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On the Semi-Local Convergence of a Noor–Waseem-like Method for Nonlinear Equations

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Abstract: The significant feature of this paper is that the semi-local convergence of high order methods for solving nonlinear equations defined on abstract spaces has not been studied extensively as done for the local convergence by a plethora of authors which is certainly a more interesting case. A process is developed based on majorizing sequences and the notion of restricted Lipschitz condition to provide a semi-local convergence analysis for the third convergent order Noor–Waseem method. Due to the generality of our technique, it can be used on other high order methods. The convergence analysis is enhanced. Numerical applications complete are used to test the convergence criteria.

Keywords: third order method; nonlinear equation; majorizing sequence; semi-local convergence

MSC: 47J25; 47H99; 49M15; 65G99

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1. Introduction

In this article we are concerned with the task of finding a solution χ^* for the nonlinear equation

$$G(x) = 0, \tag{1}$$

where $G : B \subseteq T_1 \to T_2$ is a differentiable nonlinear operator in the sense of Fréchet, T_1 and T_2 stand for Banach spaces and $B \neq \emptyset$ is an open set. A plethora of applications from applied as well as theoretical disciplines can be reduced to determining the point $\chi^* \in B$, but this task is very difficult in general. Moreover, the closed form of χ^* is hard to find unless in special cases. This forces researchers and practitioners to resort to iterative approximations to χ^*. A plethora of such approximations can be found in the literature [1–5]. Among those the most useful are the high convergence order ones. We noticed that many local convergence results exist for these methods relying Taylor expansions and derivatives of order at least one higher than the order of the method. As an example consider the third order Noor–Waseem method [6] defined by $x_0 \in \Omega$,

$$\begin{aligned} y_t &= x_t - [G'(x_t)]^{-1}G(x_t), \\ x_{t+1} &= x_t - 4[A_t]^{-1}G(x_t),\ t = 0, 1, 2, \cdots, \end{aligned} \tag{2}$$

where $A_t = 3G'\left(\frac{2x_t+y_t}{3}\right) + G'(y_t)$.

The existence of derivatives up to fourth order has been assumed although derivatives of order two and above do not appear on method. Moreover, method (2) may converge even if derivatives other than the first do not exist. Consider the academic and motivational example for $T_1 = T_2 = \Re$ and $B = [-0.5, 1.5]$ by solving the nonlinear equation

$$\varphi(u) = 0,$$

where function φ is defined as

$$\varphi(u) = \begin{cases} u^3 log(u^2) + u^5 - u^4, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0. \end{cases}$$

Then, we see that $\chi^* = 1 \in B$ and the third derivative is

$$\varphi'''(u) = 6logu^2 + 60u^2 - 24u + 22.$$

Notice that the third derivative of function φ is unbounded on B. Therefore, convergence is not assured by the results in [6]. There are no uniqueness of χ^* results or error bounds on $||x_t - x^*||, ||y_t - x_t||, ||x_{t+1} - y_t||$ that can be computed. The same observations can be made for the local results of other methods [4,6–9]. Hence, there is a need to develop results using conditions only on the first derivative that appears on these methods. These results should also provide the uniqueness of χ^* and the error bounds in advance. Moreover, they should be given for the more interesting semi-local convergence case. It turns out that these objectives can be achieved not only for (2) but for other methods too in a similar way. This is the novelty and motivation of our article. That is to expand the applicability under weaker conditions for these methods. It turns out that our error bounds are more accurate, and our convergence criteria hold even when the equivalent hypotheses in the preceding references are violated.

The remainder of this paper is organized as follows: Majorizing sequences for method (2) are introduced and studied in Section 2. The semi-local convergence is given in Section 3 for method (2). Numerical applications appear in Section 4. Concluding remarks in Section 5 complete this article.

2. Majorizing Sequences

A recall of the definition of a mozorizing sequence is needed.

Definition 1. *A nonnegative sequence $\{v_t\}$ is called majorizing for a sequence $\{w_t\}$ in a Banach space T if for all $t = 0, 1, 2, \cdots$*

$$\|w_{t+1} - w_t\| \leq v_{t+1} - v_t. \tag{3}$$

Scalar sequences are developed that majorize method (2). Let $\kappa_0 > 0, \kappa > 0, \kappa_1 > 0$ and $t \geq 0$ be given constants. Define sequence $\{u_t\}$ by $u_0 = 0, v_0 = \Omega$

$$\begin{aligned} u_{t+1} &= v_t + \frac{2\kappa(v_t - u_t)^2}{1 - \frac{\kappa_0}{6}(u_t + 2v_t)}, \\ v_{t+1} &= u_{t+1} + \frac{\kappa(u_{t+1} - u_t)^2 + 2\kappa_1(u_{t+1} - v_t)}{2(1 - \kappa_0 u_{t+1})}. \end{aligned} \tag{4}$$

Next, we present convergence criteria for sequence $\{u_t\}$.

Lemma 1. *Suppose that for all $t = 0, 1, 2, \cdots,$*

$$\kappa_0(u_t + 2v_t) < 6 \text{ and } \kappa_0 u_{t+1} < 1. \tag{5}$$

Then, sequence $\{u_t\}$ is non-decreasing, bounded from above by $\frac{1}{\kappa_0}$ and converges to its unique least upper bound $u^* \in [0, \frac{1}{\kappa_0}]$.

Proof. It follows by (4) and (5) that

$$0 \le u_t \le v_t \le u_{t+1} < \frac{1}{\kappa_0},$$

so, we conclude that $\lim_{t\to\infty} u_t = u^*$. $\square$

Stronger convergence criteria than (4) can be given but which are easier to verify as follows:

Define recurrent polynomials on the interval $[0,1)$ by

$$\begin{aligned}
p_t^{(1)}(u) &= 2\kappa u^{t-1}\Omega + \frac{\kappa_0}{6}(3(1+u+\cdots+u^t)\Omega + 2u^t\Omega) - 1,\\
p_t^{(2)}(u) &= 16\kappa^3 u^{3t-1}\Omega^3 + 8\kappa^2 u^{2t-1}\Omega^2 + \kappa_2 u^{t-1}\Omega\\
&\quad +2\kappa_0(1+u+\cdots+u^{t+1})\Omega - 2,\\
q_1(u) &= 2\kappa u - 2\kappa + \frac{\kappa_0}{6}(5u^2 - 2u),\\
q_t^{(2)}(u) &= 16\kappa^3 u^{2t+3}\Omega^2 - 16\kappa^3 u^{2t}\Omega^2 + 8\kappa^2 u^{t+2}\Omega\\
&\quad -8\kappa^2 u^t\Omega + \kappa_2 u - \kappa_2 + 2\kappa_0 u^3,
\end{aligned}$$

and

$$q_2(u) = 16\kappa^3 u^5\Omega^2 - 16\kappa^3 u^2\Omega^2 + 8\kappa^2 u^3\Omega - 8\kappa^2 u\Omega + \kappa_2 u - \kappa_2 + 2\kappa_0 u^3,$$

where $\kappa_2 = (1+8\kappa_1)\kappa$. Set also $p_2(u) = p_1^{(2)}(u)$. Notice that $q_1(0) = -2\kappa$, $q_1(1) = \frac{\kappa_0}{2}$, $q_2(0) = -\kappa_2$ and $q_2(1) = 2\kappa_0$. Hence, polynomials q_1 and q_2 have zeros in the interval $(0,1)$. Denote by α and v the smallest such zeros, respectively. These polynomials are connected.

Lemma 2. *The following items hold*

$$(i)\ p_{t+1}^{(1)}(u) = p_t^{(1)}(u) + q_1(u)u^{(t-1)}\Omega.$$

$$\textit{In particular,}\ \ p_{t+1}^{(1)}(u) = p_t^{(1)}(u) \quad \textit{at } u = \alpha.$$

$$(ii)\ q_{t+1}^{(2)}(u) \le q_t^{(2)}(u).$$

$$\textit{In particular,}\ \ q_t^{(2)}(u) \le q_2(u) \quad \textit{at } u = v.$$

and

$$(iii)\ p_{t+1}^{(2)}(u) = p_t^{(2)}(u) + q_t^{(2)}(u)u^{(t-1)}\Omega \le p_t^{(2)}(u) + q_2(u)u^{(t-1)}\Omega.$$

$$\textit{In particular,}\ \ p_{t+1}^{(2)}(u) \le p_t^{(u)}(u) \quad \textit{at } u = v.$$

Then, sequence $\{u_t\}$ is non-decreasing, bounded from above by $\frac{1}{\kappa_0}$ and converges to its unique least upper bound $u^* \in [0, \frac{1}{\kappa_0}]$.

Proof. By the definition of these polynomials we have in turn:

$$\begin{aligned}
(i)\ p_{t+1}^{(1)}(u) &= p_{t+1}^{(1)}(u) - p_t^{(1)}(u) + p_t^{(1)}(u)\\
&= 2\kappa u^t\Omega + \frac{\kappa_0}{6}(3(1+u+\cdots+u^t)\Omega + 2u^{t+1}\Omega) - 1\\
&\quad -2\kappa u^{t-1}\Omega - \frac{\kappa_0}{6}(3(1+u+\cdots+u^t)\Omega + 2u^t\Omega) + 1 + p_t^{(1)}(u)\\
&= p_t^{(1)}(u) + q_1(u)u^{t-1}\Omega,
\end{aligned}$$

so,

$$p_{t+1}^{(1)}(\alpha) = p_t^{(1)}(\alpha) + q_1(\alpha)\alpha^{(t-1)}\Omega, \; since \; q_1(\alpha) = 0.$$

$$\begin{aligned}(ii)\; q_{t+1}^{(2)}(u) - q_t^{(1)}(u) &= 16\kappa^3 u^{2t+3}\Omega^2 - 16\kappa^3 u^{2t}\Omega^2 + 8\kappa^2 u^{t+2}\Omega - 8\kappa^2 u^t \Omega \\ &= 16\kappa^3 u^{2t}\Omega^2(u-1) + 8\kappa^2 u^t \Omega(u^2-1) \\ &= 16u^3 u^{2t}\Omega^2(u-1)(u^2+u+1) + 8\kappa^2\Omega(u-1)(u+1)u^t \leq 0.\end{aligned}$$

$$\begin{aligned}(iii)\; p_{t+1}^{(2)}(u) &= 16\kappa^3 u^{3t+2}\Omega^3 + 8\kappa^2 u^{2t+1}\Omega^2 \\ &+ \kappa_2 u^t \Omega + 2\kappa_0(1+u+\cdots+u^{t+2})\Omega - 16\kappa^3 u^{3t-1}\Omega^3 \\ &- 8\kappa^2 u^{2t-1}\Omega^2 - \kappa_2 u^{t-1}\Omega - 2\kappa_0(1+u+\cdots+u^{t+1})\Omega + p_t^{(2)}(u) \\ &= p_t^{(2)}(u) + q_t^{(2)}(u)u^{(t-1)}\Omega \leq p_t^{(2)}(u) + q_1(u)u^{(t-1)}\Omega \\ &= p_t^{(2)}(u) + q_2(u)u^{(t-1)}\Omega\end{aligned}$$

□

Set

$$\begin{aligned} a &= \frac{2\kappa\Omega}{1-\frac{\kappa_0\Omega}{3}}, \; b = \frac{\kappa(u_1-u_0)^2 + 2\kappa_1(u_1-\Omega)}{2\Omega(1-\kappa_0 u_1)} \; for \; \Omega \neq 0, \\ c &= max\{a,b\} \;\; and \;\; d = min\{1-2\kappa_0\Omega, v\}.\end{aligned}$$

Lemma 3. *Suppose*

$$\kappa_0 u_1 < 1, \quad 2\kappa_0\Omega < 1, \quad p_2(\alpha) \leq 0, \tag{6}$$

and

$$0 \leq c \leq \alpha \leq d. \tag{7}$$

Notice that conditions (6) and (7) determine the smallness of Ω.

Then, sequence $\{u_t\}$ *is non-decreasing, bounded from above by* $u^{**} = \frac{\Omega}{1-\alpha}$ *and converges to its unique least upper bound* $u^* \in [0, \frac{\Omega}{1-\alpha}]$. *Moreover, the following estimates hold:*

$$0 \leq v_t - u_t \leq \alpha^t \Omega, \tag{8}$$

and

$$0 \leq u_{t+1} - v_t \leq \alpha^{t+1}\Omega. \tag{9}$$

Proof. The following items shall be shown using induction

$$0 \leq \frac{2\kappa(v_k - u_k)}{1 - \frac{\kappa_0}{6}(u_k + 2v_k)} \leq \alpha, \tag{10}$$

$$0 \leq \frac{\kappa(u_{k+1}-u_k)^2 + 2\kappa_1(u_{k+1}-v_k)}{2(1-\kappa_0 u_{k+1})} \leq \alpha(v_k - u_k), \tag{11}$$

and

$$0 \leq u_k \leq v_k \leq u_{k+1}. \tag{12}$$

These estimates hold for $k = 0$ by the expression (4), the choices of a, b, conditions (6) and (7). It follows that

$$0 \leq u_1 - v_0 \leq \alpha\Omega,$$

$$0 \le v_1 - u_1 \le \alpha(v_0 - u_0) \le \alpha\Omega,$$
$$so\ u_1 \le \Omega + \alpha\Omega = \frac{1-\alpha^2}{1-\alpha}\Omega < \frac{\Omega}{1-\alpha} = u^{**}.$$

Suppose

$$0 \le v_k - u_k \le \alpha^k\Omega,\ \ 0 \le u_{k+1} - v_t \le \alpha^{k+1}\Omega \text{ and}$$
$$u_{k+1 \le \frac{1-\alpha^{k+2}}{1-\alpha}\Omega < u^{**}},$$

hold for all $k \le t$. Then, evidently (10) holds if

$$2\kappa\alpha^k\Omega + \frac{\kappa_0}{6}\alpha\left[\frac{1-\alpha^{k+1}}{1-\alpha}\Omega + 2\left(\frac{1-\alpha^{k+1}}{1-\alpha} + \alpha^k\Omega\right)\right] - \alpha \le 0,$$

or

$$p_k^{(1)}(u) \le 0 \quad at \quad u = \alpha. \tag{13}$$

Define function $p_\infty^{(1)}$ on the interval $[0,1]$ by

$$p_\infty^{(1)}(u) = \lim_{k\to\infty} p_k^{(1)}(u). \tag{14}$$

It follows by the definition of $p_\infty^{(1)}$ and $p_k^{(1)}$ that

$$p_\infty^{(1)}(u) = \frac{\kappa_0\Omega}{2(1-u)} - 1. \tag{15}$$

Then, by Lemma 2(i) and expression (15), estimate (14) holds if

$$\frac{\kappa_0\Omega}{2(1-u)} - 1 \le 0 \ \ at \ \ u = \alpha,$$

which is true by the right hand side of (7). Notice that

$$2\frac{\kappa_0}{6}(u_k + 2v_k) \le \frac{\kappa_0}{3}3\frac{\Omega}{1-\alpha} \le 1,$$

by the choice of α, so

$$0 \le \frac{1}{1 - \frac{\kappa_0}{6}(u_k + 2v_k)} \le 2.$$

Hence, (11) holds if

$$\frac{\kappa[4\kappa(v_k-u_k)^2 + (v_k-u_k)]^2 + 8\kappa_1\kappa(v_k-u_k)^2}{2(1-\kappa_0 u_{k+1})} \le \alpha(v_k - u_k),$$

or

$$\frac{\kappa[4\kappa(v_k-u_k)+1]^2(v_k-u_k) + 8\kappa_1\kappa(v_k-u_k)}{2(1-\kappa_0 u_{k+1})} \le \alpha,$$

or

$$16\kappa^3(v_k-u_k)^3 + 8\kappa^2(v_k-u_k)^2 + \kappa(v_k-u_k) + 8\kappa_1\kappa(v_k-u_k) + 2\kappa_0 u_{k+1} - 2\alpha \le 0,$$

or

$$16\kappa^3(\alpha^k\Omega)^3 + 8\kappa^2(\alpha^k\Omega)^2 + \kappa_2\alpha^k\Omega + 2\alpha\kappa_0\frac{1-\alpha^{k+2}}{1-\alpha}\Omega - 2\alpha \leq 0,$$

or

$$p_k^{(2)}(u) \leq 0 \ \ at \ \ u = \alpha. \tag{16}$$

By Lemma 2(iii), the definition of α and v, estimate (16) holds if $p_1^{(2)}(u) \leq 0 \ at \ u = \alpha$, which is true by (13). The induction for estimate (10) and (11) is completed. Then, estimate (12) holds by expressions (4), (10) and (11). It follows that sequence $\{u_t\}$ is nondecreasing and bounded from above by u^{**} and as seen it converges to $u^* \in [0, \frac{\Omega}{1-\alpha}]$. □

3. Semilocal Convergence

Throughout this section, we prove the existence theorem for the method (2) for which the conditions H are needed. Assume:

$(h1)$ There exist $x_0 \in B, \Omega \geq 0$ such that $[G'(x_0)]^{-1} \in \delta(T_2, T_1)$ and $\|[G'(x_0)]^{-1}[G(x_0)]\| \leq \Omega$.

$(h2)$ There exists $\kappa_0 > 0$ such that for each $z \in B$

$$\|[G'(x_0)]^{-1}(G'(z) - G'(x_0))\| \quad \leq \quad \kappa_0\|z - x_0\|. \tag{17}$$

Define $B_1 = U(x_0, \frac{1}{\kappa_0}) \cap B$.

$(h3)$ There exists $\kappa > 0, \kappa_1 > 0$ such that for each $\mu \in B_1, w \in B_1$

$$\|[G'(x_0)]^{-1}(G'(\mu) - G'(w))\| \quad \leq \quad \kappa\|\mu - w\|,$$

$$and \ \|[G'(x_0)]^{-1}G'(\mu)\| \quad \leq \quad \kappa_1.$$

$(h4)$ Conditions of Lemma (1) or Lemma (3) hold, and

$(h5)$ $U[x_0, u^*] \subset B$.

Then, the following semilocal result for method (2) can be shown under conditions H.

Theorem 1. *Assume conditions H. Then, iteration $\{x_t\}$ given by method (2) is well defined in $U[x_0, u^*]$ remains in $U[x_0, u^*]$ for each $t = 0, 1, \cdots$ and converges to a solution χ^* of equation $G(x) = 0$ in $U[x_0, u^*]$. Moreover, the following assertions hold*

$$||y_t - x_t|| \leq v_t - u_t, \tag{18}$$

$$||x_{t+1} - y_t|| \leq u_{t+1} - v_t, \tag{19}$$

and

$$||\chi^* - x_t|| \leq u^* - u_t. \tag{20}$$

Proof. By condition $(h1)$ and the expression (4),

$$\|y_0 - x_0\| = \|[G'(x_0)]^{-1}[G'(x_0)]\| \leq \Omega = v_0 - u_0 \leq u^*,$$

so (17) holds for $t = 0$ and $y_0 \in U[x_0, u^*]$. Let $\mu \in U[x_0, u^*]$. Using condition $(h2)$, we get

$$\|[G'(x_0)]^{-1}(G'(\mu) - G'(x_0))\| \quad \leq \quad \kappa_0\|\mu - x_0\| \leq \kappa_0 u^* < 1,$$

Thus, the Banach lemma on linear invertible operators [1,2,7] assures that $[G'(\mu)]^{-1}$ exists and

$$\|[G'(\mu)]^{-1}G'(x_0)\| \leq \frac{1}{1-\kappa_0\|\mu - x_0\|}. \tag{21}$$

Next, we can write from the method (2)

$$\begin{aligned} x_{k+1} &= y_k + [G'(x_k)]^{-1}G(x_k) - 4[A_k]^{-1}G(x_k) \\ &= y_k + ([G'(x_k)]^{-1} - 4[A_k]^{-1})G(x_k) \\ &= y_k - (4[A_k]^{-1} - [G'(x_k)]^{-1})G(x_k) \\ &= y_k - [A_k]^{-1}(4[G'(x_k)]^{-1} - A_k)[G'(x_k)]^{-1}G(x_k). \end{aligned} \tag{22}$$

Some estimates are needed assuming inequalities (18) and (19) for all $k \leq t$

$$\begin{aligned} &4[G'(x_k)] - 3G'\left(\frac{2x_k + y_k}{3}\right) - G'(y_k) \\ &= 3[G'(x_k)] - G'\left(\frac{2x_k + y_k}{3}\right)] + [G'(x_k) - G'(y_k)], \end{aligned}$$

Hence, by conditions $(h2)$ and $(h3)$

$$\begin{aligned} &\|[G'(x_0)]^{-1}[4[G'(x_k)] - 3G'\left(\tfrac{2x_k+y_k}{3}\right) - G'(y_k)]\| \\ \leq\; & 3\kappa\|x_k - \tfrac{2x_k+y_k}{3}\| + \kappa\|y_k - x_k\| \\ \leq\; & 2\kappa\|y_k - x_k\| \leq 2\kappa(v_k - u_k), \end{aligned} \tag{23}$$

$$\begin{aligned} &\|[4G'(x_0)]^{-1}[A_k - 4G'(x_0)]\| \\ \leq\; & \frac{1}{4}[3\|[G'(x_0)]^{-1}[G'\left(\frac{2x_k + y_k}{3}\right) - G'(x_0)]\| \\ & +\|[G'(x_0)]^{-1}(G'(y_k) - G'(x_0))\|] \\ \leq\; & \frac{1}{4}\kappa_0[\|\frac{2x_k + y_k}{3} - x_0\| + \|y_k - x_0\|] \\ \leq\; & \frac{1}{4}\kappa_0[\|\frac{2\|x_k - x_0\| + \|y_k - x_0\|}{3} + \|y_k - x_0\|] \\ \leq\; & \frac{1}{4}\kappa_0[\frac{1}{3}(2u_k + v_k) + v_k] = \frac{\kappa_0}{6}(u_k + 2v_k) < 1, \end{aligned}$$

so

$$\|[A_k]^{-1}G'(x_0)\| \leq \frac{1}{1 - \frac{\kappa_0}{2}(u_k + 2v_k)}. \tag{24}$$

Hence, by expressions (4) and (22)–(24), we have

$$\|x_{k+1} - y_k\| \leq \frac{2\kappa(v_k - u_k)(v_k - u_k)}{1 - \frac{\kappa_0}{2}(u_k + 2v_k)} = u_{k+1} - v_k,$$

and

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \|x_{k+1} - y_k\| + \|y_k - x_0\| \\ &\leq u_{k+1} - v_k + v_k - u_0 = u_{k+1} \leq u^*. \end{aligned}$$

Therefore, iterate $x_{k+1} \in U[x_0, u^*]$ and inequality (19) holds for $t = 0$. Hence, iterate y_{k+1} is well defined (by (21) for $\mu = x_{k+1}$). We can write

$$\begin{aligned} G(x_{k+1}) &= G(x_{k+1}) - G(x_k) + G(x_k) \\ &= G(x_{k+1}) - G(x_k) - G'(x_k)(y_k - x_k) \\ &= G(x_{k+1}) - G(x_k) - G'(x_k)(x_{k+1} - x_k) + G'(x_k)(x_{k+1} - y_k), \end{aligned}$$

so by assumption $(h3)$ and the induction hypotheses

$$\begin{aligned} \|[G'(x_0)]^{-1}G(x_{k+1})\| &\le \tfrac{\kappa}{2}\|x_{k+1}-x_k\|^2+\kappa_1\|x_{k+1}-y_k\| \\ &\le \tfrac{\kappa}{2}(u_{k+1}-u_k\|^2+\kappa_1\|u_{k+1}-v_k\|. \end{aligned} \tag{25}$$

and

$$\begin{aligned} \|y_{k+1}-x_{k+1}\| &\le \|[G'(x_{k+1})]^{-1}G'(x_0)\|+\|[G'(x_0)]^{-1}G(x_{k+1})\| \\ &\le \frac{\kappa(u_{k+1}-u_k)^2+2\kappa_1(u_{k+1}-v_k)}{2(1-\kappa_0 u_{k+1})}=v_{k+1}-u_{k+1}, \end{aligned}$$

and

$$\begin{aligned} \|y_{k+1}-x_0\| &\le \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \\ &\le v_{k+1}-u_{k+1}+u_{k+1}-u_0=v_{k+1}\le u^*, \end{aligned}$$

which complete the induction for inequality (18) and (19). The sequence $\{u_k\}$ is fundamental as convergent and majorizes sequence $\{x_t\}$. Therefore, sequence $\{x_t\}$ is fundamental in Banach space T_1). Hence, there exists $\chi^* \in U[x_0,u^*]$ such that $\lim_{k\to\infty} x_k=\chi^*$. Then, by letting $k\to\infty$ in inequality (24), we get $G(\chi^*)=0$, where we also used the continuity of G. □

Remark 1. *(a) The limit point u^* can be replaced by $\frac{1}{\kappa_0}$ and $\frac{\Omega}{1-\alpha}$ given in closed form in Theorem 1 under the conditions of Lemmas 1 and 3, respectively.*

(b) The solution v of equation $q_2(u)=0$ depends on Ω, but it can independent of Ω as follows. Define polynomial

$$\overline{q_2}(u)=2\kappa_0 u^3+\kappa_2 u-\kappa_2.$$

Then, we have $\overline{q_2}(0)=-\kappa_2$ and $\overline{q_2}(1)=2\kappa_0$. Denote the smaller zero of equation $q_2(u)=0$ in $(0,1)$ by $\overline{v}$.

Notice also that

$$\begin{aligned} q_2(u) &= 16\kappa^3u^2\Omega^2(u^3-1)+8\kappa^2u^3\Omega(u^2-1)+\overline{q_2}(u) \\ &\le \overline{q_2}(u), \end{aligned}$$

so $\overline{v}$, $\overline{q_2}$ can replace v, q_2, respectively, in the previous results, In this case $\overline{v}$ is independent of Ω.

A uniqueness of the solution result follows.

Proposition 1. *Assume*

(1) *There exists element $\chi^* \in U[x_0,\delta_0] \subset B$ for some $\delta_0>0$ which is a simple solution for equation $G(x)=0$.*
(2) *Condition $(h2)$ holds.*
(3) *There exists $\delta_1 \ge \delta_0$ such that*

$$\frac{\kappa_0}{2}(\delta_0+\delta_1)<1. \tag{26}$$

Define $B_2=U[x_0,\delta_1]\cap B$.
Then, the element χ^* is the only solution of equation $G(x)=0$ in the set B_2.

Proof. Assume there exists $\mu^* \in B_2$ satisfying $G(\mu^*) = 0$. Define linear operator $M = \int_0^1 G'(\mu^* + \theta(\chi^* - \mu^*))d\theta$. Then, in view of condition $(h2)$ and inequality (26), we obtain in turn

$$\|[G'(x_0)]^{-1}(M - G'(x_0))\| \le \kappa_0 \int_0^1 [(1-\theta)\|\mu^* - x_0\| + \theta\|\chi^* - x_0\|]d\theta$$
$$= \frac{\kappa_0}{2}(\delta_0 + \delta) < 1.$$

Therefore, operator M is invertible. Hence, by using the identity $M(\mu^* - \chi^*) = G(\mu^*) - G(\chi^*) = 0$, we deduce that $\mu^* = \chi^*$. □

Remark 2. *Notice that not all conditions H are used in Proposition* 1*, But if they were used, then we can certainly set* $\delta_0 = u^*$.

4. Numerical Examples

Example 1. *To revert to the motivational example from the study's introductory section, dealing with the study of semilocal convergence, Let* $u_0 = 0.9955$. *The consecutive derivatives of* φ *are*

$$\begin{aligned} \varphi'(u) &= 3u^2 logu^2 + 5u^4 - 4u^3 + 2u^2, \\ \varphi''(u) &= 6ulogu^2 + 20u^3 - 12u^2 + 10u, \\ \varphi'''(u) &= 6logu^2 + 60u^2 - 24u + 22. \end{aligned}$$

It can be easily seen that φ''' *is unbounded on B. Through the conditions* $(h1)$, $(h2)$ *and* $(h3)$, *we can calculate*

$$\|[\varphi'(u_0)]^{-1}.\varphi(u_0)\| = 0.00456182 = \Omega,$$

$\kappa_0 = 21.5093, \kappa = 7.03249$ *and* $\kappa_1 = 1.3044$, *where* $B_1 = U(u_0, \frac{1}{\kappa_0}) \cap B$. *Next, we verify the conditions (5), (6) and (7) of Lemma 1 and Lemma 3 where* $\alpha = 0.714123$ *and* $v = 0.766581$. *Majorizing sequences*

$$\begin{aligned} u_t &= \{0, 0.00486441, 0.00540241, 0.00540983, 0.00540983, \cdots\} \\ v_t &= \{0.00456182, 0.00486441, 0.00539816, 0.00540983, 0.00540983, \cdots\} \end{aligned}$$

converge to $u^* \in [0, 0.159573]$. *Table 1 displays error estimates (18), (19) and (20) which are not computable in earlier studies. Nevertheless, all the assumptions of the Theorem 1 are satisfied and hence, the iteration* $\{x_t\}$ *given by scheme (2) converges to a solution* χ^* *of equation* $G(x) = 0$ *in* $U[u_0, 0.00540983]$. *Precisely, we present a technique that gives weaker sufficient semi-local convergence conditions, tighter error estimates on the distances involved, and more exact information on the solution's location.*

Table 1. Error estimates.

t	$v_t - u_t$	$u_{t+1} - v_t$	$u^* - u_t$
0	0.00456182	0.00030259	0.00540983
1	0	0.000538	0.00054542

Example 2 ([10]). *Next, in order to demonstrate the applicability of our hypothesis in a real-world scenario, take the following quartic equation, that describes the fraction of the nitrogen-hydrogen feed that gets converted to ammonia, called the fractional conversion. For 250 atm and 500C, this equation is written as follows:*

$$f(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674.$$

In this example, we wil look at the domain $B = (0.3, 0.4)$, where $x_0 = 0.3$. Through the conditions $(h1)$, $(h2)$ and $(h3)$, we can calculate

$$\|[f'(x_0)]^{-1}.f(x_0)\| = 0.02179956 = \Omega,$$

$\kappa_0 = \kappa = 1.76728$ *and* $\kappa_1 = 1.15604$, *where* $B_1 = U(x_0, \frac{1}{\kappa_0}) \cap B$. *Next, we verify the conditions (5), (6) and (7) of Lemmas 1 and 3 where* $\alpha = 0.843909$ *and* $v = 0.87568$. *Majorizing sequences*

$$\begin{aligned} u_t &= \{0, 0.0234965, 0.0260807, 0.0261157, 0.0261157, \cdots\} \\ v_t &= \{0.0217956, 0.026057, 0.0261157, 0.0261157, 0.0261157, \cdots\} \end{aligned}$$

converge to $u^* \in [0, 0.0415249]$. *Table 2 displays error estimates (18), (19) and (20). Therefore, as expected, estimates of the error are lower as initial guesses get closer to the root. Nevertheless, all the assumptions of the Theorem 1 are satisfied and hence, the iteration* $\{x_t\}$ *given by scheme (2) converges to a solution* χ^* *of equation* $G(x) = 0$ *in* $U[x_0, 0.0261157]$.

Table 2. Error estimates.

t	$v_t - u_t$	$u_{t+1} - v_t$	$u^* - u_t$
0	0.0217956	0.0017009	0.0261157
1	0.0025605	2.37×10^{-5}	0.0026192
2	3.5×10^{-5}	0	3.5×10^{-5}

Example 3 ([10]). *Let us define the function f on D_0 by*

$$f(x) = x^3 - p;$$

where $D_0 = U(x_0, 1-p)$ *and* $p \in (0, 1)$. *Set* $x_0 = 1$. *As a result, through the conditions $(h1)$, $(h2)$ and $(h3)$, we can calculate*

$$\|[f'(x_0)]^{-1}.f(x_0)\| = \frac{1-p}{3} = \Omega,$$

$$\kappa_0 = (3-p),\ \kappa = 2\min\left[(2-p), 1 + \frac{1}{\kappa_0}\right] \text{ and } \kappa_1 = (2-p)^2,$$

where $D_1 = U(x_0, \frac{1}{\kappa_0}) \cap D_0$. *Next, we verify the conditions (5), (6) and (7) of Lemmas 1 and 3 where* $\alpha = 0.848595$ *and* $v = 0.887371$. *Majorizing sequences*

$$\begin{aligned} u_t &= \{0, 0.038339, 0.047006, 0.0475302, 0.0475323, \cdots\} \\ v_t &= \{0.0333333, 0.0466847, 0.047529, 0.0475323, 0.0475323, \cdots\} \end{aligned}$$

converge to $u^* \in [0, 0.22016]$. *Table 3 displays error estimates (18), (19) and (20). Nevertheless, all the assumptions of the Theorem 1 are satisfied and hence, the iteration* $\{x_t\}$ *given by scheme (2) converges to a solution* χ^* *of equation* $G(x) = 0$ *in* $U[x_0, 0.0475323]$. *Therefore, as expected, estimates of the error are lower as initial guesses get closer to the root.*

Table 3. Error estimates.

t	$v_t - u_t$	$u_{t+1} - v_t$	$u^* - u_t$
0	0.0333333	0.0050057	0.0475323
1	0.0083457	0.000319	0.0091933
2	0.000523	1.2×10^{-6}	0.0005263

5. Conclusions

A new technique is developed using only derivatives appearing on the method to show semilocal convergence for high convergence order method (2). Earlier works have shown convergence assuming the existence of derivatives of high order which may not appear in the method. Hence, limiting its applicability. This technique also provides error bounds and uniqueness results that are not previously available. Finally, this technique is very general since it does not depend on the actual method. That is why it can be used along the same lines to extend the applicability of other methods such as single step Newton, Newton-like, Secant, Kurchatov, Stirling's or two step Traub, Newton or multistep methods [1,2,4,5,10,11].

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Nomenclature

$\kappa_0, \kappa, \kappa_1$	Lipschitz constants
$\{u_t\}, \{v_t\}$	Scalar sequences

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