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# Finite Multiple Mixed Values

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**Abstract:** In recent years, a variety of multiple zeta values (MZVs) variants have been defined and studied. One way to produce these variants is to restrict the indices in the definition of MZVs to some fixed parity pattern, which include Hoffman's multiple *t*-values, Kaneko and Tsumura's multiple *T*-values, and Xu and this paper's author's multiple *S*-values. Xu and this paper's author have also considered the so-called multiple mixed values by allowing all possible parity patterns and have studied a few important relations among these values. In this paper, we turn to the finite analogs and the symmetric forms of the multiple mixed values, motivated by a deep conjecture of Kaneko and Zagier, which relates the finite MZVs and symmetric MZVs, and a generalized version of this conjecture by the author to the Euler sum (i.e., level two) setting. We present a few important relations among these values such as the stuffle, reversal, and linear shuffle relations. We also compute explicitly the (conjecturally smallest) generating set in weight one and two cases. In the appendix, we tabulate some dimension computations for various subspaces of the finite multiple mixed values and propose a conjecture.

**Keywords:** (finite) multiple zeta values; (finite) Euler sums; (finite) multiple mixed values; (finite) multiple *t*-values; (finite) multiple *T*-values; (finite) multiple *S*-values

**MSC:** 11M32; 11B68

## 1. Introduction

### 1.1. Multiple Zeta Values and Their Finite Analogs

For any composition of positive integers $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$, we define the *multiple zeta value* (MZV) by

$$\zeta(\boldsymbol{s}) := \sum_{n_1 > \cdots > n_d > 0} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \tag{1}$$

and the *multiple zeta star value* (MZSV) by

$$\zeta^\star(\boldsymbol{s}) := \sum_{n_1 \geq \cdots \geq n_d \geq 1} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}. \tag{2}$$

These converge if and only if $s_1 \geq 2$, in which case we say that $\boldsymbol{s}$ is *admissible*. As usual, we call $|\boldsymbol{s}| := s_1 + \cdots + s_d$ the weight and $d$ the depth. These values were first systematically studied by Zagier [1] and Hoffman [2] independently.

In recent years, a lot of research has been carried out concerning the structure of the different variants of multiple zeta values due to their important applications in both mathematics and theoretical physics. For example, when we allow alternating signs to appear, we then obtain the so-called Euler sums (also called alternating MZVs), which play important roles in the study of knot theory and Witten multiple zeta functions associated with Lie algebra (see, e.g., [3,4]). If we allow not only $\pm 1$ but, more generally, $N$th roots of unity, then we can consider colored/cyclotomic MZVs of level $N$ (see [5]), which have also appeared unexpectedly in the study of Feynman diagrams [6].

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On the other hand, the modular arithmetic nature of the partial sums of MZVs was first considered by Hoffman [7] and the author of this paper [8] independently. Contrary to the classical cases above, not many variants of these sums exist. To set up the correct theoretical framework for these variants, we define the following adéle-like ring, which was first considered by Kontsevich and then applied to the $p$-adic setting by Kaneko and Zagier [9]. Let $\mathcal{P}$ be the set of primes. Set

$$\mathcal{A} := \prod_{p\in\mathcal{P}} (\mathbb{Z}/p\mathbb{Z}) \Big/ \bigoplus_{p\in\mathcal{P}} (\mathbb{Z}/p\mathbb{Z}). \tag{3}$$

Then, we can define the *finite multiple zeta values* (FMZVs) by the following:

$$\zeta_{\mathcal{A}}(\boldsymbol{s}) := \left( \sum_{p>n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \pmod{p} \right)_{p\in\mathcal{P}} \in \mathcal{A}. \tag{4}$$

In 2014, Kaneko and Zagier proposed a deep conjecture (see Conjecture 1 below for a generalization), relating these values on the $p$-adic side to MZVs on the Archimedean side via a mysterious connection. This conjecture is far from being proven, but, since then, a plethora of parallel results have been shown to hold on both sides simultaneously (see, e.g., [10–15]). In particular, for each positive integer $w \geq 2$, the element

$$\beta_w := \left( \frac{B_{p-w}}{w} \right)_{w<p\in\mathcal{P}} \in \mathcal{A} \tag{5}$$

is the finite analog of $\zeta(w)$, where $B_n$s are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

And the Fermat quotient

$$\mathfrak{q}_2 := \left( \frac{2^{p-1}-1}{p} \right)_{2<p\in\mathcal{P}} \in \mathcal{A} \tag{6}$$

is the analog of $-\zeta(\bar{1}) = \log 2$.

### *1.2. Euler Sums and Their Finite Analogs*

As is common in the study of number theory, it is often worthwhile to consider the alternating version of an interesting positive series. We now apply this idea to MZVs. For $s_1, \ldots, s_d \in \mathbb{N}$ and $\varepsilon_1, \ldots, \varepsilon_d = \pm 1$, we define the *Euler sums*

$$\zeta\begin{pmatrix} s_1, \ldots, s_d \\ \varepsilon_1, \ldots, \varepsilon_d \end{pmatrix} := \sum_{n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{\varepsilon_j^{n_j}}{n_j^{s_j}}. \tag{7}$$

For convenience, if $\varepsilon_j = -1$, then $\bar{s}_j$ is used, and, if a substring $S$ repeats $n$ times in the list, then $\{S\}^n$ is used.

To state the relations between Euler sums concisely, we define a kind of double cover of the set $\mathbb{N}$ of positive integers.

**Definition 1.** *Let $\mathbb{D}$ be the set of signed numbers $\mathbb{N} \cup \overline{\mathbb{N}}$, where*

$$\overline{\mathbb{N}} := \{\bar{s} : s \in \mathbb{N}\}.$$

*Define the absolute value function $|\cdot|$ on $\mathbb{D}$ by $|s| = |\bar{s}| = s$ for all $s \in \mathbb{N}$ and the sign function by $\operatorname{sgn}(s) = 1$ and $\operatorname{sgn}(\bar{s}) = -1$ for all $s \in \mathbb{N}$.*

For any $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{D}^d$, we define the $n$-th partial sum of the *Euler sums* by

$$\zeta_n(\boldsymbol{s}) := \sum_{n > n_1 > \cdots > n_d > 0} \prod_{j=1}^{d} \frac{\operatorname{sgn}(s_j)^{n_j}}{n_j^{|s_j|}}. \tag{8}$$

Similarly to FMZVs, finite *Euler sums* (FESs) are defined by

$$\zeta_{\mathcal{A}}(\boldsymbol{s}) := \Big(\zeta_p(\boldsymbol{s}) \pmod p\Big)_{p \in \mathcal{P}} \in \mathcal{A}. \tag{9}$$

For example, it is not hard to compute that, for all prime $p > 2$, we have (see [16], Theorem 8.2.7)

$$\zeta_p(\bar{1}) := \sum_{n=1}^{p} \frac{(-1)^n}{n} \equiv \frac{2^{p-1} - 1}{p} \pmod p.$$

Thus, $\zeta_{\mathcal{A}}(\bar{1}) = -\mathfrak{q}_2$, as defined in (6), which is the finite analog of $\zeta(\bar{1}) = -\log 2$.

In [16], Conjecture 8.6.9), the author of this paper extended the Kakeko–Zagier conjecture to the setting of the Euler sums. For $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{D}^d$, define the symmetrized version of the alternating Euler sums by

$$\zeta_*^{\mathcal{S}}(\boldsymbol{s}) := \sum_{i=0}^{d} \left( \prod_{j=1}^{i} (-1)^{|s_j|} \operatorname{sgn}(s_j) \right) \zeta_*(s_i, \ldots, s_1) \zeta_*(s_{i+1}, \ldots, s_d),$$
$$\zeta_{ш}^{\mathcal{S}}(\boldsymbol{s}) := \sum_{i=0}^{d} \left( \prod_{j=1}^{i} (-1)^{|s_j|} \operatorname{sgn}(s_j) \right) \zeta_{ш}(s_i, \ldots, s_1) \zeta_{ш}(s_{i+1}, \ldots, s_d),$$

where $\zeta_\sharp$ ($\sharp = *$ or ш) are regularized values (see [16], Proposition 13.3.8). They are called $\sharp$-*symmetric Euler sums*. If $\boldsymbol{s} \in \mathbb{N}^d$, then they are called $\sharp$-*symmetric multiple zeta values* ($\sharp$-SMZVs or simply SMZVs if $\sharp$ does not matter).

**Conjecture 1.** *For any $w \in \mathbb{N}$, let $\mathsf{FES}_w$ (resp. $\mathsf{ES}_w$) be the $\mathbb{Q}$-vector space generated by all FESs (resp. Euler sums) of weight $w$. Then, there is an isomorphism*

$$f_{\mathsf{ES}} : \mathsf{FES}_w \longrightarrow \frac{\mathsf{ES}_w}{\zeta(2)\mathsf{ES}_{w-2}},$$
$$\zeta_{\mathcal{A}}(\boldsymbol{s}) \longmapsto \zeta_{ш}^{\mathcal{S}}(\boldsymbol{s}).$$

**Remark 1.** *Note that we can replace $\zeta_{ш}^{\mathcal{S}}(\boldsymbol{s})$ by $\zeta_*^{\mathcal{S}}(\boldsymbol{s})$ (see [16], Exercise 8.7).*

To better understand this mysterious relation is the primary motivation of this paper. We mainly study a few variants of the finite analogs of Euler sums by presenting some results that are analogous to those on the Archimedean side.

## 2. Multiple Mixed Values and Their Finite Analogs

We now turn to the main object of study in this paper. For any $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d) \in \{\pm 1\}^d$, and $n \in \mathbb{N}$, we define the $n$-th partial sum of *multiple mixed values* (MMVs) by

$$M_n(\boldsymbol{s}; \boldsymbol{\varepsilon}) := \sum_{n > n_1 > \cdots > n_d > 0} \prod_{j=1}^{d} \frac{1 + \varepsilon_j (-1)^{n_j}}{2 n_j^{s_j}} = \sum_{\substack{n > n_1 > \cdots > n_d > 0 \\ n_j \equiv (1-\varepsilon_j)/2 \pmod 2}} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \tag{10}$$

and the *finite multiple mixed values* (FMMVs) by

$$M_{\mathcal{A}}(\boldsymbol{s}; \boldsymbol{\varepsilon}) := \Big(M_p(\boldsymbol{s}; \boldsymbol{\varepsilon}) \pmod p\Big)_{p \in \mathcal{P}} \in \mathcal{A}. \tag{11}$$

We call $|\boldsymbol{s}| := s_1 + \cdots + s_d$ the weight and $d$ the depth.

The motivation for defining the MMVs is to find a common generalization of a few variants of level-two MZVs, including the following. For all admissible $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$, we define the *multiple t-values* (MtVs, see [17]), *multiple T-values* (MTVs, see [18]), and *multiple S-values* (MSVs, see [19]) by

$$t(\boldsymbol{s}) := M(\boldsymbol{s}; \{-1\}^d) = \sum_{n_1 > \cdots > n_d > 0} \prod_{j=1}^{d} \frac{1}{(2n_j - 1)^{s_j}}, \tag{12}$$

$$T(\boldsymbol{s}) := M\Big(\boldsymbol{s}; (-1)^d, (-1)^{d-1}, \ldots, 1, -1\Big) = \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}, \tag{13}$$

$$S(\boldsymbol{s}) := M\Big(\boldsymbol{s}; (-1)^{d-1}, (-1)^{d-2}, \ldots, -1, 1\Big) = \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j \pmod 2}} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}, \tag{14}$$

respectively. Their finite analogs $t_{\mathcal{A}}(\boldsymbol{s})$, $T_{\mathcal{A}}(\boldsymbol{s})$, and $S_{\mathcal{A}}(\boldsymbol{s})$ are defined similarly as in (4) and (9). It is clear that

$$t_{\mathcal{A}}(\boldsymbol{s}) = \frac{1}{2^d} \sum_{\varepsilon_1, \ldots, \varepsilon_d = \pm 1} \Big( \prod_{1 \le j \le d} \varepsilon_j \Big) \zeta_{\mathcal{A}}\binom{\boldsymbol{s}}{\boldsymbol{\varepsilon}}, \quad F_{\mathcal{A}}(\boldsymbol{s}) = \frac{1}{2^d} \sum_{\varepsilon_1, \ldots, \varepsilon_d = \pm 1} \Big( \prod_{\substack{1 \le j \le d \\ 2 \mid d-j \text{ if } F = T \\ 2 \nmid d-j \text{ if } F = S}} \varepsilon_j \Big) \zeta_{\mathcal{A}}\binom{\boldsymbol{s}}{\boldsymbol{\varepsilon}}.$$

More recently, another type of level-two FMZVs has been defined by Kaneko et al. [20] as follows (after multiplying by $2^{|\boldsymbol{s}|}$ in the following):

$$\zeta_{\mathcal{A}}^{(2)}(\boldsymbol{s}) := \Bigg( \sum_{p > n_1 > \cdots > n_d > 0,\ 2 \mid n_j\, \forall j} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \Bigg)_{p \in \mathcal{P}} = \frac{1}{2^d} \sum_{\varepsilon_1, \ldots, \varepsilon_d = \pm 1} \zeta_{\mathcal{A}}\binom{\boldsymbol{s}}{\boldsymbol{\varepsilon}}. \tag{15}$$

It is also clear that

$$\zeta^{(2)}(\boldsymbol{s}) := \sum_{n_1 > \cdots > n_d > 0,\ 2 \mid n_j\, \forall j} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} = \frac{1}{2^{|\boldsymbol{s}|}} \zeta(\boldsymbol{s}). \tag{16}$$

**Definition 2.** *Let $d \in \mathbb{N}$ and $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$. We define the $\natural$-regularized MTVs ($\natural = *$ or $\sqcup\!\sqcup$) and* MSVs *by*

$$F_{\natural}(\boldsymbol{s}) := \frac{1}{2^d} \sum_{\sigma_1, \ldots, \sigma_d = \pm 1} \Big( \prod_{\substack{1 \le j \le d \\ 2 \mid d-j \text{ if } F = T \\ 2 \nmid d-j \text{ if } F = S}} \sigma_j \Big) \zeta_{\natural}(\boldsymbol{s}; \boldsymbol{\sigma}) \quad (F = T \text{ or } S).$$

*We define the $\natural$-symmetric multiple T-values (SMTVs) and $\natural$-symmetric multiple S-values (SMSVs) by*

$$F_{\natural}^{\mathcal{S}}(\boldsymbol{s}) := \begin{cases} \displaystyle\sum_{i=0}^{d} \Big( \prod_{\ell=1}^{i} (-1)^{s_\ell} \Big) F_{\natural}(s_i, \ldots, s_1) F_{\natural}(s_{i+1}, \ldots, s_d), & \text{if } d \text{ is even;} \\ \displaystyle\sum_{i=0}^{d} \Big( \prod_{\ell=1}^{i} (-1)^{s_\ell} \Big) \widetilde{F}_{\natural}(s_i, \ldots, s_1) F_{\natural}(s_{i+1}, \ldots, s_d), & \text{if } d \text{ is odd,} \end{cases}$$

*where $\widetilde{F} = S + T - F$ and we set $\prod_{\ell=1}^{0} = 1$, as usual.*

In [21], we proved that, for $\sharp = *$ or $\sqcup\!\sqcup$ and for all $\boldsymbol{s} \in \mathbb{N}^d$,

$$f_{\mathsf{ES}} T_{\mathcal{A}}(\boldsymbol{s}) = T^{\mathcal{S}}_{\sharp}(\boldsymbol{s}) \quad \text{and} \quad f_{\mathsf{ES}} S_{\mathcal{A}}(\boldsymbol{s}) = S^{\mathcal{S}}_{\sharp}(\boldsymbol{s}) \pmod{\zeta(2)}. \tag{17}$$

In comparison to the results in [19], we define the following subspaces of FES:

- FES: generated by FESs (finite Euler sums);
- FMM: generated by FMMVs (finite multiple mixed values);
- FMt: generated by FMtVs (finite multiple $t$-values);
- FMT: generated by FMTVs (finite multiple $T$-values);
- FMS: generated by FMSVs (finite multiple $S$-values);
- $\mathsf{FMZ}^{(2)}$: generated by level two FMZVs $\zeta^{(2)}_{\mathcal{A}}(\boldsymbol{s})$ defined by (15);
- FMe: generated by the $\{M_{\mathcal{A}}(\boldsymbol{s}, \varepsilon) : \boldsymbol{s} \in \mathbb{N}^d, \varepsilon \in \{\pm 1\}^d, \varepsilon_d = 1\}$;
- FMo: generated by the $\{M_{\mathcal{A}}(\boldsymbol{s}, \varepsilon) : \boldsymbol{s} \in \mathbb{N}^d, \varepsilon \in \{\pm 1\}^d, \varepsilon_d = -1\}$.

**Proposition 1.** *We have* $\mathsf{FES}_w = \mathsf{FMM}_w$ *for all* $w \in \mathbb{N}$.

**Proof.** This follows easily from the identities

$$(-1)^n = \frac{(1+(-1)^n) - (1-(-1)^n)}{2}, \quad 1 = \frac{(1+(-1)^n) + (1-(-1)^n)}{2}.$$

□

For any fixed weight $w \geq 1$, we clearly have the inclusion relations between weight $w$ pieces of the above subspaces:

$$\begin{array}{ccccccccc} & & \mathsf{FMS}_w & \subseteq & \mathsf{FES}_w & \supseteq & \mathsf{FMT}_w & & \\ & & \cap\!| & & \| & & \cap\!| & & \\ \mathsf{FMZ}^{(2)}_w & \subseteq & \mathsf{FMe}_w & \subseteq & \mathsf{FMM}_w & \supseteq & \mathsf{FMo}_w & \supseteq & \mathsf{FMt}_w \end{array}$$

Kaneko et al. ([20], Conjecture 5) conjectured that $\mathsf{FMZ}^{(2)}_w = \mathsf{FES}_w$ for all $w \in \mathbb{N}$. We further conjecture the equal signs in the second row above always hold (also see Conjecture A1).

**Conjecture 2.** *For sufficiently large weight* $w \in \mathbb{N}$, *we have*

$$\begin{array}{ccccccccc} \mathsf{FMZ}_w & \subsetneq & \mathsf{FMS}_w & \subsetneq & \mathsf{FES}_w & \supsetneq & \mathsf{FMT}_w & \supsetneq & \mathsf{FMZ}_w \\ \cap\!\nmid & & \cap\!\nmid & & \| & & \cap\!\nmid & & \cap\!\nmid \\ \mathsf{FMZ}^{(2)}_w & = & \mathsf{FMe}_w & = & \mathsf{FMM}_w & = & \mathsf{FMo}_w & = & \mathsf{FMt}_w \end{array}$$

Except for the middle vertical equal sign, all the other relations are supported by numerical evidence but no formal proofs yet. In contrast, we have the following conjectural relations between the classical subspaces of Euler sums (cf. the Venn diagram at the end of [19]).

**Problem 1.** *How can we verify numerically the inclusion* $\mathsf{FMZ}_w \subseteq \mathsf{FMS}_w \cap \mathsf{FMZ}^{(2)}_w$ *and* $\mathsf{FMZ}_w \subseteq \mathsf{FMT}_w \cap \mathsf{FMt}_w$? *We only need to find a basis in each of the four subspaces on the right and show that every FMZV can be expressed as a* $\mathbb{Q}$*-linear combination of the basis elements. Is it even possible that both (or one) of the inclusions are actually equalities?*

**Conjecture 3.** *For sufficiently large weight* $w \in \mathbb{N}$, *we have*

$$\begin{array}{ccccccccc} \mathsf{MZV}_w & \subsetneq & \mathsf{MSV}_w & \subsetneq & \mathsf{ES}_w & \supsetneq & \mathsf{MTV}_w & \supsetneq & \mathsf{MZV}_w \\ \| & & \cap\!\nmid & & \cup\!\nmid & & \cap\!\nmid & & \cap\!\nmid \\ \mathsf{MZV}^{(2)}_w & \subsetneq & \mathsf{MVe}_w & = & \mathsf{MMV}_w & \supsetneq & \mathsf{MVo}_w & \supsetneq & \mathsf{MtV}_w \end{array}$$

The left vertical equal sign is obvious by (16) and the middle vertical (strict) inclusion follows from [19], Theorem 7.1, if we assume a variant of Grothendieck's period conjecture. Again, all the other relations are supported by numerical evidence but no formal proofs yet.

In the following, we consider three classes of relations satisfied by FMMVs: stuffle relations, reversal relations, and linear shuffle relations. They play the key roles in the dimension computation of the vector space generated by FMMVs and its various subspaces.

### *2.1. Stuffle Relations*

Stuffle relations hold in all of the above subalgebras of FES appearing in Conjecture 2, except for FMT and FMS. For convenience, we make $\mathbb{D}$ a semi-group by equipping it with a commutative and associative binary operation $\oplus$ (called *O-plus*) as follows: for all $a, b \in \mathbb{D}$,

$$a \oplus b := \begin{cases} \overline{|a| + |b|}, & \text{if } \mathrm{sgn}(a) \neq \mathrm{sgn}(b); \\ |a| + |b|, & \text{if } \mathrm{sgn}(a) = \mathrm{sgn}(b). \end{cases} \tag{18}$$

For example, for all $a, b \in \mathbb{D}$,

$$\zeta_{\mathcal{A}}(a)\zeta_{\mathcal{A}}(b) = \zeta_{\mathcal{A}}(a, b) + \zeta_{\mathcal{A}}(b, a) + \zeta_{\mathcal{A}}(a \oplus b),$$

and if $a, b, c \in \mathbb{D}$

$$\begin{aligned} M_{\mathcal{A}}(a, b)M_{\mathcal{A}}(c) = {} & M_{\mathcal{A}}(a, b, c) + M_{\mathcal{A}}(a, c, b) + M_{\mathcal{A}}(c, a, b) \\ & + \delta_{\mathrm{sgn}(a),\mathrm{sgn}(c)} M_{\mathcal{A}}(a \oplus c, b) + \delta_{\mathrm{sgn}(b),\mathrm{sgn}(c)} M_{\mathcal{A}}(a, b \oplus c) \end{aligned}$$

where $\delta_{s,t}$ is the Kronecker symbol satisfying $\delta_{s,t} = 1$ if $s = t$ and $\delta_{s,t} = 0$ otherwise. Hence, the stuffing for FMMVs occurs only when the two merging components have the same sign.

**Problem 2.** *Do* FMT *and* FMS *form subalgebras of* FES*? If not, what is the first instance of a product that is not closed? Note, if the answer is negative, then it cannot be verified rigorously with the current level of knowledge. This is similar to the situation for classical* MTV *and* MSV*, and this is for the same type of reason. In the classical setting, counterexamples can only be verified numerically approximately, but they cannot be proven rigorously due to the difficulty in proving transcendence results in general. In the p-adic cases, the transcendence problem for $\mathcal{A}$-numbers may be even harder. For example, we do not even know if $\beta_w \in \mathcal{A}$ vanishes or not for any fixed odd $w \geq 3$, although, conjecturally, the density of primes $p > w$, such that $B_{p-w} \equiv 0 \pmod{p}$ should be 0.*

The question is interesting to us because the corresponding problem for the classical MTVs has an affirmative answer due to their iterated integral expression (see [18], Theorem 2.1).

Here is the process to find such a product of two FMTVs of total weight $w$ numerically, under the assumption that $\dim_{\mathbb{Q}} \mathsf{FES}_w = F_w$, which is the same as $\dim_{\mathbb{Q}} \mathsf{FMM}_w = F_w$ by Proposition 1. First, find a generating set $\mathcal{TB}_w$ of $\mathsf{FMT}_w$, which is possible by using linear shuffle, reversal, and stuffle relations. Second, expand the set into a a generating set $\mathcal{MB}_w$ of $\mathsf{FMM}_w$. If $|\mathcal{MB}_w| = F_w$, then it is a basis by the assumption $\dim_{\mathbb{Q}} \mathsf{FMM}_w = F_w$. Finally, for each product of FMTVs of weight $w$, we can express it using the conjectural basis $\mathcal{MB}_w$. If elements outside of $\mathcal{TB}_w$ are needed for such a product, then it does not lie in $\mathsf{FMT}_w$, meaning that the product of FMT is not closed.

### *2.2. Reversal Relations*

The reversal relations in the next proposition are a class of relations satisfied by the FMMVs but not by the classical MMVs.

**Proposition 2.** *For all $d \in \mathbb{N}$, $\boldsymbol{s} \in \mathbb{N}^d$, and $\boldsymbol{\varepsilon} \in \{\pm 1\}^d$,*

$$M_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}, \overleftarrow{\boldsymbol{\varepsilon}}) = (-1)^{|\boldsymbol{s}|} M_{\mathcal{A}}(\boldsymbol{s}, \boldsymbol{\varepsilon}). \tag{19}$$

*In particular, for all* $\boldsymbol{s} \in \mathbb{N}^d$,

$$t_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}\zeta_{\mathcal{A}}^{(2)}(\boldsymbol{s}), \tag{20}$$

$$T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{and} \quad S_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{if } 2|d, \tag{21}$$

$$T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{and} \quad S_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{if } 2\nmid d. \tag{22}$$

**Proof.** The relations follow immediately from the change of indices $n \to p - n$. □

**Proposition 3.** *For all* $d \in \mathbb{N}$ *and* $\boldsymbol{s} \in \mathbb{N}^d$,

$$T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{and} \quad S_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{if } 2|d, \tag{23}$$

$$T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{and} \quad S_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{if } 2\nmid d. \tag{24}$$

**Proof.** When $d$ is even, we have

$$\begin{aligned} T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) &= \sum_{i=0}^{d}\Big(\prod_{\ell=i+1}^{d}(-1)^{s_\ell}\Big)T_*(s_{i+1},\dots,s_d)T_*(s_i,\dots,s_1) \\ &= (-1)^w\sum_{i=0}^{d}\Big(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Big)T_*(s_i,\dots,s_1)T_*(s_{i+1},\dots,s_d) = (-1)^w T_*^{\mathcal{S}}(\boldsymbol{s}). \end{aligned}$$

The same argument works for SMSVs and odd $d$ cases. □

**Proposition 4.** *For all* $s \in \mathbb{N}$,

$$t_{\mathcal{A}}(s,s) = \zeta_{\mathcal{A}}^{(2)}(s,s) = \begin{cases} \mathrm{q}_2^2/2, & \textit{if } s = 1; \\ (1-2^{1-s})^2\beta_s^2/2, & \textit{if } s \geq 2, \end{cases} \tag{25}$$

*and*

$$t_{\mathcal{A}}(s,s,s) = -\zeta_{\mathcal{A}}^{(2)}(s,s,s) = \begin{cases} \mathrm{q}_2^3/6 + \beta_3/8, & \textit{if } s = 1; \\ (1-2^{1-s})^3\beta_s^3/6 + \beta_{3s}/8, & \textit{if } s \geq 2. \end{cases} \tag{26}$$

*More generally, for all* $d \in \mathbb{N}$,

$$t_{\mathcal{A}}(\{s\}^d) = -\zeta_{\mathcal{A}}^{(2)}(\{s\}^d) \in \delta_{s,1}\mathrm{q}_2^s\mathbb{Q} + \sum_{k_1+\cdots+k_\ell=d,\ell\in\mathbb{N}}\beta_{sk_1}\dots\beta_{sk_\ell}\mathbb{Q}.$$

*Moreover, we may assume that all* $k_j$*s are odd.*

**Proof.** By the stuffle relations,

$$2t_{\mathcal{A}}(s,s) = t_{\mathcal{A}}(s)^2 - t_{\mathcal{A}}(2s).$$

Thus, (25) follows from (37) and (20) by noticing that $\beta_w = 0$ if $w$ is even. Similarly, by the stuffle relations,

$$\begin{aligned} 6t_{\mathcal{A}}(s,s,s) &= t_{\mathcal{A}}(s)t_{\mathcal{A}}(s,s) - t_{\mathcal{A}}(2s,s) - t_{\mathcal{A}}(s,2s) \\ &= t_{\mathcal{A}}(s)(t_{\mathcal{A}}(s)^2 s - t_{\mathcal{A}}(2s)) - t_{\mathcal{A}}(s)t_{\mathcal{A}}(2s) + t_{\mathcal{A}}(3). \end{aligned}$$

Hence, (26) follows from (37) and (20). The general homogeneous FMtVs can be computed similarly by induction or by [7], Theorem 2.3 (see also [22,23], Lemma 5.1). The statement for $\zeta_{\mathcal{A}}^{(2)}$ is an easy application of the reversal relation (20) so that the sign in the equation is $(-1)^{sd}$. But the values could be nonzero only when both $s$ and $d$ are odd, resulting in the negative sign. □

*2.3. Linear Shuffle Relations*

The most nontrivial relations among finite MZVs and finite Euler sums are provided by the linear shuffle relations, which are closely related to the shuffle relations among the classical MZVs and Euler sums. In this subsection, we extend this to FMMVs and their alternating versions.

For any $n \in \mathbb{N}, \boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d, \boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d) \in \{\pm 1\}^d$, and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d) \in \{\pm 1\}^d$, we define the partial sums of alternating MMVs by

$$M_n(\boldsymbol{s}; \boldsymbol{\varepsilon}; \boldsymbol{\sigma}) := \sum_{n > k_1 > \cdots > k_d > 0} \frac{(1 + \varepsilon_1 (-1)^{k_1}) \sigma_1^{(2k_1 + 1 - \varepsilon_1)/4} \cdots (1 + \varepsilon_d (-1)^{k_d}) \sigma_d^{(2k_d + 1 - \varepsilon_d)/4}}{k_1^{s_1} \cdots k_d^{s_d}}. \tag{27}$$

When $n \to \infty$ and $(s_d, \sigma_d) \neq (1, 1)$, we recover the alternating MMVs studied by Xu, Yan, and the present author in [24]. By allowing $n$ to range over the set of primes, we can now define the *finite alternating multiple mixed values*

$$M_{\mathcal{A}}(\boldsymbol{s}; \boldsymbol{\varepsilon}; \boldsymbol{\sigma}) := \Big( M_p(\boldsymbol{s}; \boldsymbol{\varepsilon}; \boldsymbol{\sigma}) \pmod p \Big)_{p \in \mathcal{P}}. \tag{28}$$

It turns out that, at depth two, all the non-alternating MMVs already have special names, which we now recall. Let $d = 2$. Then, we set

$$\begin{aligned} \zeta_{\mathcal{A}}^{(2)}(\boldsymbol{s}; \boldsymbol{\sigma}) &:= M_{\mathcal{A}}(\boldsymbol{s}; 1, 1; \boldsymbol{\sigma}), & T_{\mathcal{A}}(\boldsymbol{s}; \boldsymbol{\sigma}) &:= M_{\mathcal{A}}(\boldsymbol{s}; 1, -1; \boldsymbol{\sigma}), \\ S_{\mathcal{A}}(\boldsymbol{s}; \boldsymbol{\sigma}) &:= M_{\mathcal{A}}(\boldsymbol{s}; -1, 1; \boldsymbol{\sigma}), & t_{\mathcal{A}}(\boldsymbol{s}; \boldsymbol{\sigma}) &:= M_{\mathcal{A}}(\boldsymbol{s}; -1, -1; \boldsymbol{\sigma}). \end{aligned}$$

Moreover, to save space, if an alternating sign $\sigma_j = -1$, then we put a bar on top of $s_j$ correspondingly. For example,

$$S_{\mathcal{A}}(\bar{2}, 3) = \sum_{m > n > 0, 2 \nmid m, 2 \mid n} \frac{(-1)^{(m+1)/2}}{m^2 n^3}.$$

We recall briefly the main setup for the integral expression of alternating MMVs. Let

$$\begin{aligned} \mathtt{a} &= \frac{dt}{t}, \quad \mathtt{b} = w_{+1}^{-1} := \frac{dt}{1 - t^2}, \quad \mathtt{c} = w_{-1}^{-1} := \frac{-dt}{1 + t^2}, \\ \beta &= w_{+1}^{+1} := \frac{t\, dt}{1 - t^2}, \quad \gamma = w_{-1}^{+1} := \frac{-t\, dt}{1 + t^2}. \end{aligned}$$

Set

$$w_{\sigma}^{\varepsilon_1, \varepsilon_2} := \max\{\sigma, \operatorname{sgn}(1 + \varepsilon_2 - \varepsilon_1)\} w_{\sigma}^{\varepsilon_1 \varepsilon_2} = \begin{cases} -w_{\sigma}^{\varepsilon_1 \varepsilon_2}, & \text{if } \sigma = \varepsilon_2 = -\varepsilon_1 = -1; \\ w_{\sigma}^{\varepsilon_1 \varepsilon_2}, & \text{otherwise.} \end{cases}$$

It is straightforward to deduce that alternating MMVs can be expressed by the following iterated integrals:

$$M(\boldsymbol{s}; \boldsymbol{\varepsilon}; \boldsymbol{\sigma}) = \int_0^1 w_0^{s_1 - 1} w_{\sigma_1}^{\varepsilon_1, \varepsilon_2} w_0^{s_2 - 1} w_{\sigma_1 \sigma_2}^{\varepsilon_2, \varepsilon_3} \cdots w_0^{s_r - 1} w_{\sigma_1 \sigma_2 \cdots \sigma_r}^{\varepsilon_r}. \tag{29}$$

For $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \{\pm 1\}^r$, define

$$\operatorname{sgn}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) := (-1)^{\sharp\{i < r \mid \sigma_i = \varepsilon_i = \varepsilon_{i+1} \varepsilon_{i+2} \cdots \varepsilon_r = -1\}}.$$

Then, for all $\boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $(s_1, \sigma_1) \neq (1, 1)$, we have

$$\int_0^1 w_0^{s_1 - 1} w_{\sigma_1}^{\varepsilon_1} \cdots w_0^{s_r - 1} w_{\sigma_r}^{\varepsilon_r} = \operatorname{sgn}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) M(\boldsymbol{s}; \widetilde{\boldsymbol{\varepsilon}}; \widetilde{\boldsymbol{\sigma}}) \tag{30}$$

where $\widetilde{\sigma} = (\sigma_1, \sigma_2\sigma_1, \ldots, \sigma_r\sigma_{r-1})$ and $\widetilde{\varepsilon} = (\varepsilon_1 \cdots \varepsilon_r, \varepsilon_2 \cdots \varepsilon_r, \ldots, \varepsilon_{r-1}\varepsilon_r, \varepsilon_r)$. See [24] for more details, where the definition of alternating MMVs differs from the one used in this paper by a power of 2.

We now apply the above integral expressions to our finite situations. For example,

$$\begin{aligned}\int_0^t \mathfrak{b}\beta\gamma &= \int_0^t \sum_{i\geq 0} x^{2i}\,dx \int_0^x \sum_{j\geq 0} y^{2j+1}\,dy \int_0^y \sum_{k\geq 0} (-1)^{k+1} z^{2k+1}\,dz \\ &= \sum_{i,j,k\geq 0} \frac{t^{2i+2j+2k+5}(-1)^{k+1}}{(2i+2j+2k+5)(2j+2k+4)(2k+2)} \\ &= \sum_{m>n>l>0, 2\nmid m, 2\mid n, 2\mid l} \frac{t^m(-1)^{l/2}}{mnl}.\end{aligned}$$

Taking the coefficient of $t^p$ for any prime $p$, we obtain

$$\mathrm{Coeff}_{t^p} \int_0^t \mathfrak{b}\beta\gamma = \frac{1}{p}\zeta_p^{(2)}(1,\bar{1}).$$

Then, from the shuffle relation

$$\int_0^t \mathfrak{b} \int_0^t \beta\gamma = \int_0^t \mathfrak{b} \sqcup\!\sqcup \beta\gamma = \int_0^t (\mathfrak{b}\beta\gamma + \beta\mathfrak{b}\gamma + \beta\gamma\mathfrak{b})$$

we obtain

$$p \cdot \mathrm{Coeff}_{t^p}\Big( \int_0^t \mathfrak{b} \int_0^t \beta\gamma \Big) = \zeta_p^{(2)}(1,\bar{1}) + S_p(1,\bar{1}) + t_p(\bar{1},\bar{1}).$$

Hence, we arrive at one of the linear shuffle relations:

$$\zeta_{\mathcal{A}}^{(2)}(1,\bar{1}) + S_{\mathcal{A}}(1,\bar{1}) + t_{\mathcal{A}}(\bar{1},\bar{1}) = 0. \tag{31}$$

Similarly, we can obtain the following linear shuffle relations by (30):

$$\mathfrak{b} \sqcup\!\sqcup \mathfrak{bb} \Longrightarrow 3T_{\mathcal{A}}(1,1) = 0, \tag{32}$$
$$\mathfrak{b} \sqcup\!\sqcup \mathfrak{bc} \Longrightarrow 2T_{\mathcal{A}}(1,\bar{1}) - T_{\mathcal{A}}(\bar{1},\bar{1}) = 0, \tag{33}$$
$$\gamma \sqcup\!\sqcup \mathfrak{b}\gamma \Longrightarrow (-1)^{p'} S_{\mathcal{A}}(\bar{1},\bar{1}) - 2\zeta_{\mathcal{A}}^{(2)}(\bar{1},1) = 0, \tag{34}$$
$$\gamma \sqcup\!\sqcup \gamma\mathfrak{b} \Longrightarrow 2t_{\mathcal{A}}(1,\bar{1}) + S_{\mathcal{A}}(\bar{1},\bar{1}) = 0, \tag{35}$$
$$\mathfrak{b} \sqcup\!\sqcup \gamma\beta \Longrightarrow \zeta_{\mathcal{A}}^{(2)}(\bar{1},\bar{1}) - (-1)^{p'} S_{\mathcal{A}}(\bar{1},1) - (-1)^{p'} t_{\mathcal{A}}(\bar{1},1) = 0. \tag{36}$$

Here and in the rest of this section, we put $p' = (p-1)/2$ to save space. We observe that the number of $\mathfrak{b}$ and $\mathfrak{c}$ must be either one or three in order to have nontrivial relations.

## 3. Depth-One and -Two Values

In this short section, we compute a few special forms of FMMVs of depth one and two by using the relations we have found in the previous sections.

First, we observe that, since $\zeta_{\mathcal{A}}(s) = 0$ for all $s \in \mathbb{N}$, by [16], Theorem 8.2.7,

$$\zeta_{\mathcal{A}}^{(2)}(s) = S_{\mathcal{A}}(s) = -t_{\mathcal{A}}(s) = -T_{\mathcal{A}}(s) = \frac{1}{2}\zeta_{\mathcal{A}}(\bar{s}) = \begin{cases} -\mathfrak{q}_2, & \text{if } s = 1; \\ (2^{1-s}-1)\beta_s, & \text{if } s \geq 2, \end{cases} \tag{37}$$

where $\mathfrak{q}_2$ is the Fermat quotient (6) and $\beta_s$ is given by (5). For weight one and depth one, we have the following result.

**Proposition 5.** *We have*

$$S_{\mathcal{A}}(1) = \zeta^{(2)}_{\mathcal{A}}(1) = -\mathfrak{q}_2, \quad \zeta^{(2)}_{\mathcal{A}}(\bar{1}) = S_{\mathcal{A}}(\bar{1}) = -\mathfrak{q}_2/2, \tag{38}$$

$$t_{\mathcal{A}}(1) = T_{\mathcal{A}}(1) = \mathfrak{q}_2, \qquad t_{\mathcal{A}}(\bar{1}) = T_{\mathcal{A}}(\bar{1}) = -(-1)^{p'}\mathfrak{q}_2/2 \tag{39}$$

*where we regard* $(-1)^{p'}$ *as* $\left((-1)^{p'}\right)_{p\geq 3} \in \mathcal{A}$.

**Proof.** We only need to prove that $S_{\mathcal{A}}(1) = 2S_{\mathcal{A}}(\bar{1})$. Indeed,

$$\begin{aligned}
2S_p(1) &= \sum_{k=1}^{p'} \frac{(-1)^k+1}{k} + \sum_{p'<k<p, 2|k} \frac{2}{k} \\
&= 2S_p(\bar{1}) + \sum_{k=1}^{p'} \frac{1}{k} + \sum_{0<k\leq p', 2|k} \frac{1}{p-k} + \sum_{p'<k<p, 2|k} \frac{1}{k} \\
&\equiv 2S_p(\bar{1}) + \sum_{0<k\leq p', 2|k} \frac{1}{k} + \sum_{p'<k<p, 2|k} \frac{1}{k} \pmod{p} \\
&\equiv 2S_p(\bar{1}) + S_p(1) \pmod{p}.
\end{aligned}$$

This proposition follows easily from (37) and the substitution $k \to p-k$ for $t_{\mathcal{A}}(\bar{1})$ and $T_{\mathcal{A}}(\bar{1})$. □

For depth-two values, we have the following results.

**Proposition 6.** *For all* $a, b \in \mathbb{N}$, *if* $w = a+b$ *is odd, then*

$$\zeta^{(2),\star}_{\mathcal{A}}(a,b) = -t^{\star}_{\mathcal{A}}(a,b) = \frac{1}{2}\left[2^{1-w} - 1 - (-1)^a 2^{-w}\binom{w}{a}\right]\beta_w, \tag{40}$$

$$\zeta^{(2)}_{\mathcal{A}}(a,b) = -t_{\mathcal{A}}(a,b) = \frac{1}{2}\left[1 - 2^{1-w} - (-1)^a 2^{-w}\binom{w}{a}\right]\beta_w, \tag{41}$$

$$S_{\mathcal{A}}(a,b) = T_{\mathcal{A}}(a,b) = \frac{(-1)^a}{2}\left(1-2^{-w}\right)\binom{w}{a}\beta_w. \tag{42}$$

**Remark 2.** *The formulas for* $\zeta^{(2)}_{\mathcal{A}}(s)$ *in* (37) *and for* $\zeta^{(2)}_{\mathcal{A}}(a,b)$ *in* (40) *are consistent with* [20], *Proposition 2.1. Note that the ordering in this paper is opposite to that of* [20]. *Also, our definition of* $\zeta^{(2)}_{\mathcal{A}}(\boldsymbol{s})$ *is* $2^{-|\boldsymbol{s}|}$ *times that in* [20].

**Proof.** By [16], Theorem 8.6.4,

$$\begin{aligned}
\zeta^{\star}_{\mathcal{A}}(a,b) &= \zeta_{\mathcal{A}}(a,b) = (-1)^a\binom{w}{a}\beta_w, \\
\zeta^{\star}_{\mathcal{A}}(\bar{a},\bar{b}) &= \zeta_{\mathcal{A}}(\bar{a},\bar{b}) = (-1)^a(2^{1-w}-1)\binom{w}{a}\beta_w, \\
\zeta_{\mathcal{A}}(\bar{a},b) &= \zeta_{\mathcal{A}}(a,\bar{b}) = (1-2^{1-w})\beta_w, \\
\zeta^{\star}_{\mathcal{A}}(\bar{a},b) &= \zeta^{\star}_{\mathcal{A}}(a,\bar{b}) = (2^{1-w}-1)\beta_w.
\end{aligned} \tag{43}$$

The proposition follows easily. □

## 4. Weight-Two Finite Alternating MMVs

In the previous sections, we have mainly studied FMMVs and some of their special forms. As we mentioned earlier, it is often fruitful to study the alternating version of any interesting positive series. Hence, we turn to the finite alternating MMVs and find the corresponding algebraic structures when the weight is two.

We first recall that the Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

Moreover, $E_0(0) = 1$ and, for all $j \in \mathbb{N}$,

$$E_j(0) = \frac{2^{j+1}}{j+1}\left(B_{j+1}\left(\frac{1}{2}\right) - B_{j+1}\right) = \frac{2}{j+1}(1-2^{j+1})B_{j+1} \tag{44}$$

by [16], p. 242, (8.11). The Euler numbers $E_n$ are defined by the generating function

$$\frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!},$$

which then satisfy

$$E_n = 2^n E_n\left(\frac{1}{2}\right). \tag{45}$$

Let $G$ be the traditional Catalan's constant defined by

$$G := \sum_{n\geq 1}\frac{(-1)^{n-1}}{(2n-1)^2}.$$

Then,

$$T(\bar{2}) = \sum_{n\geq 1}\frac{(-1)^n}{(2n-1)^2} = -G.$$

Motivated by the next proposition, we define the *finite Catalan's constant* by

$$G_{\mathcal{A}} := \left(\frac{E_{p-3}}{2}\right)_{3<p\in\mathcal{P}} \in \mathcal{A}.$$

**Proposition 7.** *We have*

$$T_{\mathcal{A}}(\bar{2}) = -G_{\mathcal{A}}. \tag{46}$$

**Proof.** For any prime $p \geq 5$, we have modulo $p$

$$\begin{aligned}
T_p(\bar{2}) &= \sum_{0<k<p,2\nmid k}\frac{(-1)^{(k+1)/2}}{k^2} = \sum_{0<k<p,2|k}\frac{(-1)^{(p-k+1)/2}}{(p-k)^2}\\
&\equiv (-1)^{(p+1)/2}\sum_{0<k<p,2|k}\frac{(-1)^{k/2}}{k^2}\\
&\equiv (-1)^{(p+1)/2}\sum_{n=1}^{p'}\frac{(-1)^n}{4n^2}\\
&\equiv \frac{(-1)^{(p+1)/2}}{4}\sum_{n=1}^{p'}(-1)^n n^{p-3}\\
&\equiv \frac{(-1)^{(p+1)/2}}{8}\left((-1)^{p'}E_{p-3}\left(\frac{p+1}{2}\right) + E_{p-3}(0)\right)\\
&\equiv -\frac{1}{8}E_{p-3}\left(\frac{1}{2}\right)\\
&\equiv -\frac{1}{2}E_{p-3}
\end{aligned}$$

by [16], Lemma 8.2.5 and (45) since $E_{p-3}(0) = 0$ for all odd primes $p$ by (44). This completes the proof of the proposition. □

**Proposition 8.** *We have*

$$S_{\mathcal{A}}(2) = T_{\mathcal{A}}(2) = T_{\mathcal{A}}(1,1) = 0, \quad S_{\mathcal{A}}(1,1) = -\mathfrak{q}_2^2, \quad t_{\mathcal{A}}(1,1) = \zeta_{\mathcal{A}}^{(2)}(1,1) = \frac{\mathfrak{q}_2^2}{2}. \tag{47}$$

**Proof.** The vanishing of the first three values follows from (37) and (32) quickly. Then, by Proposition 5, we obtain

$$S_{\mathcal{A}}(1,1) = S_{\mathcal{A}}(1,1) + T_{\mathcal{A}}(1,1) = S_{\mathcal{A}}(1)T_{\mathcal{A}}(1) = \mathfrak{q}_2^2.$$

By (37), we see that

$$2t_{\mathcal{A}}(1,1) = t_{\mathcal{A}}(1)^2 - t_{\mathcal{A}}(2) = \mathfrak{q}_2^2$$

and, similarly,

$$2\zeta_{\mathcal{A}}^{(2)}(1,1) = S_{\mathcal{A}}(1)^2 - S_{\mathcal{A}}(2) = \mathfrak{q}_2^2.$$

We have completed the proof of the proposition. □

**Proposition 9.** *We have*

$$T_{\mathcal{A}}(\bar{1},\bar{1}) = 2T_{\mathcal{A}}(1,\bar{1}), \quad S_{\mathcal{A}}(\bar{1},\bar{1}) = -2t_{\mathcal{A}}(1,\bar{1}), \quad t_{\mathcal{A}}(\bar{1},\bar{1}) = \zeta_{\mathcal{A}}^{(2)}(\bar{1},\bar{1}) = \frac{\mathfrak{q}_2^2}{8}. \tag{48}$$

**Proof.** The first identity is just (33) and the second is (35). The other two follow immediately from the stuffle relations by Proposition 5. □

**Proposition 10.** *We have*

$$T_{\mathcal{A}}(\bar{2}) = -G_{\mathcal{A}}, \quad T_{\mathcal{A}}(\bar{1},1) = -\frac{(-1)^{p'}}{2}G_{\mathcal{A}}, \quad S_{\mathcal{A}}(\bar{1},1) = \frac{(-1)^{p'}}{2}\mathfrak{q}_2^2 - \frac{1}{2}G_{\mathcal{A}}, \tag{49}$$

$$S_{\mathcal{A}}(\bar{2}) = (-1)^{p'}G_{\mathcal{A}}, \quad T_{\mathcal{A}}(1,\bar{1}) = \frac{1}{2}G_{\mathcal{A}}, \quad S_{\mathcal{A}}(1,\bar{1}) = -\frac{1}{2}\mathfrak{q}_2^2 + \frac{(-1)^{p'}}{2}G_{\mathcal{A}}, \tag{50}$$

$$t_{\mathcal{A}}(\bar{1},1) = -\frac{3(-1)^{p'}}{8}\mathfrak{q}_2^2 + \frac{1}{2}G_{\mathcal{A}}, \quad \zeta_{\mathcal{A}}^{(2)}(\bar{1},1) = \frac{1}{8}\mathfrak{q}_2^2 - \frac{(-1)^{p'}}{2}G_{\mathcal{A}}, \tag{51}$$

$$t_{\mathcal{A}}(1,\bar{1}) = -\frac{(-1)^{p'}}{8}\mathfrak{q}_2^2 + \frac{1}{2}G_{\mathcal{A}}, \quad \zeta_{\mathcal{A}}^{(2)}(1,\bar{1}) = \frac{3}{8}\mathfrak{q}_2^2 - \frac{(-1)^{p'}}{2}G_{\mathcal{A}}. \tag{52}$$

**Proof.** Fix a large prime $p$ so that the identities in both Propositions 8 and 9 hold for $T_p$, $S_p$, $t_p$, and $\zeta_p^{(2)}$. Throughout the rest of this proof, we drop the subscript $p$ to save space.

By stuffle relation and Proposition 5, we have

$$(-1)^{p'}\frac{\mathfrak{q}_2}{2} = S(1)T(\bar{1}) = S(\bar{1},1) + T(1,\bar{1}), \tag{53}$$

$$(-1)^{p'}\frac{\mathfrak{q}_2}{4} = S(\bar{1})T(\bar{1}) = S(\bar{1},\bar{1}) + T(\bar{1},\bar{1}) = 2T(1,\bar{1}) - 2t(1,\bar{1}), \tag{54}$$

by Proposition 9. Plugging (54) into (53),

$$S(\bar{1},1) = 3T(1,\bar{1}) - 4t(1,\bar{1}). \tag{55}$$

By the change of index $k \to p-k$ for $\zeta^{(2)}$ in (36) (or by (31)), we obtain

$$t(\bar{1},1) = (-1)^{p'}t(\bar{1},\bar{1}) - S(\bar{1},1) = (-1)^{p'}\frac{\mathfrak{q}_2^2}{8} - S(\bar{1},1) = 3t(1,\bar{1}) - 2T(1,\bar{1}) \tag{56}$$

by (54) and (55).

Next, by Proposition 5 and (54),

$$T(\bar{2}) = t(1)t(\bar{1}) - t(1,\bar{1}) - t(\bar{1},1) = -\frac{(-1)^{p'}}{2}\mathfrak{q}_2^2 - t(1,\bar{1}) - t(\bar{1},1) = -2T(1,\bar{1})$$

by (56) and (56). Thus,

$$T_{\mathcal{A}}(1,\bar{1}) = -\frac{1}{2}T_{\mathcal{A}}(\bar{2}) = \frac{1}{2}G_{\mathcal{A}}$$

by Proposition 7. On the other hand, by the index substitution $k \to p-k$, we see that

$$S_{\mathcal{A}}(\bar{2}) = -(-1)^{p'}T(\bar{2}) = (-1)^{p'}G_{\mathcal{A}}, \qquad T_{\mathcal{A}}(\bar{1},1) = -(-1)^{p'}T_{\mathcal{A}}(\bar{1},1) = -\frac{(-1)^{p'}}{2}G_{\mathcal{A}}.$$

Further, (34) yields that

$$\zeta^{(2)}(\bar{1},1) = \frac{(-1)^{p'}}{2}S(\bar{1},\bar{1}) = \frac{\mathfrak{q}_2^2}{8} - \frac{(-1)^{p'}}{2}T(\bar{1},\bar{1}) = \frac{\mathfrak{q}_2^2}{8} - (-1)^{p'}T(1,\bar{1}) = \frac{\mathfrak{q}_2^2}{8} + T(\bar{1},1) \tag{57}$$

by (54) and Proposition 9. Hence,

$$t(1,\bar{1}) = -(-1)^{p'}\zeta^{(2)}(\bar{1},1) = -\frac{(-1)^{p'}}{8}\mathfrak{q}_2^2 + T(1,\bar{1}) = -\frac{(-1)^{p'}}{8}\mathfrak{q}_2^2 + \frac{1}{2}G_{\mathcal{A}}. \tag{58}$$

All the other identities in the proposition can be now derived by applying the index substitution $k \to p-k$. □

By combining Propositions 8–10, we immediately obtain the following theorem.

**Theorem 1.** *Let* $\mathsf{FAM}_w$ *be the* $\mathbb{Q}$*-vector space generated by finite alternating MMVs of weight* $w$*. Let* $(-1)^{p'} = \left((-1)^{(p-1)/2}\right)_{3\le p\in\mathcal{P}} \in \mathcal{A}$*. Then,*

$$\mathsf{FAM}_2 = \left\langle \mathfrak{q}_2^2, G_{\mathcal{A}}^2, (-1)^{p'}\mathfrak{q}_2^2, (-1)^{p'}G_{\mathcal{A}}^2 \right\rangle. \tag{59}$$

We have carried out some extensive computations of $\mathsf{FAM}_w$ for small weights $w$ and tabulated the result in Appendix A in this paper.

## 5. Sum Formulas of FMMVs

One of the most intriguing formulas of multiple zeta values is the so-called sum formula, proven independently by Granville [25] and Zagier around 1997: for all positive integers $w > d \ge 1$, let

$$I_{w,d} := \{\boldsymbol{s} \in \mathbb{N}^d : |\boldsymbol{s}| = w\}.$$

Then,

$$\sum_{\boldsymbol{s}\in I_{w,d}} \zeta(\boldsymbol{s}) = \zeta(w).$$

There have been many variations and generalizations of this formula since then. See, for example, refs. [12,26–31]. In this section, we prove a few formulas of the same flavor for some special types of FMMVs.

### *5.1. Sum Formulas of Symmetric and Finite MTVs and MSVs of Even Depth*

The first class of formulas we consider concerns only MTVs and MSVs. They reflect clearly the philosophy that the symmetric version should provide the bridge between the $p$-adic world and the Archimedean world.

**Proposition 11.** *Suppose* $w \ge d \in \mathbb{N}$ *with* $w$ *odd and* $d$ *even. Then, for* $F = T$ *and* $S$*, we have*

$$\sum_{\boldsymbol{s}\in I_{w,d}} F_{\mathcal{A}}(\boldsymbol{s}) = 0, \qquad \sum_{\boldsymbol{s}\in I_{w,d}} F_*^{\mathcal{S}}(\boldsymbol{s}) = 0.$$

**Proof.** By reversal relation, we have

$$\sum_{\boldsymbol{s}\in I_{w,d}} T_{\mathcal{A}}(\boldsymbol{s}) = \sum_{\boldsymbol{s}\in I_{w,d}} (-1)^w T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = -\sum_{\boldsymbol{s}\in I_{w,d}} T_{\mathcal{A}}(\boldsymbol{s}) = 0.$$

The same argument works for FMSVs. For the symmetric values, by (23), we have

$$\sum_{\boldsymbol{s}\in I_{w,d}} T_*^{\mathcal{S}}(\boldsymbol{s}) = \sum_{\boldsymbol{s}\in I_{w,d}} T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^w \sum_{\boldsymbol{s}\in I_{w,d}} T_*^{\mathcal{S}}(\boldsymbol{s}) = -\sum_{\boldsymbol{s}\in I_{w,d}} T_*^{\mathcal{S}}(\boldsymbol{s}) = 0.$$

The same argument works for SMSVs. This concludes the proof of the proposition. □

*5.2. Restricted Sum Relations*

In this subsection, we prove the following results, which involve all FMMVs.

**Theorem 2.** *Let $w, d \in \mathbb{N}$ with $d \le w$. Let*

$$\begin{aligned} I_{w,d,i} &:= \{\boldsymbol{s} : |\boldsymbol{s}| = w, \operatorname{dep}(\boldsymbol{s}) = d, s_i \ge 2\} && \forall 1 \le i \le d, \\ I_{w,d,i,i} &:= \{\boldsymbol{s} : |\boldsymbol{s}| = w, \operatorname{dep}(\boldsymbol{s}) = d, s_i \ge 3\} && \forall 1 \le i \le d, \\ I_{w,d,i,j} &:= \{\boldsymbol{s} : |\boldsymbol{s}| = w, \operatorname{dep}(\boldsymbol{s}) = d, s_i \ge 2, s_j \ge 2\} && \forall 1 \le j < i \le d. \end{aligned}$$

*Then,*

$$\begin{aligned} \sum_{\boldsymbol{s}\in I_{w,d}} M_{\mathcal{A}}(\boldsymbol{s}) &= \sum_{\boldsymbol{s}\in I_{w,d}} M_{\mathcal{A}}^{\star}(\boldsymbol{s}) = 0, \\ \sum_{\boldsymbol{s}\in I_{w,d,i}} M_{\mathcal{A}}(\boldsymbol{s}) &= (-1)^{i-1}\left(\binom{w-1}{i-1} + (-1)^d \binom{w-1}{d-i}\right)\beta_w, \\ \sum_{\boldsymbol{s}\in I_{w,d,i}} M_{\mathcal{A}}^{\star}(\boldsymbol{s}) &= (-1)^{i-1}\left((-1)^d \binom{w-1}{i-1} + \binom{w-1}{d-i}\right)\beta_w, \\ \sum_{\boldsymbol{s}\in I_{w,d,i,j}} M_{\mathcal{A}}(\boldsymbol{s}) &= (-1)^d \sum_{\boldsymbol{s}\in I_{w,d,i,j}} M_{\mathcal{A}}^{\star}(\boldsymbol{s}) = \frac{1}{2} N_{w,d,i,j}\beta_w, \end{aligned}$$

*where $\beta_w$ is defined by (5) and $N_{w,d,i,j}$ is an integer explicitly given by* [31], *Theorem 3.1.*

**Proof.** For any prime $p$ and $\boldsymbol{s} \in I_{w,d}$, recall that

$$\zeta_p(\boldsymbol{s}) := \sum_{p>n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}.$$

By [7], Theorem 4.1, we have

$$\sum_{\sigma\in\mathfrak{S}_d} \zeta_p(\sigma(\boldsymbol{s})) \equiv 0 \pmod{p}$$

where $\mathfrak{S}_d$ is the group of symmetry of $d$ letters. By partitioning $I_{w,d}$ into equivalent classes under permutation, we see quickly that

$$\sum_{\boldsymbol{s}\in I_{w,d}} \zeta_{\mathcal{A}}(\boldsymbol{s}) = 0.$$

Thus,

$$\sum_{\boldsymbol{s}\in I_{w,d}} M_{\mathcal{A}}(\boldsymbol{s}) = \sum_{\boldsymbol{s}\in I_{w,d}} \sum_{0<n_1<\cdots<n_d<p} \prod_{j=1}^{d} \frac{(1+(-1)^{n_j}) + (1-(-1)^{n_j})}{2n_j^{s_j}} = \sum_{\boldsymbol{s}\in I_{w,d}} \zeta_{\mathcal{A}}(\boldsymbol{s}) = 0.$$

The second and third equations follow from the sum formulas of Saito and Wakabayashi ([12], Theorem 1.4), and the last two from [31], Theorem 3.1 immediately. □

## 6. Concluding Remarks

In this paper, motivated by the Kaneko–Zagier conjecture that relates MZVs and FMZVs, and the present author's similar conjecture relating Euler sums and finite Euler

sums (which are alternating versions of MZVs and FMZVs, respectively), we have defined and studied the finite versions of multiple mixed values (MMVs), which are further variations of MZVs, by allowing arbitrary parity patterns on the summation indices. As the set of MZVs is included in the set of Euler sums, which, in turn, is a subset of MMVs, we can hope to gain some insight on MZVs and Euler sums by studying MMVs. The same holds true for their finite analogs in the $p$-adic world. And the bridge between the two worlds is the symmetric version. Philosophically speaking, any $\mathbb{Q}$-linear relation that holds for the symmetric version (modulo $\zeta(2)$) should hold for the finite version.

By introducing several crucial algebraic relations, including the stuffle, reversal, and linear shuffle relations, which govern the structure of FMMVs and their various special forms, we were able to carry out explicit computations for generating sets in weight-one and -two cases and, more generally, analyze the dimensions of the various subspaces generated by these values. Furthermore, we proposed a couple of new conjectures (one of which is in Appendix A) based on the numerical evidence, the confirmation of which will greatly deepen our understanding of the algebraic structure and relationships among these mathematical objects.

It is well known that Euler sums have played an important role in theoretical physics. Since all of the MMVs are real numbers, one may wonder if there is any such MMV (and not an Euler sum) that appears essentially in the computation of Feynman integrals. Note that MMVs are level-four objects whose motivic structure has been investigated by Deligne in his influential paper [5], while the motivic nature of some of the Feynman integrals has been revealed in Brown's paper [32].

However, our research is limited by our current (lack of) knowledge of the still highly mysterious relations between the classical Archimedean world and the $p$-adic world—in particular, our incomplete understanding of Grothendieck's theory of motives (especially the crystalline aspect), even though we have a much better understanding of the de Rham and Betti side of the story by the seminal work of Deligne [33], Deligne and Goncharov [34], and Brown [35]. Progress in the study of motives will undoubtedly allow us to see a more holistic and complete picture of the various entities appearing in this paper.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| (F)MZV | (finite) multiple zeta values |
| (F)ES | (finite) Euler sums |
| (F)MMV | (finite) multiple mixed values |
| (F)MtV | (finite) multiple $t$-values |
| (F)MTV | (finite) multiple $T$-values |
| (F)MSV | (finite) multiple $S$-values |

## Appendix A. Dimensions of Subspaces of Finite MMVs

by Jeremy Feng, Angelina Kim, Sienna Li, Ryan Qin, Logan Wang, and J. Zhao

The following table provides the conjectural dimensions of the various subspaces of FMMs considered in this paper. We obtained these by computing the largest number of independent $\mathbb{Q}$-linear relations among these FMMVs, numerically aided by Maple with the code contained in [16], Appendix D. When a dimension is not followed by a question mark then it means we have verified it numerically by finding the correct number of basis elements.

**Table A1.** Conjectural dimensions of the various subspaces of FMMs over $\mathbb{Q}$.

| $w$ | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathsf{FMZ}_w$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 |
| $\mathsf{FMZ}_w^{(2)}$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |
| $\mathsf{FES}_w$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |
| $\mathsf{FMt}_w$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |
| $\mathsf{FMT}_w$ | 0 | 1 | 0 | 1 | 2 | 3 | 3 | 6 | 9 | 15 | 17 | 32 | 44 | 76 |
| $\mathsf{FMS}_w$ | 0 | 1 | 1 | 1 | 2 | 4 | 5 | 7 | 12 | 19 | 28 | 39 | 66 | 109 |
| $\mathsf{FMM}_w$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144? | 233? |
| $\mathsf{FMe}_w$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89? | 144? | 233? |
| $\mathsf{FMo}_w$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89? | 144? | 233? |

From numerical computation, we can formulate the following conjecture.

**Conjecture A1.** *(i) For all* $w \geq 1$,

$$\mathsf{FMZ}_w^{(2)} = \mathsf{FMe}_w = \mathsf{FMM}_w = \mathsf{FES}_w = \mathsf{FMo}_w = \mathsf{FMt}_w$$

*all have dimension* $F_w$.

*(ii) For all* $w \geq 1$,

$$\frac{\mathsf{FMZ}_w^{(2)}}{\zeta_{\mathcal{A}}^{(2)}(\{1\}^w)\mathbb{Q}} = \frac{\mathsf{FMe}_w}{\zeta_{\mathcal{A}}^{(2)}(\{1\}^w)\mathbb{Q}} = \frac{\mathsf{FES}_w}{\zeta_{\mathcal{A}}(\{\bar{1}\}^w)\mathbb{Q}} = \frac{\mathsf{FMo}_w}{t_{\mathcal{A}}(\{1\}^w)\mathbb{Q}} = \frac{\mathsf{FMt}_w}{t_{\mathcal{A}}(\{1\}^w)\mathbb{Q}}$$

*all have dimension* $F_w - 1$.

*(iii) For all* $k \geq 1$,

$$\dim_{\mathbb{Q}} \mathsf{FMT}_{2k+1} = \dim_{\mathbb{Q}} \mathsf{FMT}_{2k} + \dim_{\mathbb{Q}} \mathsf{FMT}_{2k-1}.$$

Note that Conjecture A1(ii) and [19], Theorem 7.1, can partially explain why the conjectured dimension for $\mathsf{FMM}_w$ (and $\mathsf{FMe}_w$) computed in this paper differs by 1 from the corresponding dimension for classical $\mathsf{MMV}_w$ (and $\mathsf{MVe}_w$) that was numerically computed in [19].

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