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Green's Functions for a Fractional Boundary Value Problem with Three Terms

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Abstract: We construct a Green's function for the three-term fractional differential equation $-D_{0+}^{\alpha-1}u + aD_{0+}^{\mu}u + f(t)u = h(t)$, $0 < t < b$, where $\alpha \in (2,3]$, $\mu \in (1,2]$, and f is continuous, satisfying the boundary conditions $u(0) = u'(0) = 0$, $D_{0+}^{\beta}u(b) = 0$, where $\beta \in [0,2]$. To accomplish this, we first construct a Green's function for the two-term problem $-D_{0+}^{\alpha-1}u + aD_{0+}^{\mu}u = h(t)$, $0 < t < b$, satisfying the same boundary conditions. A lemma from spectral theory is integral to our construction. Some limiting properties of the Green's function for the two-term problem are also studied. Finally, existence results are given for a nonlinear problem.

Keywords: Green's function; fractional boundary value problem

MSC: 34B15; 34B27

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1. Introduction

In this paper, we use a lemma from spectral theory to develop a Green's function for the fractional differential equation

$$-D_{0+}^{\alpha-1}u + aD_{0+}^{\mu}u + f(t)u = h(t), \quad 0 < t < b, \tag{1}$$

where $\alpha \in (2,3]$ and $\mu \in (1,2]$, satisfying the boundary conditions

$$u(0) = u'(0) = 0, \quad D_{0+}^{\beta}u(b) = 0, \tag{2}$$

where $\beta \in [0,2]$. It will be assumed throughout that f is continuous on $[0,b]$.

We first construct the Green's function for

$$-D_{0+}^{\alpha-1}u + aD_{0+}^{\mu}u = h(t), \quad 0 < t < b, \tag{3}$$

This satisfies the boundary condition in Equation (2). The limiting properties of this Green's function are studied first. Then, by using this Green's function, we can construct the Green's function for Equations (1) and (2).

Two-term fractional boundary value problems were first studied by Graef et. al [1] using spectral theory. These techniques were improved upon in [2], where the authors were able to write the Green's function corresponding to the boundary value problem

$$-D_{0+}^{\alpha-1}u + a(t)u = h(t),$$

$$u(0) = u(1) = 0,$$

where $1 < \alpha < 2$ as a series of functions. Later, in [3], these authors studied the boundary value problem

$$-D_{0+}^{\alpha-1}u + aD_{0+}^{\beta}u = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = 0,$$

where $0 \leq \beta < 1 < \alpha < 2$. They showed that a Green's function can be constructed in a closed form using generalized Mittag-Leffler functions. Recently [4], the Green's function for the boundary value problem

$$-D_{0+}^{\alpha-1}u + au = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = 0, \quad u(1) = 0,$$

where $2 < \alpha < 3$ and a is constant was constructed using alternate methods.

In this paper, we use the techniques from [3] to first construct the Green's function corresponding to Equations (2) and (3). This Green's function will be constructed using generalized Mittag-Leffler functions. Some limiting properties such as $b \to \infty$ are studied. The limiting properties of the Green's functions were studied for a one-term fractional boundary value problem in [5]. We will see that the results here are similar to the ones in [5].

The Green's function constructed for Equations (2) and (3) are then used along with the technique from [2] to construct the Green's function for Equations (1) and (2). We believe this is the first paper to study three-term fractional boundary value problems. For more works studying two-term fractional boundary value problems, see, for example, [6–8].

2. Preliminaries

For a detailed review of fractional calculus, we refer the reader to the monograph by Diethelm [9] and the book by Podlubny [10]. The following definitions and properties can be found in these references:

Definition 1. *Let $0 < \nu$ and recall the Riemann–Liouville fractional integral of a function u is defined by*

$$I_{0+}^{\nu}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds, \tag{4}$$

provided that the right-hand side exists. Moreover, let n denote a positive integer, and assume $n-1 < \alpha \leq n$. The αth Riemann–Liouville fractional derivative of the function $u : [0,\infty) \to \mathbb{R}$, denoted as $D_{0+}^{\alpha}u$, is defined as

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds = D^n I_{0+}^{n-\alpha}u(t),$$

provided that the right-hand side exists.

We need a few properties in fractional calculus to construct and analyze the family of the Green's functions. Recall that

$$I_{0+}^{\nu_1}I_{0+}^{\nu_2}u(t) = I_{0+}^{\nu_1+\nu_2}u(t) = I_{0+}^{\nu_2}I_{0+}^{\nu_1}u(t), \quad \nu_1,\nu_2 > 0, \text{ if } u \in L_1[0,b],$$

$$D_{0+}^{\nu_1}I_{0+}^{\nu_2}u(t) = I_{0+}^{\nu_2-\nu_1}u(t), \quad \text{if } 0 \leq \nu_1 \leq \nu_2, \text{ if } u \in L_1[0,b],$$

$$D_{0+}^{\alpha}I_{0+}^{\alpha}u(t) = u(t),\ 0 < t, \text{ if } u \in L_1[0,b],$$

and

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-n+(i-1)}, \quad \text{if } D_{0+}^{\alpha}u \in L_1[0,b].$$

The power rule will be employed, which states that

$$D_{0+}^{\nu_2}t^{\nu_1} = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-\nu_2)}t^{\nu_1-\nu_2}, \quad \nu_1 > -1, \nu_2 \geq 0, \tag{5}$$

where it is assumed that $\nu_2 - \nu_1$ is not a positive integer. If $\nu_2 - \nu_1$ is a positive integer, then the right hand side of Equation (5) vanishes. To see this, one can appeal to the convention that $\frac{1}{\Gamma(\nu_1+1-\nu_2)} = 0$ if $\nu_2 - \nu_1$ is a positive integer, or one can perform the calculation on the left-hand side and calculate

$$D^n t^{n-(\nu_2-\nu_1)} = 0.$$

Moreover, we state and prove the following identities, which will also be employed in Section 3:

Lemma 1. *Assume $\alpha \in (2,3]$, $\mu \in (1,2]$, and $\beta \in [0,2]$. Assume h is continuous on $[0,b]$. Then, we have the following:*

1.

$$D^{\beta}_{0^+} \int_0^t (t-s)^{(n+1)\alpha-n\mu-1} h(s)ds = \frac{\Gamma(n(\alpha-\mu)+\alpha)}{\Gamma(n(\alpha-\mu)+(\alpha-\beta))} \int_0^t (t-s)^{n(\alpha-\mu)+\alpha\beta-1} h(s)ds;$$

2.

$$D^{\beta}_{0^+} \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)} = \sum_{n=0}^{\infty} \frac{a^n D^{\beta}_{0^+} t^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)};$$

3.

$$D^{\beta}_{0^+} I^{(n+1)\alpha-n\mu}_{0^+} h = I^{(n+1)\alpha-n\mu-\beta}_{0^+} h;$$

4.

$$D^{\beta}_{0^+} \sum_{n=0}^{\infty} a^n I^{(n+1)\alpha-n\mu}_{0^+} h = \sum_{n=0}^{\infty} a^n D^{\beta}_{0^+} I^{(n+1)\alpha-n\mu}_{0^+} h = \sum_{n=0}^{\infty} a^n I^{(n+1)\alpha-n\mu-\beta}_{0^+} h.$$

Proof. Consider the cases $0 < \beta < 1$, $1 < \beta < 2$, and $\beta = 0, 1, 2$ independently. We show the details for $0 < \beta < 1$. The details for $1 < \beta < 2$ are similar, the details for $\beta = 1, 2$ are easy to verify, and the details for $\beta = 0$ are trivial. The calculations employ the Euler beta function

$$\mathcal{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We start by proving Equation (1). Now, we have

$$\begin{aligned}
&D^{\beta}_{0^+} \int_0^t (t-s)^{(n+1)\alpha-n\mu-1} h(s)ds \\
&\quad= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{1-\beta-1} \left(\int_0^s (s-r)^{(n+1)\alpha-n\mu-1} h(r)dr \right) ds \\
&\quad= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \left(\int_r^t (t-s)^{1-\beta-1} (s-r)^{(n+1)\alpha-n\mu-1} ds \right) h(r)dr \\
&\quad= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\Gamma(1-\beta)\Gamma((n+1)\alpha-n\mu)}{\Gamma((n+1)\alpha-n\mu-\beta+1)} (t-r)^{(n+1)\alpha-n\mu-\beta} h(r)dr \\
&\quad= \frac{\Gamma(n(\alpha-\mu)+\alpha)}{\Gamma(n(\alpha-\mu)+(\alpha-\beta))} \int_0^t (t-s)^{n(\alpha-\mu)+\alpha\beta-1} h(s)ds.
\end{aligned}$$

If $1 < \beta < 2$, then a similar calculation is performed. The Leibniz rule is sufficient for $\beta = 1$ or $\beta = 2$, and the calculation for $\beta = 0$ is trivial.

To prove (2), notice that

$$D_{0^+}^{\beta}\sum_{n=0}^{\infty}\frac{a^n t^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)}=\frac{d}{dt}\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{1-\beta-1}\sum_{n=0}^{\infty}\frac{a^n s^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)}ds$$
$$=\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma(1-\beta)\Gamma(n(\alpha-\mu)+\alpha)}\int_0^t(s-0)^{(n+1)\alpha-n\mu-1}(t-s)^{1-\beta-1}ds$$
$$=\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma(1-\beta)\Gamma(n(\alpha-\mu)+\alpha)}t^{(n+1)\alpha-n\mu-1+1-\beta}\int_0^1 s^{(n+1)\alpha-n\mu-1}(1-s)^{1-\beta-1}ds$$
$$=\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma(1-\beta)\Gamma(n(\alpha-\mu)+\alpha)}t^{(n+1)\alpha-n\mu-\beta}\frac{\Gamma((n+1)\alpha-n\mu)\Gamma(1-\beta)}{\Gamma((n+1)\alpha-n\mu-1+1-\beta-1+2)}$$
$$=\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta+1)}t^{(n+1)\alpha-n\mu-\beta}.$$

Since each of $\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta+1)}t^{(n+1)\alpha-n\mu-\beta}$ and $\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta)}\ t^{(n+1)\alpha-n\mu-\beta-1}$ converge uniformly on $[0,b]$, then

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta+1)}t^{(n+1)\alpha-n\mu-\beta}=\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta)}t^{(n+1)\alpha-n\mu-\beta-1}$$

and

$$D_{0^+}^{\beta}\sum_{n=0}^{\infty}\frac{a^n t^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)}=\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta)}t^{(n+1)\alpha-n\mu-\beta-1}$$
$$=\sum_{n=0}^{\infty}\frac{a^n D_{0^+}^{\beta}t^{(n+1)\alpha-n\mu-1}}{\Gamma(n(\alpha-\mu)+\alpha)}.$$

To prove (3), we see that

$$D_{0^+}^{\beta}I_{0^+}^{(n+1)\alpha-n\mu}h=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t(t-s)^{1-\beta-1}I_0^{(n+1)\alpha-n\mu}h$$
$$=\frac{d}{dt}I_{0^+}^{1-\beta}I_{0^+}^{(n+1)\alpha-n\mu}h=\frac{d}{dt}I_{0^+}^{(n+1)\alpha-n\mu+1-\beta}h$$
$$=I_{0^+}^{(n+1)\alpha-n\mu-\beta}h.$$

Finally, to prove (4), we start by noticing

$$D_{0^+}^{\beta}\int_0^t(t-s)^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\mu)}}{\Gamma((n+1)\alpha-n\mu}h(s)ds$$
$$=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t(t-s)^{1-\beta-1}\sum_{n=0}^{\infty}\frac{a^n\int_0^s(s-r)^{(n+1)\alpha-n\mu-1}}{\Gamma((n+1)\alpha-n\mu)}h(r)dsdr$$
$$=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-nu)}\int_0^t\int_0^s(t-s)^{1-\beta-1}(s-r)^{(n+1)\alpha-n\mu-1}dsdr$$
$$=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-nu)}\int_0^t\frac{\Gamma(1-\beta)\Gamma((n+1)\alpha-n\mu)}{\Gamma((n+1)\alpha-n\mu-\beta+1)}(t-r)^{(n+1)\alpha-n\mu-\beta}h(r)dr$$
$$=\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-nu-\beta+1)}\int_0^t(t-s)^{(n+1)\alpha-n\mu-\beta}h(s)ds.$$

Since each of

$$\sum_{n=0}^{\infty} \frac{a^n}{\Gamma((n+1)\alpha - nu - \beta + 1)} \int_0^t (t-s)^{(n+1)\alpha - n\mu - \beta} h(s)ds,$$

and

$$\sum_{n=0}^{\infty} \frac{a^n}{\Gamma((n+1)\alpha - nu - \beta)} \int_0^t (t-s)^{(n+1)\alpha - n\mu - \beta - 1} h(s)ds$$

converge uniformly on $[0, b]$, term by term differentiation is valid, and

$$\frac{d}{dt} \sum_{n=0}^{\infty} \frac{a^n}{\Gamma((n+1)\alpha - nu - \beta + 1)} \int_0^t (t-s)^{(n+1)\alpha - n\mu - \beta} h(s)ds$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{\Gamma((n+1)\alpha - nu - \beta)} \int_0^t (t-s)^{(n+1)\alpha - n\mu - \beta - 1} h(s)ds$$

or

$$D_{0^+}^{\beta} \sum_{n=0}^{\infty} a^n I_0^{(n+1)\alpha - n\mu} h = \sum_{n=0}^{\infty} a^n I_{0^+}^{(n+1)\alpha - n\mu - \beta} h.$$

□

The following lemma on spectral theory in Banach spaces will be integral to our construction:

Lemma 2. *Let X be a Banach space and $\mathcal{A} : X \to X$ be a linear operator with the operator norm $\|\mathcal{A}\|$ and spectral radius $r(\mathcal{A})$ of $\mathcal{A}$. Then, we have the following:*

1. $r(\mathcal{A}) < \|\mathcal{A}\|$;
2. *if* $r(\mathcal{A}) < 1$.

   Then, $(\mathcal{I} - \mathcal{A})^{-1}$ *exists, and*

$$(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n,$$

where $\mathcal{I}$ is the identity operator.

Definition 2. *The generalized Mittag-Leffler function $E_{\gamma,\delta} : \mathbb{C} \to \mathbb{C}$ is defined as*

$$E_{\gamma,\delta}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + \delta)}.$$

It is known that the generalized Mittag-Leffler function is an entire function as long as $\gamma, \delta > 0$. Notice that $E_{1,1}[z] = \exp(z)$.

As we shall observe an asymptotic property of the Green's functions of the boundary value problem in Equations (2) and (3) as functions of μ and β, respectively, we shall also make use of a further generalized Mittag-Leffler type function first studied in a special case by Le Roy [11] and, we believe, recently introduced by Gerhold [12]:

Definition 3. *Define*

$$F_{\gamma,\delta}^{(\kappa)}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + \delta)^{\kappa}}.$$

Notice that

$$F_{\gamma,\delta}^{(1)}[z] = E_{\gamma,\delta}[z].$$

Again, if $\gamma, \delta, \kappa > 0$, then $F_{\gamma,\delta}^{(\kappa)}[z]$ denotes an entire function.

The following asymptotic result will be useful for obtaining the asymptotic properties of the Green's function associated with Equations (2) and (3) as functions of μ and β, respectively:

Theorem 1 (Theorem 1 [12]). *Let $\gamma, \delta, \kappa > 0$ and $\epsilon > 0$ be arbitrary. Then, for $z \to \infty$ in the sector*

$$|\arg(z)| \leq \begin{cases} \frac{1}{2}\gamma\kappa\pi - \epsilon, & 0 < \gamma\kappa < 2, \\ \left(2 - \frac{1}{2}\gamma\kappa\right)\pi - \epsilon, & 2 \leq \gamma\kappa < 4, \\ 0, & 4 \leq \gamma\kappa, \end{cases}$$

we have the asymptotics

$$F_{\gamma,\delta}^{(\kappa)}[z] \sim \frac{1}{\gamma\sqrt{\kappa}}(2\pi)^{\frac{1-\kappa}{z}} z^{\frac{\kappa-2\delta\kappa+1}{2\gamma\kappa}} e^{\gamma z^{1/\gamma\kappa}}. \tag{6}$$

3. Green's Function for the Two-Term Problem

We look to construct the Green's function for the boundary value problem in Equations (2) and (3). Let $X = C[0, b]$ be the Banach space of continuous functions with the standard maximum norm $\|u\| = |u|_0 = \max_{t\in[0,b]} |u(t)|$. Assume u is a solution to Equations (2) and (3). Then, we have

$$-D_{0+}^{\alpha-1}u(t) + aD_{0+}^{\mu}u(t) = h(t). \tag{7}$$

and thus

$$I_{0+}^{\alpha}D_{0+}^{\mu}u = I_{0+}^{\alpha-\mu}(I_{0+}^{\mu}D_{0+}^{\mu}u) = I_{0+}^{\alpha-\mu}(u + c_0t^{\mu-1} + c_1t^{\mu-2}) = I_{0+}^{\alpha-\mu}u + \tilde{c}_0t^{\alpha-1} + \tilde{c}_1t^{\alpha-2},$$

where c_0, c_1, $\tilde{c}_0$, and $\tilde{c}_1$ are constants.

We can apply I_{0+}^{α} to both sides of Equation (7) to obtain

$$u(t) - a(I_{0+}^{\alpha-\mu}u)(t) = -(I_{0+}^{\alpha}h)(t) + c_2t^{\alpha-1} + c_3t^{\alpha-2} + c_4t^{\alpha-3}.$$

Since $u(0) = 0$, $c_4 = 0$, and since $u'(0) = 0$, $c_3 = 0$, hence

$$u(t) - a(I_{0+}^{\alpha-\mu}u)(t) = -(I_{0+}^{\alpha}h)(t) + c_2t^{\alpha-1}.$$

We can define $\mathcal{A} : X \to X$ and $\mathcal{B} : X \to X$ by

$$(\mathcal{A}u)(t) = a(I_{0+}^{\alpha-\mu}u)(t) \text{ and } (\mathcal{B}h)(t) = -(I_{0+}^{\alpha}h)(t) + c_2t^{\alpha-1}.$$

Notice that if $|a| < \Gamma(\alpha - \mu + 1)$, then

$$\begin{aligned} \|\mathcal{A}\| &= \sup_{\|u\|=1} \|\mathcal{A}u\| \\ &\leq \sup_{t\in[0,1]} \left| a \int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} ds \right| \\ &= \sup_{t\in[0,1]} \left| (aI_{0+}^{\alpha-\mu}1)(t) \right| \\ &= \frac{|a|}{\Gamma(\alpha-\mu+1)} \\ &< 1. \end{aligned}$$

Then, by Lemma 2, we obtain

$$u = \sum_{n=0}^{\infty} \mathcal{A}^n \mathcal{B}h.$$

Now, we have

$$\begin{aligned}\mathcal{A}^n\mathcal{B}h &= (\mathcal{A}^n(-I^{\alpha}_{0^+}h))(t) + c_2\mathcal{A}^n(t^{\alpha-1}) \\ &= -\frac{a^n}{\Gamma((n+1)\alpha - n\mu)}\int_0^t (t-s)^{(n+1)\alpha-n\mu-1}h(s)ds \\ &\quad + \frac{c_2 a^n\Gamma(\alpha)}{\Gamma((n+1)\alpha-n\mu)}t^{(n+1)\alpha-n\mu-1}, \quad n = 0,1,\ldots.\end{aligned}$$

We write

$$\mathcal{A}^n(t^{\alpha-1}) = \Gamma(\alpha)t^{\alpha-1}\frac{(at^{\alpha-\mu})^n}{\Gamma(n(\alpha-\mu)+\alpha)}.$$

Since the generalized Mittag-Leffler function is entire for the positive parameters $\alpha-\mu$ and α, $\sum_{n=0}^{\infty}\mathcal{A}^n(t^{\alpha-1})$ converges uniformly for $t\in[0,1]$, and

$$\sum_{n=0}^{\infty}\mathcal{A}^n(t^{\alpha-1}) = \Gamma(\alpha)t^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\mu)}}{\Gamma(n(\alpha-\mu)+\alpha)} = \Gamma(\alpha)t^{\alpha-1}E_{\alpha-\mu,\alpha}[at^{\alpha-\mu}].$$

Similarly, we write

$$\mathcal{A}^n(-I^{\alpha}_{0^+}h))(t) = \int_0^t (t-s)^{\alpha-1}\frac{(a(t-s)^{\alpha-\mu})^n}{\Gamma((n(\alpha-\mu)+\alpha)}h(s)ds.$$

The convergence of $\sum_{n=0}^{\infty}\dfrac{(a(t-s)^{\alpha-\mu})^n}{\Gamma((n(\alpha-\mu)+\alpha)}$ is uniform on the triangle, where $0\le s\le t\le b$, and so the convergence of

$$\sum_{n=0}^{\infty}(t-s)^{\alpha-1}\frac{(a(t-s)^{\alpha-\mu})^n}{\Gamma((n(\alpha-\mu)+\alpha)}h(s)$$

is uniform on the triangle $0\le s\le t\le b$. Hence, we can write

$$\begin{aligned}u(t) &= \left(\sum_{n=0}^{\infty}\mathcal{A}^n\mathcal{B}h\right)(t) \\ &= c_2\Gamma(\alpha)t^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\mu)}}{\Gamma(n(\alpha-\mu)+\alpha)} - \int_0^t (t-s)^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\beta)}}{\Gamma(n(\alpha-\mu)+\alpha)}h(s)ds \\ &= c_2\Gamma(\alpha)t^{\alpha-1}E_{\alpha-\mu,\alpha}[at^{\alpha-\mu}] - \int_0^t (t-s)^{\alpha-1}E_{\alpha-\mu,\alpha}[a(t-s)^{\alpha-\mu}]h(s)ds.\end{aligned}$$

Now, Lemma 1 can be employed to obtain

$$\begin{aligned}D^{\beta}_{0^+}u(t) &= \sum_{n=0}^{\infty}\frac{c_2a^n\Gamma(\alpha)t^{(n+1)\alpha-n\mu-\beta-1}}{\Gamma((n+1)\alpha-n\mu-\beta)} \\ &\quad -\sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\alpha-n\mu-\beta)}\int_0^t (t-s)^{(n+1)\alpha-n\mu-\beta-1}h(s)ds \\ &= c_2\Gamma(\alpha)t^{\alpha-\beta-1}\sum_{n=0}^{\infty}\frac{a^n t^{n(\alpha-\mu)}}{\Gamma(n(\alpha-\mu)+(\alpha-\beta))} \\ &\quad -\int_0^t (t-s)^{\alpha-\beta-1}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\mu)}}{\Gamma(n(\alpha-\mu)+(\alpha-\beta))}h(s)ds \\ &= c_2\Gamma(\alpha)t^{\alpha-\beta-1}E_{\alpha-\mu,\alpha-\beta}[at^{\alpha-\mu}] - \int_0^t (t-s)^{\alpha-\beta-1}E_{\alpha-\mu,\alpha-\beta}[a(t-s)^{\alpha-\mu}]h(s)ds.\end{aligned}$$

Since $D^{\beta}_{0^+}u(b) = 0$, then

$$c_2 = \int_0^1 \frac{(b-s)^{\alpha-\beta-1}E_{\alpha-\mu,\alpha-\beta}[a(b-s)^{\alpha-\mu}]}{b^{\alpha-\beta-1}\Gamma(\alpha)E_{\alpha-\mu,\alpha-\beta}[ab^{\alpha-\mu}]}h(s)ds.$$

Thus, if

$$G(\mu,\beta,b;t,s) = \begin{cases} \tilde{G}_1(\mu,\beta,b;t,s), & 0 \le t \le s \le b, \\ \tilde{G}_1(\mu,\beta,b;t,s) - \tilde{G}_2(\mu,\beta,b;t,s), & 0 \le s \le t \le b, \end{cases}$$

where

$$\tilde{G}_1(\mu,\beta,b;t,s) = \frac{t^{\alpha-1}(b-s)^{\alpha-\beta-1}E_{\alpha-\mu,\alpha}[at^{\alpha-\mu}]E_{\alpha-\mu,\alpha-\beta}[a(b-s)^{\alpha-\mu}]}{b^{\alpha-\beta-1}E_{\alpha-\mu,\alpha-\beta}[ab^{\alpha-\mu}]},$$

and

$$\tilde{G}_2(\mu,\beta,b;t,s) = (t-s)^{\alpha-1}E_{\alpha-\mu,\alpha}[a(t-s)^{\alpha-\mu}],$$

then

$$u(t) = \int_0^1 G(\mu,\beta,b;t,s)h(s)ds.$$

Working in reverse, it can be shown that if

$$u(t) = \int_0^1 G(\mu,\beta,b;t,s)h(s)ds,$$

then u satisfies Equations (2) and (3).

Thus, we obtain the following theorem:

Theorem 2. *Assume $|a| < \Gamma(\alpha-\mu+1)$. The function u satisfies Equations (2) and (3) if and only if*

$$u(t) = \int_0^b G(\mu,\beta,b;t,s)h(s)ds,$$

where

$$G(\mu,\beta,b;t,s) = \begin{cases} \tilde{G}_1(\mu,\beta,b;t,s), & 0 \le t \le s \le b, \\ \tilde{G}_1(\mu,\beta,b;t,s) - \tilde{G}_2(\mu,\beta,b;t,s), & 0 \le s \le t \le b, \end{cases}$$

where

$$\tilde{G}_1(\mu,\beta,b;t,s) = \frac{t^{\alpha-1}(b-s)^{\alpha-\beta-1}E_{\alpha-\mu,\alpha}[at^{\alpha-\mu}]E_{\alpha-\mu,\alpha-\beta}[a(b-s)^{\alpha-\mu}]}{b^{\alpha-\beta-1}E_{\alpha-\mu,\alpha-\beta}[ab^{\alpha-\mu}]},$$

and

$$\tilde{G}_2(\mu,\beta,b;t,s) = (t-s)^{\alpha-1}E_{\alpha-\mu,\alpha}[a(t-s)^{\alpha-\mu}].$$

Notice that when $\mu = \alpha-1$ and $\beta = \alpha-1$, we have

$$\tilde{G}_1(\alpha-1,\alpha-1,b;t,s) = t^{\alpha-1}E_{1,\alpha}[at]e^{-as},$$

and

$$\tilde{G}_2(\alpha-1,\alpha-1,b;t,s) = (t-s)^{\alpha-1}E_{1,\alpha}[a(t-s)].$$

In this case, $G(\alpha-1,\alpha-1,b;t,s)$ is independent of b.

Theorem 3. *Assume $\alpha-\beta > 0$ and $0 < b_0 \le b$. Then, for each $s \in [0,b_0]$, we have*

$$\lim_{b\to\infty} \frac{E_{1,\alpha-\beta}[a(b-s)]}{E_{1,\alpha-\beta}[ab]} = e^{-as}. \tag{8}$$

Proof. From Equation (6), as $b \to +\infty$, we have

$$E_{1,\alpha-\beta}[ab] \sim ab^{1-(\alpha-\beta)}e^{ab}$$

In addition, for a fixed s, we have

$$E_{1,\alpha-\beta}[a(b-s)] \sim a(b-s)^{1-(\alpha-\beta)}e^{a(b-s)} = a(b-s)^{1-(\alpha-\beta)}e^{ab}e^{-as}.$$

Therefore, Equation (8) is true. □

Theorem 4. *Assume $\alpha \in (2,3]$, $\mu = \alpha - 1$ and $\beta \in [0,2]$. Assume $0 < b_0 \leq b$. Assume $(t,s) \in [0,b_0] \times [0,b_0]$. Then, we have*

$$\lim_{b\to\infty} G(\alpha-1,\beta,b;t,s) = G(\alpha-1,\alpha-1;t,s).$$

We must point out that this result is similar to the result for the one-term problem in [5] (Theorem 2.5).

4. Green's Function for the Three-Term Problem

For the remainder of this article, $b > 0$ is fixed, and there is no need to specify a Green's function as a function of μ or β. Therefore, in particular, let $G_0(t,s) = G(\mu,\beta,b;t,s)$ be the Green's function for Equations (2) and (3). We define

$$G_n(t,s) = \int_0^b f(\tau)G_0(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n \geq 1.$$

Assume f is continuous on $[0,1]$ and define $\max_{t\in[0,b]} |f(t)| = \bar{f}$ and $M = \max_{(t,s)\in[0,b]\times[0,b]} G_0(t,s)$. Notice that

$$|G_0(t,s)| \leq M,$$
$$|G_1(t,s)| \leq b\bar{f}M^2,$$

and, in general, that

$$|G_n(t,s)| \leq b^n\bar{f}^nM^{n+1} = M(b\bar{f}M)^n.$$

Therefore, by assuming $\bar{f} < \frac{1}{bM}$, we have

$$\sum_{n=0}^{\infty} |G_n(t,s)| \leq M\sum_{n=0}^{\infty}(b\bar{f}M)^n < \infty,$$

since $b\bar{f}M < 1$.

Now, let u be a solution for Equations (1) and (2). Thus, we have

$$-D_{0+}^{\alpha-1}u(t) + aD_{0+}^{\mu}u(t) + f(t)u(t) = h(t), \quad 0 < t < b,$$

or

$$-D_{0+}^{\alpha-1}u(t) + aD_{0+}^{\mu}u(t) = h(t) - f(t)u(t), \quad 0 < t < b$$

Additionally, u satisfies the boundary conditions in Equation (2). Thus, by Theorem 2, we have

$$u(t) = \int_0^b G_0(t,s)(h(s) - f(s)u(s))ds,$$

or

$$u(t) + \int_0^b f(s)G_0(t,s)u(s)ds = \int_0^b G_0(t,s)h(s)ds. \tag{9}$$

We define $\mathcal{A}, \mathcal{B} : X \to X$ by

$$(\mathcal{A}h)(t) = \int_0^b G_0(t,s)h(s)ds,$$

and

$$(\mathcal{B}u)(t) = \int_0^b f(s)G_0(t,s)u(s)ds.$$

Then, Equation (9) becomes

$$(\mathcal{I} + \mathcal{B})u = \mathcal{A}h.$$

Since $|f(t)| \leq \bar{f} < \frac{1}{bM}$, then $\|\mathcal{B}\| = \max_{\|u\|=1} \|\mathcal{B}u\| < 1$. Hence, $r(\mathcal{B}) < 1$, and

$$u = \sum_{n=0}^{\infty} (-\mathcal{B})^n \mathcal{A}h.$$

Notice for $n = 0$ that

$$((-\mathcal{B})^n \mathcal{A}h)(t) = (\mathcal{A}h)(t) = \int_0^b G_0(t,s)h(s)ds = \int_0^b (-1)^n G_n(t,s)h(s)ds.$$

Now, assume that

$$((-\mathcal{B})^n \mathcal{A}h)(t) = \int_0^b (-1)^n G_n(t,s)h(s)ds$$

holds for $n = m \geq 0$. Then, we have

$$\begin{aligned}
((-\mathcal{B})^{m+1}\mathcal{A}h)(t) &= (-\mathcal{B}(-\mathcal{B})^m \mathcal{A}h)(t) \\
&= \int_0^b -b(\tau)G_0(t,\tau) \int_0^1 (-1)^m G_m(\tau,s)h(s)dsd\tau \\
&= \int_0^b (-1)^{m+1} \int_0^1 b(\tau)G_0(t,\tau)G_m(\tau,s)d\tau h(s)ds \\
&= \int_0^b (-1)^{m+1} G_{m+1}(t,s)h(s)ds.
\end{aligned}$$

By induction, we have

$$((-\mathcal{B})^n \mathcal{A}h)(t) = \int_0^1 (-1)^n G_n(t,s)h(s)ds$$

for all $n \in \mathbb{N}$. Therefore, since

$$\sum_{n=0}^{\infty} (-1)^n G_n(t,s)$$

converges uniformly, then

$$u(t) = \sum_{n=0}^{\infty} \int_0^1 (-1)^n G_n(t,s)h(s)ds = \int_0^1 G(t,s)h(s)ds$$

where

$$G(t,s) = \sum_{n=0}^{\infty} (-1)^n G_n(t,s).$$

Theorem 5. *Assume $|a| < \Gamma(\alpha - \mu + 1)$. Let $G_0(t,s) = G(\mu,\beta,b;t,s)$ be the Green's function for Equations (2) and (3). Define*

$$G_n(t,s) = \int_0^b f(\tau)G_0(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n \geq 1.$$

Assume f is continuous on $[0,1]$, and assume $\max_{t\in[0,b]}|f(t)| = \bar{f} < \frac{1}{bM}$, where $M = \max_{(t,s)\in[0,b]\times[0,b]} G_0(t,s)$. Then, u satisfies Equations (1) and (2) if and only if

$$u(t) = \int_0^1 G(t,s)h(s)ds$$

where

$$G(t,s) = \sum_{n=0}^{\infty}(-1)^n G_n(t,s).$$

5. The Existence of Solutions

Consider the nonlinear boundary value problem

$$-D_{0+}^{\alpha-1}u + aD_{0+}^{\mu}u + f(t)u = g(t,u), \quad 0 < t < b, \tag{10}$$

satisfying the boundary conditions of Equation (2), where a and f satisfy the conditions of Theorem 5. Here, it is assumed that $g : [0,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We define $T : X \to X$ by

$$Tu(t) = \int_0^b G(t,s)g(s,u(s))ds.$$

Theorem 6. *Let $U = \max_{(t,s)\in[0,b]\times[0,b]} G(t,s)$. Assume a and f satisfy the conditions of Theorem 5, and assume there exists an $r > 0$ such that*

$$|g(t,u)| \le \frac{r}{bU}, \quad (t,u)\in[0,b]\times[0,r].$$

Then, Equations (2) and (10) have at least one solution u with $\|u\| \le r$.

Proof. The proof is an application of Schauder's fixed point theorem. We define the set $K = \{u \in X : \|u\| \le r\}$. Then, for $u \in K$ and $t \in [0,b]$, we have

$$\begin{aligned}
|(Tu)(t)| &= \left|\int_0^b G(t,s)g(s,u(s))ds\right| \\
&\le \int_0^b |G(t,s)||g(s,u(s))|ds \\
&\le Ub\frac{r}{bU} \\
&= r.
\end{aligned}$$

Therefore, $\|Tu\| \le r$. Hence, $TK \subset K$, and $\{Tu : u \in K\}$ is uniformly bounded.

We show that $\{(Tu)' : u \in K\}$ is uniformly bounded. Since $(Tu)'(t) = \int_0^b \frac{\partial}{\partial t}G(t,s)g(s,u(s))ds$, and

$$\frac{\partial}{\partial t}G(t,s) = \sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial t}G_n(t,s),$$

consider

$$\sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial t}G_n(t,s) = \int_0^b f(\tau)\frac{\partial}{\partial t}G_0(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n \ge 1.$$

Notice that if $\left|\frac{\partial}{\partial t}G_0(t,s)\right| \le M_1$, then $\left|\frac{\partial}{\partial t}G_1(t,s)\right| \le M_1 b\bar{f}M$, and in general, the following is true:

$$\left|\frac{\partial}{\partial t}G_n(t,s)\right| \le M_1 b^n \bar{f}^n M^n$$

Therefore, since $\bar{f} < \frac{1}{bM}$, we have

$$\sum_{n=0}^{\infty}\left|\frac{\partial}{\partial t}G_n(t,s)\right| \leq M_1 \sum_{n=0}^{\infty}(b\bar{f}M)^n < \infty,$$

since $b\bar{f}M < 1$. Thus, there exists $U_1 > 0$ such that

$$U_1 \geq \left|\frac{\partial}{\partial t}G(t,s)\right|, \quad (t,s) \in [0,b] \times [0,b],$$

In addition, if $u \in K$, then $|(Tu)'(t)| \leq \frac{U_1}{U}r$, and $\{(Tu)' : u \in K\}$ is uniformly bounded. Thus, $\{Tu : u \in K\}$ is uniformly bounded, equicontinuous, and hence sequentially compact.

By Schauder's fixed point theorem , T has a fixed point in K. Hence, Equations (2) and (10) have at least one solution u with $\|u\| \leq r$. $\square$

6. An Example

Example 1. *As an example, consider the case where $\mu = \alpha - 1$ and $\beta = \alpha - 1$. Let $a = \frac{1}{2} < \Gamma(2)$ and $b = 1$. Here, for $(t,s) \in [0,1] \times [0,1]$, since $E_{1,\alpha}[t]$ is increasing as a function of t, then we have*

$$\begin{aligned}|G_0(t,s)| &\leq |\tilde{G}_1(\alpha-1,\alpha-1,1;t,s)| + |\tilde{G}_2(\alpha-1,\alpha-1,1;t,s)| \\ &\leq 2E_{1,\alpha}\left[\frac{1}{2}\right] := M.\end{aligned}$$

Thus, if

$$\max_{t\in[0,1]}|f(t)| = \bar{f} < \frac{1}{2E_{1,\alpha}\left[\frac{1}{2}\right]},$$

then the unique solution of

$$-D_{0+}^{\alpha-1}u + \frac{1}{2}D_{0^+}^{\alpha-1}u + f(t)u = h(t), \quad 0 < t < 1,$$

satisfying the boundary conditions

$$u(0) = u'(0) = 0, \quad D_{0^+}^{\alpha-1}u(1) = 0,$$

is given by

$$u(t) = \int_0^1 G(t,s)h(s)ds$$

where

$$G(t,s) = \sum_{n=0}^{\infty}(-1)^n G_n(t,s).$$

If, for example, $f(t) = \dfrac{\sin t}{4E_{1,\alpha}\left[\frac{1}{2}\right]}$, $\bar{f} = \dfrac{1}{4E_{1,\alpha}\left[\frac{1}{2}\right]} < \dfrac{1}{M}$, *then in this case, we have*

$$|G(t,s)| \leq M\sum_{n=0}^{\infty}(b\bar{f}M)^n = 2E_{1,\alpha}\left[\frac{1}{2}\right]\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2E_{1,\alpha}\left[\frac{1}{2}\right] := U, \quad (t,s) \in [0,1] \times [0,1].$$

Now, consider the nonlinear boundary value problem

$$-D_{0+}^{\alpha-1}u + \frac{1}{2}D_{0^+}^{\alpha-1}u + \frac{\sin t}{4E_{1,\alpha}\left[\frac{1}{2}\right]}u = \frac{tu^2}{5E_{1,\alpha}\left[\frac{1}{2}\right]}, \quad 0 < t < 1,$$

satisfying the boundary conditions

$$u(0) = u'(0) = 0, \quad D_{0^+}^{\alpha-1}u(1) = 0,$$

Notice that for $r = 2$, it holds that

$$\left|\frac{tu^2}{5E_{1,\alpha}\left[\frac{1}{2}\right]}\right| \le \frac{1}{E_{1,\alpha}\left[\frac{1}{2}\right]} = \frac{r}{bU}, \quad (t,u) \in [0,1] \times [0,2].$$

By Theorem 6, this boundary value problem has a solution u with $\|u\| \le 2$.

7. Conclusion

In this paper, a three-term fractional boundary value problem was studied. Spectral theory was used to calculate the Green's function for a two-term problem first. The limiting properties of this Green's function were studied. Then, the Green's function for the three-term problem was constructed. Finally, this paper considered the existence of solutions to a nonlinear three-term problem, and an example was constructed.

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