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Probabilistic Multiple-Integral Evaluation of Odd Dirichlet Beta and Even Zeta Functions and Proof of Digamma-Trigamma Reflections

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Abstract

The aim of this work was to construct explicit expressions for the summation of Dirichlet Beta functions with odd arguments and Zeta functions with even arguments. In the established literature, this is typically done using Fourier series expansions or Bernoulli numbers and polynomials. Here, instead, we achieve our goal by employing tools from probability: specifically, we introduce a generalisation of a technique based on multiple integrals and the algebra of random variables. This also allows us to increase the number of nested integrals and Cauchy random variables involved. Another key contribution is that, by generalising the exponent of Cauchy random variables, we obtain an original proof of the reflection formulae for the Digamma and Trigamma functions. These probabilistic proofs crucially utilise the Mellin transform to compute the integrals needed to determine probability density functions. It is noteworthy that, while understanding the presented topic requires knowledge of the rules for calculating multiple integrals (Fubini's Theorem) and the algebra of continuous random variables, these are concepts commonly acquired by second-year university students in STEM disciplines. Our study thus offers new perspectives on how the mathematical functions considered relate and shows the significant role of probabilistic methods in promoting comprehension of this research area, in a way accessible to a broad and non-specialist audience.

Keywords: multiple integrals; Mellin transform; polygamma functions; generalised Cauchy distributions; quotients of Cauchy-type random variables; symbolic computation

MSC: 33F10; 33B15; 44A30; 60E05; 65D20; 68W30



Academic Editor: Jay M. Jahangiri

Received: 16 June 2025

Revised: 2 August 2025

Accepted: 6 August 2025

Published: 11 August 2025

Citation: Bargellini, A.E.; Ritelli, D.; Spaletta, G. Probabilistic Multiple-Integral Evaluation of Odd Dirichlet Beta and Even Zeta Functions and Proof of Digamma-Trigamma Reflections. *Foundations* **2025**, *5*, 27. <https://doi.org/10.3390/foundations5030027>

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1. Introduction

The famous Basel problem, consisting in the exact evaluation of the sum of the harmonic series of order two:

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

was posed by Pietro Mengoli [1] in 1644, solved by Leonhard Euler in 1734 [2] and, since then, has catalysed the attention of the mathematical community. This problem has, in fact,

fascinating connections and applications in various fields, primarily because it represents a specific value ($s = 2$) of the Riemann Zeta function $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1)$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (2)$$

The relation to $\zeta(s)$ provides, in particular, crucial foundations for models and phenomena in probability and physics and serves as a valuable tool in numerical analysis and teaching.

We refrain from reviewing the large number of alternative techniques for summation of the series in (1). Our focus is on methods based on double integrals, which we generalise and extend by employing multiple integrals that allow the summation of high-order harmonic series.

We adapt the procedure in [3], calculating suitable triple and quadruple integrals and reversing the order of integration, so as to obtain:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \quad (3)$$

The same method also allows the elementary evaluation of a specific value ($s = 3$) of the Dirichlet Beta function $\beta(s)$:

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad (4)$$

$$\beta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}. \quad (5)$$

It is interesting to observe that Euler was the first to deliver an explicit calculation of $\beta(3)$ and that, in his publication [4], he essentially anticipated the summation method based on Fourier series, as mentioned in Section 2.3 of [5].

Furthermore, we prove that the summations like $\zeta(4)$ and $\beta(3)$ can be achieved by adapting the probabilistic approach in [6,7]. While doing so, we obtain a few closed-form evaluations that are already known in the literature. However, our contribution facilitates the explicit evaluation of Zeta at even arguments and Beta at odd arguments, without recourse to Fourier series or Bernoulli numbers. For the latter traditional approach, we refer to [5,8–11]. Our methods establish an intriguing link between multiple integrals and probability theory with the theory of special functions, which can be appreciated even by students at the onset of their scientific careers. Moreover, our proofs for the reflection formulae for the Digamma and Trigamma functions (i.e., the first and second logarithmic derivatives of the Gamma special function) are new, to the best of our knowledge.

We highlight that our further goal in this work is to provide elementary approaches to the exact computations presented, so that they can be used in undergraduate courses in quantitative studies, at least for students exposed to the computation of multiple integrals and the theory of continuous random variables [12]. We also observe that employing multiple integrals and ratios of random variables results in computational constraints when explicitly calculating even-indexed Zeta and odd-indexed Beta values for large arguments. Nonetheless, our objective is to present an alternative computational methodology, applicable even for small arguments, that extends to more general cases using advanced instruments such as Bernoulli polynomials and Fourier series. The Mellin transform, in par-

ticular, plays a crucial role in the evaluation of certain integrals required to determine probability density functions.

The paper is organised as follows. In Section 2, we recall some basic information from the literature and introduce the notations we use, to facilitate reading. In Section 3, we review some notions on the evaluation of $\zeta(s)$ and $\beta(s)$ via the method of multiple integrals, following the framework introduced, among others, in [3]. Section 4 is devoted to the approach based on the probabilistic technique introduced in [6], which involves quotients of Cauchy-type random variables; this yields results related to the evaluation of $\zeta(s)$ for even integers $s > 2$. Regarding the probabilistic approach, we also point out the contribution in [13], which instead uses a hyperbolic secant distribution. In Section 5, we introduce a generalisation of the Cauchy-type random variables, which allows the computation of $\zeta(s)$ and $\beta(s)$ for a generic number $s > 1$, and then, employing the approach in [6], we obtain new proofs of the reflection formulae for the Digamma and Trigamma functions. Some final comments and indications for future work are reported in the concluding Section 6.

2. Background

An exhaustive historical compendium of Euler's discovery on the Basel Problem can be found in [14]. Here, we simply recall that he relied on the infinite product representation of the sine function:

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right). \quad (6)$$

Euler's derivation of (6) was heuristic: he considered the analytic function $\sin(x)/x$ as a polynomial and extended Viète's formula, valid for polynomials only and relating the coefficient of a polynomial to sums and products of its roots. The formal proof arrived only with the contribution of Karl Weierstrass and his famous factorization theorem; refer to Chapter 15 of [15] for more details. Since then, the mathematical community has produced hundreds of other proofs, of which we only review those related to the approaches used in this work, as already mentioned in Section 1.

The first function involved in our study is the Riemann Zeta function $\zeta(s)$ defined by (1), namely, a series with non-negative terms, whose convergence is ensured by the condition $s > 1$ on the the series parameter s . Convergence also holds for all complex numbers $\sigma + i\tau$ such that $\sigma > 1$. Though not needed in this work, we recall that a treatment of $\zeta(s)$ from the point of view of complex analysis is due to Bernard Riemann, who studied its analytic continuation in his 1895 seminal paper on the distribution of prime numbers [16].

Note that, for the actual computation of $\zeta(s)$, it is possible to sum only for odd indexes:

$$\zeta(s) = \frac{2^s}{2^s - 1} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}, \quad (7)$$

since $\zeta(s) = \zeta_{\text{odd}}(s) + \zeta_{\text{even}}(s) = \zeta_{\text{odd}}(s) + \frac{1}{2^s} \zeta(s)$, that is $\zeta(s) \left(1 - \frac{1}{2^s}\right) = \zeta_{\text{odd}}(s)$, where:

$$\begin{aligned} \zeta_{\text{odd}}(s) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots, \\ \zeta_{\text{even}}(s) &= \sum_{n=0}^{\infty} \frac{1}{(2n)^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots = \frac{1}{2^s} \left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots\right) = \frac{1}{2^s} \zeta(s). \end{aligned}$$

In the evaluation of $\zeta(s)$ with even arguments, we thus employ (7), which further shows a connection (at least for s integer and positive [17]) with the second function involved in our study, namely, the Dirichlet Beta function $\beta(s)$ defined by (4).

2.1. Double-Integral Techniques for $\zeta(2)$

A double-integral technique for evaluating $\zeta(2)$ was introduced in [18] (in turn, inspired by [19]):

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy. \quad (8)$$

Expanding the integrand in (8) into a geometric series and exchanging summation and integration, the harmonic series in (1) is obtained; then, using the changes of variable:

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}},$$

and evaluating some arctangent integrals (not reported here), the equality (2) is proved.

Several researchers have considered other double integrals that allow the calculation of $\zeta(2)$; in particular, the authors of [6,20–22] made use of:

$$\int_0^\infty \int_0^\infty \frac{y}{(1+x^2y^2)(1+y^2)} dx dy, \quad (9)$$

and those in [23,24] used:

$$\int_0^1 \int_0^1 \frac{1}{1-x^2y^2} dx dy, \quad (10)$$

while [25] dealt with:

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{xy}(1-xy)} dx dy, \quad (11)$$

and finally [3] utilised:

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2y)(1+y)} dx dy. \quad (12)$$

Using any of the integrals (9), (10), (11) or (12), the summation is obtained:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \quad (13)$$

so that the identity (2) is recovered by means of (7).

To motivate the generalisations, presented in this work, of the procedure used in [3] for the computation of (12) and in [6] to evaluate (9), we recall them briefly in Sections 2.2 and 2.3; note that it suffices to obtain (13) to prove the Basel problem solution (2).

2.2. Double-Integral Technique [3] for $\zeta(2)$

In (12), we first integrate with respect to x , then to y , and use the arctan primitive in x :

$$\int_0^\infty \left(\frac{1}{1+y} \int_0^\infty \frac{dx}{1+x^2y} \right) dy = \frac{\pi}{2} \int_0^\infty \frac{dy}{\sqrt{y}(1+y)} = \frac{\pi^2}{2}. \quad (14)$$

We now revert the order of integration and use partial fractions for the variable y :

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{dy}{(1+y)(1+x^2y)} \right) dx &= \int_0^\infty \frac{1}{1-x^2} \left(\int_0^\infty \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y} \right) dy \right) dx \\ &= \int_0^\infty \frac{1}{1-x^2} \ln \left(\frac{1}{x^2} \right) dx = 2 \int_0^\infty \frac{\ln(x)}{x^2-1} dx. \end{aligned} \quad (15)$$

Equating (14) and (15), we get:

$$\int_0^{\infty} \frac{\ln(x)}{x^2 - 1} dx = \frac{\pi^2}{4}. \quad (16)$$

Even though equality (16) is known, we highlight that, here, we have obtained an elementary way to prove it. Regarding the integrand in (16), it should be specified that the singularity in $x = 1$ is removable, since:

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{x^2 - 1} = \frac{1}{2}.$$

Now, in (16), we split the integration domain into $[0, 1]$ and $[1, \infty)$, and we use the change of variable $x = 1/u$ in $[1, \infty)$:

$$\begin{aligned} \int_0^{\infty} \frac{\ln(x)}{x^2 - 1} dx &= \int_0^1 \frac{\ln(x)}{x^2 - 1} dx + \int_1^{\infty} \frac{-\ln\left(\frac{1}{x}\right)}{\left(1 - \frac{1}{x^2}\right) x^2} dx \\ &= \int_0^1 \frac{\ln(x)}{x^2 - 1} dx + \int_0^1 \frac{\ln(u)}{u^2 - 1} du. \end{aligned} \quad (17)$$

Note that the symmetry in (17) is strictly related to a probabilistic feature of this integral that will be taken into account in the following sections.

From (16) and (17) we get:

$$\int_0^1 \frac{\ln(x)}{x^2 - 1} dx = \frac{\pi^2}{8}. \quad (18)$$

Expanding $1/(1 - x^2)$ into a geometric series, as in [21], and using the monotone convergence theorem of Beppo Levi (see [26], pp. 82–84) yields:

$$\int_0^1 \frac{-\ln(x)}{1 - x^2} dx = \sum_{n=0}^{+\infty} \int_0^1 \left(-x^{2n} \ln(x) \right) dx = \sum_{n=0}^{+\infty} \int_0^1 \frac{x^{2n}}{2n+1} dx = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}. \quad (19)$$

Equating (18) and (19), we arrive again at (13).

2.3. Double-Integral Technique [6] for $\zeta(2)$

Let us now turn to the integral (9) considered in the probabilistic proof given in [6]: its computation involves the concept of non-negative Cauchy random variable X with density function:

$$f_X(x) = \chi\left([0, +\infty)\right) \frac{2}{\pi} \frac{1}{1 + x^2}, \quad (20)$$

where χ denotes the indicator function. A complete reference for random variables is [27]. Given X and Y , independent and identically distributed (i.i.d.) non-negative random variables, with density functions $f_X(x)$ and $f_Y(y)$, respectively, their quotient $T = Y/X$ has density function:

$$f_T(t) = \int_0^{\infty} x f_Y(tx) f_X(x) dx. \quad (21)$$

In the special case of a non-negative Cauchy random variable, by straightforward computation, the quotient rule (21) reduces to:

$$f_T(t) = \frac{4}{\pi^2} \int_0^{\infty} \frac{x}{(1 + x^2)(1 + t^2 x^2)} dx. \quad (22)$$

We compute the integral in (22) by the same approach as (in the inner y -integration) in (15), that is, using a partial fraction decomposition:

$$\int_0^{\infty} \frac{x}{(1+x^2)(1+t^2x^2)} dx = \frac{1}{1-t^2} \int_0^{\infty} \left(\frac{x}{1+x^2} - \frac{t^2x}{1+t^2x^2} \right) dx = \frac{\ln(t)}{t^2-1},$$

so that:

$$f_T(t) = \frac{4}{\pi^2} \frac{\ln(t)}{t^2-1}. \quad (23)$$

The function in (23) is a probability density function (PDF); therefore, recalling that this density vanishes for $t < 0$, and taking into account the independency of X and Y forming $T = Y/X$, we have:

$$\int_0^{\infty} f_T(t) dt = 1,$$

from which formula (16) can be inferred; then, (13) is recovered, once again, through (17)–(19).

2.4. Random Variables

It is worth noting that (16) can also be obtained by considering non-negative i.i.d. random variables that follow $(\sim)$ a Student's t -distribution t_ν with ν degrees of freedom:

$$X \sim \chi([0, +\infty)) \ t_\nu, \quad Y \sim \chi([0, +\infty)) \ t_\nu, \quad \nu = 1.$$

Identity (16) is again recovered, since the density function of the quotient $T = Y/X$ is:

$$f_T(t) = \frac{2}{\pi^2} \frac{\ln(t^2)}{t^2-1}.$$

Between the truncated Cauchy random variable of density function (20) and other popular random variables, we have found further relations that are interesting as they constitute a generalisation of those in [6]. Indeed, (20) can be obtained as a quotient of random variables, whose connection with $\zeta(2)$ can be explicated via a nested procedure. We present three of these relations in the list below, where:

- $N(0, 1)$ denotes a standard normal distribution of mean 0 and standard deviation 1;
 - $\text{Cauchy}(0, 1)$ is a Cauchy distribution of location parameter 0 and scale parameter 1;
 - $\text{Lévy}(0, c)$ denotes a Lévy distribution of location parameter 0 and dispersion parameter $c > 0$.
1. Form $T = Y/X$, where $X \sim \chi([0, +\infty)) \ N(0, 1)$, $Y \sim \chi([0, +\infty)) \ N(0, 1)$. Then $T \sim \chi([0, +\infty)) \ \text{Cauchy}(0, 1)$ with:

$$f_T(t) = \frac{2}{\pi} \frac{1}{1+t^2}.$$

2. As a generalisation of x^2 in (20), it is natural to take into account x^h , with $h > 2$. Let us consider, for example:

$$f_X(x) = \frac{2\sqrt{2}}{\pi} \frac{1}{1+x^4}.$$

Then:

$$f_T(t) = \int_0^{\infty} x f_X(tx) f_X(x) dx = \frac{8}{\pi^2} \int_0^{\infty} \frac{x}{(1+x^4)(1+t^4x^4)} dx = \frac{2}{\pi} \frac{1}{1+t^2}.$$

3. Form $T = Y/X$, where $X \sim \text{Lévy}(0, c)$, $Y \sim \text{Lévy}(0, c)$.
Then $T \sim \chi([0, +\infty)) \text{ Cauchy}(0, 1)$ with:

$$f_T(t) = \frac{1}{\pi} \frac{c}{\sqrt{t} (c + ct)} = \frac{1}{\pi} \frac{1}{\sqrt{t} (1+t)}.$$

Now, letting $t = u^2$, we get the PDF of a $\text{Cauchy}(0, 1)$ distribution:

$$f_T(u) = \frac{2}{\pi} \frac{1}{1+u^2},$$

since:

$$\int_0^\infty f_T(t) dt = \int_0^\infty \frac{2}{\pi} \frac{1}{1+t} \frac{1}{2\sqrt{t}} dt = \int_0^\infty \frac{2}{\pi} \frac{1}{1+u^2} du = \int_0^\infty f_T(u) du = 1.$$

This shows that we can recover the $\zeta(2)$ results using different probabilistic distributions.

3. Multiple-Integral Approach

We now extend and generalise the procedure in Section 2.2. The starting point of our approach consists in extending (12) by considering more than one y -variable, namely, y_i with $i > 1$, where all y_i multiply the quadratic term in the x -variable, so we are led to the evaluation of a multiple integral.

3.1. Triple-Integral Evaluation of $\beta(3)$

Here, we consider two y -variables, y_1 and y_2 , so that a triple integral is formed:

$$I_3 = \iiint_{[0, \infty)^3} \frac{dx dy_1 dy_2}{(1+x^2 y_1 y_2) (1+y_1) (1+y_2)}. \quad (24)$$

By integrating (24) in x first, and then in the variables y_i , we can calculate I_3 by elementary methods. The first step is straightforward since, in the variable x , we have an arctan integral:

$$I_3 = \frac{\pi}{2} \iint_{[0, \infty)^2} \frac{dy_1 dy_2}{\sqrt{y_1 y_2} (1+y_1+y_2+y_1 y_2)}.$$

Then, via the changes of variable $y_1 = u_1^2$, $y_2 = u_2^2$, we arrive at a second and third elementary (arctan) integrations:

$$I_3 = 2\pi \iint_{[0, \infty)^2} \frac{du_1 du_2}{1+u_1^2+u_2^2+u_1^2 u_2^2} = \pi^2 \int_0^\infty \frac{du_2}{1+u_2^2} = \frac{\pi^3}{2}. \quad (25)$$

We now return to (24) and modify the order of integration. This time, the first integration is performed with respect to y_1 so that, after decomposition into partial fractions, we obtain:

$$I_3 = \iint_{[0, \infty)^2} \left(\frac{2x^2 \ln(x)}{(1+x^2)(x^2 y_2 - 1)} - \frac{2 \ln(x)}{(1+x^2)(1+y_2)} + \frac{\ln(y_2)}{(x^2 y_2 - 1)(1+y_2)} \right) dx dy_2.$$

It is convenient to reorganise the double integration as shown below:

$$\begin{aligned}
 I_3 &= \iint_{[0,\infty)^2} \left(\frac{2x^2 \ln(x)}{(1+x^2)(x^2y_2-1)} - \frac{2\ln(x)}{(1+x^2)(1+y_2)} \right) dy_2 dx \\
 &+ \iint_{[0,\infty)^2} \frac{\ln(y_2)}{(x^2y_2-1)(1+y_2)} dx dy_2 \\
 &= \int_0^\infty \left[\frac{2\ln(x) \ln\left(\frac{x^2y_2-1}{1+y_2}\right)}{1+x^2} \right]_{y_2=0}^{y_2 \rightarrow \infty} dx + \int_0^\infty \left[\frac{-\ln(y_2) \tanh^{-1}(x\sqrt{y_2})}{\sqrt{y_2}(1+y_2)} \right]_{x=0}^{x \rightarrow \infty} dy_2 \quad (26) \\
 &= 4 \int_0^\infty \frac{\ln^2(x)}{1+x^2} dx + \int_0^\infty \frac{\ln(y_2)}{\sqrt{y_2}(1+y_2)} dy_2 \\
 &= 4 \int_0^\infty \frac{\ln^2(x)}{1+x^2} dx.
 \end{aligned}$$

Note that the last y_2 -integral in (26) vanishes, as it can be seen by splitting its integration interval and using the change of variable $y_2 = 1/u$ in $[1, \infty)$:

$$\begin{aligned}
 \int_0^\infty \frac{\ln(y_2)}{\sqrt{y_2}(1+y_2)} dy_2 &= \int_0^1 \frac{\ln(y_2)}{\sqrt{y_2}(1+y_2)} dy_2 + \int_1^\infty \frac{-\ln\left(\frac{1}{y_2}\right)}{\left(\frac{1}{\sqrt{y_2}} y_2\right) \left(y_2 \left(\frac{1}{y_2} + 1\right)\right)} dy_2 \\
 &= \int_0^1 \frac{\ln(y_2)}{\sqrt{y_2}(1+y_2)} dy_2 - \int_0^1 \frac{\ln(u)}{\sqrt{u}(1+u)} du = 0.
 \end{aligned}$$

Comparing the value $I_3 = \pi^3/2$ in (25) and the expression of I_3 given by (26), we obtain the identity (27), that will also be useful to evaluate $\zeta(4)$ in Section 3.2:

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2} dx = \frac{\pi^3}{8}. \quad (27)$$

At this point, splitting the integration interval and employing the change of variable $x = 1/u$ in $[1, \infty)$:

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2} dx = \int_0^1 \frac{\ln^2(x)}{1+x^2} dx - \int_1^\infty \frac{\ln^2\left(\frac{1}{x}\right)}{1+\frac{1}{x^2}} \left(\frac{-1}{x^2}\right) dx = \int_0^1 \frac{\ln^2(x)}{1+x^2} dx + \int_0^1 \frac{\ln^2(u)}{1+u^2} du,$$

the following identity is also proved:

$$\int_0^1 \frac{\ln^2(x)}{1+x^2} dx = \frac{\pi^3}{16}. \quad (28)$$

Finally, expanding $1/(1+x^2)$ into a geometric series:

$$\int_0^1 \frac{\ln^2(x)}{1+x^2} dx = \sum_{n=0}^{\infty} \int_0^1 (-1)^n x^{2n} \ln^2(x) dx = 2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3},$$

and recalling (28) and the definition (4) of the Beta function $\beta(s)$, we see that we proved identity (5):

$$\beta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^3} = \frac{\pi^3}{32}.$$

3.2. Quadruple-Integral Evaluation of $\zeta(4)$

Let us recall two useful integrals, commonly presented in standard Calculus courses:

$$\int_0^\infty \frac{\ln(x)}{1+x^2} dx = 0, \quad (29)$$

$$\int_0^\infty \frac{\ln(x)}{1+\alpha x^2} dx = \frac{-\pi}{4} \frac{\ln(\alpha)}{\sqrt{\alpha}}, \quad \alpha > 0. \quad (30)$$

To prove (29), split the integration interval and use the variable change $x = 1/u$ in $[1, \infty)$:

$$\int_0^\infty \frac{\ln(x)}{1+x^2} dx = \int_0^1 \frac{\ln(x)}{1+x^2} dx + \int_0^\infty \frac{\ln\left(\frac{1}{x}\right)}{1+\frac{1}{x^2}} \left(\frac{-1}{x^2}\right) dx = \int_0^1 \frac{\ln(x)}{1+x^2} dx - \int_0^1 \frac{\ln(u)}{1+u^2} du = 0.$$

The proof of (30) follows from (29), using the change of variable $x = u/\sqrt{\alpha}$:

$$\int_0^\infty \frac{\ln(x)}{1+\alpha x^2} dx = \int_0^\infty \left(\frac{\ln(u)}{1+u^2} - \frac{\ln(\alpha)}{2(1+u^2)} \right) \frac{1}{\sqrt{\alpha}} du = \frac{-\ln(\alpha)}{2\sqrt{\alpha}} \int_0^\infty \frac{du}{1+u^2} = \frac{-\ln(\alpha)}{2\sqrt{\alpha}} \frac{\pi}{2}.$$

At this point, we are ready to consider the quadruple integral in (31), where the value $I_4 = \pi^4/2$ can be demonstrated via simple adaptations of the computation performed for $I_3 = \pi^3/2$ in (24), that is, integrating in x first, and then in the y_i -variables:

$$I_4 = \iiint\limits_{[0,\infty)^4} \frac{dx \, dy_1 \, dy_2 \, dy_3}{(1+y_1 y_2 y_3 x^2) (1+y_1) (1+y_2) (1+y_3)} = \frac{\pi^4}{2}. \quad (31)$$

Again, we obtain interesting results by varying the order of integration, starting from any of the variables y_i . For example, here, the first integration is performed with respect to y_3 :

$$\begin{aligned} I_4 &= \iiint\limits_{[0,\infty)^3} \frac{2 \ln(x) + \ln(y_1) + \ln(y_2)}{(x^2 y_1 y_2 - 1) (1+y_1) (1+y_2)} dx \, dy_1 \, dy_2 \\ &= \iiint\limits_{[0,\infty)^3} \frac{2 \ln(x)}{(x^2 y_1 y_2 - 1) (1+y_1) (1+y_2)} dx \, dy_1 \, dy_2 \\ &\quad + \iiint\limits_{[0,\infty)^3} \frac{\ln(y_1) + \ln(y_2)}{(x^2 y_1 y_2 - 1) (1+y_1) (1+y_2)} dx \, dy_1 \, dy_2. \end{aligned} \quad (32)$$

Integrating for x the last integral in (32), which must be understood in the sense of Cauchy principal value (refer to [11], pg. 117), we can see that it vanishes since, for any $a \in \mathbb{R}$, $a \neq 0$, it holds:

$$\int_0^\infty \frac{dx}{a^2 x^2 - 1} = \frac{1}{2a} \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^{1-\varepsilon} \frac{du}{u^2 - 1} + \int_{1+\varepsilon}^\infty \frac{du}{u^2 - 1} \right) = \frac{1}{2a} \lim_{\varepsilon \rightarrow 0^+} \ln\left(\frac{\varepsilon+2}{\varepsilon-2}\right) = 0.$$

Hence, (32) becomes, after partial fraction decomposition and reorganization of the triple integration:

$$\begin{aligned} I_4 &= \iiint\limits_{[0,\infty)^3} \frac{2 \ln(x)}{(x^2 y_1 y_2 - 1) (1+y_1) (1+y_2)} dx \, dy_1 \, dy_2 \\ &= \iiint\limits_{[0,\infty)^3} \frac{2 x^2 y_1 \ln(x)}{(x^2 y_1 y_2 - 1) (1+y_1) (1+x^2 y_1)} dx \, dy_1 \, dy_2 \\ &\quad - \iiint\limits_{[0,\infty)^3} \frac{2 \ln(x)}{(1+x^2 y_1) (1+y_1) (1+y_2)} dy_2 \, dx \, dy_1. \end{aligned} \quad (33)$$

Note that the last integral in (33) vanishes by performing integration with respect to x and y_1 , in any order. The current I_4 is now integrated with respect to y_2 :

$$\begin{aligned} I_4 &= \iiint_{[0,\infty)^3} \frac{2x^2 y_1 \ln(x)}{(x^2 y_1 y_2 - 1)(1+y_1)(1+x^2 y_1)} dx dy_1 dy_2 \\ &= \iint_{[0,\infty)^2} \left[\frac{2 \ln(x) \left(\ln \left| \frac{x^2 y_1 y_2 - 1}{1+y_2} \right| \right)}{(1+x^2 y_1)(1+y_1)} \right]_{y_2=0}^{y_2 \rightarrow \infty} dx dy_1 \\ &= \iint_{[0,\infty)^2} \frac{4 \ln^2(x)}{(1+x^2 y_1)(1+y_1)} dx dy_1 + \iint_{[0,\infty)^2} \frac{2 \ln(x) \ln(y_1)}{(1+x^2 y_1)(1+y_1)} dx dy_1 \\ &= I_4^{(a)} + I_4^{(b)}. \end{aligned} \quad (34)$$

We treat the above two integrals $I_4^{(a)}$ and $I_4^{(b)}$ separately. Let us consider $I_4^{(b)}$ first, performing the integration in the order shown in (35):

$$I_4^{(b)} = \int_0^\infty \frac{2 \ln(y_1)}{1+y_1} \left(\int_0^\infty \frac{\ln(x)}{1+x^2 y_1} dx \right) dy_1. \quad (35)$$

Using (30), employing the variable change $y_1 = u^2$ and recalling the identity (27), we obtain:

$$I_4^{(b)} = -\frac{\pi}{2} \int_0^\infty \frac{\ln^2(y_1)}{(1+y_1)\sqrt{y_1}} dy_1 = -4\pi \int_0^\infty \frac{\ln^2(u)}{1+u^2} du = -\frac{\pi^4}{2}.$$

Turning to $I_4^{(a)}$, the order of (elementary) integration is shown in (36), where decomposition in partial fractions is also used, along with the consequent simplifications due to vanishing integrals:

$$\begin{aligned} I_4^{(a)} &= \int_0^\infty \left(\frac{4 \ln^2(x)}{(1+x^2 y_1)(1+y_1)} dy_1 \right) dx = \int_0^\infty \left[\frac{4 \ln^2(x) \ln \left(\frac{1+x^2 y_1}{1+y_1} \right)}{x^2 - 1} \right]_{y_1=0}^{y_1 \rightarrow \infty} dx \\ &= 8 \int_0^\infty \frac{\ln^3(x)}{x^2 - 1} dx. \end{aligned} \quad (36)$$

Putting I_4 back together:

$$I_4 = 8 \int_0^\infty \frac{\ln^3(x)}{x^2 - 1} dx - \frac{\pi^4}{2}, \quad (37)$$

and comparing the expression (37) of I_4 with the value $I_4 = \pi^4/2$ in (31), we obtain the identity (38) and observe its interesting analogy with (27):

$$\int_0^\infty \frac{\ln^3(x)}{x^2 - 1} dx = \frac{\pi^4}{8}. \quad (38)$$

Moreover, the same splitting technique that was used to obtain (28) provides:

$$\int_0^1 \frac{\ln^3(x)}{x^2 - 1} dx = \frac{\pi^4}{16}. \quad (39)$$

Expanding $1/(x^2 - 1)$ into a geometric series yields:

$$\int_0^1 \frac{\ln^3(x)}{x^2 - 1} dx = \sum_{n=0}^{\infty} \int_0^1 (-x^{2n} \ln^3(x)) dx = \sum_{n=0}^{\infty} \frac{6}{(2n+1)^4}, \quad (40)$$

where we have used the integral relation:

$$-\int_0^1 x^{2n} \ln^3(x) dx = \frac{6}{(2n+1)^4}.$$

Finally, comparing (39) and (40):

$$\sum_{n=0}^{\infty} \frac{6}{(2n+1)^4} = \frac{\pi^4}{16},$$

and recalling the definition (7) of the Zeta function $\zeta(s)$, we see that we have proved the identity (3):

$$\zeta(4) = \frac{16}{15} \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \right) = \frac{16}{15} \left(\frac{1}{6} \frac{\pi^4}{16} \right) = \frac{\pi^4}{90}.$$

This procedure can be extended to multiple integrals. For example, we can consider a quintuple integral:

$$I_5 = \int \cdots \int_{[0,\infty)^5} \frac{dx dy_1 dy_2 dy_3 dy_4}{(1+y_1 y_2 y_3 y_4 x^2) (1+y_1) (1+y_2) (1+y_3) (1+y_4)} = \frac{\pi^5}{2},$$

which leads to the evaluation of $\beta(5)$:

$$\beta(5) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

The explicit computations are omitted due to their complexity; however, the procedure follows the same steps outlined in this Section 3 and can be extended to any order.

4. Further Results on Quotients of Cauchy Distributions

Here, we illustrate a generalisation of the approaches in [6,7] achieved through the iterative application of the quotient $T = Y/X$ to Cauchy-like random variables. We show that the constructed procedure provides results comparable to those obtained with the multiple-integral technique in Section 3, and that it is computationally faster as the problem dimension increases, in particular starting from the quintuple-integral evaluation, thanks to the exploitation of the probabilistic approach.

In essence, beginning with the quotient of two random variables defined as in (20), we iteratively apply rule (21), obtaining results for the quotient of four Cauchy-like random variables; the idea is extensible to quotients of an even number of random variables. Then, we generalise further, by considering the quotient of an odd number (three, five and so on) of random variables, always defined by (20).

These generalised quotients involve integrals of the form $\int_0^{\infty} \frac{\ln^k(x)}{(x-a)(x-b)} dx$; we illustrate how to evaluate them through the general formula (45) and provide their values for $k = 1, \dots, 10$ in Table 1.

Table 1. Values of $\int_0^\infty \frac{\ln^k(x)}{(x-a)(x-b)} dx$ for $k = 1, \dots, 10$.

$\frac{-(\log a)^2 + (\log b)^2}{2a - 2b}$
$\frac{-(\log a)^3 + 2\pi^2 \log a - 2\pi^2 \log b + (\log b)^3}{3a - 3b}$
$\frac{-(\log a)^4 + 4\pi^2 (\log a)^2 - 4\pi^2 (\log b)^2 + (\log b)^4}{4a - 4b}$
$\frac{-3(\log a)^5 + 20\pi^2 (\log a)^3 + 8\pi^4 \log a - 8\pi^4 \log b - 20\pi^2 (\log b)^3 + 3(\log b)^5}{15a - 15b}$
$\frac{-(\log a)^6 + 10\pi^2 (\log a)^4 + 8\pi^4 (\log a)^2 - 8\pi^4 (\log b)^2 - 10\pi^2 (\log b)^4 + (\log b)^6}{6a - 6b}$
$\frac{-3(\log a)^7 + 42\pi^2 (\log a)^5 + 56\pi^4 (\log a)^3 + 32\pi^6 \log a - 32\pi^6 \log b - 56\pi^4 (\log b)^3 - 42\pi^2 (\log b)^5 + 3(\log b)^7}{21a - 21b}$
$\frac{-3(\log a)^8 + 56\pi^2 (\log a)^6 + 112\pi^4 (\log a)^4 + 128\pi^6 (\log a)^2 - 128\pi^6 (\log b)^2 - 112\pi^4 (\log b)^4 - 56\pi^2 (\log b)^6 + 3(\log b)^8}{24a - 24b}$
$\frac{-5(\log a)^9 + 120\pi^2 (\log a)^7 + 336\pi^4 (\log a)^5 + 640\pi^6 (\log a)^3 + 384\pi^8 \log a - 384\pi^8 \log b - 640\pi^6 (\log b)^3 - 336\pi^4 (\log b)^5 - 120\pi^2 (\log b)^7 + 5(\log b)^9}{45a - 45b}$
$\frac{-(\log a)^{10} + 30\pi^2 (\log a)^8 + 112\pi^4 (\log a)^6 + 320\pi^6 (\log a)^4 + 384\pi^8 (\log a)^2 - 384\pi^8 (\log b)^2 - 320\pi^6 (\log b)^4 - 112\pi^4 (\log b)^6 - 30\pi^2 (\log b)^8 + (\log b)^{10}}{10a - 10b}$
$\frac{-3(\log a)^{11} + 110\pi^2 (\log a)^9 + 528\pi^4 (\log a)^7 + 2112\pi^6 (\log a)^5 + 4224\pi^8 (\log a)^3 + 2560\pi^{10} \log a - 2560\pi^{10} \log b - 4224\pi^8 (\log b)^3 - 2112\pi^6 (\log b)^5 - 528\pi^4 (\log b)^7 - 110\pi^2 (\log b)^9 + 3(\log b)^{11}}{33a - 33b}$

4.1. Quotients of Four Cauchy Random Variables

As shown in [6], the quotient of two i.i.d. truncated Cauchy random variables leads to the evaluation of a series related to $\zeta(2)$. Following a similar idea, we apply the iterative procedure mentioned at the beginning of Section 4 to discover if there are other relations connected to well-known series.

We consider the quotient R of four truncated Cauchy random variables, each defined as in (20). Taking into account that the quotient of two truncated Cauchy random variables has PDF given by (23), we apply rule (21), so that:

$$f_R(t) = \frac{16}{\pi^4} \int_0^\infty \frac{x \ln(x) \ln(tx)}{(x^2 - 1)(t^2 x^2 - 1)} dx = \frac{8 \ln(t)}{3 \pi^4} \frac{\ln^2(t) + \pi^2}{t^2 - 1}. \quad (41)$$

The integral in (41) can be evaluated directly with computer algebra tools, such as those of *Mathematica*, Version 14.2 [28]; however, its computation deserves to be explored in its foundations, as it is not based on elementary techniques. Notice that the right-hand side of (41) is well defined for $t > 0$, since the singularity at $t = 1$ is apparent.

Using the properties of the logarithm and changing variable as $x^2 = u$, the integral in (41) can be rewritten as:

$$\begin{aligned} f_R(t) &= \frac{16}{\pi^4} \int_0^\infty \frac{x \ln(x) \ln(tx)}{(x^2 - 1)(t^2 x^2 - 1)} dx = \frac{2}{\pi^4} \int_0^\infty \frac{2 \ln(x) (2 \ln(t) + 2 \ln(x))}{(x^2 - 1)(t^2 x^2 - 1)} 2x dx \\ &= \frac{4 \ln(t)}{\pi^4} \int_0^\infty \frac{\ln(u)}{(u - 1)(t^2 u - 1)} du + \frac{2}{\pi^4} \int_0^\infty \frac{\ln^2(u)}{(u - 1)(t^2 u - 1)} du. \end{aligned} \quad (42)$$

The two integrals in (42), each of which should be interpreted using the Cauchy principal value due to singularities in $x = 1$ and $x = 1/t^2$, are tabulated at page 579 of [29]. The latter contains a collection of integration tables that constitute the first systematic reference and the base for the current repertoires and libraries available within the symbolic software environment presently used.

Even so, given their non-algebraic structure, the procedure for computing the two integrals in (42) deserves to be illustrated, at least in broad terms.

The starting point for their interpretation is the Mellin transform. Given a function $f : [0, +\infty] \rightarrow \mathbb{C}$, its Mellin transform:

$$\mathcal{M}_f(s) = \int_0^{+\infty} x^{s-1} f(x) dx, \quad (43)$$

is defined for those $s \in \mathbb{C}$ such that the integral (43) converges. Let us consider:

$$f(x) = \frac{1}{(x - a)(x - b)}, \quad a > b > 0,$$

whose Mellin transform is, for $s < 2$ (see [30], table 2.1.3, entry 3; see also [31], table 6.2, entry 9):

$$\mathcal{M}_f(s) = \pi \cot(\pi s) \frac{a^{s-1} - b^{s-1}}{b - a}. \quad (44)$$

Here, too, the principal value must be considered in the evaluation of the integral transform. At this point, we consider the transform (44) in terms of a definite integral and, using $s + 1$ in place of s and exploiting the periodicity of the cotangent function, we obtain:

$$\int_0^\infty \frac{x^s}{(x - a)(x - b)} dx = \pi \cot(\pi s) \frac{a^s - b^s}{b - a}. \quad (45)$$

Differentiating the identity (45)—which is ‘known’ by *Mathematica* – with respect to s (refer, for example, to Section 1.6 of [32]):

$$\int_0^\infty \frac{x^s \ln(x)}{(x-a)(x-b)} dx = \frac{\pi \left(\pi (a^s - b^s) - \sin(\pi s) \cos(\pi s) (a^s \ln(a) - b^s \ln(b)) \right)}{(a-b) \sin(\pi s)^2}. \quad (46)$$

Taking the limit for $s \rightarrow 0^+$ on both sides of (46), where the right-hand side must be expanded in the McLaurin series, we see that the first integral in (42) can be obtained from:

$$\int_0^\infty \frac{\ln(x)}{(x-a)(x-b)} = \frac{\ln^2(b) - \ln^2(a)}{2(a-b)}. \quad (47)$$

By deriving both sides of (45) a second time, and via a more intricate calculation:

$$\int_0^\infty \frac{\ln^2(x)}{(x-a)(x-b)} dx = \frac{2\pi^2 \left(\ln(a) - \ln(b) \right) - \left(\ln^3(a) - \ln^3(b) \right)}{3(a-b)}. \quad (48)$$

Formula (41) follows from (47) and (48) taking:

$$\begin{aligned} a &= 1, \quad b = \frac{1}{t^2} && \text{when } t > 1, \\ a &= \frac{1}{t^2}, \quad b = 1 && \text{when } 0 < t < 1. \end{aligned}$$

Since (41) is a PDF derived from the quotient of i.i.d. random variables, we know that [6]:

$$\begin{aligned} \frac{1}{2} &= \int_0^1 f_R(t) dt \\ &= \frac{8}{3\pi^4} \int_0^1 \frac{\ln(t) (\ln^2(t) + \pi^2)}{t^2 - 1} dt = \frac{-8}{3\pi^4} \int_0^1 \sum_{n=0}^\infty t^{2n} \ln(t) (\ln^2(t) + \pi^2) dt, \end{aligned} \quad (49)$$

where we used the fact that $1/(t^2 - 1)$ can be expanded into a geometric series. Now, we observe that:

$$\int_0^1 t^\alpha \ln^k(t) dt = \frac{(-1)^k \Gamma(k+1)}{(\alpha+1)^{k+1}}, \quad \text{for } \alpha > -1, \quad k > -1. \quad (50)$$

To go ahead with the evaluation of (49), we are interested in (50) with $k = 1, 2, 3$. For completeness, we demonstrate that (50) holds for the case $k = 3$ (the other cases can be proved in a similar way), using the variable change $\ln(t) = -z$ and a known integral result that involves the Gamma function $\Gamma(k)$ [32]:

$$\int_0^1 t^\alpha \ln^3(t) dt = \int_0^1 e^{\alpha \ln(t)} \ln^3(t) dt = - \int_0^\infty e^{-(\alpha+1)z} z^3 dz = \frac{-\Gamma(4)}{(\alpha+1)^4} = \frac{-6}{(\alpha+1)^4}.$$

At this point, using the monotone convergence theorem, as well as formula (50) with $k = 1, 2, 3$, and a straightforward partial fraction decomposition, Equation (49) becomes:

$$\frac{8}{3\pi^4} \sum_{n=0}^\infty \left(\frac{6}{(2n+1)^4} + \frac{\pi^2}{(2n+1)^2} \right) = \frac{1}{2},$$

that is:

$$\sum_{n=0}^\infty \frac{6}{\pi^4 (2n+1)^4} + \sum_{n=0}^\infty \frac{1}{\pi^2 (2n+1)^2} = \frac{3}{16}.$$

Since we can assume the identity (13), we infer that:

$$\sum_{n=0}^{\infty} \frac{6}{\pi^4 (2n+1)^4} = \frac{3}{16} - \frac{1}{8} = \frac{1}{16},$$

from which:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

The evaluation (3) of $\zeta(4)$ follows again from (7).

4.2. Quotients of Three Cauchy Random Variables

The evaluation of the quotient of an odd number of Cauchy random variables allows the explicit computation of odd-order Dirichlet Beta values. Here, we show how this approach leads to the evaluation of $\beta(3)$; then, the same procedure can be iterated to include larger odd arguments.

We consider the quotient S of three truncated Cauchy random variables, each defined as in (20). Taking into account that the quotient of two truncated Cauchy random variables has PDF given by (23), we apply rule (21), so that:

$$f_S(t) = \frac{8}{\pi^3} \int_0^{\infty} \frac{x \ln(tx)}{(x^2+1)(t^2x^2-1)} dx = \frac{1}{\pi(t^2+1)} \left(1 + \frac{4 \ln^2(t)}{\pi^2}\right). \quad (51)$$

Using the properties of the logarithm and changing variable as $x^2 = u$, the integral in (51) can be rewritten as:

$$\begin{aligned} f_S(t) &= \frac{8}{\pi^3} \int_0^{\infty} \frac{x \ln(tx)}{(x^2+1)(t^2x^2-1)} dx = \frac{2}{\pi^3} \int_0^{\infty} \frac{2 \ln(t) + 2 \ln(x)}{(x^2+1)(t^2x^2-1)} 2x dx \\ &= \frac{4 \ln(t)}{\pi^3} \int_0^{\infty} \frac{1}{(u+1)(t^2u-1)} du + \frac{2}{\pi^3} \int_0^{\infty} \frac{\ln(u)}{(u+1)(t^2u-1)} du. \end{aligned} \quad (52)$$

Even if the integral (51), as well as those in (52), can be treated symbolically by *Mathematica*, it seems appropriate to examine the steps that lead to these integral calculation, especially since the last integral in (52) is related to the identity (53), which is not tabulated in the classical repertoires:

$$\int_0^{\infty} \frac{\ln(x)}{(x+a)(x-b)} dx = \frac{\ln^2(a) - \ln^2(b) + \pi^2}{2(a+b)}, \quad a, b > 0. \quad (53)$$

Formula (53), which, as mentioned, does not appear in the repertoires, and which should be understood in the sense of the Cauchy principal value, comes from the derivation of a Mellin transform. In fact, if we consider the following (translated) Mellin transform (see [30], table 2.1.3, entry 2):

$$\int_0^{\infty} \frac{x^s}{(x+a)(x-b)} dx = \frac{\pi}{a+b} \left(\frac{a^s}{\sin(\pi s)} - b^s \cot(\pi s) \right), \quad a, b > 0, \quad s < 1, \quad (54)$$

then, we see that (53) is obtained by computing the derivative in $s = 0$ of both sides of (54). At this point, the last integral in (52) can be computed taking $a = 1$ and $b = 1/t^2$ in (53):

$$\int_0^{\infty} \frac{\ln(u)}{(u+1)(t^2u-1)} du = \frac{\pi^2 - 4 \ln^2(t)}{2(t^2+1)}, \quad (55)$$

while the remaining integral in (52) is elementary, even if we need to consider the Cauchy principal value:

$$\int_0^\infty \frac{1}{(u+1)(t^2u-1)} du = \frac{2 \ln(t)}{t^2+1}. \quad (56)$$

Formula (51) follows from (55) and (56).

Since (51) is a PDF derived from the quotient of i.i.d. random variables, we know that [6]:

$$\frac{1}{2} = \int_0^1 f_S(t) dt = \frac{1}{\pi} \int_0^1 \frac{1}{t^2+1} dt + \frac{4}{\pi^3} \int_0^1 \frac{\ln^2(t)}{t^2+1} dt. \quad (57)$$

Computing:

$$\frac{1}{\pi} \int_0^1 \frac{1}{t^2+1} dt = \frac{1}{4},$$

we see that, from (57), we have derived formula (28) in a different way.

Hence, we can replicate the integration by series, leading to the explicit evaluation (5) of $\beta(3)$.

5. Generalised Cauchy Distributions

To appreciate the results of this section, as well as for ease of reading, we briefly recall the Gamma, Digamma and Trigamma functions and, in particular, the reflection formulae satisfied by these Eulerian functions (refer, for example, to Chapters 2 and 3 of [32]).

For $x > 0$, the Gamma function is defined by the Eulerian integral of the second kind:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (58)$$

The Gamma reflection formula states:

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad \forall x \in [0, 1]. \quad (59)$$

To explain the origin of this important formula, note that it is connected to Euler's representation of $\Gamma(x)$ as an infinite product:

$$\Gamma(x) = \frac{1}{x} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^x}{1 + \frac{x}{k}}. \quad (60)$$

The infinite product (60) converges uniformly on every compact of the complex plane that does not contain negative integers. Therefore, it provides the analytical continuation of the integral (58). Formulae (59) and (60) relate to the Euler representation (6) of the sine function as an infinite product.

The Digamma function is the logarithmic derivative of Gamma:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

and admits the following series representation, where γ is the Euler–Mascheroni number:

$$\psi(x) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n-1+x} \right), \quad \gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right), \quad (61)$$

while the Digamma reflection formula follows from (59):

$$\psi(1-x) - \psi(x) = \frac{\pi}{\tan(\pi x)}, \quad \forall x \in [0, 1]. \quad (62)$$

The Trigamma function is the logarithmic derivative of Digamma. It has a series representation given by:

$$\psi_1(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1+x)^2},$$

and its reflection formula is:

$$\psi_1(1-x) + \psi_1(x) = \frac{\pi^2}{\sin^2(\pi x)}, \quad \forall x \in [0, 1]. \quad (63)$$

The proofs of the Digamma and Trigamma reflection formulae (62) and (63), existing in the literature, primarily rely on the reflection formula for the Gamma function in conjunction with its logarithmic derivatives, while our approach is direct and founded on the algebra of random variables.

5.1. Probabilistic Proof of the Digamma Reflection Formula

We start from the definition of a generalised Cauchy random variable. It is known that (see [32], pg. 97):

$$\int_0^{\infty} \frac{1}{1+x^n} dx = \frac{\pi}{n \sin(\frac{\pi}{n})}, \quad n > 1. \quad (64)$$

In what follows, we assume $n > 1$, $n \neq 2$, since we already studied the case $n = 2$ in detail.

Using normalisation in (64), we can introduce a positive random variable whose distribution can be called generalised Cauchy distribution:

$$f_X(x) = \frac{n \sin(\frac{\pi}{n})}{\pi} \frac{1}{1+x^n}.$$

Given two i.i.d. generalised Cauchy random variables X, Y , their quotient $T = Y/X$ has density:

$$f_T(t) = \frac{n \tan(\frac{\pi}{n})}{2\pi} \frac{1-t^{n-2}}{1-t^n}. \quad (65)$$

In fact, using the quotient rule (21) and the change of variable $u = x^n$, we have:

$$f_T(t) = \frac{n^2 \sin^2(\frac{\pi}{n})}{\pi^2} \int_0^{\infty} \frac{x}{(1+x^n)(1+t^n x^n)} dx = \frac{n \sin^2(\frac{\pi}{n})}{\pi^2} \int_0^{\infty} \frac{u^{\frac{2}{n}-1}}{(1+u)(1+t^n u)} du, \quad (66)$$

where the integral in (66) is a Mellin one, given explicitly by entry 1 of table 2.1.3 in [30]:

$$\int_0^{\infty} \frac{u^{\frac{2}{n}-1}}{(1+u)(1+t^n u)} du = \frac{\pi}{\sin(\frac{2\pi}{n})} \frac{1-t^{n-2}}{1-t^n}.$$

It is interesting to note that (65) is undefined for $n = 2$, which is the situation studied in [6] and in Section 4 of this paper. If we take the limit for $n \rightarrow 2$, we find formula (23). The occurrence of the logarithmic term in the limit procedure is the key element linking the case $n = 2$ to the Basel problem via the logarithmic integral in Equation (16).

Using the fact that X and Y are identically distributed, the integration of (65) yields:

$$\int_0^1 f_T(t) dt = \frac{1}{2},$$

therefore:

$$\frac{\pi}{n \tan(\frac{\pi}{n})} = \int_0^1 \frac{1-t^{n-2}}{1-t^n} dt. \quad (67)$$

Since $1/(1-t^n)$ can be expanded into a geometric series, we infer (68), using the monotone convergence theorem of Beppo Levi and integrating term by term:

$$\frac{1}{n} \frac{\pi}{\tan\left(\frac{\pi}{n}\right)} = \sum_{k=0}^{\infty} \left(\frac{1}{kn+1} - \frac{1}{kn+n-1} \right). \quad (68)$$

Formula (68) is closely related to the Digamma reflection formula (62); indeed, we can rewrite the right-hand side of (68) as follows:

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{1}{kn+1} - \frac{1}{kn+n-1} \right) &= \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{1}{k+\frac{1}{n}} - \frac{1}{k+1-\frac{1}{n}} \right) \\ &= \frac{1}{n} \sum_{h=1}^{\infty} \left(\frac{1}{(h-1)+\frac{1}{n}} - \frac{1}{(h-1)+1-\frac{1}{n}} \right) \\ &= \frac{1}{n} \sum_{h=1}^{\infty} \left(\frac{1}{h} - \frac{1}{h-\frac{1}{n}} + \frac{1}{h-1+\frac{1}{n}} - \frac{1}{h} \right) \\ &= \frac{1}{n} \left(\sum_{h=1}^{\infty} \left(\frac{1}{h} - \frac{1}{h-1+\frac{1}{n}} \right) - \sum_{h=1}^{\infty} \left(\frac{1}{h} - \frac{1}{h-1+\frac{1}{n}} \right) \right), \end{aligned} \quad (69)$$

so that, comparing (68)–(69) with (61), the Digamma reflection formula is recovered, with $x = 1/n$:

$$\frac{\pi}{\tan\left(\frac{\pi}{n}\right)} = \psi\left(1 - \frac{1}{n}\right) - \psi\left(\frac{1}{n}\right). \quad (70)$$

Note that the right-hand side of (70) is defined in $n = 2$, where it can be evaluated exactly and becomes zero; this coincides with the limit, as $n \rightarrow 2$, of the left-hand side of (70), which approaches zero through the degenerate form $1/\infty$. When $n = 1$, both sides lose their meaning, as they both diverge to ∞ .

5.2. Probabilistic Proof of the Trigamma Reflection Formula

Now, let us consider the quotient of two quotients, whose PDF is given by:

$$f_Q(t) = Q \frac{2\pi \cot\left(\frac{2\pi}{n}\right) (1-t^{n-2}) - n(1+t^{n-2}) \ln(t)}{1-t^n}, \quad Q = \frac{n \tan^2\left(\frac{\pi}{n}\right)}{4\pi^2}. \quad (71)$$

In fact, using the quotient rule (21), the identity (65) and the change of variable $u = x^n$:

$$\begin{aligned} f_Q(t) &= n Q \int_0^{\infty} \frac{x(1-x^{n-2})(1-t^{n-2}x^{n-2})}{(1-x^n)(1-t^n x^n)} dx \\ &= Q \int_0^{\infty} \frac{t^{n-2} u^{-(\frac{2}{n}-1)} + u^{\frac{2}{n}-1} - t^{n-2} - 1}{(1-u)(1-t^n u)} du \\ &= Q \frac{2\pi \cot\left(\frac{2\pi}{n}\right) (1-t^{n-2}) + n(1+t^{n-2}) \ln(t)}{1-t^n}, \end{aligned}$$

where, as in (45), we used entry 3 of table 2.1.3 in [30], while the logarithmic term comes from the Cauchy principal value of the integral:

$$\int_0^{\infty} \frac{du}{(1-u)(1-t^n u)} = \frac{n \ln(t)}{1-t^n}.$$

Exploiting again the fact that the random variables are i.i.d., we have:

$$\frac{1}{Q} \int_0^1 f_Q(t) dt = \frac{1}{2Q},$$

that is:

$$\frac{2\pi^2 \cot^2\left(\frac{\pi}{n}\right)}{n} = 2\pi \cot\left(\frac{2\pi}{n}\right) \int_0^1 \frac{1-t^{n-2}}{1-t^n} dt + n \int_0^1 \frac{(1+t^{n-2}) \ln(t)}{1-t^n} dt. \quad (72)$$

Recalling (67) and expanding $1/(1-t^n)$ into a geometric series, the passage to the limit is once again legitimised by the monotone convergence theorem, and we can rewrite the right-hand side of (72) as follows:

$$\begin{aligned} & 2\pi \cot\left(\frac{2\pi}{n}\right) \frac{\pi}{n \tan\left(\frac{\pi}{n}\right)} + n \sum_{k=0}^{\infty} \left(\frac{1}{(kn+n-1)^2} + \frac{1}{(kn+1)^2} \right) \\ &= \frac{2\pi^2}{n} \cot\left(\frac{2\pi}{n}\right) \cot\left(\frac{\pi}{n}\right) + \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{1}{\left(k+1-\frac{1}{n}\right)^2} + \frac{1}{\left(k+\frac{1}{n}\right)^2} \right) \\ &= \frac{2\pi^2}{n} \cot\left(\frac{2\pi}{n}\right) \cot\left(\frac{\pi}{n}\right) + \frac{1}{n} \left(\psi_1\left(1-\frac{1}{n}\right) + \psi_1\left(\frac{1}{n}\right) \right). \end{aligned} \quad (73)$$

Comparing (72) and (73), we see that the reflection formula (63) for the Trigamma function is recovered, with $x = \frac{1}{n}$:

$$2\pi^2 \cot^2\left(\frac{\pi}{n}\right) = 2\pi^2 \cot\left(\frac{2\pi}{n}\right) \cot\left(\frac{\pi}{n}\right) + \left(\psi_1\left(1-\frac{1}{n}\right) + \psi_1\left(\frac{1}{n}\right) \right),$$

namely:

$$\psi_1\left(1-\frac{1}{n}\right) + \psi_1\left(\frac{1}{n}\right) = \frac{\pi^2}{\sin\left(\frac{\pi}{n}\right)}.$$

6. Conclusions

In this paper, we obtained explicit forms for the summation of Riemann Zeta functions with even arguments and Dirichlet Beta functions with odd arguments, by generalising the techniques based on multiple integrals [3] and on the algebra of random variables [6,7]. This was achieved by increasing, on one hand, the number of nested integrals and, on the other hand, the number of Cauchy random variables whose quotient was calculated.

Then, by generalising the exponent of the Cauchy random variables involved, we obtained an original proof of the reflection formulae for the Digamma and Trigamma functions. These probabilistic proofs were possible thanks to integrals computed via the Mellin transform.

Author Contributions: Conceptualization, D.R.; methodology, A.E.B. and D.R.; software, A.E.B. and G.S.; validation, A.E.B. and G.S.; formal analysis, A.E.B., D.R. and G.S.; investigation, A.E.B., D.R. and G.S.; resources, G.S.; data curation, A.E.B., D.R. and G.S.; writing—original draft preparation, A.E.B., D.R. and G.S.; writing—review and editing, G.S.; visualization, A.E.B., D.R. and G.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author(s).

Acknowledgments: The authors wish to thank Yury Brychkov, whose advice was essential for the writing of this article, as well as the reviewers for useful comments and the editors for assistance in improving the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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