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The Intrinsic Exceptional Point: A Challenge in Quantum Theory

Miloslav Znojil [1,2,3]

[1] Nuclear Physics Institute, The Czech Academy of Sciences, Hlavní 130, 25068 Řež, Czech Republic; znojil@ujf.cas.cz
[2] Institute of System Science, Durban University of Technology, P.O. Box 1334, Durban 4000, South Africa
[3] Department of Physics, Faculty of Science, University of Hradec Králové, Rokitanského 62, 50003 Hradec Králové, Czech Republic

Abstract: In spite of its unbroken PT symmetry, the popular imaginary cubic oscillator Hamiltonian $H^{(IC)} = p^2 + \mathrm{i}x^3$ does not satisfy all of the necessary postulates of quantum mechanics. This failure is due to the "intrinsic exceptional point" (IEP) features of $H^{(IC)}$ and, in particular, to the phenomenon of a high-energy asymptotic parallelization of its bound-state-mimicking eigenvectors. In this paper, it is argued that the operator $H^{(IC)}$ (and the like) can only be interpreted as a manifestly unphysical, singular IEP limit of a hypothetical one-parametric family of certain standard quantum Hamiltonians. For explanation, ample use is made of perturbation theory and of multiple analogies between IEPs and conventional Kato's exceptional points.

Keywords: quantum mechanics of unitary systems; quasi-Hermitian Hamiltonians; vicinity of intrinsic exceptional points; amended perturbation theory; imaginary cubic oscillator example

1. Introduction

The concept of the so-called "intrinsic exceptional point" (IEP) was introduced by Siegl and Krejčiřík who, in their paper [1], studied the "prominent" imaginary cubic (IC) Schrödinger equation:

$$H^{(IC)}\,|\psi_n^{(IC)}\rangle = E_n^{(IC)}\,|\psi_n^{(IC)}\rangle\,, \quad n = 0, 1, \ldots\,, \quad H^{(IC)} = -\frac{d^2}{dx^2} + \mathrm{i}x^3\,. \tag{1}$$

They felt motivated by the instability of the IC spectrum under perturbations [2]. They were able to complement such a numerically supported observation by several rigorous mathematical proofs (cf. also [3]). They found that "the eigenvectors of the imaginary cubic oscillator do not form a Riesz basis" [1]. In spite of having a spectrum that is real, discrete and bounded below [4–6], the manifestly non-Hermitian IC Hamiltonian appeared not to be, in the Riesz-basis sense, diagonalizable.

Siegl, with Krejčiřík, concluded that "there is no quantum-mechanical Hamiltonian associated with it" [1]. The same authors also recalled the standard mathematical terminology, and they reformulated their conclusion: "In the language of exceptional points, the imaginary cubic oscillator possesses an 'intrinsic exceptional point"' which is, as a singularity, "much stronger than any exceptional point associated with finite Jordan blocks" [1].

These words are truly challenging, having also motivated our study of the role of IEPs in the deepest conceptual foundations of contemporary quantum physics. It makes sense to add that Siegl, with Krejčiřík, only introduced the concept via the above-cited remark, i.e., without giving a formal definition. They specified IEP as an $N = \infty$ descendant of the

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conventional exceptional point of order N (EPN, [7]). In this sense, the linear IEP differential operator $H^{(IC)}$ is really "essentially different with respect to self-adjoint Hamiltonians" [1].

The problem of the interpretation of all of the non-standard, IEP-related quantum bound-state problems remains, at present, open. In our contribution to the currently running discussion of this topic (cf. also [8]), we will study and describe, more deeply, the parallels, as well as the differences, between the two (viz., the IEP and EPN) concepts.

We will start, in Section 2, with a brief account of what is known about the linear-algebraic EPN analogs of the ordinary differential IEP Equation (1). We will consider a class of Hamiltonians (depending on a real or complex parameter g) that admit a singular EPN limit when $g \to g^{(EPN)}$. We will recall a few recent results of the studies of this problem, in which a suitable parameter-dependent N-by-N matrix quantum Hamiltonian $H^{(N)}(g)$ is considered at a finite $N < \infty$. We will emphasize the possibility and relevance of its canonical representation by an N-by-N matrix Jordan block when $g \to g^{(EPN)}$.

In the latter limit, operator $H^{(N)}(g)$ ceases to be diagonalizable and, hence, it ceases to be acceptable as an eligible quantum Hamiltonian. In Section 3, we will emphasize that many of its properties become particularly reminiscent of the IEP features of the differential operator model (1), where the corresponding Hilbert space of states is infinite-dimensional, $N = \infty$. We will remind the readers of the existing results concerning physical meaning and impact of the EPN-related finite-dimensional models. We will explain that, in many (often called "quasi-Hermitian" [9,10]) quantum models of such a type, the limiting transition $g \to g^{(EPN)}$ can be interpreted as one of the most natural realizations of a genuine quantum phase transition (cf., e.g., the description of a class of exactly solvable models of such a process in [11]).

In Section 4, the emphasis will be shifted to the $N \to \infty$ scenarios and to the existence of several very useful analogies between both the IEP and EPN singular extremes. We will point out that, in such a comparison, the key role of a methodical guide may be expected to be played by (and a possibly amended) perturbation theory. Interested readers will be recommended to find a phenomenologically oriented inspiration, as well as many related technical details, in an older paper [12]. The authors in that paper studied a fairly realistic non-Hermitian Hamiltonian, describing an N-particle Bose–Einstein condensate that is generated by a sink and a source in interaction. Using a combination of several complementary numerical, as well as analytic and perturbation methods, they managed to detect the presence of the EPN singularities in their model. They also revealed and explained that, under small perturbations, these singularities did unfold in a very specific manner.

These results appeared encouraging because, as the authors mentioned, the "further investigations" of the EPN-related problems "remain tasks for future research". In our present paper, we decided to follow just such a recommendation. In Sections 5 and 6, we will, in particular, address the main technical challenge, and we will also propose an IEP-related generalization of the well-known perturbation-theory-based description of a generic unitary quantum system near its IEP singularity. We will succeed in showing that many of the known technical tricks that are used and tested near EPN at $N < \infty$ can immediately be transferred to the quantum-dynamical scenarios in which the generic Hamiltonian $H(g)$ lies very close to its IEP limiting extreme.

In our last two Sections 7 and 8, and also in the series of six brief Appendices, we will finally complement our considerations by several quantum-physics-oriented contextual remarks.

2. Conventional Exceptional Points Associated with Finite Jordan Blocks

In [1], we read that the existence of the IEP singularity "does not restrict to the particular Hamiltonian" of Equation (1) so that some "new directions in physical interpretation"

of all of the analogous non-Hermitian quantum models have to be sought "since their properties are essentially different with respect to self-adjoint Hamiltonians" [1].

This makes the IC model important as a genuine methodical, as well as conceptual, challenge. Here, we intend to propose and advocate the idea that the resolution of the problem could be guided by another EPN-related "good basis" problem and by the existence of the parallels between quantum systems near their respective IEP and EPN singularities.

The study of these parallels could proceed in several independent directions (cf. the samples of some of them in [8] or in [13]). In what follows, we will explain that and how one of these directions could make use of perturbation expansion techniques.

2.1. The Phenomenon of EPN Degeneracy

In a review paper [14], the very first word of the abstract emphasized that every operator H that is eligible as an observable Hamiltonian of a unitary quantum system in a Schrödinger picture [15] must be diagonalizable. For any specific one-parametric family of Hamiltonians $H(g)$, such a requirement is not satisfied in the EPN limit $g \to g^{(EPN)}$. Then, the operator can consistently be treated as Hamiltonian only when $g \neq g^{(EPN)}$.

In an opposite direction of argumentation, one could recall the existence of exactly solvable quasi-Hermitian N-by-N matrix models $H^{(N)}(g)$, as shown in paper [11], for which there exists a vicinity of $g^{(EPN)}$ (i.e., a suitable compact and simply connected real or complex open domain $\mathcal{D}$ that does not, of course, contain $g^{(EPN)}$), inside which the respective quantum system is found to admit the standard physical probabilistic interpretation. For $g \in \mathcal{D}$, the diagonalizability of Hamiltonians $H^{(N)}(g)$, then, implies that we may construct all of the bound-state solutions of the so-called time-independent Schrödinger equation:

$$H^{(N)}(g)\,|\psi_n(g)\rangle = |\psi_n(g)\rangle\,E_n^{(N)}(g)\,, \quad n = 0, 1, \ldots, N-1\,. \tag{2}$$

Now, whenever the dimension of the Hilbert space of states is finite, $N < \infty$, we may immediately notice that, even in the generic non-degenerate case, all of the eigenvalues $E_n^{(N)}(g)$ and eigenvectors $|\psi_n(g)\rangle$ with $g \in \mathcal{D}$ degenerate in the ultimate (albeit manifestly unphysical) EPN limit:

$$\lim_{g\to g^{(EPN)}} E_n^{(N)}(g) = E^{(EPN)}\,, \quad \lim_{g\to g^{(EPN)}} |\psi_n(g)\rangle = |\Psi^{(EPN)}\rangle\,, \quad n = 0, 1, \ldots, N-1\,. \tag{3}$$

This leads to the following observations:

- [1] For all of the "acceptable" $g \neq g^{(EPN)}$ lying in the "physical", unitarity-compatible vicinity of the EPN value, $g \in \mathcal{D}$, the normalized eigenvectors $|\psi_n(g)\rangle$ of $H^{(N)}(g)$ are almost parallel to each other.
- [2] At the "unacceptable" value of $g = g^{(EPN)} \notin \mathcal{D}$, their set ceases to serve, as a basis suitable, say, for the purposes of perturbation theory.
- [3] At $g = g^{(EPN)}$, one can still construct a "good basis" composed of the single remaining (degenerate) eigenvector $|\psi_0(g^{(EPN)})\rangle = |\Psi_0\rangle$ and of an $(N-1)$-plet of linearly independent associated vectors $|\Psi_j\rangle$ with $j = 1, 2, \ldots, N-1$.

2.2. EPN and Modified Schrödinger Equation

The latter "good basis" can be perceived as an N-plet of column vectors. They may be arranged into the following formal N-by-N matrix:

$$\{|\Psi_0\rangle, |\Psi_1\rangle, \ldots, |\Psi_{N-1}\rangle\} := R^{(EPN)}, \tag{4}$$

which is called, usually, a transition matrix. Thus, we may introduce the two-diagonal Jordan block:

$$J^{(N)}(\eta) = \begin{pmatrix} \eta & 1 & 0 & \ldots & 0 \\ 0 & \eta & 1 & \ddots & \vdots \\ 0 & 0 & \eta & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 0 & \eta \end{pmatrix}, \tag{5}$$

and define the transition matrix as a solution of the following Schrödinger-like equation:

$$H^{(N)}(g^{(EPN)})\, R^{(EPN)} = R^{(EPN)}\, J^{(N)}(E^{(EPN)})\,. \tag{6}$$

Interested readers are recommended to find a constructive illustration of the reconstruction of the transition matrix $R^{(EPN)}$ from the Hamiltonian in [16], where the illustrative solvable Hamiltonians are real matrices that are tridiagonal and multiparametric. At $N = 2J$, one has

$$H^{(2J)}(a, b, \ldots, z) = \begin{bmatrix} 2J-1 & z & 0 & \ldots & & & & \\ -z & \ddots & \ddots & \ddots & \vdots & & & \\ 0 & \ddots & 3 & b & 0 & \ldots & & \\ \vdots & \ddots & -b & 1 & a & 0 & \ldots & \\ & \ldots & 0 & -a & -1 & b & 0 & \ldots \\ & & \ldots & 0 & -b & -3 & \ddots & \\ & & & \vdots & \ddots & \ddots & \ddots & z \\ & & & & \ldots & 0 & -z & 1-2J \end{bmatrix}, \tag{7}$$

etc., (see also Appendices A.1 and A.2 below for a few further related comments).

3. The Mechanism of the Unfolding of the EPN Degeneracy

3.1. The Hypothesis of Admissibility of at Least Some $g \approx g^{(EPN)}$

The purpose of the above-outlined choice of the basis is twofold. Firstly, it enables us to re-read our Schrödinger-like Equation (6) as an equivalent linear-algebraic relation:

$$\left[R^{(EPN)}\right]^{-1} H(g^{(EPN)})\, R^{(EPN)} = J^{(N)}(E^{(EPN)})\,, \tag{8}$$

i.e., as a definition of a canonical Jordan-block representation of any non-Hermitian Hamiltonian of interest at its EPN singularity. Secondly, the amended basis will find application in a reformulation of standard perturbation theory. In such a reformulation, the role of the unperturbed Hamiltonian will be played by its unphysical, singular EPN limit. The trick is that we use the columns of $R^{(EPN)}$ as elements of an unperturbed basis. In the overall perturbation theory spirit, the perturbed system acquires, then, a standard phenomenological interpretation for the parameters g lying inside a suitable "physical" vicinity $\mathcal{D}$ of the EPN singularity.

The latter philosophy is to be advocated and used in what follows. We will only assume the knowledge of the transition matrix $R^{(EPN)}$ at an exceptional point of finite order, and we will then extend the use of this basis to a vicinity of the singularity. This will enable us to invert the limiting process $g \to g^{(EPN)}$ and to consider the original

Hamiltonians at some $g \neq g^{(EPN)}$. Our knowledge of the transition matrix will yield the model described as a perturbation of the Jordan block matrix:

$$\left[R^{(EPN)}\right]^{-1} H^{(N)}(g)\, R^{(EPN)} = J^{(N)}(E^{(EPN)}) + \lambda\, V^{(N)}(g)\,. \tag{9}$$

A priori, we will only have to demand that the auxiliary variable $\lambda = \mathcal{O}(g - g^{(EPN)})$ remains small.

3.2. The Possibility of Keeping the Perturbed Spectrum Real

In papers [17,18], we considered the above-mentioned quantum dynamics scenarios. We studied there the criteria of smallness of perturbations $V^{(N)}$. The conditions of the stability and unitarity of the system were given there a mathematically as well as a phenomenologically consistent form.

For illustration, let us set $E^{(EPN)} = 0$, and let us consider the bound-state problem as a perturbation of its EPN limit:

$$\left[J^{(N)}(0) + \lambda\, V^{(N)}\right] |\Psi(\lambda)\rangle = \epsilon(\lambda)\, |\Psi(\lambda)\rangle\,. \tag{10}$$

With the energy levels counted by a subscript or superscript, we will never use this index, keeping it as just a dummy. We will rather introduce another subscript, which will run, say, from 1 to N, and which will number the components Ψ_j of the ket vector $|\Psi\rangle$ (we are also dropping argument λ as redundant). This convention enables us to fix the norm of $|\Psi\rangle$ by the choice of $\Psi_1 = 1$ and to define another "shifted" column vector:

$$\begin{pmatrix} \Psi_2 \\ \Psi_3 \\ \vdots \\ \Psi_N \\ \Omega_N \end{pmatrix} := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \vec{y}\,, \tag{11}$$

where Ω_N is a new auxiliary variable. Next, we notice that the N-by-N matrix

$$A = A(N,\epsilon) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \epsilon & 1 & 0 & \ddots & \vdots \\ \epsilon^2 & \epsilon & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ \epsilon^{N-1} & \ldots & \epsilon^2 & \epsilon & 1 \end{pmatrix}, \tag{12}$$

is just an inverse of the two-diagonal matrix

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\epsilon & 1 & 0 & \ddots & \vdots \\ 0 & -\epsilon & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & -\epsilon & 1 \end{pmatrix}. \tag{13}$$

Finally, we selected the first column of the matrix in Equation (10), and we denoted it by another dedicated symbol:

$$\begin{pmatrix} \epsilon - \lambda\, V_{1,1} \\ -\lambda\, V_{2,1} \\ \vdots \\ -\lambda\, V_{N,1} \end{pmatrix} := \vec{r} = \vec{r}(\lambda)\,. \tag{14}$$

All of these abbreviations convert our initial homogeneous Schrödinger Equation (10) into its equivalent matrix form:

$$(A^{-1} + \lambda\, Z)\,\vec{y} = \vec{r}\,, \tag{15}$$

or, better,

$$(I + \lambda\, A\, Z)\,\vec{y} = A\,\vec{r}\,, \tag{16}$$

where the symbol Z stands for a modified form of the matrix of perturbation:

$$V^{(N)} \;\to\; Z = \begin{pmatrix} V_{1,2} & V_{1,3} & \dots & V_{1,N} & 0 \\ V_{2,2} & V_{2,3} & \dots & V_{2,N} & 0 \\ \dots & \dots & \dots & \vdots & \vdots \\ V_{N,2} & V_{N,3} & \dots & V_{N,N} & 0 \end{pmatrix}. \tag{17}$$

In paper [17], we proved that the construction of the perturbation corrections is reduced to self-consistency condition

$$\Omega_N = 0\,. \tag{18}$$

In the same reference, interested readers may also find an explicit form of the construction in the leading-order approximation.

Its basic aspects are worth recalling because they immediately help us to clarify the meaning of the rather vague assumption of the smallness of perturbation. It is sufficient to employ the Taylor-series expansion of the resolvent that yields the following formula:

$$\vec{y}^{(solution)}(\epsilon) = A\,\vec{r} - \lambda\, A\; Z\, A\,\vec{r} + \lambda^2\, A\; Z\, A\, Z\, A\,\vec{r} - \dots\,. \tag{19}$$

Such a wave-function-representing ket vector depends on the variable parameter ϵ. But, ultimately, all of the eligible values of ϵ become fixed by Equation (18).

The latter constraint plays the role of secular equation, which has the single vector-component form

$$y_N{}^{(solution)}(\epsilon) = 0\,. \tag{20}$$

In the last step of the construction, we had to solve such an explicit transcendental equation in order to obtain all of the alternative perturbation-generated energy corrections ϵ. In a direct dependence on the model in question, the study of the roots of this equation also precisely offers the criterion of the reality of the whole spectrum in the leading-order approximation.

4. Large N and Anomalous Hamiltonians

The message to be extracted from the above-outlined EPN-based construction is that, for the purposes of transition to its IEP analog, we may try to make use of the EPN–IEP similarities. The main one will consist in the unperturbed Hamiltonian interpretation of the singular IEP operator tractable as, in some sense, a large-N descendant of its finite-N EPN analogs.

In the analysis of both of the EPN and IEP singularities, a central role is certainly played by the phenomenon of the asymptotic confluence of finitely or infinitely many eigenvectors, which are not accompanied by the confluence of the eigenvalues in the IEP case. This can be found confirmed in [8], where we read that "for matrices approaching an exceptional point, it is known [19] that the corresponding eigenvectors are tending to coalesce. For the infinite-dimensional Hilbert space (and Krein space) setup of the IC model, the eigenfunctions of the Hamiltonian having diverging projector norms and asymptotically approaching a PT phase transition region at spectral infinity signal a possible tendency toward collinearity and isotropy of an infinite number of these eigenfunctions" [8].

4.1. The Phenomenon of the Asymptotic Degeneracy of Eigenvectors

The EPN–IEP parallels are certainly incomplete. Still, in both cases, an amendment of the notation might prove useful. Here, we will follow the notation convention that was proposed in our comprehensive review paper [20]. In this spirit, the first mathematical subtlety that we will have to keep in mind is that, for a generic IEP model, the spectrum itself remains non-degenerate. Still, in a way sampled by the IC example, the generic IEP Schrödinger equation,

$$H^{(IEP)}\,|\psi_n^{(IEP)}\rangle = E_n^{(IEP)}\,|\psi_n^{(IEP)}\rangle\,, \quad n=0,1,\ldots\,, \tag{21}$$

can be considered analogous to its finite-dimensional, EPN-supporting partners.

Once the spectrum is found to be real and discrete (which is precisely the case for our illustrative IC Schrödinger Equation (1)), the same property also characterizes the formally independent Hermitian conjugate Schrödinger equation problem:

$$\left[H^{(IEP)}\right]^{\dagger}\,|\psi_n^{(IEP)}\rangle\!\rangle = E_n^{(IEP)}\,|\psi_n^{(IEP)}\rangle\!\rangle\,, \quad n=0,1,\ldots\,. \tag{22}$$

Here, our use of the "ketket" symbol $\rangle\!\rangle$ deserves an immediate comment and explanation. Mainly, because it is closely connected with its role played in the three-Hilbert-space reformulation of the conventional quantum mechanics of unitary systems, as described, e.g., in review paper [20]. For the reasons explained in our three dedicated Appendices A.4–A.6 below, the latter formalism is also—implicitly—recalled and used in our present paper. In these Appendices, interested readers may find a more extensive commentary on the entirely equivalent three-Hilbert-space version of the standard textbook quantum theory, with more emphasis put upon some of the questions of the physical probabilistic interpretation of the illustrative physical models of our present methodical interest.

The IEP-characterizing phenomenon of asymptotic degeneracy enables us to re-establish the above-mentioned analogy with the EPN form of the confluence of eigenfunctions. This phenomenon involves, first of all, the right eigenvectors $|\psi_n^{(IEP)}\rangle$ of $H^{(IEP)}$. For them, we have

$$|\psi_{M+k}^{(IEP)}\rangle \approx |\psi_{M+k+1}^{(IEP)}\rangle\,, \quad k=1,2,\ldots \tag{23}$$

at $M \gg 1$. Similarly, the degeneracy also concerns the left eigenvectors $\langle\!\langle\psi_n^{(IEP)}|$ of the same non-Hermitian operator $H^{(IEP)}$. Often, we refer to their conjugate form $|\psi_n^{(IEP)}\rangle\!\rangle$ of "ketket" eigenvectors of conjugate $\left[H^{(IEP)}\right]^{\dagger}$. In this representation, we encountered an entirely analogous IEP-related confluence of the eigenvectors

$$|\psi_{M+k}^{(IEP)}\rangle\!\rangle \approx |\psi_{M+k+1}^{(IEP)}\rangle\!\rangle\,, \quad k=1,2,\ldots\,. \tag{24}$$

In both Equations (23) and (24), the degree of confluence depends on the Hamiltonian, and it may be expected to grow with the growth of M.

The phenomenon of confluences (23) and (24) finds its formal predecessor in the finite-dimensional case in which, during the transition to singularity $g \to g^{(EPN)}$, the elements of the N-plet of eigenvectors of any preselected N-by-N Hamiltonian matrix $H = H^{(N)}(g)$ do, in fact, lose their mutual linear independence. Still, the analogy of a genuine IEP system with the IEP-mimicking $N = \infty$ EPN extreme is incomplete since, in the former case, the spectrum remains non-degenerate. A more explicit analysis, thus, is necessary.

4.2. Canonical Representation of $H^{(IEP)}$

The IEP-characterizing non-degeneracy of eigenvalues can be perceived as a serendipitous simplification of their study. Still, a decisive IEP-related difficulty results from the effective asymptotic parallelization of the unlimited number of eigenvectors.

This forces us to recall, as our main source of inspiration, Equations (5) and (6) of Section 2 above. In the IEP case, our key task can be now identified as an appropriate generalization of the transition matrices $R^{(EPN)}$, which play a key role in the perturbation theory considerations of Section 3. In other words, we have to replace Equation (6) with a modified eigenvalue-like problem:

$$H^{(IEP)}\,\mathcal{R}^{(IEP)} = \mathcal{R}^{(IEP)}\,\mathcal{J}^{(IEP)}\,, \tag{25}$$

in which the low-lying eigenstates do not require any specific attention. Thus, the conventional Jordan-block-like bidiagonal (i.e., minimally non-diagonal) canonical matrix structure of Equation (5) will only reemerge here in a infinite-dimensional submatrix of an upgraded

$$\mathcal{J}^{(IEP)} = \left(\begin{array}{cccc|cccc} E_0 & 0 & \dots & 0 & 0 & \dots & & \\ 0 & E_1 & \ddots & \vdots & \vdots & & & \\ \vdots & \ddots & \ddots & 0 & 0 & \dots & & \\ 0 & \dots & 0 & E_{K-1} & 0 & 0 & \dots & \\ \hline 0 & \dots & 0 & 0 & E_K & 1 & 0 & \dots \\ \vdots & & \vdots & 0 & 0 & E_{K+1} & 1 & \ddots \\ & & & \vdots & 0 & 0 & E_{K+2} & \ddots \\ & & & & \vdots & \vdots & \ddots & \ddots \end{array}\right). \tag{26}$$

In this arrangement, the partitioning of the basis may be characterized by projectors P (on the first K lowest eigenstates of $H^{(IEP)}$) and Q (such that the unit operator I in Hilbert space can be decomposed as $I = Q + P$).

In a certain parallel with EPN, a key technical step will be a suitable perturbation-mediated weakening or removal of the asymptotic degeneracies (23) and (24) of the asymptotic eigenstates of $H^{(IEP)}$.

5. Toward a Regularization of $H^{(IEP)}$s by Perturbation

From a purely historical point of view, the idea of "prominence" of the IEP-sampling Schrödinger Equation (1) dates back to its methodical role in field theory [21] and to the Bessis' and Zinn-Justin's empirically revealed conjecture (cf. [4], which is also cited in [5]), where the spectrum $\{E_n^{(IC)}\}$ of $H^{(IC)}$ is real, discrete and bounded from below, i.e., tractable, in principle at least, as a set of observable energy levels. In spite of the manifest non-Hermiticity of Hamiltonian $H^{(IC)}$, the model was temporarily accepted as potentially compatible with all of the principles and postulates of quantum mechanics. The corresponding technical details may be found in the review paper [22].

Unfortunately, the excitement came to an end after Siegl's and Krejčiřík's rigorous proof that the IC model cannot, in fact, be assigned any form of conventional probabilistic interpretation in a mathematically consistent manner [1]. More or less the same conclusion has been also made by Günther and Stefani in a not yet published preprint [8]. At present, in the context of the unitary evolution part of non-Hermitian quantum mechanics, the problem of a correct physical interpretation of the IC model itself remains unresolved.

Concerning future developments, we remain a bit skeptical because the IEP nature of the IC model looks, in many respects, only too similar to its much better understood (and manifestly singular and unphysical) EPN-related finite-dimensional analogs.

5.1. The EPN–IEP Differences and Parallels

The essence of the anomalous nature of any IEP-related Hamiltonian $H^{(IEP)}$ lies in the asymptotic degeneracies (23) and (24) of its, respective, right and left eigenvectors. At the same time, a weak point of the amendment of the basis, as mediated by the choice of the non-diagonal matrix (26), may be seen in the necessity of the specification of a "sufficiently large" onset $K \gg 1$ of the de-parallelization. Such a specification is just numerically and computer-precision motivated. In contrast to the above-outlined treatment of the EPN scenarios, where dimension N was fixed, the IEP-implied choice of any finite K is purely pragmatic, immanently approximative and virtually arbitrary.

We now intend to show that, surprisingly enough, the apparently more or less accidental flexibility of our choice of K can, in fact, become an important mathematical tool facilitating an EPN-resembling regularization and consistent interpretation of quantum systems near their IEP singular extreme.

First of all, there is no doubt about the necessity of transition from the less suitable unperturbed basis (composed of eigenvectors) to an "anomalous" basis resembling Equation (4). The reason for this is provided by Equations (25) and (26): only a rectification of the underlying biorthogonal or biorthonormal basis [23] can re-establish the EPN–IEP parallels—even when achieved, as well as in the latter case—at the expense of non-diagonality and non-Hermiticity of the matrix of Equation (26).

Although the EPN singularity encountered at finite matrix dimensions $N < \infty$ is, according to paper [1], perceivably weaker than its IEP analog, the essence of our present message will be complementary. Essentially, we will emphasize that one can also reveal and make a productive use of certain partial similarities between the two concepts. In particular, we propose that the above-outlined possibility and feasibility of treating the manifestly unphysical finite-dimensional singular matrices $H^{(N)}(g^{(EPN)})$ as formally acceptable unperturbed Hamiltonians is to also be transferred to the IEP context. An anomalous "good" basis composed of the columns of transition matrix should also be, *mutatis mutandis*, reconstructed from any given Hamiltonian $H^{(IEP)}$.

5.2. IEP-Unfolding Bases

The IEP (i.e., $N = \infty$) and EPN (i.e., $N < \infty$) singularities share the phenomenon of the parallelization of eigenvectors. In a small vicinity of the singularity, the analysis has to rely upon a properly adapted form of perturbation theory. Our present proposal of transfer of this idea from EPN to IEP will be inspired, therefore, by Section 3.

The parallels are, naturally, incomplete, such that, in the IEP setting, certain truly specific features have to be expected to emerge. For the purposes of clarification, let us mention that, even if we fix a finite $K \gg 1$, it remains far from obvious how to follow the analogy with Equations (2) and (4) and how to also treat $H^{(IEP)}$ as an unperturbed Hamiltonian. The reason for this is that we do not have any immediate analog of Equation (9). In the models sampled by the IC oscillator, we also miss a parameter g or λ, which would control

the form and size of perturbations needed for a phenomenologically motivated unfolding of the manifestly unphysical IEP singularity.

This being admitted, we may still be guided by the EPN dynamical scenario, as outlined in the preceding Sections 2 and 3. Therefore, in the study of the IEP systems, first of all, we should construct a good unperturbed basis in Hilbert space. The most natural IEP analog of the EPN-related Jordan-block matrix (5) is to be seen in its IEP-related amendment (26), rendering the EPN-related unperturbed Schrödinger-like Equation (6), which is replaced by its IEP-related alternative (25).

In connection with the standard and unmodified conjugate eigenvalue problems (21) and (22), the difficulty is that, in Siegl's and Krejčiřík's words, "the eigenvectors, despite possibly being complete, do not form a 'good' basis, i.e., an unconditional (Riesz) basis" [1]. Thus, the left and right eigenvectors of $H^{(IEP)}$ can only be used as a basis in the P-projected subspace of the Hilbert space. Otherwise, the EPN–IEP parallelism has to be fully taken into account, i.e., one has to recall the EPN-related definition (4) in Equation (25), as well as define, in Equation (25), its present IEP-related calligraphic symbol partner $\mathcal{R}^{(IEP)}$ as the following concatenated infinite set of column vectors:

$$\mathcal{R}^{(IEP)} = \{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{K-1}\rangle, |f_K\rangle, |f_{K+1}\rangle, |f_{K+2}\rangle, \ldots\}\,. \tag{27}$$

This array is composed of the mere first K eigenkets $|\psi_j\rangle$, which are complemented by the modified, associated-like ket vectors $|f_{K+k}\rangle$ with $k = 0, 1, 2, \ldots$.

5.3. Recurrences

The possibility (as well as, in some sense, the necessity) of the explicit construction of the latter subfamily of the new ket vectors is, in fact, the very core of our present innovation of the foundations of the formalism of quantum mechanics. Briefly, our basic message is that, in a way which parallels the EPN-related considerations of Section 3 above, our present introduction of the nontrivial IEP-motivated transition matrix of Equation (27) may be expected to play a key role in the regularization of any singular $H^{(IEP)}$ via its perturbation.

The replacement of eigenvectors $|\psi_{K+k}\rangle$ by non-eigenvectors $|f_{K+k}\rangle$ in (27) has to weaken the asymptotically increasing parallelism between the subsequent columns of the transition matrix $\mathcal{R}^{(IEP)}$. Its infinite-dimensional matrix form makes this task different from its EPN predecessor. In technical terms, the insertion of (27) may be used to reduce the nontrivial part of Equation (25) to the sequence of recurrences

$$\left(H^{(IEP)} - E^{(IEP)}_{K+m}\right)|f_{K+m}\rangle = |f_{K+m-1}\rangle\,, \quad m = 1, 2, \ldots\,, \tag{28}$$

with the initial choice of $|f_K\rangle = c_{0,0}|\psi_K\rangle$ using any $c_{0,0} \neq 0$.

The solution of these relations can be then given the form of a finite sum:

$$|f_{K+p}\rangle = \sum_{n=0}^{p} c_{p,n}\,|\psi_{K+n}\rangle\,, \quad p = 0, 1, \ldots\,, \tag{29}$$

where the leading coefficient $c_{k,k}$ is arbitrary. Now, we assume and recall the biorthonormality of the eigenbasis, yielding $\langle\langle\psi_m|\psi_n\rangle = \delta_{m,n}$ and enabling us to convert Equation (28), i.e., the recurrences for kets into the recurrences for the following coefficients:

$$c_{k,m} = (E_{K+m} - E_{K+k})^{-1} c_{k-1,m}\,, \quad m = 0, 1, \ldots, k-1\,, \quad k = 1, 2, \ldots\,. \tag{30}$$

Our freedom of choice of the highest-component coefficients $c_{k,k}$ enable us to suppress the IEP-accompanying asymptotic parallelization of the vectors of basis in Hilbert space. The goal is achieved. For every particular IEP model, we may recall the recurrences in (30) and

replace the Q-projected part of the basis composed of eigenvectors by the Q-projected part of the basis composed, up to the first item $|f_K\rangle = c_{0,0}|\psi_K\rangle$, of non-eigenvectors. And this is precisely what was done in Equation (27).

6. Constructive IEP-Perturbation Considerations

6.1. Formulation of the Problem

Günther with Stefani [8] stated that "what is still lacking" in the IEP setup "is a simple physical explanation scheme for the non-Rieszian behavior of the eigenfunction sets". We agree. We are even more skeptical because we would instead say that the expected 'simple physical explanation' making, in particular, the popular IC oscillators (1) physical need not exist at all. Indeed, we believe that a consistent physical, closed quantum system interpretation could much more easily be assigned to suitable perturbations of the "seed" IEP oscillators with uncertain interpretation (cf. [24] in this respect also).

Our belief is supported by the existence of parallels between the IEP and EPN scenarios. On this ground, one could, in fact, become able to assign a sound phenomenological meaning to many hypothetical parameter-dependent Hamiltonians $\mathcal{H}^{(new)}(\lambda)$, which are defined as certain "admissible" perturbations of the extreme IEP reference operators:

$$H^{(IEP)} \equiv \mathcal{H}^{(new)}(0).$$

In this respect, the methodical guidance, as provided by the illustrative EPN-related Equation (42), is also present in Section 7 below.

Open questions emerge when we fix a sufficiently large K, separate the Hilbert space of states into its two more or less decoupled subspaces and when we, finally, introduce a hypothetical, perturbed Hamiltonian $\mathcal{H}^{(new)}(\lambda)$ and the following IEP analog of Equation (9):

$$\left[\mathcal{R}^{(IEP)}\right]^{-1} \mathcal{H}^{(new)}(\lambda)\, \mathcal{R}^{(IEP)} = \mathcal{J}^{(IEP)} + \lambda\, \mathcal{V}\,. \tag{31}$$

The analogy with EPNs is incomplete because, here, the spectrum of the unperturbed zero-order Hamiltonian remains non-degenerate. This is a simplification that can be perceived as partially compensating the increase in the overall complexity of the IEP problem.

Incidentally, a similar simplification was also detected in the realistic EPN-supporting Bose–Hubbard model of paper [12], where, in dependence on the parameters, the authors had to use *both* the degenerate and non-degenerate versions of perturbation theory. Thus, no abstract conceptual problems have to be expected to emerge after one returns to generic IEP-related dynamics. Still, as long as the IEP-related problems are infinite-dimensional, the perturbed IEP spectrum cannot be deduced from any analog of the EPN-based implicit-definition constraint (18). The study of properties of the vicinity of the IEP singularity cannot be based on a direct reference to its EPN analog. The methods of construction have to be amended.

6.2. Structure of Solutions

In light of the $P + Q$ partitioning of matrix $\mathcal{J}^{(IEP)}$ in Equation (26), the construction of the perturbed forms of the low-lying bound states remains standard. The P-projected states may be ignored just as certain decoupled observers. Only the treatment of the "asymptotic", Q-projected components of the quantum system in question becomes difficult and singular, such as, "for instance, due to spectral instabilities" [1].

This leads to the necessity in solving the perturbed Schrödinger equation:

$$\left[\mathcal{J}^{(IEP)} + \lambda\, \mathcal{V}\right] |\psi(\lambda)\rangle = E(\lambda)\, |\psi(\lambda)\rangle, \tag{32}$$

where $\lambda \neq 0$ (such that we avoid the IEP singularity) and where we have to insert the following:

$$|\psi(\lambda)\rangle = |\psi(0)\rangle + \lambda\,|\psi^{[1]}\rangle + \lambda^2\,|\psi^{[2]}\rangle + \ldots \tag{33}$$

and

$$E(\lambda) = E(0) + \lambda\,E^{[1]} + \lambda^2\,E^{[2]}\rangle + \ldots\,. \tag{34}$$

In the light of definition (26) the P-projected part of our unperturbed Hamiltonian $\mathcal{J}^{(IEP)}$ is a diagonal matrix containing the unperturbed bound-state energy eigenvalues $E_0, E_1, \ldots E_K$. All of the related perturbed low-lying bound states can be then constructed using the conventional Rayleigh-Schrödinger perturbation theory of textbooks [15]. Therefore, for any practical purposes, it is fully acceptable just to make the choice of a sufficiently large dimension K of the P-projected subspace.

At present, in conducting conceptually more ambitious analyses of the problem, it makes sense to turn attention to the states with high-lying zero-order unperturbed energies. The related unperturbed ket vectors $|\psi(0)\rangle$ will lie in the complementary (and infinite-dimensional) Q-projected subspace of Hilbert space. The more-or-less conventional construction of its perturbed descendant, as given by Equation (33), will then, of course, possess several anomalous features.

The first anomaly is that the Q-projected part $Q\mathcal{J}^{(IEP)}Q$ of our unperturbed Hamiltonian is manifestly non-Hermitian. Even after a tentative finite-matrix truncation of the perturbed eigenvalue problem using a sufficiently large cut-off $N \gg K \gg 1$ of the Hilbert space bases, the implementation of the conventional Rayleigh-Schrödinger recipe would require a spectral representation of the unperturbed Hamiltonian operator.

Due to the specific upper-triangular structure of matrix $Q\mathcal{J}^{(IEP)}Q$, the construction of a biorthonormalized basis would be needed. Thus, the left eigenvectors of $Q\mathcal{J}^{(IEP)}Q$ (i.e., "brabras" $\langle\langle\chi_j|$ [20]) will be different from their right-eigenvector biorthogonal partners $|\chi_j\rangle$. This means that the conventional Rayleigh-Schrödinger elementary unperturbed projectors $|\chi_j\rangle\langle\langle\chi_j|$ (which are needed during the construction) will also have a practically prohibitively complicated explicit matrix structure.

One could also find another, more immediate, indication of the possible emergence of irregularities in Section 3 where Equation (20) plays the role of an ultimate transcendental equation determining all of the perturbed EPN eigenvalues as just a constraint imposed upon the very last, $N-$th component of a relevant ket vector. Needless to add that, in the IEP setting, one should consider $N \to \infty$ so that the direct analogy with EPNs is broken.

Another independent word of warning might originate from the fact that, for all of the truly high energy levels $E_{K+m}(\lambda)$, with $m \gg 1$, the use of the explicit Rayleigh-Schrödinger recipe would require the derivation of formulae, which would be m-dependent and different for the different, i.e., for the $(K+m)$-th, m-numbered excitations. Fortunately, the latter technical obstacle and difficulty has a comparatively elementary resolution because what is fully at our disposal is our choice of the value of K. We may feel free to work, exclusively, with the properly innovated Rayleigh-Schrödinger formulae that may be deduced at a single value of m, i.e., say, at $m = 0$.

This certainly simplifies our task. In a methodical setting, it will be sufficient to work with the Hamiltonian of Equation (26) at $K = 0$. Thus, one simply has to solve the following Schrödinger equation:

$$\left[\begin{pmatrix} E_0-E(\lambda) & 1 & 0 & \dots \\ 0 & E_1-E(\lambda) & 1 & \ddots \\ 0 & 0 & E_2-E(\lambda) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} + \lambda \begin{pmatrix} \mathcal{V}_{00} & \mathcal{V}_{01} & \mathcal{V}_{02} & \dots \\ \mathcal{V}_{10} & \mathcal{V}_{11} & \mathcal{V}_{12} & \ddots \\ \mathcal{V}_{201} & \mathcal{V}_{21} & \mathcal{V}_{22} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}\right] |\psi(\lambda)\rangle = 0\,, \tag{35}$$

where

$$|\psi(\lambda)\rangle = \begin{pmatrix} \psi_0(0) \\ \psi_1(0) \\ \vdots \end{pmatrix} + \sum_{k=1}^{\infty} \lambda^k \begin{pmatrix} \psi_0^{[k]} \\ \psi_1^{[k]} \\ \vdots \end{pmatrix}. \tag{36}$$

As long as $E_0 = E(0)$ and $|\psi_j(0)\rangle = 0$ at all $j \neq 0$, it makes sense to abbreviate $E_k - E(\lambda) := F_k(\lambda)$ and remember that $F_0(\lambda) = \lambda E^{[1]} + \mathcal{O}(\lambda^2)$.

In the first-order approximation we have, therefore, the following Equation:

$$\begin{pmatrix} -\lambda E^{[1]} & 1 & 0 & \dots \\ 0 & F_1(0) & 1 & \ddots \\ 0 & 0 & F_2(0) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \left[\begin{pmatrix} \psi_0(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \lambda \begin{pmatrix} \psi_0^{[1]} \\ \psi_1^{[1]} \\ \psi_2^{[1]} \\ \vdots \end{pmatrix}\right] = \tag{37}$$

$$= \lambda \begin{pmatrix} -E^{[1]}\,\psi_0(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 & o & \dots \\ F_1(0) & 1 & 0 & \ddots \\ 0 & F_2(0) & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \psi_1^{[1]} \\ \psi_2^{[1]} \\ \psi_3^{[1]} \\ \vdots \end{pmatrix} = \tag{38}$$

$$= -\lambda \begin{pmatrix} \mathcal{V}_{00} & \mathcal{V}_{01} & \mathcal{V}_{02} & \dots \\ \mathcal{V}_{10} & \mathcal{V}_{11} & \mathcal{V}_{12} & \ddots \\ \mathcal{V}_{201} & \mathcal{V}_{21} & \mathcal{V}_{22} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \psi_0(0) \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{39}$$

In the context of the conventional Rayleigh-Schrödinger perturbation-expansion recipe, this is precisely the equation that would yield the explicit formula for the Coefficient $E^{[1]}$, which is defined in terms of the matrix elements of Perturbation $\mathcal{V}$. Nevertheless, as long as our present unconventional unperturbed Hamiltonian is a non-diagonal (and also an infinite-dimensional) matrix, we have to pay the price: The left eigenvector $\langle\langle\chi(0)|$ of our unperturbed Hamiltonian is not at our disposal. We cannot use it for the standard pre-multiplication of Equation (39) from the left. This means that, without the knowledge of $\langle\langle\chi(0)|$, the first line of Equation (39) is, viz., the following relation:

$$E^{[1]} = \mathcal{V}_{00} + \psi_1^{[1]}/\psi_0(0), \tag{40}$$

which only enables us to extract the value of $E^{[1]}$ in the form of the function of an unknown Parameter $\psi_1^{[1]}$. This is the ambiguity that can be perceived as mimicking the unaccounted influence of the rest of the matrix elements of Perturbation $\mathcal{V}$.

The latter formal disadvantage is partially compensated by the presence of an easily invertible triangular matrix in Equation (39). This suggests that the role of a variable

parameter could be played, in a partial resemblance of the EPN recipe, by the energy correction $E^{[1]}$ itself. We would, then, have

$$\psi_1^{[1]} = \psi_1^{[1]}(E^{[1]}) = (E^{[1]} - \mathcal{V}_{00})\psi_0(0)\,.$$

Similarly, from the second row of Equation (39), we would obtain the value of the second wave-function component

$$\psi_2^{[1]} = (E(0) - E_1)\psi_1^{[1]}(E^{[1]}) - \mathcal{V}_{10}\psi_0(0), \tag{41}$$

etc.

7. Discussion

We can conclude that, in both the EPN- and IEP-related unitary-evolution scenarios, the properly amended form of perturbation theory seems to be able to provide, even in its leading-order form, some explicit and useful criteria of the acceptability or unacceptability of various preselected perturbations of phenomenological interest.

7.1. Benign Perturbations

Between the EPN and IEP alternatives, there still exists a crucial difference. Indeed, in a typical EPN-related analysis, our considerations usually start from our knowledge of the "physical" family of models $H(g)$. Then, the only truly difficult problem is to localize the EPN singularity, especially when the values of N are not too small. In the case of the IEP singularities, in contrast, we only know, typically, the unperturbed Hamiltonian, as sampled here by the IC operator. It is possible to conclude that this precisely makes the IEP-related models perceivably more difficult to study.

From a purely pragmatic point of view, a source of certain optimism could be drawn from the leading-order perturbation-approximation criteria. Their key strength lies in the possibility of identification of the "malign" IEP perturbations that would destroy the reality of the spectrum and would make the evolution non-unitary.

The complementary reliable identification of the "benign" perturbations is a mathematically much more difficult open problem. Incidentally, qualitatively, the same conclusions have already been obtained in a simpler EPN context. For example, in the above-mentioned study [12] of a specific Bose–Hubbard model near its EPN dynamical extreme, the authors did not insist on the reality of the spectrum. They decided to treat their mathematical results as applicable and valid in a broader, not necessarily unitary, open-system context.

In a narrower, closed-system setting, a deeper analysis was performed, and a resolution of the apparent EPN-related instability paradox was described in paper [18]. We studied, there, the exact, as well as approximate, secular equations in more detail. Our ultimate conclusion was that the necessary smallness condition specifying the class of the admissible, unitarity non-violating perturbations does not involve their upper-triangular matrix part at all. In contrast, for the perturbed EPN model in question, the lower triangular matrix part of all of the "benign" (i.e., unitarity-compatible) perturbations was shown to have the following element-dependent matrix form of condition of the sufficient smallness of λ:

$$\lambda\, V^{(N)}_{(admissible)} = \begin{bmatrix} \lambda^{1/2}\mu_{11} & 0 & \dots & 0 & 0 & 0 \\ \lambda\,\mu_{21} & \lambda^{1/2}\mu_{22} & \dots & 0 & 0 & 0 \\ \lambda^{3/2}\,\mu_{31} & \lambda\,\mu_{32} & \ddots & \vdots & \vdots & 0 \\ \lambda^{2}\mu_{41} & \lambda^{3/2}\,\mu_{42} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda\,\mu_{N-1N-2} & \lambda^{1/2}\mu_{N-1N-1} & 0 \\ \lambda^{N/2}\mu_{N1} & \lambda^{(N-1)/2}\mu_{N2} & \dots & \lambda^{3/2}\,\mu_{NN-2} & \lambda\,\mu_{NN-1} & \lambda^{1/2}\mu_{NN} \end{bmatrix}. \tag{42}$$

The matrix structure (42) may be interpreted as manifesting a characteristic anisotropy and the hierarchically ordered weights of influence of the separate matrix elements because, during the decrease in $\lambda \to 0$, all of the "benign" matrix-element parameters have to have bounded components $\mu_{j,k} = \mathcal{O}(1)$. For a more explicit explanation, we may rescale

$$\lambda\, V^{(N)}_{(admissible)} = \lambda^{1/2}\, B(\lambda)\, V^{(reduced)}\, B^{-1}(\lambda), \tag{43}$$

where $B(\lambda)$ would be a diagonal matrix with elements $B_{jj}(\lambda) = \lambda^{j/2}$ and where the whole reduced "benign" matrix of perturbation would be bounded, $V^{(reduced)}_{jk} = \mathcal{O}(1)$.

On this necessary-condition background, which is valid at all dimensions N, the samples of sufficient conditions retain a purely numerical trial-and-error character with the small-N non-numerical exceptions that were discussed in [18] for the matrix dimensions up to $N = 5$.

7.2. The IC Oscillator as a Popular Toy Model

In order to elucidate the benchmark model role of the IC IEP oscillator, let us recall paper [5], in which Bender, with Boettcher, examined a rather broad family of time-independent non-Hermitian toy-model Hamiltonians (cf. Equation (A1) in Appendix A.2 below). They felt guided by the postulate of the (antilinear) symmetry of their models:

$$\mathcal{PT} H^{(BB)} = H^{(BB)} \mathcal{PT}\,. \tag{44}$$

The linear operator $\mathcal{P}$ was treated as parity (causing the space reflection $x \to -x$), while $\mathcal{T}$ had to mimic the anti-linear time reversal.

The authors proposed to treat their operators $H^{(BB)}$ as "Hamiltonians whose spectra are real and positive" so that "these $\mathcal{PT}$-symmetric theories may be viewed as analytic continuations of conventional theories from real to complex phase space" [5]. During the subsequent wave of development of the related mathematics, it has been revealed that, in the language of functional analysis, the $\mathcal{PT}$ symmetry of Equation (44) can be re-read as pseudo-Hermiticity [14], as well as a self-adjointness, in the Krein space that is endowed with indefinite pseudo-metric $\mathcal{P}$ [25–27].

A deeper mathematical insight in the class of $\mathcal{PT}$-symmetric models was obtained. In the narrower context of quantum mechanics of closed systems, in contrast, the IEP-possessing IC model, itself, was not assigned up to now, nor are there any sufficiently consistent phenomenological interpretations yet [8]. Still, in retrospective, its temporary popularity was enormous. Its roots may be dated back to the Bessis and Zinn-Justin unpublished [4] but widely communicated [5] discovery that, in spite of the manifest non-Hermiticity of the IC Hamiltonian, its spectrum appeared to be real and bound-state-like, i.e., discrete and bounded from below.

In the extensive existing literature devoted to the study of systems with $\mathcal{PT}$ symmetry (cf., for example, the reviews of [28,29]), a great deal of attention has been paid to the non-Hermitian but still sufficiently realistic ordinary differential Hamiltonians of the form $H = T + V$, which are reminiscent of the IC oscillator in which the entirely conventional kinetic-energy term $T = -d^2/dx^2$ is combined with a suitable local complex one-dimensional potential $V = V(x)$. By many authors, the latter models were sampled by the field-theory-mimicking oscillator Hamiltonian (1), in which the purely imaginary form of the asymptotically growing potential is a truly puzzling mathematical curiosity.

This was also the feature that attracted a great deal of attention among physicists [5,22,26,28], precisely because they happened to generate the purely real, discrete and non-negative (i.e., hypothetically, observable and bound-state-like) energy spectra. Still, the ultimate verdict by mathematicians [1,8] was discouraging because they proved that the IC Hamiltonian cannot be assigned any isospectral self-adjoint avatar $\mathfrak{h}(t)$ or acceptable physical inner-product metric [1]. Thus, the rigorous mathematical analysis finally led to the loss of some of the most optimistic phenomenological expectations.

8. Summary

Not too surprisingly, the highly desirable proofs of the so-called unbroken form of $\mathcal{PT}$ symmetry (in which, by definition [22], the spectrum remains real) appeared to be, in numerous applications, strongly model-dependent. There seemed to be no universal criteria guaranteeing the existence of the unbroken $\mathcal{PT}$ symmetry in dependence on a suitable measure of degree of the non-Hermiticity of the Hamiltonian.

During the preparation of our present study, we came to the conclusion that the lack of a deeper understanding of correspondence between the (apparent) non-Hermiticity and (hidden) unitarity might have been caused by an overambitious generality of the choice of the models in the literature. For this reason, we decided to narrow the scope of our analysis, and we restricted our attention to the Hamiltonians living in a vicinity of their singular EPN or IEP extreme.

In our present paper, we have endeavored to explain how such a decision enabled us, not only to pick up a rather natural measure of the non-Hermiticity (which is characterized simply by the inverse distance of the variable physical parameter $g \in \mathcal{D}$ from its unphysical exceptional value), but also to formulate a well-defined project in which we developed and applied, consequently, some innovative and suitable perturbation-approximation techniques. In this framework, we managed to show that the unbroken $\mathcal{PT}$ symmetry of our models can, in fact, survive inside an open parametric domain on a point of boundary of which our measure of non-Hermiticity reaches its maximum. The latter point (which remains manifestly unphysical) has been shown to coincide either with the Kato's [7] exceptional point of a finite-order N or with its hypothetical IEP analog.

Such an approach has been found productive. Using certain slightly modified techniques of perturbation theory of linear operators, with finite N, we found that, paradoxically, the restriction of attention to the smallest vicinity of the singularity (in which the Hamiltonians become maximally non-Hermitian) leads to a remarkable simplification of perturbation-approximation constructions. In spite of being singular and unacceptable as observables at $g = g^{EPN}$, the special, "exceptional" non-diagonalizable operators appeared to be eligible as unperturbed Hamiltonians. In their vicinity, such that $g \in \mathcal{D}$, their diagonalizability, as well as the observability status (i.e., the standard physical status), were re-established.

In this sense, the core of our present message is that the same perturbation-regularization physical interpretation should be also attributed to the IEP models where $N = \infty$. For the purposes of an illustrative example, we chose the popular imaginary oscil-

lator Hamiltonian. Such a choice was motivated, first of all, by its long-lasting theoretical significance. It ranges from the more-or-less purely formal role of the model in mathematics and functional analysis [30] up to its deeper phenomenological significance in quantum statistics where the imaginary ϕ^3 interaction mimics the Yang–Lee edge singularity [5,21]. The same Hamiltonian also plays an important theoretical role of a benchmark model in conformal quantum field theory [31] and in Reggeon field theory [32].

In all of these contexts, our present results imply that the IEP property of the IC-like models means unphysicality. Only a suitable perturbation can reinstall the (possibly "hidden" *alias* "quasi-") unitarity and physicality. Thus, the practical realizations of the standard quantum-mechanical IC model remain, in a way and for the reasons outlined in [1], elusive, at least in the unitary-theory context [33]. At the same time, one might still expect that some of its realizations could emerge out of the realm of quantum mechanics, e.g., in optics [29].

Constructively, we also managed to clarify some of the consequences of our present perturbative regularization recipe. The emergence of qualitative, as well as quantitative, EPN–IEP parallels helped us to complement and better understand the twelve-year-old disproof of the internal mathematical consistency of the IC IEP quantum oscillator [1]. Such a clarification can be perceived as being of a fundamental importance in quantum theory. Indeed, potentially, most of our observations might also immediately be extended to many other currently popular, but still IEP-singular, non-Hermitian quantum models.

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Appendix A

Appendix A.1. Paradox of Stable Bound States in Complex Potentials

For a long time, it was believed that the locality of the real and confining potential is so strongly restrictive a constraint that the loss of the reality of $V(x)$ (i.e., of the self-adjointness of the Hamiltonian in any standard Hilbert space of states, e.g., in $L^2(\mathbb{R})$) would immediately imply the loss of the reality of the spectrum, i.e., the loss of the observability of the quantum system in question.

In 1998, in Bender's and Boettcher's pioneering letter [5], the latter belief was strongly opposed. Using a combination of methods, these authors argued that the spectrum generated by multiple *manifestly complex* local interaction potentials $V(x)$ also still appears to be strictly real and discrete, i.e., fully compatible with the conventional postulates of the quantum mechanics of the stable and unitary bound-state quantum systems.

Subsequently, the proposed amendment of the model-building paradigm has been widely accepted. For various complex $V(x)$s, rigorous [6,34], as well as numerical [35], proofs of the reality of the spectra were found and attributed to a certain "hidden form of Hermiticity" of the underlying Hamiltonians (cf., e.g., a few earlier review papers [14,22] for details).

A return to older literature (cf., e.g., review [10]) has revealed that the compatibility of the unitarity of the evolution with a manifest non-Hermiticity of the interaction can be given a comparatively elementary explanation because, whenever the Hamiltonian H in question has a real and discrete spectrum, it may be safely self-adjointed with respect to another, "correct", *ad hoc* inner product, i.e., in a modified, "physical" Hilbert space $\mathcal{H}_{phys}$.

Simultaneously, it may make sense to stay working in the initial and more user-friendly Hilbert space $\mathcal{H}_{math}$, which remains "unphysical" (i.e., formally non-equivalent) but, for some reasons, preferred.

Many years ago, many people, in fact, studied various toy models of such a type, which were characterized by the interaction that appeared *manifestly non-Hermitian* with respect to a conventional inner product. The scope of such—mostly numerical—attempts ranged from very pragmatic Dyson-inspired analyses of non-relativistic many-body systems [36,37] through to the abstract, methodically motivated considerations that concern the applicability of non-Hermitian models in the relativistic quantum field theory [4].

In a latter context, Bender's and Boettcher's results [5] proved particularly inspiring and made the idea popular. In parallel, the most elementary IC model appeared to represent a challenge in mathematics, leading, e.g., to a rigorous proof of the reality of its spectrum by Dorey et al. [6]. For this reason, the model served, for many years, as a benchmark methodical guide that inspired several new developments in relativistic quantum field theory [38], as well as in multiple other phenomenologically oriented subdomains of modern physics [29,33].

Last but not least, Bender with Boettcher extended the spectrum reality conjecture to a broad class of potentials $V^{(BB)}(x) = (\mathrm{i}x)^{\delta}\, x^2$ with arbitrary non-negative $\delta \in (0, \infty)$ and with x lying on a complex contour [5]. All of these results caused the growth of the popularity among physicists of the innovative reformulation of the quantum physics of unitary systems admitting manifestly non-Hermitian Hamiltonians. This, not too surprisingly, appeared to be paralleled with a criticism by mathematicians who referred, e.g., to the existence of counterexamples with pathological properties [2]. Incidentally, some of these counterexamples were even already known many years earlier to Dieudonné [9]).

Fortunately, an ultimate resolution of the conflict was found in a rediscovery and return to a half-forgotten but still fully relevant older review paper by Scholtz et al. [10]. In it, most of the objections by mathematicians were circumvented by an ad hoc technical assumption that one is only allowed to consider the non-Hermitian Hamiltonians (as well as any other candidates for observables) that are, as operators in Hilbert space, bounded (cf. several mathematically oriented reviews in [26] that also deal with this). Thus, one may call such a mathematically consistent version of the theory quasi-Hermitian quantum mechanics.

Appendix A.2. Beyond the Imaginary Cubic-Oscillator Potential

The above-mentioned Bender's and Boettcher's choice of Hamiltonian

$$H^{(BB)} = -\frac{d^2}{dx^2} + V_{(\delta)}(x)\,, \quad V_{(\delta)}(x) = (\mathrm{i}x)^{\delta}\, x^2\,, \quad \delta \in (0, \infty)\,, \tag{A1}$$

was motivated by their conjecture that the ubiquitous requirement of the self-adjointness of the observables might be criticized as over-restrictive. They proposed that one should consider a broader class of Hamiltonians H, for which the conventional condition of self-adjointness is replaced by the property called $\mathcal{PT}$ symmetry (cf. Equation (44)), which also has the *alias* $\mathcal{P}$ pseudo-Hermiticity:

$$\mathcal{P}H^{(BB)} = \left[H^{(BB)}\right]^{\dagger}\mathcal{P}\,. \tag{A2}$$

Under the latter assumption (cf. the comments in [25,39]), Bender with Boettcher assumed that the role of the guarantee of the reality of the spectrum of the bound-state energies (i.e., their observability in principle) can be relegated from the conventional Hermiticity to the $\mathcal{PT}$ symmetry of the system whenever such a symmetry remains spontaneously unbroken [22].

In applications, the choice of $\mathcal{PT}$-symmetric Hamiltonians is influenced by a tacit reference to the principle of correspondence (due to which H is assumed to be split into its kinetic-energy component H_{kin} and a suitable interaction term H_{int}). Moreover, the analysis is often only restricted to the single-particle one-dimensional motion with conventional $H_{kin} \sim -d^2/dx^2$ and a suitable local-interaction form of $H_{int} \sim V(x)$.

This choice has already been recommended by Bender with multiple collaborators (cf. review [22]). They have emphasized that the study of various non-Hermitian, but $\mathcal{PT}$-symmetric, quantum models with real spectra can be perceived as motivated by quantum field theory. In this context, a key role is played—which has been, in a way, proposed by Bessis with collaborators [4]—by the imaginary cubic (IC) potential $V^{(IC)}(x) = \mathrm{i}x^3$. For this reason, Siegl with Krejčiřík [1] turned their attention to the IC Hamiltonian (1), having revealed that such a model suffers unpleasant pathologies.

These pathologies appear to be too serious to be ignored. After all, Siegl with Krejčiřík only rediscovered the Dieudoné's older claim that, for such a Hamiltonian, "there is for instance no hope of building functional calculus that would follow more or less the same pattern as the functional calculus of self-adjoint operators" [9]. Siegl with Krejčiřík also listed several "pathological properties of a non-self-adjoint $H^{(IC)}$", and they have also offered a rigorous proof that these features of the IC model find a close formal analog in Kato's EPNs.

Thus, the popular toy model operator $H^{(IC)}$ is disqualified as a candidate for a quantum Hamiltonian. An independent reconfirmation of the skepticism can be found in [12,13] and also in the very first line of the abstract of review paper [14] (which requires the diagonalizability of observables). Still, several attempts were made to replace $H^{(IC)}$ with a suitable regularized alternative. Typically, the regularization has been sought in a truncation of the real line of x (cf. [40]). Unfortunately, one can hardly speak about a successful resolution of the problem because one form of unacceptability was merely replaced by another one, viz., by the complexification of the spectrum.

Appendix A.3. Beyond the Stationary Quasi-Hermitian Models

What is most characteristic for the applications of quantum mechanics, in the so-called Schrödinger picture [15], is the observability of the generator of the evolution of wave functions called "Hamiltonian". In most applications, it is required to be self-adjointed in the preselected Hilbert space $\mathcal{H}$. Then, according to Bender and Boettcher [5], the robust nature of the reality of its eigenvalues (which simply represent, in many models, the discrete bound-state energies) can be perceived as a weakness to the approach. Indeed, once we prepare, at an initial time $t = 0$, the system in a pure state (*alias* "phase"), which is described by a ket-vector $|\psi(0)\rangle \in \mathcal{H}$, we discover that the "phase" (defined by the specific set of observable aspects) cannot be changed by the evolution.

This feature would make the description of quantum phase transitions impossible. Fortunately, a change in the "phase" (e.g., an abrupt loss of the observability of the time-dependent bound-state energies) has recently been rendered possible after a conceptually straightforward transition from the Schrödinger-picture (SP) approach to a formally equivalent, albeit technically more complicated, non-Hermitian interaction picture (NIP [41]).

Some of the consequences of such a change in paradigm have also become relevant for an appropriate understanding and treatment of IEP-related considerations. Due to the lack of space for an exhaustive analysis of this problem, let us only briefly mention that, in the traditional and most popular SP framework of conventional textbooks, the unitary evolution of a closed quantum system was described in a unique preselected Hilbert space $\mathcal{H}$. People also often accept multiple additional *ad hoc* simplifying assumptions, with the

most popular one concerning the above-mentioned generator of the evolution of wave functions (e.g., $G_{(textbook)}(t) = \mathfrak{h}(t)$), which requires its self-adjointness in $\mathcal{H}$ as follows:

$$\mathfrak{h}(t) = \mathfrak{h}^{\dagger}(t)\,. \tag{A3}$$

Another traditional simplification concerns the inner product in $\mathcal{H}$, which is assumed to be time-independent [10,14].

In the generalized NIP framework, in contrast, one has to consider the following Schrödinger equation:

$$\mathrm{i}\frac{d}{dt}\,|\psi(t)\rangle = G(t)\,|\psi(t)\rangle\,, \quad |\psi(t)\rangle \in \mathcal{H}\,, \tag{A4}$$

in which the generator $G(t)$ need not represent an observable [42–46]. From the purely phenomenological point of view, such a generalization is useful. In a way, reflecting the widespread knowledge of the above-sampled differential-operator benchmark models (A1), it is still possible to introduce the energy-representing observables

$$H(t) = \triangle + V(x,t), \tag{A5}$$

which are not only non-Hermitian and manifestly time-dependent, but also different from the generator $G(t)$. In this setting, the freedom of choice between the SP or NIP framework only means that one treats the time dependence of our observables as inessential or essential, respectively.

In practice, we usually insist on the standard phenomenological and probabilistic interpretation and, in particular, on the observable energy status of the specific operator (A5). Thus, we have to keep in mind that at least some of the most popular differential operators cannot be used as benchmark models without hesitation. For this reason, a consequent constructive realization of description of the phenomenon of a genuine quantum phase transition remains to be a task for the future development of the theory.

Appendix A.4. The Question of the Unitary-Evolution Accessibility of EPNs

Due to the degeneracy of the unperturbed energy spectrum in EPN limiting the N-plet of the perturbed energy roots of the corresponding secular equation (cf., e.g., the bound-state energy roots $\epsilon_n = \epsilon_n(\lambda)$ of Equation (20) with $n = 1, 2, \ldots, N$), they need not necessarily be all real and, hence, represent observable quantities. In Refs. [12,17], for example, even some of the approximate leading-order roots were found to be complex. This observation can be reinterpreted as indicating that, even in an immediate EPN vicinity, the bounded perturbations may also still be reclassified, in unitary theory, as "inadmissibly large", thus forcing the system to perform an abrupt quantum phase transition.

Within the quantum mechanics of unitary, closed systems in its quasi-Hermitian formulation, a key to the suppression of such a quantum catastrophe lies in the construction of a correct physical inner product in Hilbert space [10]. Still, many of the truly realistic applications of the quasi-Hermitian operators may remain, in the model-building context, counterintuitive. In particular, doubts emerge in virtually all of the tentative quasi-Hermitian descriptions of the phenomenon of quantum phase transition because, traditionally, all of these processes have been treated as non-unitary, requiring an *ad hoc* effective-operator approach [47].

A feasible way out of the apparent deadlock is offered by the quasi-Hermitian quantum models, in which a given observable with a real spectrum (e.g., $\Lambda(t)$) is non-Hermitian. In such a theory (cf., e.g., the reviews of [10,14,20,22,26]), the condition of the self-adjointness

of $\Lambda(t)$s survives "in disguise" and is replaced by a formally equivalent quasi-Hermiticity condition in another Hilbert space:

$$\Lambda^{\dagger}(t)\,\Theta(t) = \Theta(t)\,\Lambda(t)\,. \tag{A6}$$

The assumption of the time dependence of the related inner-product-metric $\Theta(t)$ opens, then, the possibility of reaching a singularity via unitary evolution.

In such a case, the collapse is simply rendered possible by the coherent, simultaneous loss of the existence of the time-dependent inter-twiner $\Theta(t)$ in the critical limit of $t \to t^{(EPN)}$ or $t \to t^{(IEP)}$. One could also say that the realization of the whole process of the change in phase, i.e., of the loss of the observability of some of the measurable characteristics (i.e., of the loss of the quasi-Hermiticity of $\Lambda(t)$) is to be mediated by the metric $\Theta(t)$ in (A6), which becomes, in the limit, non-invertible and, in fact, just a rank-one operator [11].

The emergence of a fully explicit conflict between the constructive feasibility and mathematical consistency can be traced back to the 2012 paper [1], in which Siegl with Krejčiřík disproved the acceptability of a broad class of the currently popular non-Hermitian but observable Hamiltonians. The impact of the disproof was truly destructive. The currently widespread belief in the benchmark role of many non-Hermitian but still observable Hamiltonians with real spectra was shattered.

In parallel, the doubts were also thrown upon the acceptability of the specific benchmark ordinary-differential (i.e., one-dimensional, mathematical and still sufficiently user-friendly) non-Hermitian candidates for the energy-representing Hamiltonians that are decomposed into their two intuitively plausible (and stationary, as well as non-stationary) kinetic-plus potential-energy components:

$$H = -\frac{\hbar}{2m(x)}\frac{d^2}{dx^2} + V(x) \neq H^{\dagger}\,. \tag{A7}$$

The no-go theorems of paper [1] (see also [3] for further details) seemed to return us back to the older methodical analyses in which the deepest source of the mathematical difficulties was attributed to the unbounded operator nature of the most popular differential operators, as sampled by Equation (A7) (cf., e.g., [9,10,48]).

In applications, paradoxically, the latter disproofs and skepticism motivated a rapid increase in interest in the study of the so-called open quantum systems [33]. In this very traditional context, an enormous acceleration in progress (i.e., in an innovated understanding of the dynamics of resonances) was achieved due to the successful applications of the new methods of the solution of the non-Hermitian versions of Schrödinger-like evolution equations. At present, multiple branches of physics have been enriched by these tendencies, even including non-quantum ones [28,29].

Appendix A.5. A Note on the Broader Quantum Physics Framework

Even in the context of quantum physics, paradoxically, the intensification of interest in non-Hermitian Hamiltonians of closed systems has accelerated progress in the development of dedicated mathematical methods and, in particular, in the understanding of non-Hermitian operators with spectra that are not real. These developments have redirected the attention of a part of the physics community back to the traditional models in which meaningful spectra were allowed to be complex.

Our lasting interest in unitary quantum mechanics using hidden Hermitian observables has been partly motivated, in a similar way to that documented in paper [49], by the possibility of mimicking the processes of quantum phase transitions. We believe that such a

direction of analysis must necessary profit from the very recently introduced combinations of the requirements of manifest non-Hermiticity (and, especially, of hidden Hermitian forms called quasi-Hermiticity [9,10]) with some other suitable model-building options and auxiliary technical assumptions, like the time dependence of the operators and/or their $\mathcal{PT}$ symmetry (cf. [5,22,39]) or factorization (cf., e.g., [39,50,51]).

The main stream of our considerations has remained restricted to the context of quantum physics and quasi-Hermitian dynamics, in which we have insisted on the compatibility of our models with all of the basic principles of the quantum mechanics of the so-called closed and unitary systems that admit the standard probabilistic interpretation. This does not mean that the scope of the theory and its applications cannot be much broader, at least in principle. Innovations may be obtained in the representation of the states, as well as of their observable characteristics.

In the spirit of multiple relevant recent reviews, this goal can be achieved by the various physical reinterpretations of the parameter-dependent Hamiltonians $H(g)$. Even when we only admit, in a Schrödinger picture, its observable energy interpretation, it is still worth returning to Dyson's treatment [36] of such an operator as the one that is isospectral with its self-adjoint avatar $\mathfrak{h}(g)$:

$$H(g) \to \mathfrak{h}(g) = \Omega(g)\, H(g)\, \Omega^{-1}(g) = \mathfrak{h}^{\dagger}(g)\,. \tag{A8}$$

Even the metric Θ itself acquires an entirely new meaning of product

$$\Theta(g) = \Omega^{\dagger}(g)\, \Omega(g) \tag{A9}$$

of the so-called Dyson's maps, which are non-unitary and related to the quantum physics avatar $\mathfrak{h}(g)$ rather than to the non-Hermitian upper-case Hamiltonian itself.

In this spirit, Dyson introduced and treated the mappings Ω in (A8) as certain variationally motivated ad hoc multiparticle correlations. In contrast, Buslaev with Grecchi [39] offered and formulated another point of view by which these operators represent just an isospectrality-equivalent transition from a Hilbert space, which is unphysical, to another Hilbert space, which is physical. During such a transition, it is possible to distinguish and cover both open quantum systems (in which one describes resonances) and closed quantum systems (e.g., Buslaev with Grecchi studied quartic anharmonic oscillators).

In the latter, newer and less traditional case, one has to construct the auxiliary metric as product (A9). This simply amends the inner product in the mathematically friendlier and computationally preferred, but manifestly unphysical, Hilbert space $\mathcal{K}$ as follows:

$$\langle \psi_1 | \psi_2 \rangle_{\textit{in physical space}} = \langle \psi_1 | \Theta \, | \psi_2 \rangle_{\textit{in mathematical space}}\,. \tag{A10}$$

In this notation, the obligatory condition of the Hermiticity of $\mathfrak{h}(g)$ (cf. Equation (A8)) is translated into the equally obligatory condition of the quasi-Hermiticity of $H(g)$ in the mathematical Hilbert space:

$$H^{\dagger}(g)\, \Theta(g) = \Theta(g)\, H(g)\,. \tag{A11}$$

Thus, for a preselected Hamiltonian $H(g)$ with a real spectrum, its acceptability as a closed-system observable will be guaranteed either by the Hermiticity of its Ω-transformed isospectral avatar or, equivalently, by the Θ-quasi-Hermiticity property of $H(g)$ itself.

Appendix A.6. Final Note on the Notation and Outlook

In the literature devoted to the models using quasi-Hermitian observables, the notation conventions have not been united yet. Thus, the Hilbert-space metric (which we have

decided to denote by the upper-case Greek symbol Θ) can be found denoted as T (which does not mean time reversal, cf. Equation Nr. (2.2) in one of the oldest reviews of [10]); as a subscripted lower-case Greek η_+ (cf. Equation Nr. (52) in one of the more modern reviews of [14]); as $\exp(-Q)$ (cf. the 2006 paper [52]); as ρ (cf. [53]); or by the letter G (cf. Tables Nr. I and II in [54]).

The notion of the EPs (exceptional points) of our present interest emerged within the strictly mathematical theory of linear operators [7]. It has played a key role in the rigorous analysis of the criteria of convergence of perturbation series. In the context of physics, the notion was less well known, being called the Bender–Wu singularity [55]. Independently, this notion has only been found important for physicists [56,57], especially during the last twenty years, viz., during the growth of interest in the role played by the non-self-adjoint operators in several (i.e., not always just quantum) branches of phenomenology [28,29].

During the early stages of development of the latter innovative approach, the role of a benchmark illustrative example has been played by the IC (imaginary cubic) differential-operator Hamiltonian of Equation (1). Later, as we already explained above, such a choice of illustration proved to be a bit unfortunate. For proof, we cited Siegl and Krejčiřík, [1] who emphasized that, in the formal sense, the obstacles imposed by the loss of the Riesz-basis diagonalizability of the IC Hamiltonian are "much stronger than" those imposed by "any exceptional point associated with finite Jordan blocks".

The related concept of asymptotic IEP was not only very new, but also rather elusive. Even its definition, as provided by the authors, was simply implicit (see sections IID, IIIC and IV in [1]). An explanation is that their message was aimed, first of all, at the community of physicists for which the IEP IC oscillator model served as a heuristic "fons and origo" of what has been widely accepted as $\mathcal{PT}$-symmetric quantum mechanics. The same authors have also emphasized that the properties of $H^{(IC)}$ "are essentially different with respect to self-adjoint Hamiltonians, for instance, due to spectral instabilities". Thus, the main IEP-related result of [1] (viz., the proof of the existence of an IEP anomaly in the IC model) was finally formulated as an observation that "there is no quantum-mechanical Hamiltonian associated with it via ... similarity transformation".

The latter conclusion was revolutionary and opened a number of new questions concerning the necessity of finding "new directions in physical interpretation" of the model. In our present paper we, perhaps, have thrown new light on the issue, with a rather skeptical conclusion that the currently unresolved status of the twelve-year-old conceptual task of the interpretation of the IEP-related instabilities does not seem to have an easy resolution.

References

1. Siegl, P.; Krejčiřík, D. On the metric operator for the imaginary cubic oscillator. *Phys. Rev. D* **2012**, *86*, 121702(R). [CrossRef]
2. Trefethen, L.N.; Embree, M. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*; Princeton University Press: Princeton, NJ, USA, 2005.
3. Krejčiřík, D.; Siegl, P.; Tater, M.; Viola, J. Pseudospectra in non-Hermitian quantum mechanics. *J. Math. Phys.* **2015**, *56*, 103513. [CrossRef]
4. Bessis, D. (IPhT, Saclay, Paris, France). Personal communication, 1992.
5. Bender, C.M.; Boettcher, S. Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. *Phys. Rev. Lett.* **1998**, *80*, 5243. [CrossRef]
6. Dorey, P.; Dunning, C.; Tateo, R. Spectral equivalences, Bethe ansatz equations, andreality properties in $\mathcal{PT}$-symmetric quantum-mechanics. *J. Phys. A Math. Theor.* **2001**, *34*, 5679.
7. Kato, T. *Perturbation Theory for Linear Operators*; Springer: Berlin, Germany, 1966.
8. Günther, U.; Stefani, F. IR-truncated PT -symmetric ix^3 model and its asymptotic spectral scaling graph. *arXiv* **2019**, arXiv:1901.08526.

9. Dieudonné, J. Quasi-Hermitian operators. In Proceedings of the International Symposium on Linear Spaces, Jerusalem, Israel, 5–12 July 1961; Pergamon: Oxford, UK, 1961; pp. 115–122.
10. Scholtz, F.G.; Geyer, H.B.; Hahne, F.J.W. Quasi-Hermitian Operators in Quantum Mechanics and the Variational Principle. *Ann. Phys.* **1992**, *213*, 74. [CrossRef]
11. Znojil, M. Passage through exceptional point: Case study. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2020**, *476*, 20190831. [CrossRef]
12. Graefe, E.M.; Günther, U.; Korsch, H.J.; Niederle, A.E. A non-Hermitian PT symmetric Bose-Hubbard model: Eigenvalue rings from unfolding higherorder exceptional points. *J. Phys. A Math. Theor.* **2008**, *41*, 255206. [CrossRef]
13. Semorádová, I.; Siegl, P. Diverging eigenvalues in domain truncations of Schroedinger operators with complex potentials. *SIAM J. Math. Anal.* **2022**, *54*, 5064–5101. [CrossRef]
14. Mostafazadeh, A. Pseudo-Hermitian Quantum Mechanics. *Int. J. Geom. Meth. Mod. Phys.* **2010**, *7*, 1191–1306. [CrossRef]
15. Messiah, A. *Quantum Mechanics*; North Holland: Amsterdam, The Netherlands, 1961.
16. Znojil, M. Complex symmetric Hamiltonians and exceptional points of order four and five. *Phys. Rev.* **2018**, *A 98*, 032109. [CrossRef]
17. Znojil, M. Admissible perturbations and false instabilities in PT-symmetric quantum systems. *Phys. Rev. A* **2018**, *97*, 032114. [CrossRef]
18. Znojil, M. Unitarity corridors to exceptional points. *Phys. Rev. A* **2019**, *100*, 032124. [CrossRef]
19. Günther, U.; Rotter, I.; Samsonov, B. Projective Hilbert space structures at exceptional points. *J. Phys. A Math. Gen.* **2007**, *40*, 8815. [CrossRef]
20. Znojil, M. Three-Hilbert-space formulation of Quantum Mechanics. *SIGMA* **2009**, *5*, 001. arXiv:0901.0700. [CrossRef]
21. Fisher, M.E. Yang-Lee edge singularity and φ^3 field theory. *Phys. Rev. Lett.* **1978**, *40*, 1610–1613. [CrossRef]
22. Bender, C.M. Making Sense of Non-Hermitian Hamiltonians. *Rep. Prog. Phys.* **2007**, *70*, 947–1018. [CrossRef]
23. Brody, D.C. Biorthogonal quantum mechanics. *J. Phys. A Math. Theor.* **2013**, *47*, 035305. [CrossRef]
24. Mityagin, B.; Siegl, P. Local form-subordination condition and riesz basisness of root systems. *J. d'Anal. Math.* **2019**, *139*, 83–119. [CrossRef]
25. Langer, H.; Tretter, C. A Krein space approach to PT symmetry. *Czech. J. Phys.* **2004**, *54*, 1113–1120. [CrossRef]
26. Bagarello, F.; Gazeau, J.-P.; Szafraniec, F.; Znojil, M. (Eds.) *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*; Wiley: Hoboken, NJ, USA, 2015.
27. Feinberg, J.; Riser, B. Pseudo-Hermitian random-matrix models: General formalism. *Nucl. Phys. B* **2022**, *975*, 115678. [CrossRef]
28. Bender, C.M. (Ed.) *PT Symmetry in Quantum and Classical Physics*; World Scientific: Singapore, 2018.
29. Christodoulides, D.; Yang, J.-K. (Eds.) *Parity-Time Symmetry and Its Applications*; Springer: Singapore, 2018.
30. Giordanelli, I.; Graf, G.M. The Real Spectrum of the Imaginary Cubic Oscillator: An Expository Proof. *Ann. Henri Poincare* **2015**, *16*, 99–112. [CrossRef]
31. Dorey, P.; Dunning, C.; Tateo, R. From PT-symmetric quantum mechanics to conformal field theory. *Pramana-J. Phys.* **2009**, *73*, 217–239. [CrossRef]
32. Abarbanel, H.D.I.; Bronzan, J.D.; Sugar, R.L.; White, A.R. Reggeon field theory: Formulation and use. *Phys. Rep. C* **1975**, *21*, 121. [CrossRef]
33. Moiseyev, N. *Non-Hermitian Quantum Mechanics*; CUP: Cambridge, UK, 2011.
34. Shin, K.C. On the reality of the eigenvalues for a class of PT-symmetric oscillators. *Commun. Math. Phys.* **2002**, *229*, 543. [CrossRef]
35. Fernández, F.; Guardiola, R.; Ros J.; Znojil, M. Strong-coupling expansions for the PT-symmetric oscillators $V(r) = aix + b(ix)^2 + c(ix)^3$. *J. Phys. A Math. Gen.* **1998**, *31*, 10105–10112. [CrossRef]
36. Dyson, F.J. General theory of spin-wave interactions. *Phys. Rev.* **1956**, *102*, 1217–1230 [CrossRef]
37. Janssen, D.; Dönau, F.; Frauendorf, S.; Jolos, R.V. Boson description of collective states. *Nucl. Phys. A* **1971**, *172*, 145–165. [CrossRef]
38. Bender, C.M.; Milton, K.A. Nonperturbative Calculation of Symmetry Breaking in Quantum Field Theory. *Phys. Rev. D* **1997**, *55*, R3255. [CrossRef]
39. Buslaev, V.; Grecchi, V. Equivalence of unstable anharmonic oscillators and double wells. *J. Phys. A Math. Gen.* **1993**, *26*, 5541–5549. [CrossRef]
40. Bögli, S.; Siegl, P.; Tretter, C. Approximations of spectra of Schrödinger operators with complex potentials on $\mathbb{R}^d$. *Commun. Part. Diff. Equ.* **2012**, *42*, 1001–1041. [CrossRef]
41. Znojil, M. Time-dependent version of cryptohermitian quantum theory. *Phys. Rev. D* **2008**, *78*, 085003. [CrossRef]
42. Fring, A.; Moussa, M.H.Y. Unitary quantum evolution for time-dependent quasi-Hermitian systems with non-observable Hamiltonians. *Phys. Rev. A* **2016**, *93*, 042114. [CrossRef]
43. Znojil, M. Non-Hermitian interaction representation and its use in relativistic quantum mechanics. *Ann. Phys.* **2017**, *385*, 162–179. [CrossRef]
44. Khantoul, B.; Bounames, A.; Maamache, M. On the invariant method for the time-dependent non-Hermitian Hamiltonians. *Eur. Phys. J. Plus* **2017**, *132*, 258. [CrossRef]

45. Bishop, R.F.; Znojil, M. Non-Hermitian coupled cluster method for non-stationary systems and its interaction-picture reinterpretation. *Eur. Phys. J. Plus* **2020**, *135*, 374. [CrossRef]
46. Ju, C.-Y.; Miranowicz, A.; Minganti, F.; Chan, C.-T.; Chen, G.-Y.; Nori, F. Einstein's Quantum Elevator: Hermitization of Non-Hermitian Hamiltonians via a generalized vielbein Formalism. *Phys. Rev. Res.* **2022**, *4*, 023070. [CrossRef]
47. Feshbach, H. Unified theory of nuclear reactions. *Ann. Phys.* **1958**, *5*, 357–390. [CrossRef]
48. Bagarello, F. Algebras of unbounded operators and physical applications: A survey. *Rev. Math. Phys.* **2007**, *19*, 231–272. [CrossRef]
49. Znojil, M. Quantum catastrophes: A case study. *J. Phys. A Math. Theor.* **2012**, *45*, 444036. [CrossRef]
50. Znojil, M. Composite quantum Coriolis forces. *Mathematics* **2023**, *11*, 1375. [CrossRef]
51. Znojil, M. Hybrid form of quantum theory with non-Hermitian Hamiltonians. *Phys. Lett. A* **2023**, *457*, 128556. [CrossRef]
52. Jones, H.F.; Mateo, J. An Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ Potential. *Phys. Rev. D* **2006**, *73*, 085002. [CrossRef]
53. Fring, A.; Frith, T. Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians. *Phys. Rev. A* **2017**, *95*, 010102(R). [CrossRef]
54. Ju, C.Y.; Miranowicz, A.; Chen, Y.N.; Chen, G.Y.; Nori, F. Emergent parallel transport and curvature in Hermitian and non-Hermitian quantum mechanics. *Quantum* **2024**, *8*, 1277. [CrossRef]
55. Alvarez, G. Bender-Wu branch points in the cubic oscillator. *J. Phys. A Math. Gen.* **1995**, 28, 4589–4598. [CrossRef]
56. Heiss, W.D. Exceptional points—Their universal occurrence and their physical significance. *Czech. J. Phys.* **2004**, *54*, 1091–1100. [CrossRef]
57. Heiss, W.D. The physics of exceptional points. *J. Phys. A Math. Theor.* **2012**, *45*, 444016. [CrossRef]

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