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Extending the Applicability of an Efficient Eighth-Order Method for Solving Equations

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Abstract

The convergence order of higher-order iterative methods for solving systems of nonlinear equations was analyzed using Taylor series expansion, which typically requires the computation of higher-order derivatives not inherently part of the method. This dependency limits the method's applicability and increases the computational cost. The distinctiveness of our work lies in the development of improved convergence theorems that rely solely on first-order derivatives. The proposed approach offers a stronger framework than existing methods by incorporating details about the convergence region's radius and providing precise error estimates. Furthermore, we explore semi-local convergence, which holds greater significance as it allows the identification of the specific domain where the iterative sequence remains valid. The theoretical findings are substantiated through suitable numerical illustrations.

Keywords: nonlinear equations; iterative methods; operator equations; convergence

1. Introduction

In applied science and technology, many complex problems can be reformulated as systems of nonlinear equations of the form

$$F(x) = 0,$$

where $F : Q \subset Q_1 \to Q_2$ is a function that is differentiable in the Fréchet sense. Here, Q_1 and Q_2 denote complete normed linear spaces, and Q represents a non-empty, open, and convex subset.

Closed-form solutions to such nonlinear problems are generally difficult to obtain, making iterative procedures the preferred approach for finding approximate solutions. Among these, Newton's method is one of the most widely used techniques due to its quadratic convergence and efficiency in solving equations of the form above. Over the years, substantial advancements in mathematical sciences have led to the introduction of several higher-order iterative algorithms for nonlinear equations [1–11].

Despite their theoretical appeal, many of these advanced methods face practical limitations due to the requirement of evaluating second or higher-order derivatives. The computational burden of calculating F'' at each iteration often reduces their effectiveness in real-world scenarios. Moreover, several convergence proofs rely on Taylor series ex-

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pansions, which themselves demand derivatives of orders exceeding those used in the iterative scheme.

The study of local and semi-local convergence behavior offers deep insights into the performance of iterative methods, helping to determine convergence regions, establish error bounds, and identify the domain of uniqueness for solutions. Recent investigations [12–19] have focused on analyzing these aspects for efficient iterative algorithms, leading to important findings on convergence radii, accuracy estimates, and broader applicability. Such analyses play a crucial role in guiding the choice of suitable initial approximations and enhancing the overall reliability of numerical methods.

In this paper, we examine an existing class of iterative methods that consist of three consecutive steps. The aim of the study is to establish convergence theorems for these methods, building upon the theoretical framework presented in an earlier work [10].

The method is defined for $x_0 \in \Omega$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ A_n &= F'(x_n)^{-1}(F'(x_n) - F'(y_n)), \\ z_n &= y_n - \left[I + \left(I + \frac{5}{4}A_n\right)A_n\right]F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - \left[I + (\left(I + \frac{3}{2}A_n\right)A_n\right]F'(x_n)^{-1}F(z_n). \end{aligned} \tag{1}$$

The local convergence order eight is shown in [10] under the Taylor expansion series approach for $Q_1 = Q_2 = \mathbb{R}^m$ and provided that $F^{(8)}$ exists which is not on the method. However, the method (1) can still achieve convergence provided that only the derivative F' exists. To clarify this idea, we present a motivating example in which the function F is defined over the interval $Q = [-0.5, 1.5]$ as follows:

$$F(x) = \begin{cases} \frac{1}{3}x^3 \ln(x) + 8x^5 - 8x^4, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0 \end{cases}. \tag{2}$$

It can be seen that the solution is $x^* = 1 \in Q$, and the third derivative is expressed as

$$F'''(x) = \frac{11}{3} - 192x + 480x^2 + 2\ln(x).$$

It is clear that F''' is unbounded over the interval Q. Consequently, based on the results in [10], convergence cannot be assured in all cases. Therefore, the local convergence analysis should rely solely on the information available within the method (1), namely F and F'. Such an analysis serves as the primary motivation for the present study.

The new local convergence analysis presented in this study uses only conditions on the operators on the method, i.e., F and F'. Moreover, the more challenging semi-local analysis of convergence not studied before for the method (1) is provided based on majorizing sequences. The analyses are provided also using generalized continuity conditions used to control the derivative and the results hold for Banach space valued operators. The same approach can be used to extend the applicability of other methods along the same lines [1–11].

References [13–15,17,19,20] also develop convergence analyses within a similar operator-theoretic framework. The eighth-order method in [10] has a distinct internal structure that requires a tailored construction of the associated control sequences and corresponding majorizing conditions. In particular, the present study establishes, for the first time, a complete local and semi-local convergence theory for this specific scheme under generalized continuity assumptions, providing explicit convergence radii and refined a priori error estimates in Banach spaces.

The structure of this paper is outlined as follows: Section 2 presents the local convergence analysis, while Section 3 discusses the semi-local convergence results. Section 4 provides numerical experiments that demonstrate the applicability of the established convergence theorems. Finally, Section 5 offers concluding observations and remarks.

2. Local Convergence

In this section, we assume that a solution $x^* \in Q$ exists for the equation $F(x) = 0$. Let $M = [0, +\infty)$. Additionally, we make the following assumptions:

(C_1) There exists a function $w_0 : M \to M$ which is continuous and non-decreasing so that $w_0(t) - 1 = 0$ admits a smallest solution. Denote such solution by ρ_0 and let $M_0 = [0, \rho_0)$.

(C_2) There exist a continuous and non-decreasing function $w : M_0 \to M$ such that $g_1(t) - 1 = 0$ where

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)\mathrm{d}\theta}{1 - w_0(t)}.$$

Denote such a solution by r_1.

(C_3) The equation $w_0(g_1(t)t) - 1 = 0$ admits a smallest solution in $(0, \rho_0)$. Denote such a solution by ρ_1 and $M_1 = [0, \rho_1)$.

(C_4) The equation $g_2(t) - 1 = 0$ admits a least solution in $M_1 - \{0\}$, denoted as r_2, where $g_2 : M_1 \to M$ is given as

$$g_2(t) = \left[\frac{\int_0^1 w((1-\theta)g_1(t)t)\mathrm{d}\theta}{1 - w_0(g_1(t)t)} + \frac{\bar{w}(t)(1 + \int_0^1 w_0(\theta g_1(t)t)\mathrm{d}\theta)}{(1 - w_0(t))(1 - w_0(g_1(t)t))}\right.$$
$$\left. + \frac{\bar{w}(t)}{(1 - w_0(t))^2}\left(1 + \frac{5}{4}\frac{\bar{w}(t)}{1 - w_0(t)}\right)\left(1 + \int_0^1 w_0(\theta g_1(t)t)\mathrm{d}\theta\right)\right] g_1(t),$$

where,

$$\bar{w}(t) = \begin{cases} w((1 + g_1(t))t) \\ \text{or} \\ w_0(t) + w_0(g_1(t)t). \end{cases}$$

(C_5) The equation $w_0(g_2(t)t) - 1 = 0$ admits a smallest solution in $M_1 - \{0\}$. Denote such a solution by ρ_2 and $M_2 = [0, \rho_2)$.

(C_6) The equation $g_3(t) - 1 = 0$ admits a least solution in $M_2 - \{0\}$, denoted as r_3, where $g_3 : M_2 \to M$ is given as

$$g_3(t) = \left[\frac{\int_0^1 w((1-\theta)g_2(t)t)\mathrm{d}\theta}{1 - w_0(g_2(t)t)} + \frac{\bar{\bar{w}}(t)\left(1 + \int_0^1 w_0(\theta g_2(t)t)\mathrm{d}\theta\right)}{(1 - w_0(t))(1 - w_0(g_2(t)t))}\right.$$
$$\left. + \frac{\bar{w}(t)}{(1 - w_0(t))^2}\left(1 + \frac{3\bar{w}(t)}{2(1 - w_0(t))}\right)\left(1 + \int_0^1 w_0(\theta g_2(t)t)\mathrm{d}\theta\right)\right] g_2(t),$$

where,

$$\bar{\bar{w}}(t) = \begin{cases} w((1 + g_2(t))t) \\ \text{or} \\ w_0(t) + w_0(g_2(t)t). \end{cases}$$

(C_7) There exists a linear operator $\mathscr{B} : Q_1 \to Q_2$ such that $\mathscr{B}^{-1} \in \mathscr{L}(Q_2, Q_1)$, which is the space of linear continuous operators mapping Q_2 into Q_1.

(C_8) $\|\mathscr{B}^{-1}(F'(x) - \mathscr{B})\| \leq w_0(\|x - x^*\|)$ for all $x \in Q$.
Notice that by the assumptions (C_1) and (C_8), $w_0(\|x - x^*\|) < 1$. Thus, $F'(x)^{-1} \in \mathscr{L}(Q_2, Q_1)$ by the standard Banach perturbation Lemma [21,22] involving linear operators.

(C_9) $\|\mathscr{B}^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|)$ for all $x \in S_0 = S(x^*, \rho_0) \cap Q$, and

(C_{10}) $S[x^*, r] \subset Q$, where $r = \min\{r_i\}$, $i = 1, 2, 3$.

Note:

1. In practice the smallest of the two versions of functions $\bar{w}$ and $\bar{\bar{w}}$ are chosen.
2. Throughout this section, it is assumed that the quantities involved in the estimates lie in the interval where the functions are defined, ensuring the validity of all inequalities used.

The calculations requiring these assumptions for $x_0 \in S(x^*, r) - \{x^*\}$ and induction are in turn:

$$
\begin{aligned}
y_n - x^* &= x_n - x^* - F(x_n)^{-1} F(x_n), \\
\|y_n - x^*\| &\leq \frac{\int_0^1 w((1-\theta)\|x_n - x^*\| \mathrm{d}\theta \|x_n - x^*\|}{1 - w_0(\|x_n - x^*\|)} \\
&\leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \\
z_n - x^* &= y_n - x^* - F'(y_n)^{-1} F(y_n) + \left(F'(y_n)^{-1} - F'(x_n)^{-1}\right) F(y_n) \\
&\quad - \left(I + \frac{5}{4} A_n\right) A_n F'(x_n)^{-1} F(y_n), \\
\|z_n - x^*\| &\leq \Bigg[\frac{\int_0^1 w((1-\theta)\|y_n - x^*\|)\mathrm{d}\theta}{1 - w_0(\|y_n - x^*\|)} \\
&\quad + \frac{\bar{w}_n \left(1 + \int_0^1 w_0(\theta\|y_n - x^*\|)\mathrm{d}\theta\right)}{(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|y_n - x^*\|))} \\
&\quad + \frac{\bar{w}_n}{(1 - w_0(\|x_n - x^*\|))^2} \left(1 + \frac{5}{4} \frac{\bar{w}_n}{1 - w_0(\|x_n - x^*\|)}\right) \\
&\quad \times \left(1 + \int_0^1 w_0(\theta\|y_n - x^*\|)\mathrm{d}\theta\right) \Bigg] \|y_n - x^*\| \\
&\leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \\
x_{n+1} - x^* &= z_n - x^* - F'(z_n)^{-1} F(z_n) \\
&\quad + F'(z_n)^{-1}(F'(x_n) - F'(z_n)) F'(x_n)^{-1} F(z_n) \\
&\quad - \left(I + \frac{3}{2} A_n\right) A_n F'(x_n)^{-1} F(z_n), \\
\|x_{n+1} - x^*\| &\leq \Bigg[\frac{\int_0^1 w((1-\theta)\|z_n - x^*\|)\mathrm{d}\theta}{1 - w_0(\|z_n - x^*\|)} \\
&\quad + \frac{\bar{\bar{w}}_n \left(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)\mathrm{d}\theta\right)}{(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|z_n - x^*\|))} \\
&\quad + \frac{\bar{w}_n}{(1 - w_0(\|x_n - x^*\|))^2} \left(1 + \frac{3}{2} \frac{\bar{w}_n}{(1 - w_0(\|x_n - x^*\|))}\right) \\
&\quad \times \left(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)\mathrm{d}\theta\right) \Bigg] \|z_n - x^*\| \\
&\leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|.
\end{aligned}
$$

Hence, we showed by induction:

Theorem 1. *Given the assumptions* (C_1)*–*(C_{10})*, it is established that* $\{x_j\} \subset S(x^*, r)$ *and converges to* x^* *as* j *tends to infinity, provided that the initial value* x_0 *lies in the set* $S(x^*, r) - \{x^*\}$*.*

We now provide a result that establishes the uniqueness of the solution in the context of local convergence.

Proposition 1. *Assume that there exists a solution* $x^{**} \in S(x^*, \rho_3)$ *of the equation* $F(x) = 0$*, where* $\rho_3 > 0$*.*

Furthermore, assume the condition in (C_8) *is satisfied within the ball* $S(x^*, \rho_3)$*, and there exists a larger radius* $\rho_4 \geq \rho_3$ *such that*

$$\int_0^1 w_0(\theta\rho_4)\mathrm{d}\theta < 1. \tag{3}$$

Let $S_1 = Q \cap S[x^*, \rho_4]$*. Then,* x^* *is the unique solution of the equation* $F(x) = 0$ *within the set* S_1*.*

Proof. Define the linear operator $\mathscr{V} = \int_0^1 F'(x^* + \theta(x^{**} - x^*))\mathrm{d}\theta$. Utilizing condition in (C_8) and (3), we can deduce the following:

$$\begin{aligned}\|F'(x^*)^{-1}(\mathscr{V} - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^{**} - x^*\|)\mathrm{d}\theta \\ &\leq \int_0^1 w_0(\theta\rho_4)\mathrm{d}\theta \\ &< 1.\end{aligned}$$

Therefore, $\mathscr{V}^{-1} \in \mathscr{L}(Q_2, Q_1)$, and based on the approximation

$$x^{**} - x^* = \mathscr{V}^{-1}(F(x^{**}) - F(x^*)) = \mathscr{V}^{-1}(0) = 0,$$

we conclude that $x^{**} = x^*$. $\square$

3. Semi-Local Convergence

The semi-local convergence analysis is carried out using a majorizing sequence technique. A scalar sequence $\{\alpha_n\}$ is constructed to act as an upper bound for the iteration error, allowing the operator convergence problem to be reduced to the study of a scalar recurrence relation.

Analogous to the local analysis but with the role of x^*, "w" is exchanged by x_0, "γ" functions which are developed below.

Assume:

(H_1) There exists a continuous and non-decreasing function $\gamma_0 : M \to M$ such that $\gamma_0(t) - 1 = 0$ admits a smallest solution. Denote such solution by R. Let $M_3 = [0, R)$ and $S_2 = S(x_0, R) \cap Q$.

(H_2) Same as (C_7).

(H_3) $\|\mathscr{B}^{-1}(F'(x) - \mathscr{B})\| \leq \gamma_0(\|x - x_0\|)$ for all $x \in Q$.
As in (C_8) and because of (H_1) for $x = x_0$, $\gamma_0(0) \leq 1$, so $F'(x_0)^{-1} \in \mathscr{L}(Q_2, Q_1)$. Thus, the norm $\|F'(x_0)^{-1}F(x_0)\|$ exists.

(H_4) There exists continuous and non-decreasing function $\gamma : M_3 \to M$ so that

$$\|\mathscr{B}^{-1}(F'(y) - F'(x))\| \leq \gamma(\|y - x\|) \quad \text{for all} \quad x, y \in S_2.$$

The majorant sequence $\{\alpha_n\}$ is defined for $\alpha_0 = 0$, $\beta_0 \geq \|F'(x_0)^{-1}F(x_0)\|$ and each $n = 0, 1, 2, \dots$ by

$$\begin{aligned}
\lambda_n &= \int_0^1 \gamma((1-\theta)(\beta_n - \alpha_n))\mathrm{d}\theta(\beta_n - \alpha_n), \\
\bar{\gamma}_n &= \begin{cases} \gamma(\beta_n - \alpha_n) \\ \text{or} \\ \gamma_0(\alpha_n) + \gamma_0(\beta_n), \end{cases} \\
\gamma_n &= \beta_n + \left[1 + \left(1 + \frac{5}{4}\frac{\bar{\gamma}_n}{1 - \gamma_0(\alpha_n)}\right)\frac{\bar{\gamma}_n}{1 - \gamma_0(\alpha_n)}\right]\frac{\lambda_n}{1 - \gamma_0(\alpha_n)} \\
\mu_n &= \left(1 + \int_0^1 \gamma_0(\beta_n + \theta(\gamma_n - \beta_n))\mathrm{d}\theta\right)(\gamma_n - \beta_n) + \lambda_n, \\
\alpha_{n+1} &= \gamma_n + \left[1 + \left(1 + \frac{3}{2}\frac{\bar{\gamma}_n}{1 - \gamma_0(\alpha_n)}\right)\frac{\bar{\gamma}_n}{1 - \gamma_0(\alpha_n)}\right]\frac{\mu_n}{1 - \gamma_0(\alpha_n)}, \\
\delta_{n+1} &= \int_0^1 \gamma((1-\theta)(\alpha_{n+1} - \alpha_n))\mathrm{d}\theta(\alpha_{n+1} - \alpha_n) \\
&\quad + (1 + \gamma_0(\alpha_n))(\alpha_{n+1} - \beta_n)
\end{aligned} \tag{4}$$

and

$$\beta_{n+1} = \alpha_{n+1} + \frac{\delta_{n+1}}{1 - \gamma_0(\alpha_{n+1})}.$$

As in the local case, $\gamma_0(\|x_0 - x_0\|) = \gamma_0(0) < \gamma_0(R) \leq 1$. Thus, $F'(x_0)^{-1} \in \mathscr{L}(Q_2, Q_1)$ and the iterate β_0 is well defined.

(H_6) There exists $R_0 \in [0, R)$ such that for all $m = 0, 1, 2, \dots$

$$\gamma_0(\alpha_m) < 1 \quad \text{and} \quad \alpha_m \leq R_0.$$

It follows by this assumption and (4) that

$$0 \leq \alpha_m \leq \beta_m \leq \gamma_m \leq \alpha_{m+1} < R_0$$

and the sequence $\{\alpha_m\}$ is convergent to its least upper bound $R^* \in [0, R_0]$. This limit is unique.

(H_7) $S[x_0, R^*] \subset Q$.

The sequence $\{\alpha_m\}$ is constructed so that the subsequent conditions are satisfied:

$$
\begin{aligned}
F(y_n) &= F(y_n) - F(x_n) - F'(x_n)(y_n - x_n), \\
\|L^{-1}F(y_n)\| &\leq \int_0^1 \gamma((1-\theta)\|y_n - x_n\|)\mathrm{d}\theta\|y_n - x_n\| \\
&\leq \lambda_n, \\
\text{so,} & \\
\|z_n - y_n\| &\leq \left[1 + \left(1 + \frac{5}{4}\frac{\bar{\gamma}_n}{1-\gamma_0(\alpha_n)}\right)\frac{\bar{\gamma}_n}{1-\gamma_0(\alpha_n)}\right]\frac{\lambda_n}{1-\gamma_0(\alpha_n)} \\
&\leq \gamma_n - \beta_n, \\
F(z_n) &= F(z_n) - F(y_n) + F(y_n), \\
\|L^{-1}F(z_n)\| &\leq \left(1 + \int_0^1 \gamma_0(\beta_n + \theta(\gamma_n - \beta_n))\mathrm{d}\theta\right)(\gamma_n - \beta_n) + \lambda_n \\
&= \mu_n \\
\|x_{n+1} - z_n\| &\leq \left[1 + \left(1 + \frac{3}{4}\frac{\bar{\gamma}_n}{1-\gamma_0(\alpha_n)}\right)\frac{\bar{\gamma}_n}{1-\gamma_0(\alpha_n)}\right]\frac{\mu_n}{1-\gamma_0(\alpha_n)} \\
&\leq \alpha_{n+1} - \gamma_n, \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - y_n), \\
\|L^{-1}F(x_{n+1})\| &\leq \int_0^1 \gamma((1-\theta)(\alpha_{n+1} - \alpha_n))\mathrm{d}\theta(\alpha_{n+1} - \alpha_n) \\
&\quad + (1 + \gamma_0(\alpha_n))(\alpha_{n+1} - \beta_n) = \delta_{n+1} \\
\text{and} & \\
\|y_{n+1} - x_{n+1}\| &\leq \|F'(x_{n+1})^{-1}L\|\|L^{-1}F(x_{n+1})\| \\
&\leq \frac{\delta_{n+1}}{1-\gamma_0(\alpha_{n+1})} \\
&= \beta_{n+1} - \alpha_{n+1}.
\end{aligned} \tag{5}
$$

Thus, the iterates $\{x_j\}, \{y_j\}, \{z_j\} \subset S(x_0, R^*)$ and are Cauchy in Banach space Q_1. Hence, there exists $x^* \in S[x_0, R^*]$ so that $\lim_{j\to\infty} x_j = x^*$. Moreover, by (5) $F(x^*) = 0$. Furthermore, by the estimate

$$\|x_{j+k} - x_j\| \leq \alpha_{j+k} - \alpha_j,$$

the useful items

$$\|x^* - x_j\| \leq R^* - \alpha_j$$

become available.

Hence, we arrive at:

Theorem 2. *Subject to the conditions (H_1)–(H_7), the sequence $\{x_m\}$ converges towards a solution $x^* \in S[x_0, R^*]$ of the equation $F(x) = 0$.*

We establish the uniqueness of the solution domain in the following proposition.

Proposition 2. *Assume the following conditions:*

(i) There exists a solution $\bar{x}^*$ of the equation $F(x) = 0$ in $S(x_0, R_1)$ for some $R_1 > 0$.

(ii) Condition (H_3) holds on $S(x_0, R_1)$.

(iii) There exists $R_2 > R_1$ such that

$$\int_0^1 \gamma_0((1-\theta)R_1 + \theta R_2)\mathrm{d}\theta < 1.$$

Set $S_4 = Q \cap S[x_0, R_2]$.

Then, the only point in the domain S_4 *that satisfies the equation* $F(x) = 0$ *is* $\bar{x}^*$.

Proof. Let us assume that there exists $x^{'} \in S_4$ such that $F(x) = 0$. Conditions (ii) and (iii) allow us to obtain the following inequality:

$$\begin{aligned}
\|F'(x_0)^{-1}(\mathscr{X} - F'(x_0))\| &\leq \int_0^1 \gamma_0((1-\theta)\|\bar{x}^* - x_0\| + \theta\|x^{'} - x_0\|)\mathrm{d}\theta \\
&\leq \int_0^1 \gamma_0((1-\theta)R_1 + \theta R_2)\mathrm{d}\theta \\
&< 1,
\end{aligned}$$

where $\mathscr{X} = \int_0^1 F'(\bar{x}^* + \theta(x^{'} - \bar{x}^*))\mathrm{d}\theta$. Hence, we conclude that $x^{'} = \bar{x}^*$. □

Remark 1.

(i) *In condition* (H_7)*, the limit point* R^* *can be replaced by* R.

(ii) *Under all the assumptions* (H_1)–(H_7)*, let* $\bar{x}^* = x^*$ *and* $R_1 = R^*$ *in Proposition 2.*

4. Numerical Examples

Example 1. *Consider the following system of differential equations:*

$$F_1'(x_1) = e^{x_1}, \qquad F_2'(x_2) = (e-1)x_2 + 1, \qquad F_3'(x_3) = 1,$$

subject to the initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. *Define the mapping* $F = (F_1, F_2, F_3)$ *with* $Q_1 = Q_2 = \mathbb{R}^3$ *and* $Q = S[0, 1]$. *It is clear that the point* $x^* = (0, 0, 0)^T$ *satisfies the given system.*

For any vector $x = (x_1, x_2, x_3)^T \in Q$*, the operator* F *is explicitly given by*

$$F(x) = \left(e^{x_1} - 1,\ \frac{e-1}{2}x_2^2 + x_2,\ x_3\right)^T.$$

The Jacobian matrix of F *at* x *takes the form*

$$F'(x) = \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & (e-1)x_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Evaluating at x^**, we obtain* $F'(x^*) = I$*, the identity matrix.*

To verify the local convergence assumptions, conditions (C_1)–(C_{10}) *must be satisfied. This is achieved by selecting the functions* $w_0(t) = (e-1)t$ *and* $w(t) = e^{\frac{1}{e-1}}t$*, along with the radius* $\rho_0 = 0.581977$. *Accordingly, we define the set* $S_0 = S \cap S(x^*, \rho_0)$. *The convergence radius corresponding to method (1) is summarized in Table 1.*

Example 2. *Let* $Q_1 = Q_2 = Q = \mathbb{R}$. *Consider the mapping* $F : Q \to Q$ *defined by*

$$F(x) = \sin x.$$

Its derivative is given by

$$F'(x) = \cos x.$$

The fixed point of this operator is $x^* = 0$.

To verify the convergence assumptions (C_1)–(C_{10})*, we take*

$$w_0(t) = w(t) = t, \qquad \rho_0 = 1,$$

and define the set $S_0 = S \cap S(x^*, \rho_0)$*. The corresponding convergence radii for the proposed methods are reported in Table 1.*

Table 1. Estimates for Examples 1, 2 and 3.

Radii	r_1	r_2	r_3	$r = min\{r_i\}$
Example 1	0.382692	0.212347	0.175791	0.175791
Example 2	0.666667	0.369262	0.305102	0.305102
Example 3	0.047619	0.0280224	0.0240001	0.0240001

Example 3. *Let* $H[0,1]$ *denote the Banach space of all continuous real-valued functions defined on the interval* $[0,1]$*, equipped with the maximum norm. We assume that the domain and codomain coincide, that is,*

$$Q_1 = Q_2 = H[0,1].$$

We introduce a nonlinear operator $F : Q = S[0,1] \to H[0,1]$ *defined by*

$$F(w)(x) = w(x) - 7\int_0^1 x\, s\, w(s)^3\, ds, \qquad x \in [0,1].$$

Since the integrand is continuous for every $w \in S[0,1]$*, it follows that* $F(w) \in H[0,1]$*. Hence,* $F : S[0,1] \to H[0,1]$ *is well defined.*

By differentiating the operator F*, we obtain its Fréchet derivative at a point* w*, which is given by*

$$F'(w)[u](x) = u(x) - 21\int_0^1 x\, s\, w(s)^2\, u(s)\, ds, \qquad u \in H[0,1].$$

This expression follows directly from differentiation under the integral sign and shows that $F'(w)$ *is a bounded linear operator.*

It is easy to verify that the point $x^* = 0$ *satisfies the equation, and hence represents a solution of the operator equation* $F(w) = 0$*.*

To establish the convergence framework, we have to verify that conditions (C_1) *to* (C_{10}) *are satisfied. We estimate the derivative variation under the maximum norm* $\|\cdot\|$ *on* $H[0,1]$*.*

For $w \in S[0,1]$*, taking the maximum norm and using* $|x| \le 1$*,* $|s| \le 1$*, we obtain*

$$\|(F'(w) - F'(0))[u](x)\| \le 21\|w\|^2\|u\| \int_0^1 xs\, ds.$$

Since

$$\sup_{x\in[0,1]} \int_0^1 xs\, ds = \sup_{x\in[0,1]} \frac{x}{2} = \frac{1}{2},$$

it follows that

$$w_0(t) = \frac{21}{2}t.$$

Similarly, bounding the derivative variation between two points $w_1, w_2 \in S[0,1]$ *gives*

$$w(t) = 21t.$$

With these choices, the required bounds follow immediately. The corresponding convergence radii for the proposed method are reported in Table 1. We obtain

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$$r = 0.0240001.$$

This value characterizes the neighborhood in which the local convergence of the iterative scheme is guaranteed.

Example 4. *We consider the nonlinear system*

$$F(z) = \left(z_j - \cos\left(2z_j - \sum_{i=1}^{m} z_i\right)\right)_{1\leq j\leq m}.$$

For this problem, we choose the parameter $m = 4$. Thus, four scalar cosine evaluations are required in computing the vector function F, and four scalar sine evaluations arise in the computation of the Jacobian matrix F'.

The initial approximation is selected using non-symmetric initial values,

$$z^{(0)} = (0.51,\ 0.6,\ 0.39,\ 0.49)^T.$$

Solving the system yields the solution

$$z^* \approx (0.5149,\ 0.5149,\ 0.5149,\ 0.5149)^T.$$

The error bounds corresponding to the iterative method is reported in the corresponding Table 2.

Example 5. *Let $F(x) = (f_1(x), \dots, f_n(x))$ be a vector-valued mapping, where each component is defined by*

$$f_i(x) = x_{(i)} + 1 - 2\ln\left(1 + \sum_{\substack{j=1\\ j\neq i}}^{n} x_{(j)}\right), \qquad i = 1, 2, \dots, n,$$

for $n = 8$. The starting vector is chosen using non-symmetric initial values,

$$x_0 = (2.1,\ 3.2,\ 4.3,\ 5.1,\ 6.7,\ 7.3,\ 8.9,\ 9.34)^T.$$

The computed root of the system is

$$x^* = (6.7539323119353585 94\dots, \dots, 6.7539323119353585 94\dots)^T.$$

Numerical experiments using the iterative scheme (1) show that convergence to x^* is attained after 3 iterations.

Table 2. Estimates for Examples 4 and 5.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	$\|x_4 - x^*\|$
Example 4	$1.53251558 \times 10^{-1}$	$4.10659022 \times 10^{-3}$	$6.65293223 \times 10^{-5}$	$6.65293223 \times 10^{-5}$
Example 5	7.391972×10^{0}	4.690178×10^{-8}	0.000000×10^{0}	$0.00000000e \times 10^{0}$

5. Conclusions

In this work, we developed a new analytical framework for establishing both local and semi-local convergence of iterative schemes with high orders of convergence. A key feature of the proposed approach is that it relies exclusively on the derivatives explicitly involved in the iterative method, rather than assuming the availability of higher-order derivatives that are not present in the method. This distinguishes our analysis from many existing studies, where such assumptions restrict the practical scope of the results.

The proposed technique enables the derivation of convergence radii, error estimates, and uniqueness of solutions under milder and more realistic conditions. These results significantly broaden the theoretical understanding of high-order methods and address gaps left by earlier analyses. Furthermore, the framework is method-independent, which makes it highly adaptable and versatile. As a result, it can be directly employed to analyze and enhance a wide class of higher-order iterative methods, including both single-step and multi-step schemes, as reported in the related literature [1–11].

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Abbreviations

The following abbreviations are used in this manuscript:

S(x,y)	Open ball centered at x with radius y.
S[x,y]	Closed ball centered at x with radius y.

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