on the interval on which it is positive. Since $0.0017854 + 0.001x_3 > 0$, the equation $F(A, x_3) = 0$ has at most one non-negative solution. Taking account of the boundary case $x_3 = 0$, the maximization problem of Firm 3 therefore has a unique solution for every $(x_1, x_2) \in [0, 2]^2$. We denote it by $b_3(x_1, x_2)$.

The simultaneous post-cartel Cournot–Nash equilibrium for these market primitives is $(x_1^N, x_2^N, x_3^N) \approx (0.445044, 0.445044, 0.394932)$, whereas an anticipatory post-cartel equilibrium will be verified below at $(x_1^A, x_2^A, x_3^A) \approx (0.399996904, 0.399996904, 0.390000416)$.

These profiles concern two different games and therefore do not contradict one another.

We now consider the anticipatory model. Fix the quantity y of one of the two leaders and define the induced payoff of the other leader by

$$\psi_y(x) = \frac{x}{\big(1 + x + y + b_3(x,y)\big)^5} - 0.0004201x, \qquad x \in [0,2].$$

Because the response of Firm 3 is uniquely determined, the implicit function theorem gives

$$\frac{\partial b_3}{\partial x}(x,y) = -\frac{P'(Q) + b_3(x,y)P''(Q)}{2P'(Q) + b_3(x,y)P''(Q) - 0.001}, \qquad Q = x + y + b_3(x,y),$$

whenever the response is interior. Consequently,

$$\psi_y'(x) = P(Q) + xP'(Q)\left(1 + \frac{\partial b_3}{\partial x}(x,y)\right) - 0.0004201.$$

A symmetric solution of the two induced first-order equations was enclosed by interval Newton iteration in $x_1^A, x_2^A \in [0.3999968, 0.3999971]$ and $x_3^A \in [0.3900002, 0.3900007]$.

In particular, $(x_1^A, x_2^A, x_3^A) \approx (0.399996904, 0.399996904, 0.390000416)$.

The interval computation isolates the reported symmetric stationary solution within the displayed box. It is not used to establish global uniqueness of the anticipatory equilibrium. In particular, asymmetric anticipatory equilibria elsewhere in the strategy space are not excluded.

It remains to verify that the stationary quantities of Firms 1 and 2 are global maximizers of their induced payoffs. By symmetry, it is sufficient to consider ψ_y for $y \in [0.3999968, 0.3999971]$. Outward-rounded interval evaluation of ψ_y', using subintervals of width at most 10^{-4}, gives the following bounds:

Range of x	Verified conclusion
$[0, 0.39]$	$\psi_y'(x) \geq 3.4 \times 10^{-4}$
$[0.39, 0.41]$	$-3.7 \times 10^{-2} \leq \psi_y''(x) \leq -3.0 \times 10^{-2}$
$[0.41, 0.75]$	$\psi_y'(x) \leq -3.2 \times 10^{-4}$
$[0.75, 1.25]$	$\psi_y'(x) \leq -3.0 \times 10^{-3}$
$[1.25, 2]$	$\psi_y'(x) \leq -1.8 \times 10^{-3}$

Thus, ψ_y' is positive on $[0, 0.39]$, strictly decreasing through zero on $[0.39, 0.41]$, and negative on $[0.41, 2]$. Therefore, ψ_y first increases and then decreases on its entire strategy set. Its stationary point in $[0.3999968, 0.3999971]$ is consequently its unique global maximizer.

The same argument applies to Firm 2. Consequently, the reported symmetric profile is an anticipatory post-cartel equilibrium. This conclusion establishes the global optimality of each leader's quantity against the reported quantity of the other leader, but it does not imply global uniqueness of the anticipatory equilibrium.

Its rounded representation is $(x_1^A, x_2^A, x_3^A) \approx (0.4, 0.4, 0.39)$.

The profiles $(0.445044, 0.445044, 0.394932)$ and $(0.399996904, 0.399996904, 0.390000416)$ approximate equilibria of two different games. The first is the unique equilibrium of the simultaneous post-cartel Cournot model, whereas the second is a symmetric equilibrium of the anticipatory extension. Consequently, the uniqueness conclusion in Example 3 is not contradicted.

At this anticipatory equilibrium, the aggregate output is $Q^A = x_1^A + x_2^A + x_3^A \approx 1.189994224$. Therefore, $P(Q^A) \approx 0.0198509$ and $P'(Q^A) \approx -0.0453216$.

The marginal costs at this equilibrium are $c_1'(x_1^A) = c_2'(x_2^A) = 0.0004201$ and $c_3'(x_3^A) = 0.0017854 + 0.001x_3^A \approx 0.0021754$. Consequently, $c_i'(x_i^A) < c_3'(x_3^A)$ for $i = 1, 2$. Thus, Firms 1 and 2 possess a technological advantage over Firm 3.

We next examine the strategic effect. The interior first-order condition of Firm 3 is $P(Q) + x_3P'(Q) - c_3'(x_3) = 0$. Moreover, at the equilibrium,

$$2P'(Q^A) + x_3^A P''(Q^A) - c_3''(x_3^A) \approx -0.0432174 \neq 0.$$

Therefore, the implicit function theorem applies and gives

$$\frac{\partial b_3}{\partial x_i} = -\frac{P'(Q) + x_3P''(Q)}{2P'(Q) + x_3P''(Q) - c_3''(x_3)}, \qquad i = 1, 2.$$

At the anticipatory equilibrium, $\partial b_3/\partial x_1 = \partial b_3/\partial x_2 \approx 0.0718 > 0$. Thus, an increase in the production of either Firm 1 or Firm 2 induces Firm 3 to increase its production. This response creates a strategic effect that works against the production advantage of Firms 1 and 2.

Nevertheless, $x_i^A(1 + \partial b_3/\partial x_i) - x_3^A \approx 0.0387 > 0$, $i = 1, 2$. Furthermore,

$$P'(Q^A)\left[x_i^A\left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^A\right] \approx -0.0017553 = c_i'(x_i^A) - c_3'(x_3^A),$$

in agreement with Proposition 5.

At the reported anticipatory equilibrium, $x_1^A = x_2^A \approx 0.4 > x_3^A \approx 0.39$. Therefore, the technological advantage of Firms 1 and 2 dominates the opposing strategic effect generated by the increasing response of Firm 3. The example illustrates that technological and strategic forces need not work in the same direction and that the observed ordering of equilibrium quantities depends on their relative strengths.

5.3. Strategic Asymmetry with a Decreasing Response of Firm 3

Corollary 6 shows that, under identical convex production costs, a decreasing response of Firm 3 gives Firms 1 and 2 a strategic production advantage in the anticipatory model. The following example illustrates this result. Since all three firms have the same technology, the resulting asymmetry is purely strategic rather than technological.

Example 6. *Let* $X_1 = X_2 = [0, 40]$ *and* $X_3 = [0, 45]$. *Consider the inverse demand function* $P(Q) = \max\{100 - Q, 0\}$ *and identical cost functions* $c_1(x) = c_2(x) = c_3(x) = 10x$.

The truncated linear inverse demand is continuous and non-negative on $\mathbb{R}_+$. All the equilibrium quantities obtained below lie in the region $Q < 100$, where $P(Q) = 100 - Q$ and $P'(Q) = -1$. The truncation therefore does not alter any of the calculations, but it ensures that the inverse demand function is economically meaningful on its entire domain.

On the relevant positive-price region, this model belongs to the linear-quadratic class covered by Lemma 2.

For fixed $x_1, x_2 \in [0, 40]$, the profit of Firm 3 is $\Pi_3(x_1, x_2, x_3) = x_3(100 - x_1 - x_2 - x_3) - 10x_3$ throughout the relevant positive-price region. This function is strictly concave in x_3, and its unique maximizer is $b_3(x_1, x_2) = 45 - (x_1 + x_2)/2$. Since $x_1 + x_2 \leq 80$, one has $5 \leq b_3(x_1, x_2) \leq 45$, so the response is feasible throughout $X_1 \times X_2$.

Moreover, $\partial b_3/\partial x_1 = \partial b_3/\partial x_2 = -1/2 < 0$. Thus, an increase in the production of either Firm 1 or Firm 2 induces Firm 3 to reduce its optimal production.

The induced payoff of Firm i, $i = 1, 2$, is $\Psi_i(x_i, x_j) = x_i(100 - x_i - x_j - b_3(x_i, x_j)) - 10x_i$, where $j \neq i$. The corresponding leader best responses are

$$B_1(x_2) = \text{proj}_{[0,40]}\left(45 - \frac{x_2}{2}\right), \qquad B_2(x_1) = \text{proj}_{[0,40]}\left(45 - \frac{x_1}{2}\right).$$

Since projection onto a closed interval is non-expansive, the joint best-response mapping $B(x_1, x_2) = (B_1(x_2), B_2(x_1))$ is a contraction in the maximum norm with contraction constant $1/2$. Hence, the anticipatory game has a unique equilibrium. This conclusion is also reflected in the strict diagonal dominance of the coefficient matrix of the interior first-order system. Substitution of b_3 gives $\Psi_i(x_i, x_j) = x_i(45 - \frac{x_i + x_j}{2})$. This function is continuous and strictly concave in x_i. Hence, the hypotheses of Theorem 6 are satisfied.

The unique fixed point is interior and therefore satisfies

$$45 - x_1 - \frac{x_2}{2} = 0, \qquad 45 - x_2 - \frac{x_1}{2} = 0.$$

Its unique solution is $x_1^* = x_2^* = 30$. Consequently, $x_3^* = b_3(x_1^*, x_2^*) = 15$.

Thus, the unique anticipatory equilibrium is $(x_1^*, x_2^*, x_3^*) = (30, 30, 15)$, and $x_1^* = x_2^* > x_3^*$. Since the three cost functions are identical, this ordering cannot be explained by technological differences. It is generated by the decreasing response of Firm 3 and agrees with Corollary 6.

Indeed, $x_i^*(1 + \partial b_3/\partial x_i) = 30(1 - \frac{1}{2}) = 15 = x_3^*$. Since the marginal costs are identical and constant, both sides of the identity in Proposition 5 are zero.

Comparison with the full-cartel and simultaneous Cournot–Nash benchmarks. To distinguish the three market structures, we use the superscripts M, N, and A for the full-cartel, simultaneous Cournot–Nash, and anticipatory post-cartel regimes, respectively.

Under the full-cartel regime, aggregate profit is $\Pi^M(Q) = Q(100 - Q) - 10Q = Q(90 - Q)$. The cartel-optimal aggregate output is $Q^M = 45$, and the corresponding aggregate profit is $\Pi^M = 2025$. Under an equal allocation, the individual quantities and profits are $x_i^M = 15$ and $\Pi_i^M = 675$, respectively, for $i = 1, 2, 3$.

In the simultaneous three-firm Cournot–Nash model, the symmetric equilibrium is $x_1^N = x_2^N = x_3^N = 22.5$, with aggregate output $Q^N = 67.5$. Each firm earns $\Pi_i^N = 506.25$.

In the anticipatory post-cartel equilibrium, $(x_1^A, x_2^A, x_3^A) = (30, 30, 15)$, and hence $Q^A = 75$. Firms 1 and 2 earn $\Pi_1^A = \Pi_2^A = 30(25 - 10) = 450$, whereas Firm 3 earns $\Pi_3^A = 15(25 - 10) = 225$. The comparison is summarized in Table 1.

Table 1. Comparison of the three market structures.

Market Structure	Quantities	Firm 1 Profit	Firm 2 Profit	Firm 3 Profit
Full cartel	$(15, 15, 15)$	675	675	675
Simultaneous Cournot	$(22.5, 22.5, 22.5)$	506.25	506.25	506.25
Anticipatory post-cartel	$(30, 30, 15)$	450	450	225

Firm 3 therefore earns less in the anticipatory post-cartel equilibrium than under either the equal-share cartel allocation or simultaneous Cournot competition. Accordingly, this example does not describe an endogenously profitable deviation by Firm 3. The cartel breakdown is treated as exogenously given and may result, for example, from regulatory intervention, contractual failure, or other external circumstances. The term departing firm identifies the institutional origin of Firm 3 and does not imply that leaving the cartel is profitable.

For completeness, consumer surplus and total welfare can also be compared. With the linear inverse demand $P(Q) = 100 - Q$, consumer surplus is $CS(Q) = Q^2/2$. Under the

full-cartel regime, $CS^M = 1012.5$ and $W^M = 3037.5$. Under simultaneous Cournot–Nash competition, $CS^N = 2278.125$ and $W^N = 3796.875$. Under the anticipatory post-cartel equilibrium, $CS^A = 2812.5$ and $W^A = 3937.5$.

Thus, the anticipatory structure produces the largest aggregate output and consumer surplus among the three regimes, but it substantially reduces the profit of the departing firm. The example illustrates that a strategic production advantage for Firms 1 and 2 does not imply that cartel departure is profitable for Firm 3.

If the equilibrium could not be computed explicitly, the same ordering would follow from parts (iii) and (iv) of Proposition 6. Indeed, $\partial b_3/\partial x_i = -1/2 < 0$ throughout the entire strategy region, so every interior anticipatory equilibrium in any verified enclosing region would satisfy $x_i^* > x_3^*$, $i = 1, 2$.

5.4. Local Comparison When the Equilibrium Is Not Explicitly Known

Proposition 6 permits equilibrium quantities to be compared whenever a verified compact region $U \subseteq X_1 \times X_2$ containing an equilibrium is available. The following consequence provides sufficient conditions stated directly in terms of the marginal costs and the response of Firm 3.

Corollary 7. *Let $U \subseteq X_1 \times X_2$ be a compact region containing an interior anticipatory post-cartel equilibrium (x_1^*, x_2^*), and set $x_3^* = b_3(x_1^*, x_2^*)$. Assume that $P'(Q) < 0$ on the corresponding aggregate-output region and that the differentiability assumptions of Proposition 5 are satisfied. For a fixed $i \in \{1, 2\}$, the following statements hold:*

1. *If $c_i'(x_i) < c_3'(b_3(x_1, x_2))$ and $\frac{\partial b_3}{\partial x_i}(x_1, x_2) \leq 0$ for every $(x_1, x_2) \in U$, then $x_i^* > x_3^*$.*
2. *If $c_i'(x_i) > c_3'(b_3(x_1, x_2))$ and $\frac{\partial b_3}{\partial x_i}(x_1, x_2) \geq 0$ for every $(x_1, x_2) \in U$, then $x_i^* < x_3^*$.*

Proof. Under the assumptions of part (i), the inequalities hold, in particular, at (x_1^*, x_2^*). By Proposition 5, $c_i'(x_i^*) < c_3'(x_3^*)$ implies

$$x_i^* \left(1 + \frac{\partial b_3}{\partial x_i}(x_1^*, x_2^*)\right) > x_3^*.$$

Since $\partial b_3/\partial x_i(x_1^*, x_2^*) \leq 0$, the expression on the left does not exceed x_i^*. Therefore, $x_i^* > x_3^*$.

Similarly, under the assumptions of part (ii), Proposition 5 gives

$$x_i^* \left(1 + \frac{\partial b_3}{\partial x_i}(x_1^*, x_2^*)\right) < x_3^*.$$

Because $\partial b_3/\partial x_i(x_1^*, x_2^*) \geq 0$, the expression on the left is at least x_i^*, and hence $x_i^* < x_3^*$. □

The result does not require the exact equilibrium quantities to be known. For example, suppose that $U = I_1 \times I_2$, where I_1 and I_2 are verified intervals containing the equilibrium quantities of Firms 1 and 2. It is sufficient to obtain interval bounds for $b_3(U)$, $c_i'(I_i)$, $c_3'(b_3(U))$, and $\partial b_3/\partial x_i(U)$. If these bounds verify one of the two sets of inequalities in Corollary 7, the corresponding ordering of x_i^* and x_3^* follows without an explicit solution of the equilibrium problem.

When the technological and strategic effects have opposite signs, neither part of Corollary 7 may apply. In that case, the more general residual $\mathcal{H}_i$ from Proposition 6 can be evaluated throughout the relevant subset of U. If its interval enclosure excludes zero, one of the two possible quantity orderings is ruled out. The conclusion then applies to every interior anticipatory post-cartel equilibrium contained in U, including when the equilibrium is not unique.

5.5. Illustration of the Combined Effect of Technology and Strategy

The preceding examples considered cases in which the technological and strategic effects either reinforced one another or in which one of the two effects was absent. Proposition 5 shows, however, that the two mechanisms may also work in opposite directions. The following example illustrates a decreasing response of Firm 3, which favors larger quantities by Firms 1 and 2, combined with a substantial technological disadvantage of those firms. The technological effect proves to be stronger and reverses the ordering suggested by the strategic effect alone.

Example 7. *Let* $X_1 = X_2 = [0, 20]$ *and* $X_3 = [0, 100/3]$. *Consider the inverse demand function* $P(Q) = \max\{100 - Q, 0\}$ *and the cost functions* $c_1(x) = c_2(x) = 5x^2$ *and* $c_3(x) = \frac{1}{2}x^2$.

The inverse demand is continuous and non-negative on $\mathbb{R}_+$. All admissible aggregate quantities and all equilibrium quantities considered below lie in the region $Q < 100$, where $P(Q) = 100 - Q$ and $P'(Q) = -1$.

The cost functions are increasing and strictly convex. Firms 1 and 2 have marginal costs $c_1'(x) = c_2'(x) = 10x$, whereas Firm 3 has marginal cost $c_3'(x) = x$. Thus, Firms 1 and 2 face a substantial technological disadvantage.

The numerical coefficients define a normalized illustrative market. They are chosen to make the technological and strategic effects operate in opposite directions while preserving explicit solutions. They are not obtained from an empirical calibration.

On the relevant positive-price region, this model belongs to the linear-quadratic class covered by Lemma 2.

For fixed quantities x_1 and x_2, the profit of Firm 3 is $\Pi_3(x_1, x_2, x_3) = x_3(100 - x_1 - x_2 - x_3) - \frac{1}{2}x_3^2$. This function is strictly concave in x_3, and its unique maximizer is $b_3(x_1, x_2) = (100 - x_1 - x_2)/3$.

For $(x_1, x_2) \in [0, 20]^2$, one has $20 \leq b_3(x_1, x_2) \leq 100/3$. Hence, the response is feasible, interior, continuous, and single-valued throughout the leaders' strategy space. Moreover, $\partial b_3/\partial x_1 = \partial b_3/\partial x_2 = -1/3 < 0$. Thus, an expansion by either Firm 1 or Firm 2 induces Firm 3 to reduce its production.

For $i, j \in \{1, 2\}$, $i \neq j$, the induced payoff of Firm i is $\Psi_i(x_i, x_j) = x_i P(x_i + x_j + b_3(x_i, x_j)) - 5x_i^2$. Since $P(x_i + x_j + b_3(x_i, x_j)) = \frac{200}{3} - \frac{2}{3}(x_i + x_j)$, we obtain

$$\Psi_i(x_i, x_j) = x_i\left(\frac{200}{3} - \frac{2}{3}(x_i + x_j)\right) - 5x_i^2.$$

The strict concavity of the induced payoffs gives the leader best-response functions

$$B_1(x_2) = \mathrm{proj}_{[0,20]}\left(\frac{200 - 2x_2}{34}\right), \qquad B_2(x_1) = \mathrm{proj}_{[0,20]}\left(\frac{200 - 2x_1}{34}\right).$$

Therefore, the joint best-response mapping $B(x_1, x_2) = (B_1(x_2), B_2(x_1))$ is a contraction in the maximum norm with contraction constant $1/17$. It consequently has a unique fixed point. Equivalently, the coefficient matrix of the interior first-order system is strictly diagonally dominant.

The unique fixed point is interior and satisfies $200 - 34x_1 - 2x_2 = 0$ and $200 - 34x_2 - 2x_1 = 0$. Hence, $x_1^* = x_2^* = 50/9$. The induced payoff is continuous and strictly concave with respect to x_i. Therefore, the hypotheses of Theorem 6 are satisfied, and the anticipatory game possesses an equilibrium.

The interior optimality conditions for Firms 1 and 2 are $200 - 34x_1 - 2x_2 = 0$ and $200 - 34x_2 - 2x_1 = 0$. Their unique solution is $x_1^* = x_2^* = 50/9$. The corresponding quantity of Firm 3 is $x_3^* = b_3(x_1^*, x_2^*) = 800/27$.

Consequently, $x_1^* = x_2^* = 50/9 < 800/27 = x_3^*$.

At the equilibrium, $c_1'(x_1^*) = c_2'(x_2^*) = 500/9$, whereas $c_3'(x_3^*) = 800/27$. Thus, $c_i'(x_i^*) > c_3'(x_3^*)$ for $i = 1, 2$.

On the other hand,

$$x_i^*\left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^* = \frac{50}{9}\left(1 - \frac{1}{3}\right) - \frac{800}{27} = -\frac{700}{27}.$$

Since $P'(Q^*) = -1$, it follows that

$$P'(Q^*)\left[x_i^*\left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^*\right] = \frac{700}{27}.$$

Moreover, $c_i'(x_i^*) - c_3'(x_3^*) = 500/9 - 800/27 = 700/27$, in exact agreement with Proposition 5.

The decreasing response of Firm 3 constitutes a strategic advantage for Firms 1 and 2. If the technologies were identical, this effect would favor the ordering $x_i^* > x_3^*$. In the present example, however, Firms 1 and 2 have substantially higher marginal costs. Their technological disadvantage dominates the favorable strategic effect, and the actual equilibrium ordering is $x_1^* = x_2^* < x_3^*$.

This example demonstrates that the sign of $\partial b_3/\partial x_i$ alone does not determine the ordering when the firms have different technologies. Similarly, marginal costs alone do not describe the strategic mechanism. The equilibrium allocation is determined by the balance between the two effects, as expressed by Proposition 5.

5.6. Strategic Asymmetry with an Increasing Response of Firm 3

The preceding examples considered a decreasing response of Firm 3. We now examine the complementary strategic regime in which an increase in the quantity of either Firm 1 or Firm 2 induces Firm 3 to increase its own optimal production. Under identical convex technologies, Corollary 6 predicts that Firms 1 and 2 produce less than Firm 3 at an interior anticipatory equilibrium.

This example is also important because it demonstrates that an increasing response of Firm 3 is attainable under positive and strictly decreasing inverse demand, convex costs, and the generalized concavity conditions used in the paper.

Example 8. *Let* $X_1 = X_2 = X_3 = [0, 2]$, *and let* $P(Q) = 1/(1 + Q)^{3.1084}$. *Assume that the three firms have the same cost function* $c_i(x) = 0.0029845x + \frac{0.0001086}{2}x^2$, $i = 1, 2, 3$.

The numerical coefficients are chosen to construct an interior equilibrium at which the response of Firm 3 is locally increasing. They are not obtained from an empirical calibration and are not intended to represent a particular industry. Their purpose is to demonstrate that the positive response regime identified in Corollary 6 is non-vacuous.

The inverse demand function is positive, continuously differentiable, and strictly decreasing on $\mathbb{R}_+$. Moreover, $P(Q)^{-1/3} = (1 + Q)^{3.1084/3}$ is convex because $3.1084/3 > 1$. Hence, P is $(-1/3)$-concave. The common cost function is increasing and strictly convex.

For fixed $x_1, x_2 \in [0, 2]$, the profit of Firm 3 is $\Pi_3(x_1, x_2, x_3) = x_3 P(x_1 + x_2 + x_3) - c_3(x_3)$.

For notational convenience, denote by a the exponent 3.1084 appearing in the inverse demand function $P(Q)$; thus, $a = 3.1084$. Let $A = 1 + x_1 + x_2 \in [1, 5]$. The marginal profit of Firm 3 is

$$G_A(x_3) = \frac{A - (a-1)x_3}{(A + x_3)^{a+1}} - 0.0029845 - 0.0001086x_3,$$

and

$$G_A'(x_3) = \frac{a\big((a-1)x_3 - 2A\big)}{(A + x_3)^{a+2}} - 0.0001086.$$

Whenever $G_A(x_3) \geq 0$, one necessarily has $A - (a-1)x_3 > 0$, and hence $G_A'(x_3) < 0$. Therefore, G_A is strictly decreasing as long as it is non-negative and can cross zero at most once. Moreover, $G_A(0) \geq 5^{-a} - 0.0029845 > 0$.

At $x_3 = 2$, the first fraction in $G_A(2)$ is nonpositive if $A \leq 2(a-1)$. If $A > 2(a-1)$, it is bounded above by

$$\frac{5 - 2(a-1)}{\big(2(a-1)+2\big)^{a+1}} < 0.000431 < 0.0032017.$$

Thus, $G_A(2) < 0$ for every $A \in [1,5]$. Consequently, G_A has exactly one zero in $(0,2)$. The profit of Firm 3 is strictly increasing before this zero and strictly decreasing after it, so the individual optimization problem has a unique global maximizer.

Denote this maximizer by $b_3(x_1, x_2)$. It is characterized by

$$P(Q) + x_3 P'(Q) - c_3'(x_3) = 0,$$

where $Q = x_1 + x_2 + x_3$.

The nondegeneracy expression associated with this condition is

$$2P'(Q) + x_3 P''(Q) - c_3''(x_3). \tag{5}$$

Whenever (5) is nonzero, the implicit function theorem gives

$$\frac{\partial b_3}{\partial x_i} = -\frac{P'(Q) + x_3 P''(Q)}{2P'(Q) + x_3 P''(Q) - c_3''(x_3)}, \qquad i = 1,2.$$

Firms 1 and 2 maximize the induced payoffs $\Psi_i(x_1, x_2) = \Pi_i(x_1, x_2, b_3(x_1, x_2))$, $i = 1,2$. We next verify that the quantities reported below are global, rather than merely local, maximizers of these induced payoffs.

Fix the quantity y of one leader and define

$$\psi_y(x) = xP\big(x + y + b_3(x,y)\big) - c(x), \qquad x \in [0,2].$$

Let

$$F(x,y,z) = P(x+y+z) + zP'(x+y+z) - c'(z).$$

The interior response $z = b_3(x,y)$ satisfies the equation $F(x,y,b_3(x,y)) = 0$. Since $F_z = 2P'(Q) + zP''(Q) - c''(z)$, implicit differentiation gives

$$\frac{\partial b_3}{\partial x} = -\frac{P'(Q) + zP''(Q)}{2P'(Q) + zP''(Q) - c''(z)}.$$

Differentiating the implicit equation once more gives

$$\frac{\partial^2 b_3}{\partial x^2} = -\frac{F_{xx} + 2F_{xz}\frac{\partial b_3}{\partial x} + F_{zz}\left(\frac{\partial b_3}{\partial x}\right)^2}{F_z},$$

where

$$F_{xx} = P''(Q) + zP'''(Q), \qquad F_{xz} = 2P''(Q) + zP'''(Q),$$

and

$$F_{zz} = 3P''(Q) + zP'''(Q) - c'''(z).$$

In the present example $c'''(z) = 0$.

The first and second derivatives of the induced payoff are therefore

$$\psi_y'(x) = P(Q) + xP'(Q)(1 + b_x) - c'(x)$$

and

$$\psi_y''(x) = 2P'(Q)(1+b_x) + x\Big[P''(Q)(1+b_x)^2 + P'(Q)b_{xx}\Big] - c''(x),$$

where $Q = x + y + b_3(x,y)$.

Interval Newton iteration applied to the two symmetric induced first-order equations and the first-order condition of Firm 3 gives

$$x_1^*, x_2^* \in [0.9394729, 0.9394734], \qquad x_3^* \in [0.9864461, 0.9864467].$$

Thus,

$$x_1^* = x_2^* \approx 0.939473147, \qquad x_3^* \approx 0.986446371.$$

It remains to verify global optimality. By symmetry, it is sufficient to consider ψ_y for $y \in [0.9394729, 0.9394734]$. Outward-rounded interval arithmetic on subintervals of $[0,2]$ of width at most 10^{-3}, enclosing the follower response and its first two derivatives through $F = 0$, gives

Range of x	Verified enclosure of $\psi_y''(x)$
$[0, 0.5]$	$[-0.126, -0.0269]$
$[0.5, 1.0]$	$[-0.0307, -0.00712]$
$[1.0, 1.5]$	$[-0.00883, -0.00189]$
$[1.5, 2]$	$[-0.00293, -0.000389]$

Thus $\psi_y''(x) < 0$ throughout $[0,2]$, uniformly for the stated range of y. In addition, $\psi_y'(0) \in [0.0393, 0.0396]$ and $\psi_y'(2) \in [-0.00363, -0.00348]$. Hence ψ_y' is strictly decreasing and has exactly one zero in $(0,2)$, which is the unique global maximizer of ψ_y.

The same argument applies to both leaders. Consequently, the computed profile is an anticipatory post-cartel equilibrium, and

$$(x_1^*, x_2^*, x_3^*) \approx (0.939473147, 0.939473147, 0.986446371).$$

The corresponding aggregate output is $Q^* \approx 2.865392665$.

We now apply the enclosing-region criterion without using the pointwise comparison of the approximate equilibrium quantities. Let

$$U = [0.9394729, 0.9394734] \times [0.9394729, 0.9394734].$$

Outward-rounded interval evaluation of the follower equation on U gives

$$b_3(U) \subseteq [0.9864461, 0.9864467].$$

For $(x_1, x_2) \in U$, set $x_3 = b_3(x_1, x_2)$ and $Q = x_1 + x_2 + x_3$. Outward-rounded interval evaluation gives

$$P'(Q) + x_3 P''(Q) \in [0.0005827, 0.0005829]$$

and

$$2P'(Q) + x_3 P''(Q) - c_3''(x_3) \in [-0.011552, -0.011551].$$

In particular, the denominator in the implicit-derivative formula does not vanish anywhere on the enclosing region. Therefore, b_3 is continuously differentiable on U, and

$$\frac{\partial b_3}{\partial x_i}(x_1, x_2) = -\frac{P'(Q) + x_3 P''(Q)}{2P'(Q) + x_3 P''(Q) - c_3''(x_3)} \in [0.05044, 0.05047], \qquad i = 1, 2,$$

throughout U.

Thus,

$$\frac{\partial b_3}{\partial x_i}(x_1, x_2) > 0, \qquad (x_1, x_2) \in U, \quad i = 1, 2.$$

Since all three firms have the same convex cost function, part (iv) of Proposition 6 applies. Consequently, every interior anticipatory post-cartel equilibrium contained in U satisfies

$$x_1^* < x_3^*, \qquad x_2^* < x_3^*.$$

This ordering is therefore obtained from a uniform sign verification on the enclosing region and does not rely on a pointwise comparison of the numerically approximated equilibrium quantities.

For confirmation, at the reported equilibrium,

$$\frac{\partial b_3}{\partial x_1}(x_1^*, x_2^*) = \frac{\partial b_3}{\partial x_2}(x_1^*, x_2^*) \approx 0.050451 > 0,$$

and

$$x_1^* = x_2^* \approx 0.939473 < 0.986446 \approx x_3^*.$$

Hence, the ordering predicted by Proposition 6 agrees with the computed equilibrium profile. Because the technologies are identical, the quantity asymmetry is generated entirely by the strategic effect.

The comparison identity can also be verified numerically. At the equilibrium,

$$x_i^* \left(1 + \frac{\partial b_3}{\partial x_i}\right) - x_3^* \approx 0.0004242, \qquad i = 1, 2.$$

Since $P'(Q^*) \approx -0.0120257$, the left-hand side of the identity in Proposition 5 is approximately -5.1013×10^{-6}. On the other hand,

$$c_i'(x_i^*) - c_3'(x_3^*) = 0.0001086(x_i^* - x_3^*) \approx -5.1013 \times 10^{-6}.$$

Thus, the two sides agree to the reported numerical precision.

The example confirms that the increasing-response branch of Corollary 6 is economically and mathematically attainable. More importantly, it illustrates how Proposition 6 can be used to determine the ordering of equilibrium quantities from a verified enclosing region when no closed-form equilibrium is available.

6. Discussion

The analysis distinguishes two market structures that may follow the exogenous breakdown of an initially complete cartel. In the baseline post-cartel Cournot model, all three firms choose quantities simultaneously and independently. Firm 3 has no follower status, and its identity as the former cartel defector does not create strategic asymmetry. The set-valued formulation retains all profit-maximizing quantities and separates three logically distinct questions: existence of equilibrium, single-valuedness of individual best responses, and uniqueness of equilibrium. The existence argument is an application of the classical Debreu–Glicksberg–Fan framework rather than a new fixed-point principle.

The anticipatory extension is a different equilibrium model. Firms 1 and 2 choose their quantities simultaneously while incorporating the single-valued response $b_3(x_1, x_2)$ of Firm 3 into their induced payoffs. The resulting two-leader–one-follower structure requires a separate existence argument because quasi-concavity of the original profit functions does not automatically carry over to the induced leader payoffs. If the follower response were

genuinely set-valued, an additional optimistic, pessimistic, or other selection rule would be required.

Proposition 5 separates the technological effect, represented by differences in marginal costs, from the strategic effect generated by $\partial b_3/\partial x_i$. These effects may reinforce, offset, or reverse one another. Moreover, Proposition 6 shows that their implications for equilibrium quantity ordering can be established uniformly on a verified enclosing region even when no closed-form equilibrium is available, as demonstrated in the final numerical example.

The comparison among the full-cartel, simultaneous Cournot–Nash, and anticipatory regimes also clarifies the interpretation of cartel departure. In the linear example, Firm 3 earns less in the anticipatory equilibrium than under either the equal-share cartel allocation or simultaneous Cournot competition. Hence, the model does not imply that departure is endogenously profitable. The breakdown of the cartel is taken as exogenously given and may result from regulatory intervention, contractual failure, changes in market conditions, or other external circumstances. The term *failed cartel* identifies the institutional origin of the subsequent market rather than asserting that unilateral departure must benefit the departing firm.

The same example illustrates that profit and welfare comparisons need not move in the same direction. The anticipatory regime generates the largest aggregate output, consumer surplus, and total welfare among the three regimes considered there, while substantially reducing the profit of Firm 3. These calculations are illustrative rather than general welfare results. Nevertheless, they show why the distributional effects on individual firms should be distinguished from the effects on consumers and aggregate welfare.

6.1. Empirical and Competition-Policy Implications

The distinction between technological and strategic asymmetry suggests an empirical identification problem closely related to the literature on the estimation of conduct parameters [59–61]. Observed quantity correlations do not by themselves identify strategic responses because firms' quantities are jointly determined. Estimation of $\partial b_3/\partial x_i$ would therefore require exogenous variation, such as firm-specific input-cost shocks, capacity restrictions, plant outages, or regulatory changes, together with institutional evidence supporting the assumed timing of decisions.

Conditional on such identification, the equilibrium identity in Proposition 5 provides a way to distinguish technological from strategic asymmetry. Persistent quantity differences associated with marginal-cost or capacity differences support a technological interpretation, whereas different quantities among firms with similar observable technologies may reflect the anticipated response of Firm 3. In general, both mechanisms may be present and should be assessed jointly.

This distinction is also relevant for competition policy. A comparatively large output or market share does not necessarily reveal a technological advantage; it may instead reflect a favorable strategic position. Likewise, the formal breakdown of a cartel does not establish that the subsequent market is an ordinary symmetric Cournot game. The appropriate counterfactual depends on whether firms act simultaneously or whether some firms systematically anticipate the response of another producer.

These implications remain provisional. The present model does not provide a complete empirical identification strategy, and the numerical examples are not industry calibrations. An empirical application would require market-specific data, valid instruments, independently justified timing, and suitable measures of firms' marginal costs.

6.2. Limitations and Directions for Future Research

Several limitations delimit the scope of the conclusions. The analysis concerns a homogeneous-good triopoly with a single departing firm. The baseline existence result treats the cartel breakdown as exogenously given, while the anticipatory extension requires a continuous single-valued response of Firm 3 and quasi-concavity of the induced leader payoffs. The paper does not model endogenous cartel formation, the profitability or stability of the initial deviation, or optimistic and pessimistic equilibria under a set-valued follower response.

The welfare comparison is derived for a particular linear example and should not be interpreted as a general welfare ranking of the three market regimes. A systematic welfare analysis would require conditions under which the ordering of aggregate output, consumer surplus, firms' profits, and total welfare is preserved across broader classes of inverse demand and cost functions. Similarly, the empirical implications identified above remain theoretical until they are evaluated using market-specific data and a credible identification strategy.

Extensions to larger markets, multiple departing firms, differentiated products, capacity constraints, dynamic or repeated interaction, stochastic demand, and endogenous coalition formation remain natural directions for further research. Another important extension would allow Firm 3 to possess a genuinely set-valued response in the anticipatory model and would compare the equilibria generated by optimistic, pessimistic, or alternative selection rules. These developments would broaden the applicability of the framework while preserving the distinction between simultaneous post-cartel competition and hierarchical strategic interaction established in the present paper.

7. Conclusions

This paper develops an equilibrium analysis of market structures arising after the breakdown of an initially complete cartel. Its central distinction is between a simultaneous post-cartel Cournot model and a separate anticipatory single-valued extension. In the simultaneous model, all three firms act independently and choose quantities at the same time. Their optimal decisions may be non-unique and are therefore represented by set-valued best-response correspondences. Classical results from parametric optimization and fixed-point theory then provide sufficient conditions for the existence of a post-cartel Cournot–Nash equilibrium without requiring differentiability or uniquely determined reaction functions.

When the best responses are singleton-valued, they define continuous reaction functions, but this property alone does not guarantee uniqueness of equilibrium. Additional generalized-concavity or diagonal-strict-concavity conditions are needed. The analysis therefore separates three logically different questions: whether an equilibrium exists, whether each individual optimization problem has a unique solution, and whether the resulting equilibrium is itself unique.

In the anticipatory extension, Firms 1 and 2 maximize their individual induced payoffs while anticipating the uniquely determined response of Firm 3. A separate existence theorem is obtained under continuity and quasi-concavity assumptions on these induced payoffs. Within this model, equilibrium quantity differences may be generated both by technological asymmetry and by the strategic response of Firm 3. The theoretical identities and illustrative examples show that these two effects may reinforce, offset, or reverse one another.

Overall, the results provide a mathematically explicit framework for distinguishing the consequences of cartel breakdown from those of an additional anticipatory leader–follower structure. They also demonstrate the usefulness of set-valued best-response

methods in the simultaneous model and clarify the additional assumptions required when the analysis moves to a single-valued hierarchical setting. The comparison among the full-cartel, simultaneous Cournot–Nash, and anticipatory regimes in the linear benchmark example additionally illustrates how the different market structures may affect firms' profits, consumer surplus, and total welfare. In that example, the anticipatory regime produces the highest aggregate output, consumer surplus, and total welfare, while considerably reducing the profit of the departing firm.

This welfare comparison is illustrative and does not establish a general ranking of the three regimes. Deriving conditions under which similar welfare orderings hold for broader classes of inverse demand and cost functions remains an important direction for future research. The proposed framework may also serve as a basis for studying endogenous cartel instability, competition-policy effects, and more general oligopoly models involving multiple leaders, multiple followers, or non-unique optimal responses.

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