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Classifying Sets of Type (4,*n*) in PG(3,*q*)

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Abstract: In the present work, we classify sets of type (4,*n*) in PG(3,*q*). We prove that PG(3,*q*), apart from the planes of PG(3,3), contains only sets of type (4,*n*) with standard parameters. Thus, somewhat surprisingly, we conclude that there are no sets of type (4,*n*) in PG(3,*q*), $q \neq 3$, with non-standard parameters.

Keywords: two-character sets; two-intersection sets; projective linear two-weight codes; strongly regular graphs

MSC: 05B25; 51E20

1. Introduction

A *k*-set of type (*m*,*n*) in PG(3,*q*) is a set of *k* points which intersects every plane in *m* or *n* points and in which both values occur. The integers *m* and *n* are called *intersection numbers* and such a set is also called *a two-intersection set*. With every two-intersection set, there is associated a projective two-weight linear code and a strongly regular graph; see [1,2]. If $n = m + q$, then the two intersection numbers are said to be *standard*; see [3] (p. 231) and [4] (p. 378). The origin of the idea of this research is a paper by Vito Napolitano and Domenico Olanda [5] (p. 396) where they ask if PG(3,*q*) contains sets of type (3,*h*) with nonstandard parameters. Since sets of type (*m*,*n*) in PG(3,q) are completely determined for $m = 1, 2, 3$ in [6–8], in this paper we consider the case (4,*n*). We classify sets of type (4,*n*) in PG(3,*q*). As a consequence, we conclude that, apart from the planes of PG(3,3), sets of type (4,*n*) in PG(3,*q*) only have standard parameters.

2. Material and Methods

Let us consider $S_{r,q} := \mathrm{PG}(r,q)$, the projective space of dimension *r* and order *q*, where $q = p^h$, is a prime power. Let us denote by $\theta_d := \sum_{i=0}^{d} q^i$ the size of a subspace S_d and by $\gamma_{r,d} := \prod_{j=0}^{d} \frac{\theta_{r-j}}{\theta_{d-j}}$ the number of subspaces of dimension *d* of $S_{r,q}$. Let *K* denote a *k*-set, i.e., a set of *k* points. As usually, for each integer *i* such that $0 \leq i \leq \theta_d$, let us call as *i–subspace of dimension d* a subspace S_d of $S_{r,q}$ meeting *K* in exactly *i* points. Moreover, for any integers *i* and *j* such that $0 \leq i \leq \theta_j$ and $0 \leq j \leq r$, let us denote by $t_i^j = t_i^j(K)$ the number of *i*–subspaces of dimension *j* of $S_{r,q}$. The numbers t_i^j are called the *characters* of *K* with respect to the subspaces of dimension *j*; see [9]. Double counting the number of subspaces of dimension *j* of $S_{r,q}$, ($h = 0$), the number of pairs point-subspace of dimension *j*, (*P*,*S*), where $P \in K \cap S$, ($h = 1$), and the number of pairs ($\{P, Q\}$, *S*) where $\{P, Q\} \subset K \cap S$, ($h = 2$), the following equations hold; cf. [9,10]:

$$\sum_{i=0}^{\theta_j} \binom{i}{h} t_i^j = \binom{k}{h} \gamma_{r,j} \; h = 0,\ 1,\ 2 \tag{1}$$

Moreover, for any point *P* let us consider the characteristic function of the set *K*, $x(P) = \begin{cases} 0 \; if \; P \notin K \\ 1 \; if \; P \in K \end{cases}$. For any integers *i* and *j* such that $0 \leq i \leq \theta_j$ and $0 \leq j \leq r$, let us denote

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by $u_i^j = u_i^j(P)$ the number of i–subspaces of dimension j of $S_{r,q}$ through the point P. The numbers u_i^j are called the *characters* of the point P with respect to the subspaces of dimension j; see [9]. Double counting the number of subspaces of dimension j of $S_{r,q}$ through the point P, ($h = 0$) and the number of pairs $(\{P, Q\}, S)$, where $Q \in K \cap S$ and S is a subspace of dimension j of $S_{r,q}$ through the point P, ($h = 1$), we get the following equations; cf. [9,10]:

$$\sum_{i=0}^{\theta_j} \binom{i - x(P)}{h} u_i^j = \binom{k - x(P)}{h} \gamma_{r-h-1,j-h-1} \; h = 0, \; 1 \tag{2}$$

Let $m_1, m_2, \ldots, m_s$ be s integers such that $0 \leq m_1 < m_2 <, \ldots, < m_s \leq \theta_d$. A set K is said to be *of class* $[m_1, m_2, \ldots, m_s]_j$ if $t_i^j \neq 0$ only if $i \in \{m_1, m_2, \ldots, m_s\}$. Moreover, K is said to be *of type* $(m_1, m_2, \ldots, m_s)_j$ if $t_i^j \neq 0$ if and only if $i \in \{m_1, m_2, \ldots, m_s\}$. The integers $m_1, m_2, \ldots, m_s$ are called *intersection numbers* with respect to the dimension j.

3. Results

Let K be a k-set of type $(4,n)_2$ of PG(3,q), with $n \geq 5$. Equation (1) becomes

$$\begin{cases} t_4^2 + t_n^2 = (q+1)(q^2+1) \\ 4t_4^2 + nt_n^2 = k(q^2+q+1) \\ 12t_4^2 + n(n-1)t_n^2 = k(k-1)(q+1) \end{cases} \tag{3}$$

We get the equation

$$(q+1)k^2 - k[(n+4)(q^2+q+1) - q^2] + 4n(q^2+1)(q+1) = 0 \tag{4}$$

which can be written as the equation

$$[(q+1)k - (3q^2+4q+4)]k = n[(q^2+q+1)k - 4(q^2+1)(q+1)] \tag{5}$$

Let $p \notin K$, Equations (2) become

$$\begin{cases} u_4^2 + u_n^2 = q^2 + q + 1 \\ 4u_4^2 + nu_n^2 = k(q+1) \end{cases}$$

and we have that

$$\begin{cases} u_4^2 = \frac{n(q^2+q+1) - k(q+1)}{n-4} - \frac{q^2}{n-4} \\ u_n^2 = \frac{k(q+1) - 4(q^2+q+1)}{n-4} + \frac{q^2}{n-4} \end{cases}$$

Thus, $(n-4) \mid q^2$.
We prove the following:

Lemma 1. *In PG(3,2) a k-set of type (4,n) is the complementary set either of one line or of one elliptic quadric.*

Proof. The Equation (4), for $q = 2$, becomes $3k^2 - 7nk - 24k + 60n = 0$. Since $n - 4 \mid 4$, and $n \leq 7$, we have that $n - 4 \in \{1, 2\}$. Since for $n = 5$ the equation $3k^2 - 59k + 300 = 0$ has complex solutions, we have that $n = 6$. Thus, we have either a 10-set or a 12-set of type (4,6) whose complementary set is either a 5-set or a 3-set of type (1,3), i.e., either an elliptic quadric or one line; see [6]. □

Lemma 2. *In PG(3,3) a k-set of type (4,n) is either a plane, or the whole space PG(3,3), or a hyperbolic quadric, or, more generally, the union of four pairwise skew lines, or the union of the edges of a tetrahedron without its vertices with one external line ℓ.*

Proof. The Equation (4), for $q = 3$, becomes $4k^2 - 13nk - 43k + 160n = 0$. Since $n - 4 \mid 9$, we have that $n - 4 \in \{1, 3, 9\}$. Since for $n = 5$ the equation $k^2 - 27k + 200 = 0$ has complex solutions, we have that $n \in \{7, 13\}$. If $n = 7$, we have a 16-set K of type (4,7) with, by Equation (1), $t_4^2 = 24$ and $t_7^2 = 16$. Let π denote a 7-plane. The 7-set $K \cap \pi$ is a blocking set, because a 0-line is contained in four 4-planes, necessarily. We claim that K contains at least one line. Suppose, on the contrary, that K contains no line. Thus, by Equation (1), the blocking 7-set $K \cap \pi$ contains exactly 6 3-lines. There are two types of 4-planes: those with three collinear points and those without three collinear points. Since PG(2,3) contains a unique blocking 7-set with exactly three 2-lines and any 2-line is contained in exactly two 7-planes, we have that $t_2^1 = 24$ and, by Equation (1), $t_0^1 = 10$, $t_1^1 = 64$, and $t_3^1 = 32$. Let us denote by x the number of 4-planes with three collinear points. Double counting the pairs 3-line–4-plane (ℓ,π) where $\ell \subset \pi$, we get $x = 32$, a contradiction because $t_4^2 = 24$. Let ℓ denote a line contained in K. The set $K - \ell$ is a 12-set of type (3,6) of PG(3,3), and so it is either the union of three pairwise skew lines or the edges of a tetrahedron without its vertices; see [5] (p. 401). Since for $n = 13$ the equation $k^2 - 53k + 520 = 0$ has two integer solutions $k_1 = 13$ and $k_2 = 40$, we get either a plane or the whole space PG(3,3). □

Now, let us suppose $q \geq 4$. Thus, in any 4-plane there is at least one 0–line ℓ_0. Let us denote by a the number of 4–planes through ℓ_0. Computing the size k by the planes through ℓ_0, we obtain the equation

$$k = 4a + n(q + 1 - a) \tag{6}$$

Moreover, in any 4-plane, there is at least one 1-line ℓ_1. Let us denote by b the number of 4–planes through ℓ_1. Computing the size k by the planes through ℓ_1, we get the equation

$$k = 1 + 3b + (n - 1)(q + 1 - b) \tag{7}$$

Equating the Expressions (6) and (7), we obtain the equation

$$(n - 4)(a - b) = q$$

And we get that $(n - 4) \mid q$. Thus, $n = 4 + p^e$ with $0 \leq e \leq h$.
Therefore, $n - 4 \leq q \Rightarrow n \leq q + 4$, and the Equation (5) becomes

$$[(q + 1)k - (3q^2 + 4q + 4)]k \leq (q + 4)[(q^2 + q + 1)k - 4(q^2 + 1)(q + 1)] \tag{8}$$

We obtain

$$[k - 4(q + 1)]\cdot[(q + 1)k - (q + 1)\cdot(q^2 + 3q - 2) - 6] \leq 0$$

Therefore, we have that

$$4(q+1) \ \leq \ k \leq \ q^2 + 3q \ - \ 2 + \frac{6}{q+1} \tag{9}$$

where the equalities hold if and only if $n = q + 4$.

Thus, we have the following:

Proposition 1. *For $q \geq 4$, a set of type (4,q + 4) in PG(3,q) either has size k = 4(q + 1) or it is one of the eight inequivalent 39-sets of type (4,9) in PG(3,5).*

Proof. Since in (9) the equalities hold if and only if $n = q + 4$, either $k = 4(q + 4)$, or $q + 1 \mid 6$. Therefore, for $q \geq 4$, $q = 5$ and K is one of the 8 inequivalent 39-sets of type (4,9) classified in [11] (p. 106). □

Now, we prove

Theorem 1. *In PG(3,q), a k-set of type (4,n) has standard parameters, i.e., n = q + 4.*

Proof. Since by Lemmas 1 and 2, apart from the planes of PG(3,3), the assertion is true for $q = 2$ and $q = 3$, let us suppose, on the contrary, that $q > 3$ and $n < q + 4$. Thus, $0 \leq e < h$ and $k > 4(q + 1)$. Since $q > 3$, a 4-plane π contains at least one external line ℓ_0. Let u denote the number of n-planes through the line ℓ_0. Computing the size k by the planes through ℓ_0, we get $k = 4(q + 1 - u) + nu = 4(q + 1) - 4u + (4 + p^e)u = 4(q + 1) + up^e$. Since $k > 4(q + 1)$, we have that $u > 0$. So, there is an integer u such that $k = 4(q + 1) + up^e$ with $0 < u \leq q$.

We claim that $u = 1 \Rightarrow n = q + 4$. Indeed, suppose $u = 1$. We have that $n = 4 + p^e$, $k = 4(q + 1) + p^e$ and Equation (5) becomes

$$4\left(q + 4 + p^e + \frac{p^e}{q}\right)(4 + p^e)(3 + p^e) \tag{10}$$

which implies that $q \mid 4p^e$. So, if $p \neq 2$, $n = q + 4$ because $n - 4 = p^e \mid q$. If $p = 2$, since $q > 3$, $q = 2^h$ with $h \geq 2$, $2^h \mid 2^{e+2}$ and $2^e \mid 2^h$.

Therefore, $e + 1 \leq h \leq e + 2$. We have that either $e = h - 1$ or $e = h - 2$.

If $e = h - 1$, $p^e = 2^{h-1}$ and $q = 2^h$, and Equation (10) becomes

$$(2^{h+1} + 2^3 + 2^h + 1) = (2 + 2^{h-2})(3 + 2^{h-2})$$

which is not possible because $2^{h+1} + 2^3 + 2^h + 1$ is odd, but $(2 + 2^{h-2})(3 + 2^{h-2})$ is even.

If $e = h - 2$, $p^e = 2^{h-2}$ and $q = 2^h$, and Equation (10) becomes

$$(2^{h+2} + 2^4 + 2^h + 1) = (2^2 + 2^{h-2})(3 + 2^{h-2})$$

which is not possible because $2^{h+2} + 2^4 + 2^h + 1$ is odd, but $(2^2 + 2^{h-2})(3 + 2^{h-2})$ is even.

Now, we claim that $u < q$. Indeed, suppose $u = q$. We have that $n = 4 + p^e$, $k = 4(q + 1) + qp^e$, and Equation (5) becomes

$$4p^{h-e}(q + 1 + p^e) = 4 + p^e - q^2 \tag{11}$$

which implies that $4 + p^e - q^2 > 4qp^{h-e} \Rightarrow 4qp^{h-e} + q^2 < 4 + p^e$, which is not possible.

Then u is an integer such that $k = 4(q + 1) + up^e$ with $1 < u < q$.

Substituting $q = p^h$, Equation (5) becomes

$$(4p^{2h-e} + up^h + 8u - 4p^h - up^{h+e} + 4p^{h-e} - u^2p^e - up^e - 4)p^{h-e} = u(u - 1)$$

Therefore, $p^{h-e} \mid u(u - 1)$.

Since u and $u - 1$ are coprime, either $p^{h-e} \mid u - 1$ or $p^{h-e} \mid u$.

Let us suppose that $p^{h-e} \mid u - 1$.

Let us denote by d the integer such that $u = 1 + dp^{h-e}$. We have that $k = 4(q + 1) + dq + p^e$. Since $1 < u < q$, we have that $0 < d < p^e$. Therefore, $0 < e < h$. Substituting $n = 4 + p^e$ and $k = 4(q + 1) + dq + p^e$ into the Equation (5), we get

$$(dp^e - d^2 - d - 4)p^{2h-e} + [(p^e - d + 3)p^e - (d^2 + 8d + 4)]p^{h-e} - (d + 4) = 0 \tag{12}$$

Which implies that $p^{h-e} \mid d + 4$.

If $d + 4 = p^{h-e}$, $d = p^{h-e} - 4$, and we have that Equation (12) becomes

$$-p^{3h-2e} + 7p^{2h-e} - p^{2h-2e} - 4p^{h+e} + p^{2e} - 17p^h + p^{2h-e} + 7p^e + 11 = 0 \tag{13}$$

Thus, $p^e \mid 11 \Rightarrow p = 11$ and $e = 1$, and Equation (13) becomes

$$(209 - 61 \cdot 11^h) = (11^h - 87)11^{2h-2} \tag{14}$$

a contradiction for $h \geq 2$.

Therefore, $p^{h-e} < d + 4 = \frac{u-1}{p^{h-e}} + 4 < \frac{p^h}{p^{h-e}} + 4 = p^e + 4 \Rightarrow h - e \leq e \Rightarrow h \leq 2e$.

We write Equation (12) as

$$[dp^h + p^e - (d^2 + d + 4)p^{h-e} - 2d + 3](p^h + 1) - [(7p^{h-e} - 1)d + p^e + 7] = 0 \quad (15)$$

Therefore, $(p^h + 1) \mid [(7p^{h-e} - 1)d + p^e + 7] > 0$. It follows that there is an integer $x > 0$ such that

$$x = \frac{\left(7p^{h-e} - 1\right)d + p^e + 7}{p^h + 1}. \quad (16)$$

Since $d < p^e$, we have that $(7p^{h-e} - 1)d + p^e + 7 < (7p^{h-e} - 1)p^e + p^e + 7 = 7p^h + 7 = 7(p^h + 1)$.
Thus, $0 < x < \frac{7(p^h+1)}{p^h+1} = 7 \Rightarrow x \in \{1, 2, 3, 4, 5, 6\}$.
Substituting $[(7p^{h-e} - 1)d + p^e + 7] = x(p^h + 1)$ in the Equation (15) we get

$$dp^h + p^e - (d^2 + d + 4)p^{h-e} - 2d + 3 - x = 0 \quad (17)$$

$h - e \leq e \Rightarrow p^{h-e} \mid -2d + 3 - x = -2(d + 4) + 11 - x$. So, $p^{h-e} \mid 11 - x$, because $p^{h-e} \mid d + 4$.
Since $x \in \{1, 2, 3, 4, 5, 6\}$, let us consider the possible cases.
$x = 1 \Rightarrow p^{h-e} \mid 11 - 1 = 10 \Rightarrow h - e = 1$ and either $p = 2$ or $p = 5$.
If $p = 2$, by Equations (16) and (17), we get

$$13d + 6 = 2^{h-1}$$
$$(2d + 1)2^{h-1} - 2(d^2 + d + 4) - 2d + 2 = 0$$

Which gives

$$(2d + 1)(13d + 6) - 2(d^2 + d + 4) - 2d + 2 = 0$$

And, finally, $24d^2 + 21d = 0$, which has no integer solution different from zero.
If $p = 5$, by Equations (16) and (17), we get

$$17d + 3 = 2 \cdot 5^{h-1}$$
$$(5d + 1)5^{h-1} - 5(d^2 + d + 4) - 2d + 2 = 0$$

Which gives

$$(5d + 1)(17d + 3) - 10(d^2 + d + 4) - 4d + 4 = 0$$

And, finally, $25d^2 + 6d - 11 = 0$, which has no integer solutions.
$x = 2 \Rightarrow p^{h-e} \mid 11 - 2 = 9 = 3^2 \Rightarrow p = 3$ and either $h - e = 1$ or $h - e = 2$.
If $h - e = 1$, by Equations (16) and (17), we get

$$4d + 1 = 3^{h-1}$$
$$(3d + 1)3^{h-1} - 3(d^2 + d + 4) - 2d + 1 = 0$$

Which gives

$$(3d + 1)(4d + 1) - 3(d^2 + d + 4) - 2d + 1 = 0$$

And, finally, $9d^2 + 2d + 2 = 0$, which has complex roots.
If $h - e = 2$, by Equations (16) and (17), we get

$$62d + 5 = 17 \cdot 3^{h-2}$$
$$(9d + 1)3^{h-2} - 9(d^2 + d + 4) - 2d + 1 = 0$$

Which gives

$$(9d + 1)(62d + 5) - 153(d^2 + d + 4) - 34d + 17 = 0$$

And, finally, $405d^2 - 46d - 590 = 0$, which has no integer solutions.
$x = 3 \Rightarrow p^{h-e} \mid 11 - 3 = 8 = 2^3 \Rightarrow p = 2$ and either $h - e = 1$ or $h - e = 2$ or $h - e = 3$.

If $h - e = 1$, by Equations (16) and (17), we get

$$13d + 4 = 5 \cdot 2^{h-1}$$
$$(2d + 1)2^{h-1} - 2(d^2 + d + 4) - 2d = 0$$

Which gives

$$(2d + 1)(13d + 4) - 10(d^2 + d + 4) - 10d = 0$$

And, finally, $16d^2 + d - 36 = 0$, which has no integer solutions.
If $h - e = 2$, by Equations (16) and (17), we get

$$27d + 4 = 11 \cdot 2^{h-2}$$
$$(4d + 1)2^{h-2} - 4(d^2 + d + 4) - 2d = 0$$

Which gives

$$(4d + 1)(27d + 4) - 44(d^2 + d + 4) - 22d = 0$$

And, finally, $64d^2 - 23d - 172 = 0$, which has no integer solutions.
If $h - e = 3$, by Equations (16) and (17), we get

$$55d + 4 = 23 \cdot 2^{h-3}$$
$$(8d + 1)2^{h-3} - 8(d^2 + d + 4) - 2d = 0$$

Which gives

$$(8d + 1)(55d + 4) - 184(d^2 + d + 4) - 46d = 0$$

And, finally, $256d^2 - 143d - 732 = 0$, which has no integer solutions.
$x = 4 \Rightarrow p^{h-e} \mid 11 - 4 = 7 \Rightarrow h - e = 1$ and $p = 7$.
By Equations (16) and (17), we get

$$16d + 1 = 9 \cdot 7^{h-1}$$
$$(7d + 1)7^{h-1} - 7(d^2 + d + 4) - 2d - 1 = 0$$

Which gives

$$(7d + 1)(16d + 1) - 63(d^2 + d + 4) - 18d - 9 = 0$$

And, finally, $49d^2 - 58d - 260 = 0$, which has no integer solutions.
$x = 5 \Rightarrow p^{h-e} \mid 11 - 5 = 6 \Rightarrow h - e = 1$ and either $p = 2$ or $p = 3$.
If $p = 2$, by Equations (16) and (17), we get

$$13d + 2 = 9 \cdot 2^{h-1}$$
$$(2d + 1)2^{h-1} - 2(d^2 + d + 4) - 2d - 2 = 0$$

Which gives

$$(2d + 1)(13d + 2) - 18(d^2 + d + 4) - 18d - 18 = 0$$

And, finally, $8d^2 - 19d - 88 = 0$, which has no integer solutions.
If $p = 3$, by Equations (16) and (17), we get

$$10d + 1 = 7 \cdot 3^{h-1}$$
$$(3d + 1)3^{h-1} - 3(d^2 + d + 4) - 2d - 2 = 0$$

Which gives

$$(3d + 1)(10d + 1) - 21(d^2 + d + 4) - 14d - 14 = 0$$

And, finally, $9d^2 - 22d - 97 = 0$, which has no integer solutions.

$x = 6 \Rightarrow p^{h-e} \mid 11 - 6 = 5 \Rightarrow h - e = 1$ and $p = 5$.
By Equations (16) and (17), we get

$$34d + 1 = 29 \cdot 5^{h-1}$$
$$(5d + 1)5^{h-1} - 5(d^2 + d + 4) - 2d - 3 = 0$$

Which gives

$$(5d + 1)(34d + 1) - 145(d^2 + d + 4) - 58d - 87 = 0$$

And, finally, $25d^2 - 164d - 666 = 0$, which has no integer solutions.
Let us suppose that $p^{h-e} \mid u$.
Let us denote by d the integer such that $u = dp^{h-e}$. We have that $k = 4(q + 1) + dq$. Since $1 < u < q$, we have that $0 < d < p^e$. Therefore, $0 < e < h$. Substituting $n = 4 + p^e$ and $k = 4(q + 1) + dq$ into the Equation (4), we get

$$(p^e d - d^2 - d - 4)\, p^{2h-e} + [(d + 4)p^e - (d^2 + 8d + 4)]\, p^{h-e} + (d + 4)p^e = 0 \tag{18}$$

Which implies that $p^{h-e} \mid d + 4$. If $d + 4 = p^{h-e}$, we have that $d = p^{h-e} - 4$ and Equation (18) becomes

$$p^{3h-2e} - 7p^{2h-e} + p^{2h-2e} - p^{2h} + 4p^{h+e} + 15p^h - 13 = 0 \tag{19}$$

Since $1 \leq e < h$, we have that $p^h \mid 13$, i.e., $p^h = 13$, $p = 13$, and $h = 1$, a contradiction.
Thus, we have that $p^{h-e} \mid d + 4$ and $p^{h-e} < d + 4$.
Therefore $p^{h-e} \leq \frac{d+4}{2} \leq d + 1 \leq p^e \Rightarrow h - e \leq e \Rightarrow h \leq 2e$.
Since $q = p^h$, we write Equation (19) as

$$[(dp^e - d^2 - d - 4)p^{h-e} + 4](p^h + 1) - (7p^{h-e} - 1)d = 0 \tag{20}$$

Therefore, $p^h + 1 \mid (7p^{h-e} - 1)d > 0$. Let us denote by x the positive integer such that

$$x = \frac{\left(7p^{h-e} - 1\right)d}{p^h + 1}. \tag{21}$$

Since $0 < d < p^e$, we have that $(7p^{h-e} - 1)d < (7p^{h-e} - 1)p^e = (7p^h - p^e) < 7p^h < 7(p^h + 1)$.
Thus, $0 < x < \frac{7\left(p^h+1\right)}{p^h+1} = 7$ and $x \in \{1, 2, 3, 4, 5, 6\}$.
Putting $(7p^{h-e} - 1)d = x(p^h + 1)$ into Equation (20), we get

$$(dp^e - d^2 - d - 4)p^{h-e} + 4 - x = 0. \tag{22}$$

Therefore,

$$p^{h-e} \mid 4 - x. \tag{23}$$

Since $1 \leq e < h$ and $x \in \{1, 2, 3, 4, 5, 6\}$, the possible cases are $x \in \{1, 2, 4, 6\}$. Let us consider the possible cases.
$x = 1 \Rightarrow p^{h-e} \mid 4 - 1 = 3 \Rightarrow p = 3$ and $h - e = 1 \Rightarrow h = e + 1$. Substituting $x = 1$, by Equations (21) and (22), we get

$$20d - 1 = 3 \cdot 3^{h-1}$$
$$3^{h-1}d - d^2 - d - 3 = 0$$

Which gives

$$\frac{20d - 1}{3}d - d^2 - d - 3 = 0$$

And, finally, $17d^2 - 4d - 9 = 0$, which has no integer solutions, a contradiction.

$x = 2 \Rightarrow p^{h-e} \mid 4 - 2 = 2 \Rightarrow p = 2$ and $h - e = 1 \Rightarrow h = e + 1$. Putting $x = 2$, by Equations (21) and (22), we obtain

$$13d - 2 = 4 \cdot 2^{h-1}$$
$$2^{h-1}d - d^2 - d - 3 = 0$$

Which gives

$$\frac{13d-2}{4}d - d^2 - d - 3 = 0$$

And, finally, $3d^2 - 2d - 4 = 0$, which has no integer solutions, a contradiction.
$x = 4$. Substituting $x = 4$, by Equations (21) and (22), we get

$$(7d - 4p^e)p^{h-e} = d + 4$$
$$dp^e = d^2 + d + 4$$

so, $dp^e = d^2 + d + 4 \Rightarrow d \mid 4 \Rightarrow k \equiv 0 \bmod 4 \Rightarrow n = 4 + p^e \Rightarrow n \equiv 0 \bmod 4$. Since $n = 4 + p^e$ we have that $p^e \equiv 0 \bmod 4$. Thus, $p = 2$ and $d \in \{2, 4\}$, because $d \mid 4$ and $2^{h-e} \mid d + 4$. If $d = 2$, we have that $2 \cdot 2^e = 2^2 + 2 + 4 = 10$, a contradiction. If $d = 4$, we have that $4 \cdot 2^e = 4^2 + 4 + 4 = 24$, a contradiction.
$x = 6 \Rightarrow p^{h-e} \mid 4 - 6 = -2 \Rightarrow p = 2$ and $h - e = 1 \Rightarrow h = e + 1$. Putting $x = 6$, by Equations (21) and (22), we get

$$13d - 6 = 12 \cdot 2^{h-1}$$
$$2^{h-1}d - d^2 - d - 5 = 0$$

Which gives

$$\frac{13d-6}{12}d - d^2 - d - 5 = 0$$

And, finally, $d^2 - 18d - 60 = 0$, which has no integer solutions. This contradiction completes the proof. □

Theorem 2 has the following, somewhat surprising, consequence.

Corollary 1. *In PG(3,q), $q \neq 3$, there is no set of type (4,n) with non-standard parameters.*

Thus, in view of the previous Theorem, from now on we will assume that $n = q + 4$.
Now, we prove the

Theorem 2. *In PG(3,q), with $q \geq 7$, but $q \neq 9$, a k-set K of type (4,q + 4) is the set of points on four pairwise skew lines.*

Proof. By Proposition 3, if $q \geq 7$, the size k is equal to $4(q + 4)$. Since there are 4-planes, then there are 0-lines. Computing the size k by the planes through a 0-line ℓ_0, if x denotes the number of $(q + 4)$-planes through ℓ_0, we get

$$k = x(q + 4) + 4(q + 1 - x) = 4(q + 4),$$

which gives $x = 0$. Thus, a $(q + 4)$-plane contains no 0-line. Let π denote a $(q + 4)$-plane. The $(q + 4)$-set $K \cap \pi$ is a blocking set, i.e., a set of points which meets every line. A blocking set of a projective plane with size less than $q + \sqrt{q} + 1$ contains one line; cf. [10] (p. 338). Since $q + 4 \leq q + \sqrt{q} + 1$ leads to $q \leq 9$, we get that, for $q \geq 11$, a $(q + 4)$-plane contains at least one line ℓ. Moreover, since in PG(2,7) a blocking set with size less than 12 contains one line (cf. [10] (p. 348)), we get that, for $q = 7$, a $(q + 4)$-plane contains at least one line ℓ. Since any plane through ℓ is a $(q + 4)$-plane, the set $K - \ell$ is a $3(q + 1)$-set of type $(3,q + 3)$,

and, by Theorem II of [5] (p. 396), $K - \ell$ is the set of points on three pairwise skew lines. Thus, K is the set of points on four pairwise skew lines. □

Everything is known for the remaining cases $q \in \{4, 5, 9\}$.

We recall that a *partial spread* of PG(3,q) is a set S of lines for which every point is incident with at most one line of S. A partial spread is called *maximal* if S is not contained in a larger partial spread. A *hole* of S is a point which is not incident with any line of S. Taking into account Theorem 2.21 of [12] (p. 1145), we have that, up to projective equivalence, for $q \in \{4, 5, 9\}$, there are four $4(q + 4)$-sets K of type $(4,q + 4)$ which contain no lines:

- The two inequivalent sets of holes of the maximal partial spreads of size 13 in PG(3,4) classified via a computer search by L. H. Soicher in [13].
- The set of holes of a maximal partial spread of size 22 in PG(3,5); cf. [14].
- The Baer subspace PG(3,3) in PG(3,9).

A $4(q + 4)$-set K of type $(4,q + 4)$ which contains at least one line ℓ, for $q \in \{4, 5, 9\}$, is the union of the line ℓ with a $3(q + 3)$-set K' of type $(3,q + 3)$. Thus, taking into account the classification of [5] (p. 396), K is either the set of points of four pairwise skew lines or the union of the line ℓ with PG(3,2) embedded in PG(3,4).

4. Discussion

Our aim is to understand in depth the case $m = 4$, in order to find some combinatorial features that can potentially give some evidence about the existence or non-existence of two-intersection sets with non-standard parameters. We hope to come back to such a discussion in a forthcoming article soon.

5. Conclusions

As a consequence we conclude that, apart from the plane of PG(3,3), there is just one set of type (m,n), with $m \leq 4$, with non-standard parameters: a 39-set of type (3,7) in PG(3,8); cf. [15]. It is associated with a quasi-cyclic projective linear code over GF(8) of length 39, dimension 4, and weights 32 and 36. This is quite striking and justifies the question posed by Napolitano and Olanda in [5] (p. 396).

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