SUPPLEMENTARY INFORMATION

Deriving an Electric Wave Equation from Weber's Electrodynamics

Qingsong Li [1,*] and Simon Maher [2]

[1] Independent Researcher, Sugar Land, TX 77479, USA
[2] Department of Electrical Engineering and Electronics, University of Liverpool, Liverpool L69 3GJ, UK; ee0u61a9@liverpool.ac.uk
* Correspondence: qingsong.li.geo@gmail.com

Further details relating to the integrations performed in the main manuscript are given here:

1. Integration on spherical shell of radius r for the expression below

$$\int (\hat{r} \cdot \vec{v})\,\hat{r}\,ds$$

where $\hat{r}$ is a unit vector pointing from origin to shell, $\vec{v}$ is a constant vector.

$$\hat{r} = \sin\theta \cos\varphi\,\hat{x} + \sin\theta \sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$
$$ds = r^2 \sin\theta\,d\varphi d\theta$$

where θ and φ are angles in spherical coordinates.

$$\int (\hat{r} \cdot \vec{v})\,\hat{r}\,ds = \int_0^{2\pi} d\varphi \int_0^{\pi} \left(\sin\theta \cos\varphi\,v_x + \sin\theta \sin\varphi\,v_y + \cos\theta\,v_z\right)(\sin\theta \cos\varphi\,\hat{x} + \sin\theta \sin\varphi\,\hat{y}$$
$$+ \cos\theta\,\hat{z})r^2 \sin\theta\,d\theta = \frac{4}{3}\pi r^2 \left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right) = \frac{4}{3}\pi r^2 \vec{v} \qquad (A1)$$

2. Integration on spherical shell of radius r for the expression below

$$\int (\hat{r} \cdot \vec{v})^2 (\hat{r} \cdot \vec{u})\,\hat{r}\,ds$$

where $\vec{u}$ is a constant vector.

$$\hat{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$$

Thus

$$\int (\hat{r} \cdot \vec{v})^2 (\hat{r} \cdot \vec{u})\,\hat{r}\,ds$$
$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \left(\sin\theta \cos\varphi\,v_x + \sin\theta \sin\varphi\,v_y + \cos\theta\,v_z\right)^2 \left(\sin\theta \cos\varphi\,u_x\right.$$
$$\left. + \sin\theta \sin\varphi\,u_y + \cos\theta\,u_z\right)(\sin\theta \cos\varphi\,\hat{x} + \sin\theta \sin\varphi\,\hat{y} + \cos\theta\,\hat{z})r^2 \sin\theta\,d\theta$$
$$= \frac{4}{15}\pi r^2 \left((\vec{v} \cdot \vec{v})\vec{u} + 2(\vec{v} \cdot \vec{u})\vec{v}\right) \qquad (A2)$$

3. Integration in equation (6)

$$\vec{F}_c = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \left(2\rho_-(\vec{0}) + 2\vec{r} \cdot \nabla\rho_-(\vec{0}) - 2\rho\right)\vec{r}\,ds\,dr$$

Because of spherical symmetry, $\int \vec{r}\,ds = \int -\vec{r}\,ds = 0$. Thus

$$\vec{F}_c = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \left(2\vec{r} \cdot \nabla\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$

Using equation (A1), we obtain

$$\vec{F}_c = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV\,2r^2 dr\,\frac{4}{3}\pi r^2 \nabla\rho_-(\vec{0}) = -\frac{2r}{3\varepsilon_0}\rho_-(\vec{0})dV\nabla\rho_-(\vec{0})dr$$

4. Integration in equation (10)

$$\vec{F}_v = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\cdot\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right) - \frac{3}{2r^2}\left(\vec{r}\cdot\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right)^2\right)\left(\rho_-(\vec{0}) + \vec{r}\right.$$
$$\left.\cdot\nabla\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$
$$+\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\cdot\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right.$$
$$\left.-\frac{3}{2r^2}\left(\vec{r}\cdot\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right)^2\right)\left(2\rho - \rho_-(\vec{0}) - \vec{r}\cdot\nabla\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$

We drop the higher order terms, which has a product of $\vec{r}\cdot\nabla\rho_-(\vec{0})$ and $\vec{r}\cdot\nabla\vec{v}_-(\vec{0})$.

$$\vec{F}_v = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\cdot\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right.$$
$$\left.-\frac{3}{2r^2}\left(\vec{r}\cdot\left(\vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right)^2\right)\left(\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$
$$+\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\cdot\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right.$$
$$\left.-\frac{3}{2r^2}\left(\vec{r}\cdot\left(2\vec{v}_-(\vec{0}) + \vec{r}\cdot\nabla\vec{v}_-(\vec{0})\right)\right)^2\right)\left(2\rho - \rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$
$$+\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(2\vec{v}_-(\vec{0})\right)\cdot\left(2\vec{v}_-(\vec{0})\right) - \frac{3}{2r^2}\left(\vec{r}\cdot\left(2\vec{v}_-(\vec{0})\right)\right)^2\right)\left(-\vec{r}\right.$$
$$\left.\cdot\nabla\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$

Because of spherical symmetry, $\int \vec{r}\,ds = \int -\vec{r}\,ds = 0$. Thus the first and second terms of the above equation equal zero.

$$\vec{F}_v = \frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{1}{c^2}\left(\left(2\vec{v}_-(\vec{0})\right)\cdot\left(2\vec{v}_-(\vec{0})\right) - \frac{3}{2r^2}\left(\vec{r}\cdot\left(2\vec{v}_-(\vec{0})\right)\right)^2\right)\left(-\vec{r}\cdot\nabla\rho_-(\vec{0})\right)\vec{r}\,ds\,dr$$

Using equation (A1 & A2), we obtain

$$\vec{F}_v = \frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \frac{4}{c^2}\left(\vec{v}_-(\vec{0})\cdot\vec{v}_-(\vec{0})\right)r^2dr\frac{-4}{3}\pi r^2\nabla\rho_-(\vec{0})$$
$$+\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \frac{4}{c^2}\frac{3}{2r^2}r^2dr\frac{4}{15}\pi r^2\left(\left(\vec{v}_-(\vec{0})\cdot\vec{v}_-(\vec{0})\right)\nabla\rho_-(\vec{0})\right.$$
$$\left.+ 2\left(\vec{v}_-(\vec{0})\cdot\nabla\rho_-(\vec{0})\right)\vec{v}_-(\vec{0})\right)$$
$$= -\frac{14r}{15\varepsilon_0}\rho_-(\vec{0})dV \frac{1}{c^2}\left(\vec{v}_-(\vec{0})\cdot\vec{v}_-(\vec{0})\right)\nabla\rho_-(\vec{0})dr$$
$$+\frac{4r}{5\varepsilon_0}\rho_-(\vec{0})dV \frac{1}{c^2}\left(\vec{v}_-(\vec{0})\cdot\nabla\rho_-(\vec{0})\right)\vec{v}_-(\vec{0})dr$$

5. Integration in equation (14)
$$\vec{F}_a = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2\rho}{c^2}\vec{r}\cdot\left(\vec{r}\cdot\nabla\vec{a}_-(\vec{0})\right)\vec{r}\,ds\,dr$$
$$-\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2}{c^2}\left(\rho_+(\vec{0}) + \vec{r}\cdot\nabla\rho_+(\vec{0})\right)\vec{r}\cdot\vec{a}_-(\vec{0})\vec{r}\,ds\,dr$$

We drop the higher order terms, which has a product of $\vec{r}\cdot\nabla\rho_+(\vec{0})$ and $\vec{r}\cdot 2\vec{a}_-(\vec{0})$.

$$\vec{F}_a = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2\rho}{c^2}\vec{r}\cdot\left(\vec{r}\cdot\nabla\vec{a}_-(\vec{0})\right)\vec{r}\,ds\,dr$$
$$-\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2}{c^2}\left(\rho_+(\vec{0})\right)\vec{r}\cdot\vec{a}_-(\vec{0})\vec{r}\,ds\,dr$$

Because of spherical symmetry, $\int \vec{r}\cdot(\vec{r}\cdot\vec{u})\vec{r}\,ds = \int(-\vec{r})\cdot\left((-\vec{r})\cdot\vec{u}\right)(-\vec{r})ds = 0$. Thus

$$\vec{F}_a = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2}{c^2}\left(\rho_+(\vec{0})\right)\vec{r}\cdot\vec{a}_-(\vec{0})\vec{r}\,ds\,dr$$

We also use the approximation $\rho_+(0) \approx \rho$, and use equation (A1). Then

$$\vec{F}_a = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV \int \frac{2}{c^2}\rho\vec{r}\cdot\vec{a}_-(\vec{0})\vec{r}\,ds\,dr = -\frac{1}{4\pi\varepsilon_0 r^3}\rho_-(\vec{0})dV\frac{2}{c^2}\rho r^2 dr\frac{4}{3}\pi r^2\vec{a}_-(\vec{0})$$

$$= -\frac{2r}{3\varepsilon_0}\rho_-(\vec{0})dV\frac{\rho}{c^2}\vec{a}_-(\vec{0})dr$$