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Existence and Uniqueness Results for (k, ψ) -Caputo Fractional Boundary Value Problems Involving Multi-Point Closed Boundary Conditions

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Abstract

In this paper, we investigate a new class of nonlinear fractional boundary value problems (BVPs) involving (k, ψ) -Caputo fractional derivative operators subject to multipoint closed boundary conditions. Such a formulation of boundary data generalizes classical closure constraints in terms of nonlocal dependence of the unknown function at several interior points, giving rise to a flexible mechanism for describing physical and engineering phenomena governed by nonlocal and memory effects. The proposed problem is first transformed into an equivalent fixed-point formulation, enabling the application of standard analytical tools. Results concerning the existence and uniqueness of solutions to the problem are obtained through the application of fixed-point principles, specifically those of Banach, Krasnosel'skiĭ, and the Leray–Schauder nonlinear alternative. The obtained results extend and generalize several known findings. Illustrative examples are presented to demonstrate the applicability of the theoretical findings. Moreover, the introduction incorporates a succinct review of boundary value problems associated with fractional differential equations and inclusions subject to closed boundary conditions.

Keywords: fractional calculus; (k, ψ) -Caputo derivative; nonlocal boundary conditions; fixed-point theorems; existence and uniqueness; multipoint closed boundary conditions



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1. Introduction

Fractional calculus represents a profound and dynamically developing field within the area of mathematical analysis, devoted to extending the classical notions of differentiation and integration beyond integer orders. Instead of being confined to whole-number operations, this theory introduces differential and integral operators of arbitrary fractional or even complex order. Such a generalization provides a powerful and adaptable mathematical framework for modeling processes whose present states depend not only on current conditions but also on their historical evolution. Consequently, fractional calculus serves as an essential tool for capturing memory and hereditary effects that are intrinsic to many natural and engineered systems. Its influence spans a broad spectrum of scientific and technological areas, including anomalous diffusion in physics, viscoelasticity and control theory in engineering, population dynamics in biology, and stochastic modeling in finance.

Comprehensive exposition and development of the subject can be found in the monographs by Samko et al. [1], Gorenflo and Mainardi [2], Podlubny [3], Hilfer [4], and Kilbas et al. [5].

Fractional boundary value problems (BVPs) have gained prominence for their ability to model systems exhibiting memory and nonlocal effects. Investigations span various boundary formulations, including Steklov-type, coupled integral, and fractional integral constraints [6]. The growing emphasis on nonlocal conditions reflects their superior capacity to represent spatially distributed and temporally cumulative phenomena compared with classical local models. Such conditions naturally arise in applications such as reservoir simulation, anomalous diffusion, viscoelasticity, and wave propagation in heterogeneous media [7]. Consequently, fractional BVPs with nonlocal constraints form a rapidly evolving field with broad scientific and engineering applications.

Closed boundary conditions are essential in fluid dynamics modeling, representing configurations with zero mass flux across domain boundaries (for example, see Chapter 9 in [8] and Section 9.8.2 in [9]). They describe impermeable or insulated surfaces and include the free-slip variant, which permits tangential motion while prohibiting normal flow. Such formulations are vital in modeling gravitational effects, radiative transfer, elastic wave propagation, and radiation heat transfer, and are widely applied in computational fluid dynamics, image deblurring, and transport phenomena in structured materials like honeycomb lattices [10–15]. The foregoing citations highlight some representative applications of closed boundary conditions, where similar structural features arise, even if the formulations are not identical to the one we treat in our work. This also assists the readers—especially those who are not already familiar with these boundary conditions—in understanding why such problems are of interest and where they naturally appear. Proper implementation of these conditions ensures physical consistency, numerical stability, and analytical accuracy in the simulation and analysis of complex flow systems.

We now present a concise review of boundary value problems involving fractional differential equations and inclusions under closed boundary conditions.

1.1. Fractional Differential Equations with Ordinary Closed Boundary Conditions

The boundary value problems of fractional differential equations and inclusions with closed boundary conditions were initiated in [16], where the authors studied the following problem

$$\begin{cases} {}^C D^\mu y(t) = f(t, y(t)), & t \in J := [0, T], \\ y(T) = p_1 y(0) + p_2 Ty'(0), \\ Ty'(T) = q_1 y(0) + q_2 Ty'(0), \end{cases} \quad (1)$$

where ${}^C D^\mu$ denotes the Caputo fractional derivative of order μ , $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $p_1, p_2, q_1, q_2 \in \mathbb{R}$. In [16] also were studied and the corresponding inclusion problem

$$\begin{cases} {}^C D^\mu y(t) \in F(t, y(t)), & t \in J := [0, T], \\ y(T) = p_1 y(0) + p_2 Ty'(0), \\ Ty'(T) = q_1 y(0) + q_2 Ty'(0), \end{cases} \quad (2)$$

where $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a compact-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Existence results were proved in the single-valued case by using degree theory, while the existence results for the multi-valued case were established by applying Leray-Schauder

nonlinear alternative in the convex-valued multi-valued maps and by Covitz and Nadler fixed point theorem for non-convex-valued multi-valued maps.

Across several contributions [17–20], the authors analyzed an integro-differential equation characterized by a right Caputo fractional derivative and nonlinearities of mixed nature, including both usual and Riemann–Liouville–type integral components

$${}^C D_{T-}^{\mu} y(t) + \lambda I_{T-}^{\rho} I_{0+}^{\sigma} f(t, y(t)) = g(t, y(t)), \quad t \in J := [0, T], \quad (3)$$

where ${}^C D_{T-}^{\mu}$ denotes the right Caputo fractional derivative of order $\mu \in (1, 2]$, I_{T-}^{ρ} and I_{0+}^{σ} denote the right and left Riemann–Liouville fractional integrals of orders $\rho, \sigma > 0$ respectively, $\rho + \sigma < \mu$, $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions and $\lambda, \in \mathbb{R}$, supplemented with

- nonlocal closed boundary conditions of the form

$$\begin{cases} y(T) = p_1 y(\xi) + p_2 Ty'(\xi), \\ Ty'(T) = q_1 y(\xi) + q_2 Ty'(\xi), \quad 0 < \xi < T, \end{cases} \quad (4)$$

in [17], where $p_i, q_i \in \mathbb{R}$, $i = 1, 2$,

- multipoint variant of closed boundary conditions of the form

$$\begin{cases} y(T) = \sum_{i=1}^m (p_i y(\xi_i) + T q_i y'(\xi_i)), \\ Ty'(T) = \sum_{i=1}^m (r_i y(\xi_i) + T v_i y'(\xi_i)), \end{cases} \quad (5)$$

in [18], where $p_i, q_i, r_i, v_i \in \mathbb{R}$, $i = 1, 2, \dots, m$, $\xi_i \in (0, T)$,

- nonlocal closed integral boundary conditions of the form

$$\begin{cases} y(T) = \int_0^{\xi} (p_1 y(s) + p_2 Ty'(s)) ds, \\ Ty'(T) = \int_0^{\xi} (q_1 y(s) + q_2 Ty'(s)) ds, \end{cases} \quad (6)$$

in [19], where $p_i, q_i \in \mathbb{R}$, $i = 1, 2$, and

- nonlocal multipoint sub-strips closed type boundary conditions

$$\begin{cases} y(T) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) + T \sum_{j=1}^{n-2} \beta_j \int_{\theta_j}^{\eta_j} y(s) ds, \\ Ty'(T) = \sum_{i=1}^{m-2} \delta_i y'(\xi_i) + T \sum_{j=1}^{n-2} \gamma_j \int_{\theta_j}^{\eta_j} y'(s) ds, \end{cases} \quad (7)$$

in [20], respectively, where $\alpha_i, \beta_j, \delta_j, \gamma_j \in \mathbb{R}$, $\xi_1 < \xi_2 < \dots < \xi_{m-2} < \dots < \theta_1 < \eta_1 > \theta_2 < \eta_2 < \dots < \theta_{n-2} < \eta_{n-2}$, $\xi_i, \theta_j, \eta_j \in (0, T)$, $i = 1, 2, \dots, m-2$, $j = 1, 2, \dots, n-2$.

Existence and uniqueness results were proved in the problems (3) and (4), (3)–(5), (3)–(6) and (3)–(7) by using Leray–Schauder nonlinear alternative and Banach and Krasnosel’skii’s fixed point theorems.

1.2. Fractional Differential Systems with Ordinary Closed Boundary Conditions

In [21,22] were investigated the following system of nonlinear Caputo fractional differential equations

$$\begin{cases} {}^C D^{q_1} x(t) = f(t, x(t), y(t)), & t \in J := [0, T], \\ {}^C D^{q_2} y(t) = g(t, x(t), y(t)), & t \in J := [0, T], \end{cases} \quad (8)$$

where ${}^C D^{q_1}, {}^C D^{q_2}$ denote the Caputo fractional derivatives of order $q_1, q_2, 1 < q_1, q_2 < 2$, respectively, and $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions supplemented with

- coupled closed boundary conditions:

$$\begin{cases} x(T) = p_1 y(0) + q_1 T y'(0), & T x'(T) = r_1 y(0) + s_1 T y'(0), \\ y(T) = p_2 x(0) + q_2 T x'(0), & T y'(T) = r_2 x(0) + s_2 T x'(0), \end{cases} \quad (9)$$

in [21], where $p_i, q_i, r_i, s_i \in \mathbb{R}, i = 1, 2$ and

- multi-point nonlocal coupled closed boundary conditions:

$$\begin{cases} x(T) = \sum_{i=1}^m (\alpha_i y(\xi_i) + \beta_i T y'(\xi_i)), & T x'(\xi_i) = \sum_{i=1}^m (\gamma_i y(\xi_i) + \delta_i T y'(\xi_i)), \\ y(T) = \sum_{i=1}^m (u_i x(\xi_i) + v_i T x'(\xi_i)), & T y'(T) = \sum_{i=1}^m (\rho_i x(\xi_i) + \omega_i T x'(\xi_i)), \end{cases} \quad (10)$$

in [22], respectively, where $\alpha_i, \beta_i, \gamma_i, u_i, v_i, \rho_i, \omega_i \in \mathbb{R}, \xi_i \in (0, T), i = 1, 2, \dots, m$.

In [23] the authors studied the existence of solutions for a coupled system of nonlinear sequential fractional differential equations:

$$\begin{cases} ({}^C D^q + k_1 {}^C D^{q-1})x(t) = f(t, x(t), y(t)), & t \in J := [0, T], T > 0, \\ ({}^C D^p + k_2 {}^C D^{p-1})y(t) = g(t, x(t), y(t)), & t \in J := [0, T], T > 0, \end{cases} \quad (11)$$

where ${}^C D^q, {}^C D^p$ denote the Caputo fractional derivative operators of order $q \in (1, 2]$ and $p \in (1, 2]$, respectively, and $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions subject to the closed coupled boundary conditions given by

$$\begin{cases} x(T) = \alpha_1 y(0) + \beta_1 T y'(0), & T x'(T) = \gamma_1 y(0) + \delta_1 T y'(0), \\ y(T) = \alpha_2 x(0) + \beta_2 T x'(0), & T y'(T) = \gamma_2 x(0) + \delta_2 T x'(0), \end{cases} \quad (12)$$

where $\alpha_i, \beta_i, \delta_i, \gamma_i \in \mathbb{R}, i = 1, 2$.

The existence and uniqueness results for the problem (8) and (9), (8)–(10) and (11) and (12) were proved by applying the Leray-Schauder alternative, and Banach's fixed point theorem.

1.3. Impulsive Fractional Differential Equations with Ordinary Closed Boundary Conditions

In [24,25] investigated the existence of solutions for the following impulsive fractional differential equations with closed boundary conditions:

$$\begin{cases} {}^C D^\mu y(t) = f(t, y(t)), & t \in J' := [0, T] \setminus \{t_1, t_2, \dots, t_p\}, \\ \Delta y(t_k) = I_k(y(t_k)), \quad \Delta y'(t_k) = I_k^*(y(t_k)), & k = 1, 2, \dots, p, \\ y(T) = p_1 y(0) + p_2 T y'(0), \\ T y'(T) = q_1 y(0) + q_2 T y'(0), \end{cases} \quad (13)$$

where ${}^C D^\mu$ denotes the Caputo fractional derivative of order μ , $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $J = [0, T]$, $0 = t_0 < t_1 < \dots < t_k < \dots < t_p < t_{p+1} = T$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+)$ and $y(t_k^-)$ denote the right and left limits of $y(t)$ at $t = t_k$, $k = 1, 2, \dots, p$, respectively. $\Delta y'(t_k)$ have a similar meaning for $y'(t)$.

Existence and uniqueness results were proved for the problem (13) via Schauder, Schaefer, Banach and Burton-Kirk fixed point theorems.

In the work [26], the authors examined an impulsive boundary value problem defined by a nonlinear multi-term Caputo-type fractional q -difference equation accompanied by ordinary closed boundary conditions described as

$$\begin{cases} \lambda {}^C F_q^\alpha y(t) + (1 - \lambda) {}^C F_q^\beta y(t) = f(t, y(t)), & t \in \mathcal{J} = [0, T], t \neq t_k, k = 1, 2, \dots, p, \\ \Delta y(t_k) = I_k(y(t_k)), \quad \Delta y'(t_k) = I_k^*(y(t_k)), & k = 1, 2, \dots, p, \\ y(T) = p_1 y(0) + p_2 T y'(0), \\ T y'(T) = q_1 y(0) + q_2 T y'(0), \end{cases} \quad (14)$$

where ${}^C D_q^\omega$ denotes the Caputo fractional q derivative of order ω , $(\omega = \alpha, \beta)$, $0 < q < 1$, $1 < \alpha < 2$ and $0 < \beta < 1$ such that $\alpha - \beta > 1$, $0 < \lambda \leq 1$, $p_1, q_1 \in \mathbb{R}$, $i = 1, 2$ and $f : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Moreover, I_k, I_k^*, Δ are as in problem (13).

Existence and uniqueness results for the problem (14) were established with the help of Schaefer and Banach fixed point theorems.

1.4. Fractional Differential Equations with Fractional Closed Boundary Conditions

Recently, the work [27] addressed fractional differential equations endowed with mixed nonlinearities and nonlocal closed boundary conditions of fractional order, given by

$$\begin{cases} {}^C D_{0+}^\theta \varphi(\varkappa) + \lambda \mathcal{I}_{0+}^\sigma v_1(\varkappa, \varphi(\varkappa)) = v_2(\varkappa, \varphi(\varkappa)), & 0 \leq \varkappa \leq 1, \\ \varphi(1) = \mu_1 \varphi(\xi) + \mu_2 ({}^C D_{0+}^\gamma \varphi)(\xi), \\ ({}^C D_{0+}^\gamma \varphi)(1) = \delta_1 \varphi(\xi) + \delta_2 ({}^C D_{0+}^\gamma \varphi)(\xi), & 0 < \xi \leq 1, \end{cases} \quad (15)$$

where ${}^C D_{0+}^\theta, {}^C D_{0+}^\gamma$ denote the Caputo fractional derivative operators of order $1 < \theta \leq 2$ and $0 < \gamma \leq 1$, respectively, $\mathcal{I}_{0+}^\sigma$ represent the Riemann-Liouville integral operator of order $0 < \sigma \leq 1$, $v_1, v_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous functions and $\mu_i, \delta_i, \lambda \in \mathbb{R}$ for all $i = 1, 2$.

Existence and uniqueness results for the problem (15) were established with the help of Leray-Schauder alternative and fixed point theorems due to Banach and Krasnosel'skiĭ.

Let us give some comparative remarks on the boundary conditions considered in the literature. The work reviewed in this section illustrates several distinct classes of closed boundary conditions that appear in fractional differential equations and inclusions. Although their analytic frameworks differ, these formulations share a common structure: they impose linear constraints coupling function and its fractional (or classical) derivatives values at one or more points in the interval.

Ordinary closed boundary conditions (e.g., [16]) involve only the endpoints and form the simplest framework; multipoint variants (e.g., [18]) extend this framework by incorporating evaluations at interior points, while integral closed conditions (e.g., [19]) impose

nonlocal constraints through integrals of the solution or its derivative. Impulsive closed conditions (e.g., [24,25]) add jump discontinuities, and fractional closed conditions (e.g., [27]) involve fractional-order operators within the boundary constraints themselves. Finally, coupled systems (e.g., [21–23]) introduce interdependence between multiple components, requiring compatibility across several closed-type conditions simultaneously.

Despite their differences, all these formulations share a unifying feature: each induces a boundary operator that links endpoint data to interior or derivative information in a linear but nonstandard way. This structural similarity explains why classical tools of the fixed point theory such as the Leray–Schauder alternative, degree theory, and fixed-point methods appear repeatedly in the literature.

1.5. A New Problem on Fractional Differential Equations with Fractional Closed Boundary Conditions

Keeping in mind that all the work related to closed boundary conditions mentioned in the previous section only involved Caputo fractional derivative operators, in the present paper, we study the existence and uniqueness of solutions for a (k, ψ) -Caputo type fractional differential equation complemented with the multi-point fractional closed boundary conditions. Precisely, we investigate the following problem

$$\begin{cases} {}^{k,C}D_{a^+}^{\theta;\psi} \varphi(\varkappa) = v(\varkappa, \varphi(\varkappa)), & a \leq \varkappa \leq b, \\ \varphi(b) = \sum_{i=1}^m \left(\eta_i \varphi(\xi_i) + b \mu_i {}^{k,C}D_{a^+}^{\sigma_i;\psi} \varphi(\xi_i) \right), \\ b {}^{k,C}D_{a^+}^{\gamma;\psi} \varphi(b) = \sum_{i=1}^m \left(p_i \varphi(\xi_i) + b q_i {}^{k,C}D_{a^+}^{\sigma_i;\psi} \varphi(\xi_i) \right), \end{cases} \quad (16)$$

where ${}^{k,C}D_{a^+}^{\theta;\psi}$, ${}^{k,C}D_{a^+}^{\gamma;\psi}$ denote the (k, ψ) -Caputo fractional derivative operators of orders $1 < \frac{\theta}{k} < 2$ and $0 < \frac{\gamma}{k} < 1$, respectively, while ${}^{k,C}D_{a^+}^{\sigma_i;\psi}$ represents the (k, ψ) -Caputo fractional derivative operator of order $0 < \frac{\sigma_i}{k} < 1$ for all $i = 1, 2, \dots, m$, for each $k > 0$. Moreover, $a < \xi_1 < \xi_2 < \dots < \xi_m < b$, $a < \zeta_1 < \zeta_2 < \dots < \zeta_m < b$, $\eta_i, \mu_i, p_i, q_i \in \mathbb{R}$ for all $i = 1, 2, \dots, m$ and $v : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Notice that, for specific choices of ψ and k , the (k, ψ) -Caputo fractional derivative reduces to well-known operators. In particular, it coincides with the classical Caputo fractional derivative when $\psi(\varkappa) = \varkappa$ and $k = 1$, while the choice $\psi(\varkappa) = \log \varkappa$ and $k = 1$ yields the Caputo–Hadamard fractional derivative. Moreover, we emphasize that our objective for studying the problem (16) is to combine (i) the general (k, ψ) -Caputo fractional differential equation and (ii) multi-point closed fractional boundary conditions; a combination which not addressed in the existing literature so far. This underscores the gap our paper fills and highlights the novelty of the present results.

Now let us compare our work with the one obtained in [18]. Firstly, as we have already emphasized above, the problem considered here is indeed motivated by the one studied in [18]. However, the present problem does not merely replace the classical derivative y' in the boundary conditions with a Caputo-type fractional derivative. Instead, we investigate a significantly more general framework based on the (k, ψ) -Caputo fractional derivative, which includes as special cases a wide variety of existing operators such as the classical Caputo derivative, the Riemann–Liouville derivative, etc., depending on the choice of k and auxiliary functions ψ . This level of generality is not addressed in [18], nor in related literature. Secondly, the incorporation of multi-point closed fractional boundary conditions within the (k, ψ) -Caputo framework is entirely new. As mentioned before, the literature on closed fractional boundary conditions is very limited. To the best of our knowledge, only one paper ([27]) deals with such conditions for Caputo-type operators, and even that work

focuses solely on the standard Caputo derivative and single-point nonlocal formulation. Our study extends this area in two substantial directions:

- From Caputo to (k, ψ) -Caputo derivatives, vastly expanding the class of admissible fractional dynamics;
- From single-point to multi-point closed fractional boundary conditions, resulting in a more general and technically richer boundary value problem.

Thirdly, although we use fixed-point techniques (as does a large portion of the fractional differential equations literature), the functional setting, the operator properties, and the specific boundary operator corresponding to multi-point closed fractional conditions in the (k, ψ) -Caputo context require nontrivial adaptations of the analytical framework. These steps are not straightforward extensions, because the k and transformation ψ introduce additional structural complexity which must be handled carefully in the analysis. Fourthly, the results obtained are not merely technical variations but lead to new general theorems that unify, extend, and in many cases properly generalize several earlier existence and uniqueness results for closed fractional boundary value problems. Many existing results in the literature are recovered as special cases of our theorems. This type of generalization is common and valuable in fractional calculus, where new operators aim to capture more realistic memory kernels or nonlocal behaviors.

First of all, we reformulate the boundary value problem as an equivalent fixed-point problem. Subsequently, we derive our existence results through the application of Krasnosel'skiĭ's fixed point theorem and Leray–Schauder's nonlinear alternative, while the uniqueness result is accomplished by means of Banach's fixed point theorem. It is well established that the techniques of modern analysis provide a powerful framework for developing the existence theory for both initial and boundary value problems. We highlight that the results obtained in this work are not only novel within the considered framework but also specialize to several new results as direct consequences.

The remainder of the paper is organized as follows. Section 1 presents some basic definitions and lemmas, as well as, an auxiliary lemma that enables the conversion of the nonlinear problem into an equivalent fixed-point formulation. Section 2 is devoted to the main theoretical results, while Section 3 provides illustrative examples that demonstrate their applicability. Finally, Section 4 describes some noteworthy conclusions and potential directions for further study.

2. Preliminaries

This section is devoted to introducing the essential definitions and lemmas on (k, ψ) -fractional calculus to obtain the main results.

Definition 1 ([28]). For $\vartheta \in \mathbb{C}$ with a positive real part and $k \in \mathbb{R}^+$, the k -gamma function is defined by

$$\Gamma_k(\vartheta) = \int_0^\infty \tau^{\vartheta-1} e^{-\frac{\tau^k}{k}} d\tau.$$

Moreover, the following relations hold:

$$\Gamma(\vartheta) = \lim_{k \rightarrow 1} \Gamma_k(\vartheta), \quad \Gamma_k(\vartheta) = k^{\frac{\vartheta}{k}-1} \Gamma\left(\frac{\vartheta}{k}\right) \quad \text{and} \quad \vartheta \Gamma_k(\vartheta) = \Gamma_k(\vartheta + k).$$

Definition 2 ([29]). Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Furthermore, let ψ denote an increasing and positive function on $(a, b]$ whose derivative ψ' exists and is continuous on (a, b) .

Then, the (k, ψ) -Riemann–Liouville fractional integrals of a function φ with respect to another function ψ on $[a, b]$ of order ϑ and $k > 0$ is defined by

$${}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \varphi(\varkappa) = \frac{1}{k\Gamma_k(\vartheta)} \int_a^{\varkappa} \psi'(\tau) (\psi(\varkappa) - \psi(\tau))^{\frac{\vartheta}{k}-1} \varphi(\tau) d\tau, \quad \varkappa > a.$$

Definition 3 ([30]). Let $\vartheta, k \in \mathbb{R}^+$, $\psi \in C^n([a, b], \mathbb{R})$ be such that ψ is increasing and $\psi'(\varkappa) > 0$ for all $\varkappa \in [a, b]$ and $\varphi \in C^n([a, b], \mathbb{R})$. Then, the (k, ψ) -Caputo fractional derivative of order ϑ for a function φ is defined by

$${}^{k,C}D_{a^+}^{\vartheta;\psi} \varphi(\varkappa) = \frac{1}{k\Gamma_k(nk - \vartheta)} \int_a^{\varkappa} \psi'(\tau) (\psi(\varkappa) - \psi(\tau))^{nk - \frac{\vartheta}{k} - 1} \left(\frac{k}{\psi'(\tau)} \frac{d}{d\tau} \right)^n \varphi(\tau) d\tau$$

where $n = \left\lceil \frac{\vartheta}{k} \right\rceil$ is the ceiling function of $\frac{\vartheta}{k}$.

Lemma 1 ([30]). Let $\vartheta, k \in \mathbb{R}^+$ and $n = \left\lceil \frac{\vartheta}{k} \right\rceil$. Suppose that $\varphi \in C^n([a, b], \mathbb{R})$, Then,

$$\left({}^k\mathcal{I}_{a^+}^{\vartheta;\psi} {}^{k,C}D_{a^+}^{\vartheta;\psi} \varphi \right)(\varkappa) = \varphi(\varkappa) - \sum_{j=0}^{n-1} \frac{(\psi(\varkappa) - \psi(a))^j}{\Gamma_k(jk + k)} \left[\left(\frac{k}{\psi'(\varkappa)} \frac{d}{d\varkappa} \right)^j \varphi(\varkappa) \right]_{\varkappa=a}.$$

Lemma 2 ([30]). Let $\vartheta_1, \vartheta_2, k \in \mathbb{R}^+$ with $\vartheta_2 > \vartheta_1$. Then,

$${}^{k,C}D_{a^+}^{\vartheta_1;\psi} {}^k\mathcal{I}_{a^+}^{\vartheta_2;\psi} \varphi(\varkappa) = {}^k\mathcal{I}_{a^+}^{\vartheta_2 - \vartheta_1;\psi} \varphi(\varkappa).$$

Lemma 3 ([30]). Let $\vartheta, k \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that $\frac{\mu}{k} > -1$. Then,

$${}^{k,C}D_{a^+}^{\vartheta;\psi} (\psi(\varkappa) - \psi(a))^{\frac{\mu}{k}} = \frac{\Gamma_k(\mu + k)}{\Gamma_k(\mu + k - \vartheta)} (\psi(\varkappa) - \psi(a))^{\frac{\mu - \vartheta}{k}}.$$

The next lemma addresses a linear version of Equation (16) and plays a key role in reformulating Equation (16) as a fixed-point problem.

Lemma 4. Let $\Delta \neq 0$, $k > 0$, $1 < \frac{\vartheta}{k} < 2$, $0 < \frac{\gamma}{k} < 1$, $a < \xi_1 < \xi_2 < \dots < \xi_m < b$, $a < \zeta_1 < \zeta_2 < \dots < \zeta_m < b$, $\eta_i, \mu_i, p_i, q_i \in \mathbb{R}$, $0 < \frac{\sigma_i}{k} < 1$ for all $i = 1, 2, \dots, m$ and $\phi \in L^1(a, b)$, ${}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \phi \in C^1[a, b]$, and $\varphi \in C^2([a, b], \mathbb{R})$. Then the linear problem

$$\begin{cases} {}^{k,C}D_{a^+}^{\vartheta;\psi} \varphi(\varkappa) = \phi(\varkappa), & a \leq \varkappa \leq b, \\ \varphi(b) = \sum_{i=1}^m \left(\eta_i \varphi(\xi_i) + b \mu_i {}^{k,C}D_{a^+}^{\sigma_i;\psi} \varphi(\zeta_i) \right), \\ b {}^{k,C}D_{a^+}^{\gamma;\psi} \varphi(b) = \sum_{i=1}^m \left(p_i \varphi(\xi_i) + b q_i {}^{k,C}D_{a^+}^{\sigma_i;\psi} \varphi(\zeta_i) \right), \end{cases} \quad (17)$$

is equivalent to the integral equation

$$\begin{aligned} \varphi(\varkappa) = & {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \phi(\varkappa) + \frac{1}{\Delta} \left[Q_4 - Q_3 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \phi(b) \right. \\ & \left. - \sum_{i=1}^m \left(\eta_i {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \phi(\xi_i) + b \mu_i {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i;\psi} \phi(\zeta_i) \right) \right\} \\ & + \frac{1}{\Delta} \left[Q_1 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} - Q_2 \right] \left\{ b {}^k\mathcal{I}_{a^+}^{\vartheta - \gamma;\psi} \phi(b) \right. \end{aligned}$$

$$- \sum_{i=1}^m \left(p_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b q_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right) \Bigg\}, \quad (18)$$

where

$$\begin{aligned} Q_1 &= \sum_{i=1}^m \eta_i - 1, \\ Q_2 &= \sum_{i=1}^m \left(\eta_i \frac{(\psi(\xi_i) - \psi(a))}{\Gamma_k(2k)} + b \mu_i \frac{(\psi(\xi_i) - \psi(a))^{1 - \frac{\sigma_i}{k}}}{\Gamma_k(2k - \sigma_i)} \right) - \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)}, \\ Q_3 &= \sum_{i=1}^m p_i, \\ Q_4 &= \sum_{i=1}^m \left(p_i \frac{(\psi(\xi_i) - \psi(a))}{\Gamma_k(2k)} + b q_i \frac{(\psi(\xi_i) - \psi(a))^{1 - \frac{\sigma_i}{k}}}{\Gamma_k(2k - \sigma_i)} \right) - b \frac{(\psi(b) - \psi(a))^{1 - \frac{\gamma}{k}}}{\Gamma_k(2k - \gamma)}, \end{aligned}$$

and

$$\Delta = Q_1 Q_4 - Q_2 Q_3.$$

Proof. Applying the (k, ψ) -Riemann fractional integral ${}^k \mathcal{I}_{a^+}^{\vartheta; \psi}$ to both sides of Equation (17) and using Lemma 1, we obtain

$$\varphi(\varkappa) = {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\varkappa) + c_0 + \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} c_1 \quad (19)$$

where $c_0 = [\varphi(\varkappa)]_{\varkappa=a}$ and $c_1 = \left[\left(\frac{k}{\psi'(\varkappa)} \frac{d}{d\varkappa} \right) \varphi(\varkappa) \right]_{\varkappa=a}$. Now, by applying the multi-point closed boundary conditions in Equation (17) and using the notations together with Lemma 2, we obtain the following system

$$\begin{cases} Q_1 c_0 + Q_2 c_1 = {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(b) - \sum_{i=1}^m \left(\eta_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b \mu_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right), \\ Q_3 c_0 + Q_4 c_1 = b {}^k \mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} \phi(b) - \sum_{i=1}^m \left(p_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b q_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right). \end{cases} \quad (20)$$

Solving the system (20) for c_0 and c_1 , we find that

$$\begin{aligned} c_0 &= \frac{1}{\Delta} \left[Q_4 \left\{ {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(b) - \sum_{i=1}^m \left(\eta_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b \mu_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right) \right\} \right. \\ &\quad \left. - Q_2 \left\{ b {}^k \mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} \phi(b) - \sum_{i=1}^m \left(p_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b q_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right) \right\} \right], \\ c_1 &= \frac{1}{\Delta} \left[Q_1 \left[b {}^k \mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} \phi(b) - \sum_{i=1}^m \left(p_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b q_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right) \right] \right. \\ &\quad \left. - Q_3 \left\{ {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(b) - \sum_{i=1}^m \left(\eta_i {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\xi_i) + b \mu_i {}^k \mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} \phi(\xi_i) \right) \right\} \right]. \end{aligned}$$

Substituting the above values of c_0 and c_1 in (19), we obtain the solution (18). The converse of the lemma can be obtained after applying the operator ${}^{k,C} D_{a^+}^{\vartheta; \psi}$ on (18) and using the result: ${}^{k,C} D_{a^+}^{\vartheta; \psi} {}^k \mathcal{I}_{a^+}^{\vartheta; \psi} \phi(\varkappa) = \phi(\varkappa)$ (a special case of Lemma 2) together with the arguments employed in [30]. On the other hand, it is easy to verify that $\varphi(\varkappa)$ given by (18) satisfies the boundary conditions in (17). The proof is finished. $\square$

3. Main Results

Let $X = C([a, b], \mathbb{R})$ represent the Banach space of all continuous functions from $[a, b]$ to $\mathbb{R}$ furnished with the norm $\|\varphi\| = \sup\{|\varphi(\varkappa)| : \varkappa \in [a, b]\}$.

Following Lemma 4, we introduce an operator $T : X \rightarrow X$ by

$$\begin{aligned} T(\varphi)(\varkappa) = & {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \nu(\varkappa, \varphi(\varkappa)) + \frac{1}{\Delta} \left[Q_4 - Q_3 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \nu(b, \varphi(b)) \right. \\ & \left. - \sum_{i=1}^m \left(\eta_i {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \nu(\xi_i, \varphi(\xi_i)) + b \mu_i {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i;\psi} \nu(\xi_i, \varphi(\xi_i)) \right) \right\} \\ & + \frac{1}{\Delta} \left[Q_1 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} - Q_2 \right] \left\{ b {}^k\mathcal{I}_{a^+}^{\vartheta-\gamma;\psi} \nu(b, \varphi(b)) \right. \\ & \left. - \sum_{i=1}^m \left(p_i {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} \nu(\xi_i, \varphi(\xi_i)) + b q_i {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i;\psi} \nu(\xi_i, \varphi(\xi_i)) \right) \right\}. \end{aligned} \quad (21)$$

For convenience, we set some positive constants:

$$\begin{aligned} \Theta_1 &= \frac{1}{|\Delta|} \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\}, \\ \Theta_2 &= \frac{1}{|\Delta|} \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\}, \\ \Omega_1 &= \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} [1 + \Theta_1], \\ \Omega'_1 &= \Omega_1 - \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)}, \\ \Omega_2 &= |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta-\gamma}{k}}}{\Gamma_k(\vartheta - \gamma + k)} \Theta_2, \\ \Omega_3 &= |b| \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta-\sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} [|\mu_i| \Theta_1 + |q_i| \Theta_2], \\ \Omega_4 &= \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} [|\eta_i| \Theta_1 + |p_i| \Theta_2], \\ \Omega &= \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4, \\ \Omega' &= \Omega'_1 + \Omega_2 + \Omega_3 + \Omega_4. \end{aligned} \quad (22)$$

In the following theorem, we establish the uniqueness result for the (k, ψ) -Caputo fractional boundary value problem equipped with the multi-point closed boundary conditions (16), by applying the Banach contraction mapping principle [31].

Theorem 1. Let $\nu : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume that the following condition is satisfied:

(A₁) For all $\varkappa \in [a, b]$ and $\varphi_1, \varphi_2 \in \mathbb{R}$, there exists a real constant $\mathcal{L}$ such that

$$|\nu(\varkappa, \varphi_1) - \nu(\varkappa, \varphi_2)| \leq \mathcal{L} |\varphi_1 - \varphi_2|.$$

Then, the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (16) has a unique solution on $[a, b]$, provided that

$$\mathcal{L}\Omega < 1, \quad (23)$$

where Ω is given by (22).

Proof. To verify the hypotheses of Banach's contraction mapping principle, we consider a closed ball $B_r = \{\varphi \in X : \|\varphi\| \leq r\}$ with $r \geq \frac{\mathcal{M}\Omega}{1-\mathcal{L}\Omega}$ where $\mathcal{M} = \sup_{0 \leq \varkappa \leq b} \{|\nu(\varkappa, 0)| : \varkappa \in [0, b]\} < \infty$. By using the assumption (A_1) , we obtain

$$|\nu(\varkappa, \varphi(\varkappa))| \leq |\nu(\varkappa, \varphi(\varkappa)) - \nu(\varkappa, 0)| + |\nu(\varkappa, 0)| \leq \mathcal{L}|\varphi(\varkappa)| + \mathcal{M} \leq \mathcal{L}\|\varphi\| + \mathcal{M} \leq \mathcal{L}r + \mathcal{M}.$$

First, we show that $T(B_r) \subset B_r$. For each $\varphi \in B_r$, we have

$$\begin{aligned} |T(\varphi)(\varkappa)| &\leq {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\varkappa, \varphi(\varkappa))| + \frac{1}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(b, \varphi(b))| \right. \\ &\quad \left. + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi(\xi_i))| + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |\nu(\xi_i, \varphi(\xi_i))| \right) \right\} \\ &\quad + \frac{1}{|\Delta|} \left[|Q_1| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| {}^k\mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} |\nu(b, \varphi(b))| \right. \\ &\quad \left. + \sum_{i=1}^m \left(|p_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi(\xi_i))| + |b| |q_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |\nu(\xi_i, \varphi(\xi_i))| \right) \right\} \\ &\leq (\mathcal{L}r + \mathcal{M}) \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + \frac{\mathcal{L}r + \mathcal{M}}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} \right] \\ &\quad \times \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + \sum_{i=1}^m \left(|\eta_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |\mu_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ &\quad + \frac{\mathcal{L}r + \mathcal{M}}{|\Delta|} \left[|Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta - \gamma}{k}}}{\Gamma_k(\vartheta - \gamma + k)} \right. \\ &\quad \left. + \sum_{i=1}^m \left(|p_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |q_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ &= (\mathcal{L}r + \mathcal{M}) \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[1 + \frac{1}{|\Delta|} \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right] \\ &\quad + |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta - \gamma}{k}}}{|\Delta| \Gamma_k(\vartheta - \gamma + k)} \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} (\mathcal{L}r + \mathcal{M}) \\ &\quad + \frac{1}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[|\eta_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\ &\quad \left. + |p_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] (\mathcal{L}r + \mathcal{M}) \\ &\quad + \frac{|b|}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \left[|\mu_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\ &\quad \left. + |q_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] (\mathcal{L}r + \mathcal{M}) \\ &= (\mathcal{L}r + \mathcal{M}) (\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4) \end{aligned}$$

$$= (\mathcal{L}r + \mathcal{M}) \Omega \leq r,$$

which implies that $\|T(\varphi)\| \leq r$, and consequently $T(B_r) \subset B_r$.

Next, we will prove that the operator T is a contraction. For $\varphi_1, \varphi_2 \in B_r$ and $\varkappa \in [a, b]$ by the assumption (A_1) , we obtain

$$\begin{aligned} & |T(\varphi_1)(\varkappa) - T(\varphi_2)(\varkappa)| \\ & \leq {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} |\nu(\varkappa, \varphi_1(\varkappa)) - \nu(\varkappa, \varphi_2(\varkappa))| + \frac{1}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} \right] \\ & \quad \times \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} |\nu(b, \varphi_1(b)) - \nu(b, \varphi_2(b))| + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right. \right. \\ & \quad \left. \left. + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i;\psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right) \right\} + \frac{1}{|\Delta|} \left[|Q_1| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \\ & \quad \times \left\{ |b| {}^k\mathcal{I}_{a^+}^{\vartheta-\gamma;\psi} |\nu(b, \varphi_1(b)) - \nu(b, \varphi_2(b))| + \sum_{i=1}^m \left(|p_i| {}^k\mathcal{I}_{a^+}^{\vartheta;\psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right. \right. \\ & \quad \left. \left. + |b| |q_i| {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i;\psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right) \right\} \\ & \leq \mathcal{L} \|\varphi_1 - \varphi_2\| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + \frac{\mathcal{L} \|\varphi_1 - \varphi_2\|}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} \right] \\ & \quad \times \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + \sum_{i=1}^m \left(|\eta_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |\mu_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta-\sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ & \quad + \frac{\mathcal{L} \|\varphi_1 - \varphi_2\|}{|\Delta|} \left[|Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta-\gamma}{k}}}{\Gamma_k(\vartheta - \gamma + k)} \right. \\ & \quad \left. + \sum_{i=1}^m \left(|p_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |q_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta-\sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ & = \mathcal{L} \Omega \|\varphi_1 - \varphi_2\|, \end{aligned}$$

which implies that

$$\|T(\varphi_1) - T(\varphi_2)\| \leq \mathcal{L} \Omega \|\varphi_1 - \varphi_2\|.$$

Hence, by condition (23), the operator T is a contraction. Therefore, the operator T has a unique fixed point in the ball B_r by applying the Banach contraction mapping principle. As a result, the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (16) has a unique solution on $[a, b]$. The proof is completed. $\square$

The first existence result for problem (16) is established in the following theorem by means of Leray-Schauder nonlinear alternative.

Lemma 5 (Leray-Schauder nonlinear alternative [32]). *Let X be a Banach space, C a closed, convex subset of X , U an open subset of C and $0 \in U$. Suppose that $T : \overline{U} \rightarrow C$ is a continuous, compact (that is, $T(U)$ is a relatively compact subset of C) map. Then either*

- (i) T has a fixed point in $\overline{U}$, or
- (ii) there is a $\varphi \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $\varphi = \lambda T(\varphi)$.

Theorem 2. Let $v : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ denote a continuous function, subject to the following assumptions:

(A₂) For all $\varkappa \in [a, b]$ and $\varphi \in \mathbb{R}$, one can define a continuous, non-decreasing function $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ together with a positive continuous function $\omega \in C([a, b], \mathbb{R}^+)$ such that

$$|v(\varkappa, \varphi)| \leq \omega(\varkappa) \delta(|\varphi|);$$

(A₃) There exist a positive constant $\mathcal{K}$ such that

$$\frac{\mathcal{K}}{\|\omega\| \delta(\mathcal{K}) \Omega} > 1, \quad (24)$$

where Ω is given by (22).

Then the (k, ψ) -Caputo fractional boundary value problem supplemented with multi-point closed boundary conditions (16) has at least one solution on $[a, b]$.

Proof. Consider the operator T defined by (21). Observe that the continuity of v implies the continuity of T . At this stage, we establish that the operator T maps any bounded subset of X into a bounded subset of X . Assume that $B_\rho = \{v \in X : \|v\| \leq \rho\}$, where ρ is a fixed number, is a bounded set in X . Then, we have

$$\begin{aligned} |T(\varphi)(\varkappa)| &\leq {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(\varkappa, \varphi(\varkappa))| + \frac{1}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(b, \varphi(b))| \right. \\ &\quad \left. + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(\xi_i, \varphi(\xi_i))| + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |v(\xi_i, \varphi(\xi_i))| \right) \right\} \\ &\quad + \frac{1}{|\Delta|} \left[|Q_1| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| {}^k\mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} |v(b, \varphi(b))| \right. \\ &\quad \left. + \sum_{i=1}^m \left(|p_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(\xi_i, \varphi(\xi_i))| + |b| |q_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |v(\xi_i, \varphi(\xi_i))| \right) \right\}. \\ &\leq \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[1 + \frac{1}{|\Delta|} \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right] \|\omega\| \delta(\|\varphi\|) \\ &\quad + |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta - \gamma}{k}}}{|\Delta| \Gamma_k(\vartheta - \gamma + k)} \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \|\omega\| \delta(\|\varphi\|) \\ &\quad + \frac{1}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[|\eta_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\ &\quad \left. + |p_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] \|\omega\| \delta(\|\varphi\|) \\ &\quad + \frac{|b|}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \left[|\mu_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\ &\quad \left. + |q_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] \|\omega\| \delta(\|\varphi\|) \\ &= \|\omega\| \delta(\|\varphi\|) \Omega. \end{aligned}$$

Therefore, $\|T(\varphi)\| \leq \|\omega\| \delta(\rho) \Omega$, which indicates that the operator T is uniformly bounded.

At this stage, we establish that the operator T maps any bounded subset of X into an equicontinuous subset of X . Let $\varkappa_1, \varkappa_2 \in [a, b]$ with $\varkappa_1 < \varkappa_2$ and $\varphi \in B_\rho$. Then we have

$$\begin{aligned} & |T(\varphi)(\varkappa_2) - T(\varphi)(\varkappa_1)| \\ & \leq \frac{1}{k\Gamma_k(\vartheta)} \left| \int_a^{\varkappa_1} \psi'(\tau) \left((\psi(\varkappa_2) - \psi(\tau))^{\frac{\vartheta}{k}-1} - (\psi(\varkappa_1) - \psi(\tau))^{\frac{\vartheta}{k}-1} \right) \nu(\tau, \varphi(\tau)) d\tau \right. \\ & \quad + \int_{\varkappa_1}^{\varkappa_2} \psi'(\tau) (\psi(\varkappa_2) - \psi(\tau))^{\frac{\vartheta}{k}-1} \nu(\tau, \varphi(\tau)) d\tau \left. + \frac{(\psi(\varkappa_2) - \psi(\varkappa_1))}{|\Delta| \Gamma_k(2k)} \left[|Q_3| \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(b, \varphi(b))| \right. \right. \right. \\ & \quad + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi(\xi_i))| + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i; \psi} |\nu(\xi_i, \varphi(\xi_i))| \right) \left. \right\} + |Q_1| \left\{ |b| \right. \\ & \quad \times {}^k\mathcal{I}_{a^+}^{\vartheta-\gamma; \psi} |\nu(b, \varphi(b))| + \sum_{i=1}^m \left(|p_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi(\xi_i))| + |b| |q_i| {}^k\mathcal{I}_{a^+}^{\vartheta-\sigma_i; \psi} |\nu(\xi_i, \varphi(\xi_i))| \right) \left. \right\} \left. \right] \\ & \leq \frac{\|\omega\| \delta(\rho)}{\Gamma_k(\vartheta+k)} \left\{ 2(\psi(\varkappa_2) - \psi(\varkappa_1))^{\frac{\vartheta}{k}} + \left| (\psi(\varkappa_2) - \psi(a))^{\frac{\vartheta}{k}} - (\psi(\varkappa_1) - \psi(a))^{\frac{\vartheta}{k}} \right| \right\} \\ & \quad + \frac{\|\omega\| \delta(\rho)}{|\Delta| \Gamma_k(2k)} \left\{ \sum_{i=1}^m \left(|b| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta-\sigma_i}{k}}}{\Gamma_k(\vartheta-\sigma_i+k)} \left[|\mu_i| |Q_3| + |q_i| |Q_1| \right] \right. \right. \\ & \quad + \left. \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)} \left[|\eta_i| |Q_3| + |p_i| |Q_1| \right] \right) + \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)} \left. \right\} (\psi(\varkappa_2) - \psi(\varkappa_1)), \end{aligned}$$

which implies that

$$|T(\varphi)(\varkappa_2) - T(\varphi)(\varkappa_1)| \rightarrow 0, \quad \text{as } \varkappa_1 \rightarrow \varkappa_2,$$

independently of $\varphi \in B_\rho$. Hence, the operator T is equicontinuous. In view of the Arzelà–Ascoli theorem, this implies that T is completely continuous.

Finally, we show that the set of all solutions to equation $\varphi = \lambda T(\varphi)$ is bounded for $\lambda \in (0, 1)$. For any $\varkappa \in [a, b]$, and following calculations similar to the first step above, we have

$$|\varphi(\varkappa)| = \lambda |T(\varphi)(\varkappa)| \leq \|\omega\| \delta(\|\varphi\|) \Omega,$$

which leads to

$$\frac{\|\varphi\|}{\|\omega\| \delta(\|\varphi\|) \Omega} \leq 1.$$

In view of (A_3) , we can find a positive constant $\mathcal{K}$ satisfying $\|\varphi\| \neq \mathcal{K}$. After that, we define

$$U = \{\varphi \in X, \|\varphi\| < \mathcal{K}\}.$$

Obviously, $T : \overline{U} \rightarrow X$ is continuous and completely continuous. In view of the choice of U there is no $\varphi \in \partial U$ such that $\varphi = \lambda T(\varphi)$ for $\lambda \in (0, 1)$. By the Leray–Schauder nonlinear alternative, T has a fixed point $\varphi \in \overline{U}$, which is a solution of the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (16). This finishes the proof. $\square$

Now, we apply Krasnosel'skiĭ's fixed point theorem to present the second existence result for problem (16).

Lemma 6 ([33]). Let B be a bounded, convex, closed, and nonempty subset of the Banach space X . Let T_1 and T_2 be two operators such that

- (i) $T_1(\varphi_1) + T_2(\varphi_2) \in B$ whenever $\varphi_1, \varphi_2 \in B$,
- (ii) T_1 is compact and continuous,
- (iii) T_2 is a contraction mapping.

Then, there exists $w \in B$ such that $w = T_1(w) + T_2(w)$.

Theorem 3. Let $v : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the assumption (A_1) . Furthermore, we assume that:

(A_4) For all $\varkappa \in [a, b]$ and $\varphi \in \mathbb{R}$, there exists a continuous function $z \in C([a, b], \mathbb{R}^+)$ such that

$$|v(\varkappa, \varphi)| \leq z(\varkappa).$$

Then, the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (16) has at least one solution on $[a, b]$, provided that $\mathcal{L}\Omega' < 1$, where Ω' is given by (22).

Proof. Let $\|z\| = \sup_{a \leq \varkappa \leq b} |z(\varkappa)|$ and consider the closed ball $B_\varrho = \{\varphi \in \mathbb{R} : \|\varphi\| \leq \varrho\}$ with $\varrho \geq \|z\| \Omega$. We decompose the operator T defined by (21) as $T = T_1 + T_2$ on B_ϱ , where T_1 and T_2 are defined as follows:

$$T_1(\varphi)(\varkappa) = {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} v(\varkappa, \varphi(\varkappa)), \quad \varkappa \in [a, b],$$

and

$$\begin{aligned} T_2(\varphi)(\varkappa) = & \frac{1}{\Delta} \left[Q_4 - Q_3 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} v(b, \varphi(b)) \right. \\ & \left. - \sum_{i=1}^m \left(\eta_i {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} v(\xi_i, \varphi(\xi_i)) + b \mu_i {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} v(\xi_i, \varphi(\xi_i)) \right) \right\} \\ & + \frac{1}{\Delta} \left[Q_1 \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} - Q_2 \right] \left\{ b {}^k\mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} v(b, \varphi(b)) \right. \\ & \left. - \sum_{i=1}^m \left(p_i {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} v(\xi_i, \varphi(\xi_i)) + b q_i {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} v(\xi_i, \varphi(\xi_i)) \right) \right\}, \quad \varkappa \in [a, b]. \end{aligned}$$

At this stage, we establish that the operators T_1 and T_2 satisfy the hypotheses of Krasnosel'skii's fixed point theorem. The proof is organized into three steps.

Step I. In the first step, we show that $T_1(\varphi_1) + T_2(\varphi_2) \in B_\varrho$. For any $\varphi_1, \varphi_2 \in B_\varrho$, we find that

$$\begin{aligned} & |T_1(\varphi_1)(\varkappa) + T_2(\varphi_2)(\varkappa)| \\ & \leq {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(\varkappa, \varphi_1(\varkappa))| + \frac{1}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(b, \varphi_2(b))| \right. \\ & \quad \left. + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |v(\xi_i, \varphi_2(\xi_i))| + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |v(\xi_i, \varphi_2(\xi_i))| \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{|\Delta|} \left[|Q_1| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b|^k \mathcal{I}_{a^+}^{\vartheta-\gamma;\psi} |\nu(b, \varphi_2(b))| \right. \\
 & \left. + \sum_{i=1}^m \left(|p_i|^k \mathcal{I}_{a^+}^{\vartheta;\psi} |\nu(\xi_i, \varphi_2(\xi_i))| + |b| |q_i|^k \mathcal{I}_{a^+}^{\vartheta-\sigma_i;\psi} |\nu(\xi_i, \varphi_2(\xi_i))| \right) \right\}. \\
 & \leq \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[1 + \frac{1}{|\Delta|} \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right] \|z\| \\
 & + |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta-\gamma}{k}}}{|\Delta| \Gamma_k(\vartheta - \gamma + k)} \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \|z\| \\
 & + \frac{1}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \left[|\eta_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\
 & \left. + |p_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] \|z\| \\
 & + \frac{|b|}{|\Delta|} \sum_{i=1}^m \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta-\sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \left[|\mu_i| \left\{ |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_4| \right\} \right. \\
 & \left. + |q_i| \left\{ |Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right\} \right] \|z\| \\
 & = \|z\| \Omega \leq \varrho.
 \end{aligned}$$

Therefore $\|T_1(\varphi_1) + T_2(\varphi_2)\| \leq \varrho$, which shows that $T_1(\varphi_1) + T_2(\varphi_2) \in B_\varrho$.

Step II. In this part, we demonstrate the compactness of T_1 by means of the Arzelà–Ascoli theorem. As ν is continuous, the operator T_1 is consequently continuous. We begin by proving that T_1 is uniformly bounded. For any $\varphi \in B_\varrho$, we have

$$|T_1(\varphi)(\varkappa)| \leq \frac{1}{k\Gamma_k(\vartheta)} \int_a^\varkappa \psi'(\tau) (\psi(\varkappa) - \psi(\tau))^{\frac{\vartheta}{k}-1} |\nu(\tau, \varphi(\tau))| d\tau \leq \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \|z\|$$

and consequently

$$\|T_1(\varphi)\| \leq \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \|z\|,$$

which implies that the operator T_1 is uniformly bounded on B_ϱ . Now, we show that T_1 is equicontinuous. Let $\varkappa_1, \varkappa_2 \in [a, b]$ such that $\varkappa_1 < \varkappa_2$. For any $\varphi \in B_\varrho$, we have

$$\begin{aligned}
 & |T_1(\varphi)(\varkappa_2) - T_1(\varphi)(\varkappa_1)| \\
 & \leq \frac{1}{k\Gamma_k(\vartheta)} \left| \int_a^{\varkappa_1} \psi'(\tau) \left((\psi(\varkappa_2) - \psi(\tau))^{\frac{\vartheta}{k}-1} - (\psi(\varkappa_1) - \psi(\tau))^{\frac{\vartheta}{k}-1} \right) \nu(\tau, \varphi(\tau)) d\tau \right. \\
 & \quad \left. + \int_{\varkappa_1}^{\varkappa_2} \psi'(\tau) (\psi(\varkappa_2) - \psi(\tau))^{\frac{\vartheta}{k}-1} \nu(\tau, \varphi(\tau)) d\tau \right| \\
 & \leq \frac{1}{\Gamma_k(\vartheta + k)} \left\{ 2 (\psi(\varkappa_2) - \psi(\varkappa_1))^{\frac{\vartheta}{k}} + \left| (\psi(\varkappa_2) - \psi(a))^{\frac{\vartheta}{k}} - (\psi(\varkappa_1) - \psi(a))^{\frac{\vartheta}{k}} \right| \right\} \|z\|,
 \end{aligned}$$

which implies that

$$|T_1(\varphi)(\varkappa_2) - T_1(\varphi)(\varkappa_1)| \rightarrow 0, \quad \text{as } \varkappa_1 \rightarrow \varkappa_2,$$

independently of $\varphi \in B_\varrho$. Thus, T_1 is equicontinuous. Hence, T_1 is compact on B_ϱ by the Arzelà–Ascoli theorem.

Step III. In the final step, we demonstrate that the operator T_2 is a contraction mapping. Assuming (A_1) and using (23), for any $\varkappa \in [a, b]$ and $\varphi_1, \varphi_2 \in B_\varrho$, we have

$$\begin{aligned} & |T_2(\varphi_1)(\varkappa) - T_2(\varphi_2)(\varkappa)| \\ & \leq \frac{1}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(b, \varphi_1(b)) - \nu(b, \varphi_2(b))| \right. \\ & \quad + \sum_{i=1}^m \left(|\eta_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right. \\ & \quad \left. \left. + |b| |\mu_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right) \right\} \\ & \quad + \frac{1}{|\Delta|} \left[|Q_1| \frac{(\psi(\varkappa) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| {}^k\mathcal{I}_{a^+}^{\vartheta - \gamma; \psi} |\nu(b, \varphi_1(b)) - \nu(b, \varphi_2(b))| \right. \\ & \quad + \sum_{i=1}^m \left(|p_i| {}^k\mathcal{I}_{a^+}^{\vartheta; \psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right. \\ & \quad \left. \left. + |b| |q_i| {}^k\mathcal{I}_{a^+}^{\vartheta - \sigma_i; \psi} |\nu(\xi_i, \varphi_1(\xi_i)) - \nu(\xi_i, \varphi_2(\xi_i))| \right) \right\} \\ & \leq \frac{\mathcal{L} \|\varphi_1 - \varphi_2\|}{|\Delta|} \left[|Q_4| + |Q_3| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} \right] \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} \right. \\ & \quad \left. + \sum_{i=1}^m \left(|\eta_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |\mu_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ & \quad + \frac{\mathcal{L} \|\varphi_1 - \varphi_2\|}{|\Delta|} \left[|Q_1| \frac{(\psi(b) - \psi(a))}{\Gamma_k(2k)} + |Q_2| \right] \left\{ |b| \frac{(\psi(b) - \psi(a))^{\frac{\vartheta - \gamma}{k}}}{\Gamma_k(\vartheta - \gamma + k)} \right. \\ & \quad \left. + \sum_{i=1}^m \left(|p_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} + |b| |q_i| \frac{(\psi(\xi_i) - \psi(a))^{\frac{\vartheta - \sigma_i}{k}}}{\Gamma_k(\vartheta - \sigma_i + k)} \right) \right\} \\ & = \mathcal{L} (\Omega'_1 + \Omega_2 + \Omega_3 + \Omega_4) \|\varphi_1 - \varphi_2\| \\ & = \mathcal{L} \Omega' \|\varphi_1 - \varphi_2\|, \end{aligned}$$

which shows that $\|T_2(\varphi_1) - T_2(\varphi_2)\| \leq \mathcal{L} \Omega' \|\varphi_1 - \varphi_2\|$, and consequently the operator T_2 is a contraction.

Therefore, since all the hypotheses of Krasnosel'skii's fixed point theorem are met, it follows that the (k, ψ) -Caputo fractional boundary value problem with multi-point closed boundary conditions (16) possesses at least one solution on $[a, b]$. $\square$

Remark 1. By reversing the roles of the operators T_1 and T_2 in the aforementioned result, one can establish a second result by enforcing the condition

$$\mathcal{L} \frac{(\psi(b) - \psi(a))^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k)} < 1,$$

instead of $\mathcal{L} \Omega' < 1$.

4. Numerical Examples

In this section, we present examples that illustrate the practical application of our theoretical results. Consider the following (k, ψ) -Caputo fractional derivative involving the multi-point fractional closed boundary conditions of the form

$$\begin{cases} {}^{k,C}D_{a^+}^{\vartheta;\psi} \varphi(\varkappa) = \nu(\varkappa, \varphi(\varkappa)), & 1 \leq \varkappa \leq 2, \\ \varphi(2) = \frac{8}{5} \varphi\left(\frac{5}{4}\right) + \frac{9}{7} \varphi\left(\frac{4}{3}\right) + \frac{1}{2} {}^{6/5,C}D_{a^+}^{\frac{3}{5};\varkappa^2} \varphi\left(\frac{5}{3}\right) + \frac{1}{4} {}^{6/5,C}D_{a^+}^{\frac{2}{5};\varkappa^2} \varphi\left(\frac{7}{4}\right), \\ 2 {}^{6/5,C}D_{a^+}^{\frac{1}{5};\varkappa^2} \varphi(2) = \frac{8}{15} \varphi\left(\frac{5}{4}\right) + \frac{3}{5} \varphi\left(\frac{4}{3}\right) + 2 {}^{6/5,C}D_{a^+}^{\frac{3}{5};\varkappa^2} \varphi\left(\frac{5}{3}\right) + 3 {}^{6/5,C}D_{a^+}^{\frac{2}{5};\varkappa^2} \varphi\left(\frac{7}{4}\right), \end{cases} \quad (25)$$

where $a = 1$, $b = 2$, $\psi(\varkappa) = \varkappa^2$, $k = \frac{6}{5}$, $\vartheta = \frac{3}{2}$, $\gamma = \frac{1}{5}$, $m = 2$, $\sigma_1 = \frac{3}{5}$, $\sigma_2 = \frac{2}{5}$, $\zeta_1 = \frac{5}{4}$, $\zeta_2 = \frac{4}{3}$, $\zeta_3 = \frac{5}{3}$, $\zeta_4 = \frac{7}{4}$, $\eta_1 = \frac{8}{5}$, $\eta_2 = \frac{9}{7}$, $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{1}{8}$, $p_1 = \frac{8}{15}$, $p_2 = \frac{3}{5}$, $q_1 = 1$ and $q_2 = \frac{3}{2}$.

Using the given data, it is found that $Q_1 \approx 1.88571$, $Q_2 \approx 0.167405$, $Q_3 \approx 1.13333$, $Q_4 \approx 3.5914$, $\Delta = Q_1 Q_4 - Q_2 Q_3 \approx 6.58261$, $\Theta_1 \approx 0.97601$, $\Theta_2 \approx 0.741602$, $\Omega_1 \approx 5.48257$, $\Omega'_1 \approx 2.708$, $\Omega_2 \approx 3.85521$, $\Omega_3 \approx 4.19189$, $\Omega_4 \approx 1.54245$, $\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \approx 15.07212$ and $\Omega' = \Omega'_1 + \Omega_2 + \Omega_3 + \Omega_4 \approx 12.29755$.

Illustration of Theorem 1. Assume that the nonlinear function $\nu : [1, 2] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\nu(\varkappa, \varphi) = \frac{e^{-(\varkappa-1)^2}}{4\sqrt{\varkappa+63}} \left(\frac{\varphi^2 + 2|\varphi|}{1 + |\varphi|} \right) + \varkappa^2 + 1. \quad (26)$$

Observe that ν satisfies the Lipschitz condition since

$$|\nu(\varkappa, \varphi_1) - \nu(\varkappa, \varphi_2)| \leq \frac{1}{16} |\varphi_1 - \varphi_2|, \text{ for } \varkappa \in [1, 2], \varphi_1, \varphi_2 \in \mathbb{R}.$$

It is clear that the function ν satisfies the Lipschitz condition (A_1) in Theorem 1 with $\mathcal{L} = \frac{1}{16}$. Furthermore, we obtain $L\Omega \approx 0.942007 < 1$, implying that the inequality in (23) is fulfilled. Therefore, the assumptions of Theorem 1 hold, and therefore the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (25), corresponding to the function in (26), has a unique solution in $[1, 2]$.

Illustration of Theorem 2. Assume that the function $\nu : [1, 2] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\nu(\varkappa, \varphi) = \frac{1}{8(\varkappa+1)^2} \left(\frac{\varphi^4}{1 + \varphi^2} + \cos^2 \varkappa^4 \right) \quad (27)$$

Then we have

$$\nu(\varkappa, \varphi) \leq \frac{1}{8(\varkappa+1)^2} (\varphi^2 + 1).$$

Then, we choose $\omega(\varkappa) = \frac{1}{8(\varkappa+1)^2}$ and $\delta(|\varphi|) = \varphi^2 + 1$. Then, we have $\|\omega\| = \frac{1}{32}$ and there exists a constant $\mathcal{K} \in (0.7053, 1.41783)$ that satisfies the inequality in (24) of Theorem 2. Consequently, Theorem 2 ensures that the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (25), with the function defined in (27), has at least one solution on $[1, 2]$.

Illustration of Theorem 3. Assume that the function $\nu : [1, 2] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\nu(\varkappa, \varphi) = \frac{1}{\sqrt{169 + \ln \varkappa}} \tan^{-1} |\varphi| + \sin^2 \varkappa + 77. \quad (28)$$

Then we have

$$\nu(\varkappa, \varphi) \leq \frac{\pi}{2\sqrt{169 + \ln \varkappa}} + \sin^2 \varkappa + 77 := z(\varkappa).$$

Moreover, ν satisfies the Lipschitz condition as

$$|\nu(\varkappa, \varphi_1) - \nu(\varkappa, \varphi_2)| \leq \frac{1}{13} |\varphi_1 - \varphi_2|.$$

It is clear that the function ν satisfies the Lipschitz condition (A_1) in Theorem 1 with $\mathcal{L} = \frac{1}{13}$. Then, the relation $\mathcal{L}\Omega' \approx 0.94596 < 1$ holds. According to Theorem 3, the (k, ψ) -Caputo fractional boundary value problem involving the multi-point closed boundary conditions (25), with ν given by (28), possesses at least one solution on $[1, 2]$.

5. Conclusions

In this article, we investigated a new class of nonlinear fractional boundary value problems involving the (k, ψ) -Caputo fractional derivative operators under multipoint closed fractional boundary conditions. Even though standard fixed-point approaches were applied to demonstrate the existence and uniqueness results for the problem under consideration, yet our results are novel in the given configuration and generalize several known results in the literature, particularly the ones involving the Caputo and Riemann–Liouville derivative operators with left-sided, right-sided and mixed-type formulations.

Furthermore, the developed framework highlights the versatility of the (k, ψ) -Caputo fractional derivative in capturing diverse dynamical behaviors within fractional-order systems. The theoretical findings presented here not only enrich the mathematical theory of fractional boundary value problems but also offer a foundation for future investigations involving numerical methods, stability analysis, and applications to complex systems in physics and engineering.

In our future work, we plan to explore analogous problems incorporating delay effects, stochastic perturbations, or hybrid fractional operators, as well as extending the current results to higher-dimensional and variable-order settings.

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