

Article

On Relative Stability for Strongly Mixing Sequences

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Abstract

We consider a class of strongly mixing sequences with infinite second moment. This class contains important GARCH processes that are applied in econometrics. We show the relative stability for such processes and construct a counterexample. We apply these results and obtain a new CLT without the requirement of exponential decay of mixing coefficients, and provide a counterexample to this as well.

Keywords: relative stability; central limit theorem; α coefficient

MSC: 60F05; 60G10; 60G42

1. Introduction

Let $\{X_k : k \in \mathbb{Z}\}$ be a strictly stationary sequence defined on a probability space $(\Omega, \mathcal{F}, P)$. In the case of independent random variables, A. Ya. Khinchine in 1926 stated the problem of the relation between the Central Limit Theorem (CLT) for centered summands and the Law of Large Numbers (LLN) (see p. 421 in [1]). Let $U_2(x) = E(|X_1|^2 I(|X_1| \leq x))$, where $I(A)$ is the indicator of the set A , and

$$b_n^2 = \sup\{x > 0; nU_2(x) \geq x^2\}. \quad (1)$$

It turns out that if $\{X_k\}$ is centered, independent and identically distributed (i.i.d.), then the CLT holds if and only if (see §28 in [2])

$$\frac{X_1^2 + \cdots + X_n^2}{b_n^2} \rightarrow_P 1,$$

where $\rightarrow_P$ denotes convergence in probability. The convergence in probability of normalized sums to 1 was named relative stability in 1936 by A. Ya. Khinchine. One of the equivalent conditions for relative stability is the slow variation (in the sense of Karamata) of $U_2(x)$ (see Th. 8.8.1 in [3]). The latter remains true for uniformly strong mixing sequences (see p. 436 in Vol. I, [4]). Here, we concentrate on strongly mixing sequences such that $E(X_1^2) = \infty$. Recall the following measure of dependence

$$\alpha(n) = \alpha_n = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty) = \sup\{|P(B \cap A) - P(A)P(B)|; A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_n^\infty\},$$

where $\mathcal{F}_k^m = \sigma(\{X_i : -\infty \leq k \leq i \leq m \leq \infty\})$ (in notations, definitions, and other explanations, we, in principle, follow [4]). We say that $\{X_k\}$ is strongly (or α -) mixing if $\lim_{n \rightarrow \infty} \alpha(n) = 0$. One of the aims of this paper is to give criteria for relative stability applicable to the CLT. The second aim of this paper is to give an example of a strongly



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mixing sequence that fails to be relatively stable with $\{b_n\}$ defined in (1). Thus, the situation is different from that of uniformly strong mixing sequences.

The paper is organized as follows. In Section 2 the main results are presented. These are new relative stability results for α -mixing sequences with an infinite second moment in terms of the quantile function (see Ch. 10 in [4]) and a counterexample to this result. To the authors' knowledge, there are no such counterexamples in the existing literature. In Section 3, we apply results from the previous section. We obtain a new CLT for martingale differences with infinite variance and CLT with heavy tailed marginals without the assumption of the exponential decay of α_n . Such results are important in modelling volatility phenomena in econometric time series (see [5]).

2. Main Results

Set

$$Q_Z(u) = \inf\{t \in \mathbb{R} : P(Z > t) \leq u\} \quad \text{and} \quad Y_{kn} = X_k^2 I(|X_k| \leq b_n).$$

The following statement gives a sufficient condition for relative stability in the case of α -mixing.

Theorem 1. Suppose $U_2(x)$ is slowly varying such that $E(X_1^2) = \infty$. If

$$n \sum_{k=1}^n \int_0^{\alpha_k} Q_{Y_{1n}}^2(u) du = O(b_n^4) \quad (2)$$

then $\{X_k^2\}$ is relatively stable.

Note that in the case of a finite second moment and the CLT, the “borderline” conditions are formulated regarding the quantile function (see Ch. 10 in [4]).

Proof. THEOREM 1

Write

$$\left(\sum_{k=1}^n x_k\right)^2 = \sum_{k=1}^n x_k^2 + 2 \sum_{v=2}^n \sum_{k=1}^{v-1} x_k x_v.$$

By this and Theorem 1.1 in [6], Theorem 8.1.3 on p. 332 in [3], and (2), we get

$$\begin{aligned} \text{Var}\left(\sum_{k=1}^n \frac{Y_{kn}}{b_n^2}\right) &\leq \frac{n \text{Var}(Y_{1n})}{b_n^4} + \frac{4}{b_n^4} \sum_{v=2}^n \sum_{k=1}^{v-1} \int_0^{\alpha_{v-k}} Q_{Y_{1n}}^2(u) du \\ &\leq o(1) + \frac{4n}{b_n^4} \sum_{k=1}^n \int_0^{\alpha_k} Q_{Y_{1n}}^2(u) du < \infty. \end{aligned}$$

Therefore $\{\frac{\sum_{k=1}^n Y_{kn}}{b_n^2}\}$ is uniformly integrable, so by using the Markov–Bernstein blocking technique (see Sec. 1.17 in Vol. I, [4]), Theorem 2 on p. 140 in [2], and the truncation argument, we obtain Theorem 1. $\square$

Next, consider a strictly stationary Markov chain $\{\xi_k\}$ with a countable number of states and a positive functional f defined on it. Suppose $\{\xi_k\}$ is aperiodic and irreducible and $(\pi_1, \pi_2, \dots)$ is the initial (stationary) distribution. By Theorem 7.7 on p. 212 in Vol. I, [4] $\{f(\xi_k)\}$ is α -mixing. We will follow §16 in [7]. Set

$$\tau_1(\omega) = \min\{n \geq 1; \xi_n(\omega) = i\},$$

$$\tau_{k+1}(\omega) = \min\{n \geq \tau_k(\omega); \xi_n(\omega) = i, \xi_\nu(\omega) \neq i, \tau_k(\omega) \leq \nu < n\}.$$

Let

$$l_n(\omega) = \min\{k \geq 0; \tau_k(\omega) \leq n < \tau_{k+1}(\omega)\}$$

and put $\rho_k = \tau_{k+1} - \tau_k$. It is well-known that $\{\rho_k\}$ is an i.i.d. sequence, $E(\rho_k) = \frac{1}{\pi_i}$, $\frac{l_n}{n} \rightarrow_{\text{a.s.}} \pi_i$, $\frac{\tau_{l_n}}{n} \rightarrow_{\text{a.s.}} 1$, where $\rightarrow_{\text{a.s.}}$ denotes almost sure convergence. We have

$$P(\rho_k = n) = f_{ii}^{(n)} = P(\xi_\nu \neq i, 0 < \nu < n, \xi_n = i | \xi_0 = i).$$

Finally, define

$$Y_k = \sum_{\nu=\tau_k}^{\tau_{k+1}-1} f(\xi_\nu), \quad Y'_n = \sum_{\nu=0}^{\min\{n, \tau_1\}-1} f(\xi_\nu), \quad Y''_n = I_{[l_n \geq 1]} \sum_{\nu=\tau_{l_n}}^n f(\xi_\nu).$$

We have the following dissection formula

$$S_n = \sum_{k=1}^n f(\xi_k) = Y'_n + \sum_{k=1}^{l_n-1} Y_k + Y''_n, \quad (3)$$

where $\{Y_k\}$ is an i.i.d. random sequence and $Y'_n < \infty$ (a.s.), Y''_n is bounded in probability. Thus, by the Khinchine relative stability theorem (cf. Theorem 8.8.1 on p. 373 in [3]), we get Theorem 2.

Theorem 2. Suppose $\{\xi_k\}$ is a strictly stationary Markov chain taking countable number states and f is a positive functional defined on it. Assume $\{\xi_k\}$ is aperiodic and irreducible. Then, $\{f(\xi_k)\}$ is relatively stable if and only if $E(Y_1 I(Y_1 \leq x))$ is a slowly varying function in the sense of Karamata.

In Theorem 2, we cannot replace Y_1 with the marginal.

Theorem 3. There exists an α -mixing homogeneous Markov chain $\{X_k^2\}$ with $E(X_1^2 I(X_1^2 \leq x))$ slowly varying which is not relatively stable with b_n^2 .

Proof. THEOREM 3

Consider the Markov chain $\{\xi_k\}$ with state space $\{1, 2, \dots\}$ and the transition matrix

$$\begin{pmatrix} 1 - \frac{1^2 \ln 3}{2^2 \ln 2} & \frac{1^2 \ln 3}{2^2 \ln 2} & 0 & 0 & \dots \\ 1 - \frac{2^2 \ln 4}{3^2 \ln 3} & 0 & \frac{2^2 \ln 4}{3^2 \ln 3} & 0 & \dots \\ 1 - \frac{3^2 \ln 5}{4^2 \ln 4} & 0 & 0 & \frac{3^2 \ln 5}{4^2 \ln 4} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Set $\mu = 1 + \frac{1}{\ln 2} \sum_{k \geq 1} \frac{\ln(k+2)}{(k+1)^2}$. The stationary initial vector π equals

$$\left(\frac{1}{\mu}, \frac{\ln 3}{\mu 2^2 \ln 2}, \frac{\ln 4}{\mu 3^2 \ln 2}, \frac{\ln 5}{\mu 4^2 \ln 2}, \dots \right)$$

and by Theorem 7.7 on p. in Vol. I, [4] the chain $\{\xi_k\}$ is α -mixing. Further,

$$P(\{\xi_n = n + m\} \cap \{\xi_0 = n\}) = \frac{\ln(n + m + 1)}{\mu(n + m)^2 \ln 2}$$

and therefore for $m = n$ we have

$$\liminf_n n^2 \alpha_n \geq \frac{1}{4\mu \ln 2} \liminf_n \ln(2n + 1) = \infty$$

(cf. Example 7.11 on pp. 217–218 in Vol. I, [4]).

Take $f(x) = \frac{x}{\sqrt{\ln x}}$. Set $X_k = \sqrt{f(\xi_k)}$. Write $g(x) \sim h(x)$ when $\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = 1$. We have

$$\begin{aligned} x^2 P(X_1 > x) &= x^2 \sum_{\frac{k}{\sqrt{\ln k}} > x^2} \frac{\ln(k+1)}{\mu k^2 \ln 2} \\ &\sim x^2 \sum_{k > 2x^2 \ln x} \frac{\ln(k+1)}{\mu k^2 \ln 2} \sim x^2 \frac{\ln(2x^2 \ln x)}{2\mu \ln 2 \cdot x^2 \ln x} \sim \frac{\ln x}{\mu \ln 2 \cdot \ln x} = \frac{1}{\mu \ln 2} \end{aligned}$$

and therefore $E(X_1^2 I(X_1 \leq x)) \sim \frac{\ln x}{\mu \ln 2}$. Thus the normalizing sequence in Theorem 1 is $b_n^2 = c'_n = nE(X_1^2 I(X_1 \leq c'_n)) \sim \frac{n \ln n}{\mu \ln 2}$.

On the other hand,

$$\begin{aligned} f_{11}^{(n)} &= \{\xi_v \neq 1, 0 < v < n, \xi_n = 1 | \xi_0 = 1\} \\ &= \frac{1^2 \ln 3}{2^2 \ln 2} \cdot \frac{2^2 \ln 4}{3^2 \ln 3} \cdots \frac{(n-1)^2 \ln(n+1)}{n^2 \ln n} \cdot \left(1 - \frac{n^2}{(n+1)^2} \frac{\ln(n+2)}{\ln(n+1)}\right) \\ &= \frac{\ln(n+1)}{n^2 \ln 2} - \frac{\ln(n+2)}{(n+1)^2 \ln 2} \sim \frac{2 \ln(n+1)}{n^3 \ln 2} \end{aligned}$$

and $P(Y_1 = \sum_{v=1}^{n-1} \frac{v}{\sqrt{\ln v}}) = f_{11}^{(n)}$. Thus

$$\begin{aligned} P(Y_1 > x) &= \sum_{\sum_{v=1}^{n-1} \frac{v}{\sqrt{\ln v}} > x} \frac{2 \ln(n+1)}{n^3 \ln 2} \sim \sum_{\frac{n^2}{\sqrt{\ln n}} > 2x} \frac{2 \ln(n+1)}{n^3 \ln 2} \\ &\sim \sum_{n > 2\sqrt{x\sqrt{\ln x}}} \frac{2 \ln(n+1)}{n^3 \ln 2} \sim \frac{2 \ln(2\sqrt{x\sqrt{\ln x}})}{8x\sqrt{\ln x} \ln 2} \sim \frac{\sqrt{\ln x}}{8x \ln 2} \end{aligned}$$

and therefore $E(Y_1 I(Y_1 \leq x)) \sim \int_1^x \frac{\sqrt{\ln y}}{8y \ln 2} dy \sim \frac{\ln^{\frac{3}{2}} x}{12 \ln 2}$. Further, the normalizing sequence in Theorem 2 is $c_n \sim \frac{n}{12 \ln 2} \ln^{\frac{3}{2}} n$. Thus by Theorem 2 we have

$$\frac{12 \ln 2}{n \ln^{\frac{3}{2}} n} \sum_{k=1}^n Y_k \rightarrow_P 1 \quad \text{and} \quad \frac{12 \mu \ln 2}{n \ln^{\frac{3}{2}} n} \sum_{k=1}^n Y_k \rightarrow_P 1.$$

So

$$\frac{12 \mu \ln 2}{n \ln^{\frac{3}{2}} n} \sum_{k=1}^n X_k^2 = \frac{12 \mu \ln 2}{n \ln^{\frac{3}{2}} n} \sum_{k=1}^n f(\xi_k) \rightarrow_P 1.$$

Whence for c'_n from Theorem 1

$$\frac{\mu \ln 2}{n \ln n} \sum_{k=1}^n X_k^2 = \frac{\sqrt{\ln n}}{12} \cdot \frac{12 \mu \ln 2}{n \ln^{\frac{3}{2}} n} \sum_{k=1}^n f(\xi_k) \rightarrow_P \infty.$$

Therefore $\{X_k^2\}$ is not relatively stable with $b_n^2 = c'_n$. $\square$

The following result is a consequence of Theorem 3.

Corollary 1. *There exists a strongly mixing (martingale difference) sequence $\{X_k\}$ with $E(X_1^2 I(|X_1| \leq x))$ slowly varying, $E(X_1^2) = \infty$, for which CLT fails to hold under normalizing b_n .*

Proof. COROLLARY 1

Suppose $\{r_k\}$ is a sequence of the Rademacher functions independent of $\{X_k\}$. Now one can, via Theorem 3, obtain an example of a martingale difference sequence $\{r_k X_k\}$ that is strongly mixing and fails to satisfy CLT while the truncated second moment of its marginal varies slowly. $\square$

3. CLT via Principle of Conditioning

The Principle of Conditioning (PC) says that if we transfer the conditions of a limit theorem for row-wise independent random variables in such a way that

- (i) the expectations are replaced by conditional expectations with respect to the past;
- (ii) the convergence of numbers is replaced by convergence in the probability of random variables appearing in the conditions;

then still the conclusion will hold for adapted sequences (see the proof of Th. 2.2.4 in [8]). For example, the Lévy theorem states that if $E(X_1) = 0$, $E(X_1^2) = 1$, then CLT holds for $\{X_k\}$. Thus if $(X_k, \mathcal{F}_k)$ is an adapted sequence and $E(X_k | \mathcal{F}_{k-1}) = 0$ and

$$\frac{E(X_1^2 | \mathcal{F}_0) + \cdots + E(X_n^2 | \mathcal{F}_{n-1})}{n} \rightarrow_P 1$$

then CLT holds, too (because $n = E(X_1^2) + \cdots + E(X_n^2)$). From the latter the Billingsley–Ibragimov theorem follows. However, by this method, only necessary conditions transfer to adapted sequences. Compared with the Markov–Bernstein method, the advantage of the PC is that we avoid the calculation of the dependence coefficient (which might not be easy).

We can apply PC for dependent random variables (e.g., martingale differences (cf. [9])).

Theorem 4. Suppose that $(X_k, \mathcal{F}_k)$ is an identically distributed, adapted sequence with $E(X_1^2 I(|X_1| \leq x))$ slowly varying function (s.v.) and such that $E(X_k | \mathcal{F}_{k-1}) = 0$. If $\{X_k^2\}$ is relatively stable with b_n^2 , then CLT holds.

Proof. THEOREM 4

Set $X_{nk} = E(X_k I(|X_k| \leq b_n))$, $b_n^2 \sim nE(X_1^2 I(|X_1| \leq b_n))$. Since $nP(|X_1| > b_n) \rightarrow 0$, we have to prove that

$$b_n^{-2} \sum_{k=1}^n E((X_{nk} - E(X_{nk} | \mathcal{F}_{k-1}))^2 | \mathcal{F}_{k-1}) \rightarrow_P 1$$

or equivalently,

$$b_n^{-2} \sum_{k=1}^n (E(X_{nk}^2 | \mathcal{F}_{k-1}) - E^2(X_{nk} | \mathcal{F}_{k-1})) \rightarrow_P 1. \quad (4)$$

Now, observe that by Karamata’s theorem (see Theorem 8.1.2 in [3])

$$\frac{x E(|X_1| I(|X_1| > x))}{E(X_1^2 I(|X_1| \leq x))} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

hence

$$b_n^{-2} \sum_{k=1}^n E^2(X_{nk} | \mathcal{F}_{k-1}) \leq b_n^{-2} \sum_{k=1}^n |E(X_{nk} | \mathcal{F}_{k-1})| E|X_k - X_{nk}| \rightarrow_{L^1} 0$$

since $|E(X_{nk} | \mathcal{F}_{k-1})| \leq E|X_k| I(|X_k| > b_n)$. Again, by the Karamata theorem (see Theorem 8.1.3 on p. 332 in [3])

$$\frac{E(X_1^4 I(|X_1| \leq x))}{x^2 E(X_1^2 I(|X_1| \leq x))} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

so we have

$$\begin{aligned} b_n^{-4} E\left(\sum_{k=1}^n (X_{nk}^2 - E(X_{nk}^2 | \mathcal{F}_{k-1}))^2\right) &= b_n^{-4} \sum_{k=1}^n E(X_{nk}^2 - E(X_{nk}^2 | \mathcal{F}_{k-1}))^2 \\ &\leq \frac{2n}{b_n^4} E(X_1^4 I(|X_1| \leq b_n)) \leq C \frac{E(X_1^4 I(|X_1| \leq b_n))}{b_n^2 E(X_1^2 I(|X_1| \leq b_n))} \rightarrow 0. \end{aligned}$$

Thus (4) holds since $\{X_k^2\}$ is r.s. with b_n^2 . To prove the conditional Lindeberg condition

$$b_n^{-2} \sum_{k=1}^n E((X_{nk} - E(X_{nk} | \mathcal{F}_{k-1}))^2 I(|X_{nk} - E(X_{nk} | \mathcal{F}_{k-1})| > \epsilon b_n) | \mathcal{F}_{k-1}) \rightarrow_P 0 \quad (5)$$

we use the inequality

$$2^{-p} E(|X - Y|^p I(|X - Y| > 2z)) \leq E(|X|^p I(|X| > z)) + E(|Y|^p I(|Y| > z))$$

with $p = 2$. Thus

$$b_n^{-2} E\left|\sum_{k=1}^n E(X_{nk}^2 I(|X_{nk}| > \frac{\epsilon}{4} b_n) | \mathcal{F}_{k-1})\right| \leq \frac{n}{b_n^2} E(X_1^2 I(|X_1| \in (\frac{\epsilon}{4} b_n, b_n])) \rightarrow_n 0$$

and

$$\begin{aligned} b_n^{-2} E\left|\sum_{k=1}^n E(E^2(X_{nk} | \mathcal{F}_{k-1}) I(|E(X_{nk} | \mathcal{F}_{k-1})| > \frac{\epsilon}{4} b_n) | \mathcal{F}_{k-1})\right| \\ \leq b_n^{-2} \sum_{k=1}^n E(E^2(X_{nk} | \mathcal{F}_{k-1})) \leq \frac{n}{b_n} E(|X_1| I(|X_1| > b_n)) \rightarrow_n 0. \end{aligned}$$

So (5) is fulfilled and by the PC (see Th. 4.7 in [10]) we get $b_n^{-1} S_n \rightarrow \mathcal{N}(0, 1)$ in distribution. $\square$

Suppose $x^2 P(|X_1| > x) \sim C$, $C \in (0, \infty)$, as $x \rightarrow \infty$. It is not difficult to see that $b_n^2 \sim C n \ln n$. Thus by Theorem 1 and Theorem 4 we have

Theorem 5. Suppose that $(X_k, \mathcal{F}_k)$ is a strictly stationary, adapted sequence such that $E(X_k | \mathcal{F}_{k-1}) = 0$ and $x^2 P(|X_1| > x) \sim C$ as $x \rightarrow \infty$. If $n\alpha_n = O(\ln^3 n)$ then CLT holds.

This is since (2) holds for $\{X_k^2\}$. This result is applicable to GARCH processes (see, e.g., [11]). However, GARCH processes are strongly mixing at an exponential rate. Corollary 1 says that for a slower-than-exponential rate, the CLT may fail.

Proof. THEOREM 5

Set $U_4(x) = E(X_1^4 I(X_1 \leq x))$. By the Hölder inequality and 10.13(III) on p. 319 in Vol. I, [4] we get

$$\int_0^{\alpha_k} \mathcal{Q}_{Y_{1n}}^2(u) du = \int_0^1 I_{[0, \alpha_k)}(u) \mathcal{Q}_{Y_{1n}}^2(u) du \leq \sqrt{\alpha_k U_4(b_n)}.$$

Using the formula

$$U_4(x) = -x^4 P(|X_1| > x) + 4 \int_0^x y^3 P(|X_1| > y) dy$$

and de l'Hôpital's rule we have $U_4(x) \sim Cx^2$. Whence

$$\frac{n}{b_n^4} \sum_{k=1}^n \int_0^{\alpha_k} \mathcal{Q}_{Y_{1n}}^2(u) du \leq \frac{\text{const.}}{\sqrt{n \ln^3 n}} \sum_{k=1}^n \sqrt{\frac{\ln^3 k}{k}} = O(1)$$

so that (2) holds. Thus $\{X_k^2\}$ is relatively stable and we deduce Theorem 5 from Theorem 4. $\square$

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