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Revisiting the Definition of Vectors—From 'Magnitude and Direction' to Abstract Tuples

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Abstract: Vectors are almost always introduced as objects having magnitude and direction. Following that idea, textbooks and courses introduce the concept of a vector norm and the angle between two vectors. While this is correct and useful for vectors in two- or three-dimensional Euclidean space, these concepts make no sense for more general vectors, that are defined in abstract, non-metric vector spaces. This is even the case when an inner product exists. Here, we analyze how several textbooks are imprecise in presenting the restricted validity of the expressions for the norm and the angle. We also study one concrete example, the so-called 'vector-based sustainability analytics', in which scientists have gone astray by mistaking an abstract vector for a Euclidean vector. We recommend that future textbook authors introduce the distinction between vectors that have and that do not have magnitude and direction, even in cases where an inner product exists.

Keywords: vector; Euclidean norm; magnitude; direction; angle; inner product; sustainability vector; vector-based sustainability analytics

1. Introduction

The concept of a vector was developed by a gradual process, with seminal contributions by Josiah Willard Gibbs and Oliver Heaviside [1]. Gibbs [2] writes that "magnitude and direction taken together constitutes what is called a vector". Heaviside [3] introduced "vectorial algebra" to be able to treat "such magnitudes as displacement, velocity, acceleration, force, momentum, electric current, etc., which have direction as well as size, and which are fully specified by statement of the size and direction".

Both sources have a clearly recognizable present-day echo in textbooks on physics and engineering. For instance, ref. [4] states that "a vector can be thought of as an object that has both a length (or magnitude) and a direction in space", ref. [5] opens by defining vectors as objects that "have direction as well as magnitude (length)", and the authors in [6] use the term to refer to "all quantities that have a direction". In addition, a much-used mathematical resource for engineers [7] describes a vector as "a quantity that has both magnitude and direction". And, likewise, ref. [8] uses the term "to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction". The list goes on: [9] states that vectors "capture both the strength of a quantity and the direction in which it acts", and [10] that a vector "is a quantity that requires both a magnitude ($\geq$0) and a direction in space to specify it completely".

Textbooks with a more-restricted, mathematical audience in mind sometimes take a different approach. Ref. [11], for instance, differentiates between the "algebraic and geometric points of view", and defines a vector in the algebraic way as an ordered pair

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(in the case of two components) or an ordered triple (in the case of three components) of numbers. Only in the geometric approach does it mention "objects having 'magnitude and direction.'" The same applies to [12], which presents an interpretation of a vector "as a point in $\mathbb{R}^m$", adding that "quantities such as force and velocity, that is, quantities that possess both magnitude and direction ... are best represented by arrows". Ref. [13] provides a short historical introduction to vectors, in which three different presentations are mentioned:

- geometric, in which vectors are considered to be represented by arrows, having length and direction;
- analytic, which is based on component-wise calculations;
- axiomatic, which is a completely abstract approach, based on a vector space.

The text then continues with the analytic approach, defining a vector as an ordered tuple of real numbers: $(a_1, a_2, \ldots, a_n)$. In a later axiomatic chapter, the vector concept is generalized to complex vector spaces and vector-based functions. The geometric point of view receives less attention, but there is an explicit section on the "geometric interpretation", in which a "geometric vector" is defined in terms of an initial point (A) and a terminal point (B), written as $\overrightarrow{AB}$. Such "geometric vectors are especially convenient for representing certain physical quantities such as force, displacement, velocity, and acceleration, which possess both magnitude and direction".

While for some topics an algebraic and a geometric definition are just two ways to come to the same result (think of the trigonometric functions), in the case of vectors there is an important difference. Geometric vectors are typically two- or three-dimensional objects that live in physical space, which is a Euclidean space, endowed with a Euclidean norm. That is, given a two-dimensional vector $\mathbf{a} = (a_1, a_2)$, its norm (length, size, magnitude) is given by

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}. \tag{1}$$

The norm is easily generalized to vectors in higher-dimensional spaces:

$$\|\mathbf{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2}. \tag{2}$$

Analytically defined vectors lack any geometric representation, and concepts like length, size, and magnitude do not make sense, unless explicitly defined. Often, such a norm is obtained by defining the vector space to be a normed vector space. Such a norm next induces a distance function, which turns the vector space into a metric space. But, it requires the explicit definition of a norm, of which the Euclidean version in Equation (2) is just one example.

In engineering and in many branches of physics, vectors often represent directed quantities in physical Euclidean, space, such as velocities, forces, and electric fields, for which a Euclidean norm is a suitable choice. In other fields of science, vectors are conveniently used to represent other types of composite quantities. In economic input–output analysis [14], the final demand vector represents a consumer's or economy's demand for products of different kinds. The vector $(30, 40)$, for instance, might represent a demand for USD 30 of meat and USD 40 of gasoline. We can envisage a two-dimensional coordinate system, with an axis for USD of meat and a perpendicular axis for USD of gasoline, in which a point with coordinates $(30, 40)$ indicates the demand by a certain consumer. If we draw a line (or arrow) between the origin $(0, 0)$ and the point $(30, 40)$, this line clearly has a length (magnitude) and it also has an angle with the agricultural axis (direction). But, this length and angle have no economic meaning. Although the Euclidean length could be calculated

as USD 50, it makes more sense to report a total consumption of USD 70. And, when the products are expressed in different units, say 30 pounds of meat and 40 gallons of gasoline, this total of 70 makes no sense either. Clearly, a more abstract space than our physical space does not automatically possess the feature of a norm that exists for a geometric vector. The same is true of price vectors [15], vectors that represent amounts or concentrations of chemicals [16], and age-stratified population vectors [17]. Such quantities are conveniently represented with vectors, because there are certain operations and properties available in linear algebra that turn out to be useful. An example from ecology is the Leslie matrix [17], which models how a vector at time t changes into a vector at time $t+1$, where the components of the vector indicate the number of individuals of a certain age. Another example comes from macro-economics, where a matrix inverse can be used to calculate the equilibrium solution of a vector whose components represent the national income (Y) and interest rate (r) [18]. It goes without saying that these authors would object to addressing the length of this solution vector, in the form $\sqrt{Y^2 + r^2}$.

So, while there are (at least) two ways to teach vectors, the two forms are not fully equivalent: the geometric approach allows for the definition and interpretation in terms of a length, while the analytic approach does not automatically do so. Unfortunately, many of the textbooks are not explicit about this restriction. They either start from the 'direction and magnitude' point of view, or they define vectors as tuples, but then, along the way, introduce the notion of a length or norm. For instance, ref. [19] defines the norm of a vector just as "a real-valued function", which satisfies a certain condition, without issuing any warning that it may not always be possible or meaningful to apply. Likewise, ref. [12] introduces the inner product as "an important scalar function of two real vectors ... which leads directly to the norm", which "represents the geometric idea of 'length' of the vector". Ref. [20] defines a vector as an array of scalars, but then moves on to define the norm as a number that satisfies certain axioms, like non-negativity, homogeneity, and the triangular inequality. And, [21] is hardly better in this regard; it just defines a norm (in a similar axiomatic way), adding that "a vector space together with a norm is called a normed space", and that "for some types of objects, a norm of an object may be called its 'length' or its 'size'". The essential point that there is an important class of vector space without a norm remains implicit, at best.

So, we see that many textbooks assume that all vectors are Euclidean vectors, and that another group of textbooks remains almost silent about the difference between the vectors with and without a norm. In the next two sections, we will develop the argument more precisely, turning our attention to situations with an inner product but without a norm. In Section 4, we examine one real-life case from the applied literature in detail.

In the remainder of this article, we will (throughout most of the article) write vectors in bold lowercase print, e.g., $\mathbf{a}$, and its components in the italic form the same letter: a_1, a_2, etc. We will restrict our analysis to n-dimensional vector spaces where all components are real numbers. We restrict the discussion to real vectors because these are the ones that are most frequently discussed in textbooks that introduce vectors in physics and engineering classes. For the inner product, we will use the special case of the dot product (some sources (e.g., [13,21]) make a distinction between the inner product and the dot product, while in other sources (e.g., [7,11]) the two are synonymous. Ref. [22] refers to the dot product as the "standard inner product". Ref. [19] defines only the inner product, but also uses dot products. Also, the term 'scalar product' is used in several textbooks [5,8]), given by $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$; for the norm of a vector, we will only consider the Euclidean case, so $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.

2. Dot Products 'Within' and 'Between' Vector Spaces

The standard procedure for defining a vector norm proceeds through the definition of the dot product. The dot product of two vectors, $\mathbf{x}$ and $\mathbf{y}$, both in $\mathbb{R}^n$, is defined as

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i. \tag{3}$$

The norm of any of these vectors, say $\mathbf{x}$, is then simply

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}}. \tag{4}$$

That functions correctly in a context-free mathematical setting. But, let us consider the case of a vector of prices, $\mathbf{p} \in \mathbb{R}^n$, in which component i, so p_i, is the price of product i, in USD/kilogram, and a vector of quantities, $\mathbf{q} \in \mathbb{R}^n$, in which q_i is the quantity of product i, in kilogram. Then, the dot product $\mathbf{p}\cdot\mathbf{q}$ is a perfectly defined quantity, which represents the total sales, in USD. But, the norm of $\mathbf{q}$, $\|\mathbf{q}\|$, has no interpretation as the total quantity of products, and likewise, $\|\mathbf{p}\|$ has no interpretation as an overall price.

To understand this, let us consider a fundamental property of a vector space, namely addition. It does make sense to add two quantity vectors $\mathbf{q}$. For instance, if $\mathbf{q}_1$ represents the quantity sold on day 1 and $\mathbf{q}_2$ the quantity sold on day 2, $\mathbf{q}_1 + \mathbf{q}_2$ represents the total quantity sold in two consecutive days. In the same style, a price vector $\mathbf{p}_1$ will change after prices have increased by an amount $\mathbf{p}_2$ into a new price vector $\mathbf{p}_1 + \mathbf{p}_2$. But, adding a price vector $\mathbf{p}$ and a quantity vector $\mathbf{q}$ is meaningless. We can interpret this by assuming that, although both $\mathbf{p}$ and $\mathbf{q}$ are vectors in one mathematical vector space $\mathbb{R}^n$, they live in distinct real-world vector spaces. Let us indicate these real-world vector spaces as P and Q. The fundamental property of vector addition is valid for vectors within P and also for vectors within Q, but vectors in P and Q cannot be mutually added, even though a mathematician would argue that $P = \mathbb{R}^n$ and $Q = \mathbb{R}^n$, so $P = Q$.

For the dot product, the situation is opposite. It is not possible to define a meaningful dot product within P and neither within Q, but it does make sense to define a dot product between P and Q as $\mathbf{p}\cdot\mathbf{q} = p_1q_1 + p_2q_2 + \cdots + p_nq_n$.

Let us define this more formally. We will indicate a topological space without a metric as a non-metric space. Suppose X is a non-metric n-dimensional vector space and Y is another non-metric n-dimensional vector space. Then, we may define the dot product $\mathbf{x}\cdot\mathbf{y}: X \times Y \to \mathbb{R}$ as $\sum_{i=1}^{n} x_i y_i$. Because the inner product takes two arguments from different, non-metric vector spaces, it is not possible to evaluate $\mathbf{x}\cdot\mathbf{x}$ or $\mathbf{y}\cdot\mathbf{y}$, and therefore this dot product does not induce a norm.

Clearly, not all combinations of X and Y are endowed with a meaningful dot product. One requirement is that the components are somehow aligned. If component 5 of $\mathbf{p}$ represents the price of wheat and component 5 of $\mathbf{q}$ the quantity of steel, the expression p_5q_5 makes no sense. And, if $\mathbf{p}$ represents the price per kilogram and $\mathbf{q}$ the number of days that the product was kept in a warehouse, the inner product is also devoid of an interpretation.

The act of creating a dot product between two vectors of 'distinct' vector spaces is actually something that is usual in physics too. The expression for energy, E, is often written as $E = \mathbf{F}\cdot\mathbf{s}$, where $\mathbf{F}$ is the force (following usual conventions, here we write a capital $\mathbf{F}$ for the force vector. Similarly, we write a capital $\mathbf{B}$ for the magnetic field vector) and $\mathbf{s}$ is the displacement. In this case, the norms of $\mathbf{F}$ and $\mathbf{s}$ still make sense, but in another example, in Maxwell's equations, we find expressions like $\nabla\cdot\mathbf{B} = 0$, where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\mathbf{B}$ is the magnetic field. Here, the norm of $\mathbf{B}$ still makes sense, but there is no norm of ∇,

and even this cannot be regarded as an element of $\mathbb{R}^n$. Further, no physicist will add such vectors as $\mathbf{F} + \mathbf{s}$ or $\nabla + \mathbf{B}$.

3. Direction and Angles

So far, we have discussed the topic of magnitude, for which the norm is a conventional measure. In addition, many textbooks mention the aspect of direction as a second quality of vectors, besides magnitude. For a vector $\mathbf{x}$ in two-dimensional Euclidean space, we can express the angle with the first axis as $\arctan\left(\frac{x_2}{x_1}\right)$, provided $x_1 \neq 0$, and possibly accounting for the case that $x_1, x_2 < 0$, in which the notation $\text{atan2}(x_2, x_1)$ is sometimes used. Formally speaking, this expression requires that x_2 and x_1 have the same dimension. For instance, if x_1 and x_2 are both expressed in meters, their ratio is dimensionless, and the inverse tangent can be evaluated. If, however, x_1 is a length in meters and x_2 is a time in seconds, their ratio has a dimension, and the inverse tangent cannot be calculated [23]. But, also in cases where the dimensions of x_1 and x_2 are equal, there is not always a meaningful interpretation possible. Consider the case of a price vector $(2,\ 10)$, where the first component is the price (in USD/kilogram) of rice and the second component is the price (also in USD/kilogram) of cheese. The angle is approximately 79 degrees, but it has no obvious economic interpretation.

For a vector in three or more dimensions, we need to specify more than one angle to uniquely describe the direction, so a vector of this order cannot uniquely be measured by one number for magnitude and one number for direction. That is logical, because a vector in three-dimensional space has three degrees of freedom, and two numbers do not suffice to uniquely describe it. In physics, two-dimensional vectors are often expressed using polar coordinates, as $(r, \phi) = \left(\sqrt{x_1^2 + x_2^2}, \arctan\left(\frac{x_2}{x_1}\right)\right)$, while in three dimensions, spherical coordinates are widely used, for instance (r, θ, ϕ), where θ and ϕ represent two angles, for which different conventions exist. Altogether, for vectors in n-dimensional Euclidean space, the magnitude of a vector is conveniently expressed in one number using $\sqrt{\sum_{i=1}^{n} x_i^2}$, while the direction requires $n - 1$ angles, typically involving inverse cosines or tangents of expressions that involve the norm. For vectors without a norm, both the magnitude and the concept of a direction make no sense. But, a geometrical rendering of such a vector as a line clearly has a length and a direction. We should take care to consider such a drawing as a representation only, and not derive conclusions from such a physical representation. After all, such a physical line has a width too, but this is also just an artifact of the representation.

Related to the angle (or angles) of a vector $\mathbf{x}$ is the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$, which is usually expressed in terms of the dot product and two norms:

$$\theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}\right). \tag{5}$$

This concept of the angle between two vectors is different from that of the angle of a single vector, because the angle between two vectors in n-dimensional space is a single number, while there is no unique angle concept for a single vector. To emphasize this difference, we will speak of the direction of a vector, which requires $n - 1$ angles with a well-defined reference frame, and of the angle between two vectors, which is just one number, independent of a coordinate system.

Clearly, the expression for the angle between two vectors is only valid when both the dot product and the vector norm are defined. Vectors that have no magnitude, because no norm has been defined, can also not be subject to the pairwise measurement of angles.

4. Vector-Based Sustainability

In this section, we will study a concrete example that demonstrates how an unjustified 'magnitude and direction' interpretation of a vector may lead to incorrect science. In describing this example, we will focus on the mathematical principles, ignoring several details.

The unsustainability of our production–consumption system is by now recognized by most governments and large companies. As such, sustainability has become an important principle of public policy and business strategy. It is generally perceived as a multidimensional concept, involving the natural environment, social well-being, economic progress, and perhaps even more aspects [24]. A popular incarnation is the triple bottom line [25], in which the environmental, economic, and social pillars together determine the sustainability or unsustainability of a situation or action. It has been popularized as "people, planet, profit", with a later refurbishing as "people, planet, prosperity" [24]. The same three pillars reoccur in other places, such as the life cycle sustainability assessment [26,27].

The need to monitor sustainability, forecast the effects of policy, and assess the sustainability of companies and their products has spawned the development of sustainability indicators. Of these, there are many types and variations. For instance, the United Nations' Sustainable Development Goals are measured with no less than 231 indicators [28].

One such set of indicators has been discussed and applied extensively in the context of chemical engineering. It was introduced by the authors of [29] as "vector-based sustainability analytics" and applied by the authors of [30–37]. A characteristic feature of this approach is its explicit formulation in terms of vector algebra. This opens up the option to connect it with more advanced techniques, such as eigenvalues [37]. Another advantage is that it is formulated in a clear mathematical way, enabling a critical study.

Vector-based sustainability, as described in [29], is based on a "sustainability vector function, which is characterized by its magnitude and direction in a triple-bottom-line-based 3D space". The method relies on "vector analysis techniques, as any sustainability performance improvement problem involves the direction and extent of sustainability status change". So, explicitly, the concepts of magnitude and direction are included in the approach. Ref. [29] refers to [38] and to [39], both of which do not use the term 'vector', but instead define a three-dimensional sustainability space, which is referred to as a "sustainability cube", with the three coordinate components representing scores on economic, environmental, and social aspects.

It is worthwhile to present the basics of the approach. In doing so, we will adapt the notation because the sources ([29–39]) deviate, and because not all details (such as a possible time-dependency) are important for the present discussion. For an industrial system, three scores are calculated, for the economic performance (s_E), for the environmental performance (s_V), and for the social performance (s_L). These three scores together form a triplet, denoted as $\mathbf{s} = (s_\mathrm{E}, s_\mathrm{V}, s_\mathrm{L})$ and interpreted as the system's "sustainability vector" [29] (the original source [28] uses the notation $\vec{S} = \langle E, V, L \rangle$). The indicators are defined such that $\mathbf{s} = (0, 0, 0)$ indicates "no sustainability" [31], while $\mathbf{s} = (1, 1, 1)$ indicates "complete sustainability" [31] (Ref. [36] refers to these two states as the "nadir" and "ideal" states).

Ref. [38] next introduces the "overall sustainability level", S, as

$$S = \frac{1}{\sqrt{3}} \|(s_\mathrm{E}, s_\mathrm{V}, s_\mathrm{L})\|. \tag{6}$$

The symbol $\|\cdot\|$ is not defined in the text, but given that it appears to work on a vector-like quantity $\mathbf{s} = (s_\mathrm{E}, s_\mathrm{V}, s_\mathrm{L})$, with a maximum of $(1, 1, 1)$, and that a factor $1/\sqrt{3}$ is present

because it is "highly desirable that the overall sustainability level, S, is also normalized", we can conjecture that it represents the Euclidean norm of the vector:

$$\|(s_E, s_V, s_L)\| = \sqrt{s_E^2 + s_V^2 + s_L^2}. \tag{7}$$

Indeed, ref. [29], using a slightly different notation, use this formula, simply referring to $\|\cdot\|$ as "the norm". Older sources ([38,39]) also use the notation $\|\cdot\|$, despite the fact that they do not employ terms like 'vector', 'length', 'size', 'magnitude', or 'norm'.

In addition to the norm to express the magnitude of the sustainability vector, ref. [29] introduces an expression for the direction, not for the sustainability vector as such, but for the sustainability vector vis-à-vis another vector, which they refer to as the "optimal sustainability state", $(1,1,1)$ (Ref. [31] uses (α, β, γ) instead of $(1,1,1)$ to indicate the position of the situation of complete sustainability). The paper discusses this in the context of a "development imbalance angle, which quantifies the deviation of the sustainability status of a system under study from the totally balanced development path". This angle would then take the form

$$\theta = \arccos\left(\frac{\mathbf{s}\cdot\mathbf{s}_\infty}{\|\mathbf{s}\|\|\mathbf{s}_\infty\|}\right), \tag{8}$$

where $\mathbf{s}$ is our notation for the actual value of the sustainability vector and $\mathbf{s}_\infty$ denotes the optimal one (the original paper [28] uses various, almost similar, notations: $\vec{S}_\infty(t)$, $\vec{S}(\infty)$, $\vec{S}_\infty$, S_∞, $\vec{S}(t_\infty)$ and $S(t_\infty)$. We think all these symbols denote the "most ideal sustainability state", namely $(1,1,1)$). If $\mathbf{s}_\infty = (1,1,1)$, the expression becomes

$$\theta = \arccos\left(\frac{s_E + s_V + s_L}{\sqrt{3\left(s_E^2 + s_V^2 + s_L^2\right)}}\right). \tag{9}$$

Both the overall sustainability level, S, and the imbalance angle, θ, are based on the norm of a vector for which it is not a priori clear whether the concept of magnitude makes any sense, apart from in a geometrical picture. Ref. [36] suggests that "the magnitude of the vector" and the "cosine angle of the vector from the ideal" follow directly from the fact that the three coordinates are presented as a vector, because these are "based on the characteristics of the vector function". Likewise, ref. [32] claims that "a sustainability vector is characterized by its magnitude and direction" and [31] states that "the magnitude of the vector was calculated to score the absolute sustainability performance of the alternative chemical processes, the angle was used to quantify the relative deviation of their sustainability performance from the totally sustainable direction". All of these sources appear to be based on the misleading textbook mnemonic that a vector has magnitude and direction. It will be clear that this true for some vectors, but not for all vectors. The sustainability vector, as defined in [29–39], defines the state of a system in terms of three variables: economic, environmental, and social, but there is no good argument for assigning an interpretation to the magnitude or direction of the state of that system.

This does not necessarily mean that the three components of the sustainability vector should be kept apart forever. Several proposals for the weighting of the three sustainability pillars have been proposed in the literature, [40–42], typically applying a weighted sum:

$$S = w_E s_E + w_V s_V + w_L s_L. \tag{10}$$

Here, w_E, w_V, and w_L are weighting factors, which are based on subjective priorities, for instance derived from surveys among experts. The interesting connection with vector

analysis is that these weighting factors may be stacked into a vector $\mathbf{w} = (w_E, w_V, w_L)$, and that the overal sustainabilibity score then takes the form of an inner product:

$$S = \mathbf{w} \cdot \mathbf{s}. \tag{11}$$

Here, again we meet a case of an inner product between two vector spaces that are both in $\mathbb{R}^3$, but that should nevertheless be kept seperate, because expressions like $\mathbf{w} + \mathbf{s}$ do not make sense. An important difference with the approach in [29–39] is that explicitly defined weighting factors are needed, instead of an unweighted squared summation. This may be seen as a disadvantage, but it is actually logical that an explicit normative choice of the relative importance of economy vis-à-vis environment vis-à-vis social must be part of the procedure [43].

5. Discussion and Conclusions

When introduced in the context of physical, Euclidean, three-dimensional space, vectors are endowed with the property of direction, which sets them apart from the non-directional scalar numbers that only have magnitude. Indeed, when a textbook on elementary physics defines the work W as the force F multiplied by the distance s over which this force is applied, all three quantities are ordinary scalars. However, when we recognize the possibility that the force and the distance may be unaligned, we should consider them as directed quantities and take their inner product, $\mathbf{F} \cdot \mathbf{s}$. The directed quantities are then introduced to the student as a generalization of the ordinary number concept, and the name 'vector' is attached to this concept. The angle (θ) between two such vectors then matters: for fully aligned vectors the calculation simplifies to Fs or $\|\mathbf{F}\| \|\mathbf{s}\|$, while for orthogonal vectors, the work is simply 0. All cases in between are found using $\mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos\theta$. Vectors are then specified as pairs or triples of numbers, and the student learns how to do the calculations, using inner products and norms. The mnemonic 'a vector has magnitude and direction' helps to impregnate students' brains with grasping this more advanced calculus.

Mnemonics, however, often have a limited validity. A well-known case is the one that says that square roots can only be taken from non-negative numbers. When textbooks introduce imaginary and complex numbers, they often spend several pages explaining the different logic, and indeed, quite a few students find it a hard, counterintuitive topic. Here, we argue that more care is needed to prepare students for understanding the concept of a vector of an abstract, ordered tuple. A concrete vector may still be imagined as a point in n-dimensional space, and when $n = 2$ or 3, it may be shown in a figure, using a point, line, or arrow. But, the mnemonic that a vector has magitude and direction is not helpful and even counterproductive in understanding the idea of an abstract vector space.

We argue that vectors are better introduced as ordered pairs, then as ordered triples, and finally as ordered tuples. In some cases, a meaningful operation can be carried out, by forming the inner product. And, in some cases, a meaningful norm and/or angle can be computed.

We also argue that there are important cases of vector spaces in which an inner product between two vector spaces is possible, but without an inner product, let alone a norm, within one and the same vector space. We discussed the example of a sustainability vector $\mathbf{s}$ and a weighting vector $\mathbf{w}$, for which the expression $\mathbf{w} \cdot \mathbf{s}$ is useful, but for which expressions like $\mathbf{w}_1 \cdot \mathbf{w}_2$, $\mathbf{s}_1 \cdot \mathbf{s}_2$, $\|\mathbf{w}\|$ and $\|\mathbf{s}\|$ are meaningless. The standard textbooks of vector algebra are silent about such cases. Yet even in physics and engineering, we have seen that $\mathbf{F} \cdot \mathbf{s}$ is useful, while $\mathbf{s}_1 \cdot \mathbf{s}_2$ is not.

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