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Extending King's Method for Finding Solutions of Equations

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Abstract: King's method applies to solve scalar equations. The local analysis is established under conditions including the fifth derivative. However, the only derivative in this method is the first. Earlier studies apply to equations containing at least five times differentiable functions. Consequently, these articles provide no information that can be used to solve equations involving functions that are less than five times differentiable, although King's method may converge. That is why the new analysis uses only the operators and their first derivatives which appear in King's method. The article contains the semi-local analysis for complex plane-valued functions not presented before. Numerical applications complement the theory.

Keywords: King's method; semi-local convergence; fourth convergence order

MSC: 49M15; 65J15; 65G99

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1. Introduction

In this article, the function $F : \Omega \subset T \longrightarrow T$ is differentiable, where $T = \mathbb{R}$ or $T = \mathbb{C}$ and Ω is an open nonempty set.

The nonlinear equation

$$F(x) = 0 \tag{1}$$

is studied in this article. An analytic form of a solution x^* is preferred. However, this form is not always available. So, mostly iterative solution methods have been applied to approximate the solution x^*.

In particular, King's [1] fourth-order method (KM) has been used;

$$\begin{aligned} u_0 \in \Omega,\ v_n &= u_n - F'(u_n)^{-1}F(u_n) \\ u_{n+1} &= v_n - A_n^{-1}(F(u_n) + \gamma F(v_n))F'(u_n)^{-1}F(v_n), \end{aligned} \tag{2}$$

where $\gamma \in T$ is a parameter and $A_n = F(u_n) + (\gamma - 2)F(v_n)$.

As motivation consider the real function

$$\mu(s) = \begin{cases} 0 & if\ s = 0 \\ s^5 - s^4 + s^3 \log s^2 & if\ s \neq 0. \end{cases}$$

This definition gives

$$\mu'''(s) = 6 \log s^2 + 60s^2 - 24s + 22.$$

However, then, the third derivative is unbounded. So, the convergence of KM is not assured by previous analyses in [1–8].

This is the case, since Taylor series requiring derivatives of high order (not in KM) are utilized in the analysis for convergence. This is a common observation for other methods, such as Traub's, Jarratt's, and the Kung–Traub method to mention some [2,3,5–10]. On the top of these concerns, some other problems exist with earlier studies. No computable data are provided for distances $\|u_{n+1} - u_n\|$ or $\|u_n - x^*\|$ or the uniqueness and location of solution x^*.

All these concerns are addressed utilizing conditions involving only the first derivative in the method (2) [9–16].

The next four sections include semi-local analysis, local analysis, the experiment, and conclusions, respectively.

2. Semi-Local Analysis

Set L_0, L, L_1, L_2, δ and η to be positive parameters. Set $L_3 = \frac{LL_2}{2}$, and $L_4 = \frac{\delta|\gamma|L^2}{4}$. Let the sequence $\{t_n\}$ be given as

$$\begin{aligned} t_0 &= 0, s_0 = \eta, \\ t_{n+1} &= s_n + \frac{[L_3 + L_4(s_n - t_n)](s_n - t_n)^3}{(1 - p_n)(1 - L_0 t_n)} \\ s_{n+1} &= t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2L_1(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}, \end{aligned} \tag{3}$$

where $p_n = L_2(t_n + |\gamma - 2|(s_n - \eta))$. Sequence $\{t_n\}$ shall be shown to be majorizing for KM.

Lemma 1. *Suppose*

$$t_n < \frac{1}{L_0} \text{ and } p_n < 1. \tag{4}$$

Then, the following assertions hold

$$t_n \le s_n \le t_{n+1} \tag{5}$$

and

$$\lim_{n \longrightarrow \infty} t_n = t^* \le \frac{1}{L_0}, \tag{6}$$

where t^* is the unique least upper bound of sequence $\{t_n\}$.

Proof. Assertions (5) and (6) follow immediately by (3) and (4). □

Another result is given for the sequence $\{t_n\}$ using stronger conditions but which are easier to verify than (4). However, first, we need to introduce some concepts. Let

$$a = (L_3 + L_4\eta)\eta^2, \; b = \frac{Lt_1^2 + 2L_1(t_1 - \eta)}{2(1 - L_0 t_1)\eta},$$

and

$$c = \max\{a, b\}.$$

Develop polynomials defined on the interval $[0, 1)$ as

$$f_n^{(1)}(t) = 2(L_3 + L_4 t^n \eta) t^{2n-1} \eta^2 + L_0(1 + t)(1 + t + \ldots + t^{n-1})\eta - 1,$$

$$g_n^{(1)}(t) = 2(L_3 + L_4 t^{n+1}\eta) t^{n+1} \eta - 2(L_3 + L_4 t^n \eta) t^{n-1} \eta + L_0(1 + t),$$

$$g_1(t) = g_1^{(1)}(t),$$

and

$$
\begin{aligned}
f_n^{(2)}(t) \quad = \quad & L[4(L_3+L_4t^n\eta)(t^n\eta)^2+1]^2t^{n-1}\eta \\
& +8L_1(L_3+L_4t^n\eta)t^{2n-1}\eta^2 \\
& +2L_0(1+t)(1+t+\ldots+t^n)\eta-2.
\end{aligned}
$$

Moreover, set

$$g_2(t)=g_2^{(2)}(t).$$

Notice that polynomials g_1 and g_2 are independent of n. In particular, say

$$g_1(t)=2L_4t\eta^2(t^3-1)+2L_3\eta(t^2-1)+L_0(1+t).$$

Then, condition $g_1(t)\geq 0$ needed in the next Lemma holds if

$$2L_4t\eta^2(1-t^3)+2L_3\eta(1-t^2)\leq L_0(1+t).$$

The left side of this estimate is a positive multiple of η. However, the right side of it is positive but independent of η. So, this estimate certainly holds for sufficiently small η. The same observation is made for polynomial g_2 and condition $g_2(t)\geq 0$.

An auxiliary result connects these polynomials.

Lemma 2. *The following items hold:*

(i) $f_{n+1}^{(1)}(t)-f_n^{(1)}(t)=g_n^{(1)}(t)t^n\eta$;
(ii) $g_{n+1}^{(1)}(t)\geq g_n^{(1)}(t)$;
(iii) $g_{n+1}^{(1)}(t)-f_n^{(1)}(t)\geq g_1(t)t^n\eta$, *if* $g_1(t)\geq 0$; *and*
(iv) $f_{n+1}^{(2)}(t)-f_n^{(2)}(t)\geq g_2(t)t^{n-1}\eta$, *if* $g_2(t)\geq 0$.

Proof. By the definition of these polynomials, we get in turn:

(i)

$$
\begin{aligned}
f_{n+1}^{(1)}(t) &= f_{n+1}^{(1)}(t)-f_n^{(1)}(t)+f_n^{(1)}(t) \\
&= 2(L_3+L_4t^{n+1}\eta)t^{2n+1}\eta^2+L_0(1+t)(1+t+\ldots+t^n)\eta-1 \\
&\quad -2(L_3+L_4t^n\eta)t^{2n-1}\eta^2-L_0(1+t)(1+t+\ldots+t^{n-1})\eta+f_n^{(1)}(t)+1 \\
&= f_n^{(1)}(t)+g_n^{(1)}(t)t^n\eta;
\end{aligned}
$$

(ii)

$$
\begin{aligned}
g_{n+1}^{(1)}(t)-g_n^{(1)}(t) &= 2(L_3+L_4t^{n+2}\eta)t^{n+2}\eta \\
&\quad -2(L_3+L_4t^{n+1}\eta)t^n\eta-2(L_3+L_4t^{n+1})\eta)t^{n+1}\eta \\
&\quad +2(L_3+L_4t^n\eta)t^{n-1}\eta \\
&= 2[(L_3+L_4t^{n+2}\eta)t^3-(L_3+L_4t^{n+1}\eta)t \\
&\quad -(L_3+L_4t^{n+1}\eta)t^2+(L_3+L_4t^n\eta)]t^{n-1}\eta \\
&= 2(t-1)^2(t+1)(L_3+L_4\eta t^n(t^2+t+1))t^{n-1}\eta\geq 0.
\end{aligned}
$$

(iii) This estimate follows immediately from the first two;
(iv) It follows similarly from the definition of polynomials g_2 and $f_n^{(2)}$, since $t\in[0,1)$.
□

Define the parameters

$$\beta_1 = \frac{1 - L_0\eta}{1 + L_0\eta},\ \beta_2 = \frac{1 - 2L_0\eta}{1 + 2L_0\eta},\ \beta_3 = \frac{1 - 2L_2\eta}{1 + 2L_2\eta(1 + 2|\gamma - 2|)},$$

$$\beta = \min\{\beta_1, \beta_2, \beta_3\}$$

and

$$M = 2\max\{L_0, L_2\}.$$

Notice that $\beta \in (0, 1)$.

Lemma 3. *Suppose:*

$$L_0 t_1 < 1, \tag{7}$$

$$M\eta < 1, \tag{8}$$

$$c \leq \alpha \leq \beta, \tag{9}$$

$$g_1(t) \geq 0 \text{ at } t = \alpha \tag{10}$$

and

$$g_2(t) \geq 0 \text{ at } t = \alpha \tag{11}$$

hold for some $\alpha \in (0, 1)$. Then, sequence $\{t_n\}$ is convergent to t^*. Notice, criteria (7)–(11) determine the "smallness" of η to force convergence of the method.

Proof. Mathematical induction is used to show

$$0 \leq \frac{[L_3 + L_4(s_m - t_m)](s_m - t_m)^3}{(1 - p_m)(1 - L_0 t_m)} \leq \alpha, \tag{12}$$

$$0 \leq \frac{L(t_{m+1} - t_m)^2 + 2L_1(t_{m+1} - s_m)}{2(1 - L_0 t_{m+1})} \leq \alpha(s_m - t_m) \tag{13}$$

and

$$t_m \leq s_m \leq t_{m+1}. \tag{14}$$

These estimates are true for $m = 0$ by (7) or (8) and the definition of sequence $\{t_m\}$. Then, it follows $0 \leq t_1 - s_0 \leq \alpha(s_0 - t_0) = \alpha\eta$ and $0 \leq s_1 - t_1 \leq \alpha(s_0 - t_0) = \alpha\eta$. Suppose:

$$0 \leq t_{m+1} - s_m \leq \alpha(s_m - t_m) \leq \alpha^{m+1}\eta \tag{15}$$

and

$$0 \leq s_{m+1} - t_{m+1} \leq \alpha(s_m - t_m) \leq \alpha^{m+1}\eta. \tag{16}$$

Then,

$$\begin{aligned}
t_{m+1} &\leq s_m + \alpha^{m+1}\eta \leq t_m + \alpha^m\eta + \alpha^{m+1}\eta \\
&\leq s_{m-1} + 2\alpha^m\eta + \alpha^{m+1}\eta \\
&\vdots \\
&\leq s_1 + 2\alpha^2\eta + \ldots + 2\alpha^m\eta + \alpha^{m+1}\eta \\
&\leq t_1 + \alpha\eta + 2\alpha^2\eta + \ldots + 2\alpha^m\eta + \alpha^{m+1}\eta \\
&= \eta + 2\alpha\eta(1 + \alpha + \ldots + \alpha^{m-1})\eta + \alpha^{m+1}\eta \\
&= \eta\frac{(1 + \alpha)(1 - \alpha^{m+1})}{1 - \alpha} \\
&< \eta\frac{1 + \alpha}{1 - \alpha} = t^{**}.
\end{aligned} \tag{17}$$

Evidently, (12) holds if

$$2(L_3 + L_4\alpha^m\eta)(\alpha^m\eta)^2 + L_0\alpha\frac{(1+\alpha)}{1-\alpha}(1-\alpha^m)\eta - \alpha \leq 0 \tag{18}$$

or

$$f_m^{(1)}(t) \leq 0 \text{ at } t = \alpha. \tag{19}$$

Define

$$f_\infty^{(1)}(t) = \lim_{m\longrightarrow\infty} f_m^{(1)}(t). \tag{20}$$

It can be shown instead from Lemma 2 that

$$f_\infty^{(1)}(t) \leq 0 \text{ at } t = \alpha. \tag{21}$$

However, by (15) and (20),

$$f_\infty^{(1)}(t) = \frac{L_0(1+t)\eta}{1-t} - 1. \tag{22}$$

Then, (21) holds by (10) and (22). Moreover, instead of (13), we can show

$$\frac{[L(t_{n+1} - s_n + s_n - t_n)^2 + 2L_1\left(\frac{(L_3+L_4(s_n-t_n)](s_n-t_n)^3}{(1-p_n)(1-L_0t_n)}\right)}{2(1-L_0t_{m+1})} \leq \alpha(s_n - t_n), \tag{23}$$

since

$$\frac{1}{1-L_0t_m} \leq 2, \tag{24}$$

$$\frac{1}{1-p_m} \leq 2 \tag{25}$$

and

$$\begin{aligned} 0 &\leq t_{m+1} - t_m \\ &\leq (1+\alpha)(s_m - t_m) \end{aligned} \tag{26}$$

hold. Indeed, (24) holds if

$$2L_2t_m \leq 2L_2\frac{(1+\alpha)\eta}{1-\alpha} \leq 1$$

or

$$\alpha \leq \frac{1-2L_0\eta}{1+2L_0\eta}.$$

However, this holds because of the choice of β_2 and (9). Moreover, estimate (25) holds if

$$2L_2\left(|\gamma - 2|[\frac{(1+\alpha)\eta}{1-\alpha} - \eta] + \frac{(1+\alpha)\eta}{1-\alpha}\right) \leq 1,$$

which is true by the choice of β_3 and (9). Then, (23) holds if

$$\begin{aligned} &L[4(L_3 + L_4(s_n - t_n))(s_n - t_n)^2 + 1]^2(s_n - t_n) \\ &+8L_1(L_3 + L - 4(s_n - t_n))(s_n - t_n)^2 \leq \alpha \end{aligned}$$

or

$$\begin{aligned} &L[4(L_3 + L_4\alpha^n\eta)(\alpha^n\eta)^2 + 1]\alpha^{n-1}\eta \\ &+8L_1(L_3 + L - 4\alpha^n\eta)\alpha^{2n-1}\eta^2 - 1 \leq 0 \end{aligned} \tag{27}$$

or

$$f_m^{(2)}(t) \leq 0 \text{ at } t = \alpha. \tag{28}$$

or

$$f(t) \leq 0 \text{ at } t = \alpha.$$

However, this holds by (11). By sequence $\{t_m\}$, (12) and (13), the estimate (14) also holds. Therefore, the induction for estimates (12)–(14) is terminated. Hence, $\{t_m\}$ is bounded by t^{**}, which is non-decreasing. Hence, it converges to t^*. $\square$

The semi-local convergence analysis of KM uses conditions (H). Suppose that there exist:

(H1) $u_0 \in \Omega$, $\eta \geq 0, \delta \geq 0 : F'(u_0) \neq 0, A_0 \neq 0$, $\|F'(u_0)^{-1}F(u_0)\| \leq \eta$ and $\|A_0^{-1}F'(u_0)\| \leq \delta$;

(H2) $L_0 > 0 : \|F'(u_0)^{-1}(F'(v) - F'(u_0))\| \leq L_0\|v - u_0\|$ for all $v \in \Omega$. Set $\Omega_0 = U(u_0, \frac{1}{L_0}) \cap \Omega$;

(H3) $L > 0, L_1 > 0, L_2 > 0 : \|F'(u_0)^{-1}(F'(v) - F'(u))\| \leq L\|v - u\|$,

$$\|F'(u_0)^{-1}F'(v)\| \leq L_1$$

and

$$\|A_0^{-1}F'(v)\| \leq L_2,$$

for all $v, w \in \Omega_0$;

(H4) The conditions in Lemma 1 or in Lemma 3 are true;

(H5) $U[u_0, t^*] \subset \Omega$.

Theorem 1. *Assume conditions H hold. Then, KM is well defined in $U(u_0, t^*)$, lies in $U(u_0, t^*)$, for all $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in U[u_0, t^*]$ of Equation (1), so*

$$\|v_m - u_m\| \leq s_m - t_m \tag{29}$$

and

$$\|u_{m+1} - v_m\| \leq t_{m+1} - s_m. \tag{30}$$

Proof. We have by $\{t_n\}$ and (H1)

$$\|v_0 - u_0\| = \|F'(u_0)^{-1}F(u_0)\| \leq \eta = s_0 - t_0 < t^*.$$

So, (29) is true if $m = 0$ and $v_0 \in U(u_0, t^*)$. Pick $u \in U(u_0, t^*)$. By (H1), (H2) and t^*, then

$$\|F'(u_0)^{-1}(F'(u_0) - F'(u))\| \leq L_0\|u_0 - u\| \leq L_0 t^* < 1.$$

That is $F'(u) \neq 0$ with

$$\|F'(u)^{-1}F'(u_0)\| \leq \frac{1}{1 - L_0\|u - u_0\|}. \tag{31}$$

By the Banach lemma on functions [11–13], iteration u_1 is well-defined. Suppose $u_k, v_k \in U(u_0, t^*)$. Then, we can write

$$u_{k+1} - v_k = A_k^{-1}(F(u_k) + \gamma F(v_k))F'(u_k)^{-1}F(v_k). \tag{32}$$

By (H1), (H3), we get

$$\begin{aligned}
\|A_0^{-1}(A_k - A_0)\| &\leq \|A_0^{-1}(F(u_0) - F(u_k))\| \\
&\quad + |\gamma - 2|\|A_0^{-1}(F(v_0) - F(v_k))\| \\
&\leq \|\int_0^1 A_0^{-1}F'(u_0 + \theta(u_k - u_0))d\theta\|\|u_k - v_0\| \\
&\quad + |\gamma - 2|\|\int_0^1 A_0^{-1}(F'(v_0 + \theta(v_k - v_0))d\theta\|\|v_k - v_0\| \\
&\leq L_2(\|u_k - u_0\| + |\gamma - 2|\|v_k - v_0\|) \\
&\leq \bar{p}_k \leq p_k = L_2(t_k + |\gamma - 2|(s_k - \eta)) < 1,
\end{aligned}$$

so $A_k \neq O$ and

$$\|A_k^{-1}A_0\| \leq \frac{1}{1-p_k}. \tag{33}$$

Then, by (H3), (3), (31) (for $u = u_0$), (32) and (33), we obtain

$$\begin{aligned} \|u_{k+1}-v_k\| &\leq \|A_k^{-1}A_0\|[\|A_0^{-1}F(u_k)\| + |\gamma|\|A_0^{-1}F'(u_0)\| \\ &\quad \times\|F'(u_0)^{-1}F(v_k)\|]\|F'(u_k)^{-1}F'(u_0)\|\|F'(u_0)^{-1}F(v_k)\| \\ &\leq \frac{[L_2((s_k-t_k)+\delta|\gamma|\frac{L}{2}(s_k-t_k)^2]\frac{L}{2}(s_k-t_k)^2}{(1-p_k)(1-L_0t_k)} \\ &= t_{k+1}-s_k, \end{aligned} \tag{34}$$

so (30) holds, where we also used that (29) and (30) hold for all k smaller than $n-1$. We also get

$$\begin{aligned} \|F'(u_0)^{-1}F(u_k)\| &\leq \|F'(u_0)^{-1}F'(u_k)(v_k-u_k)\| \\ &\leq L_1\|v_k-u_k\| \leq L_1(s_k-t_k), \end{aligned} \tag{35}$$

$$\begin{aligned} F(v_k) &= F(v_k)-F(u_k)+F(u_k) \\ &= \int_0^1 F'(u_k+\theta(v_k-u_k))d\theta(v_k-u_k)-F'(u_k)(v_k-u_k), \end{aligned}$$

and

$$\|F'(u_0)^{-1}F(v_k)\| \leq \frac{L}{2}(s_k-t_k)^2 \tag{36}$$

We also have

$$\|u_{k+1}-u_0\| \leq \|u_{k+1}-v_k\|+\|v_k-u_0\| \leq (t_{k+1}-s_k)+(s_k-t_0) = t_{K+1} < t^*,$$

so $u_{k+1} \in U(u_0,t^*)$. Then, we write

$$\begin{aligned} F(u_{k+1}) &= F(u_{k+1})-F(u_k)+F(u_k) \\ &= F(u_{k+1})-F(u_k)-F'(u_k)(v_k-u_k) \\ &= F(u_{k+1})-F(u_k)-F'(u_k)(u_{k+1}-u_k)+F'(u_k)(u_{k+1}-v_k) \\ &= \int_0^1 (F'(u_k+\theta(u_{k+1}-u_k))d\theta-F'(u_k))(u_{k+1}-u_k) \\ &\quad +F'(u_k)(u_{k+1}-v_k). \end{aligned} \tag{37}$$

By (H3), we get

$$\begin{aligned} \|F'(u_0)^{-1}F(u_{k+1})\| &\leq \frac{L}{2}\|u_{k+1}-u_k\|^2+L_1\|u_{k+1}-v_k\| \\ &\leq \frac{L}{2}(t_{k+1}-t_k)^2+L_1(t_{k+1}-s_k). \end{aligned} \tag{38}$$

Then, by the first substep of KM

$$\begin{aligned} \|v_{k+1}-u_{k+1}\| &\leq \|F'(u_{k+1})^{-1}F'(u_0)\|\|F'(u_0)^{-1}F(u_{k+1})\| \\ &\leq \frac{\frac{L}{2}(t_{k+1}-t_k)^2+L_1(t_{k+1}-s_k)}{1-L_0t_{k+1}} \\ &= s_{k+1}-t_{k+1}, \end{aligned} \tag{39}$$

and

$$
\begin{aligned}
\|v_{k+1}-u_0\| &\leq \|v_{k+1}-u_{k+1}\|+\|u_{k+1}-u_0\| \\
&\leq s_{k+1}-t_{k+1}+t_{k+1}-t_0=s_{k+1}<t^*.
\end{aligned}
$$

Therefore, (29) holds and $v_{k+1} \in U[u_0,t^*]$. The induction is finished. So, $\{u_k\}$ is Cauchy in T. Hence, there exists $x^* \in U[u_0,t^*]$ such that $\lim_{k\longrightarrow\infty} x_n = x^*$. By letting k approach ∞ in (35), $F(x^*)=0$. □

Notice that $\frac{1}{L_0}$ under conditions of Lemma 1 or $\frac{(1+\alpha)\eta}{1-\alpha}$ under conditions of Lemma 3 provided in closed form may be used for t^* in Theorem 1.

Proposition 1. *Suppose*

(1) *The point $b \in U[u_0,r_0] \subset \Omega$ is a solution of Equation (1) with $F'(b) \neq 0$, and condition (H2) holds;*

(2) *Point $r \geq r_0$ exists:*

$$L_0(r+r_0)<2. \tag{40}$$

Set $\Omega_1 = U[u_0,r] \cap \Omega$. Then, b uniquely solves Equation (1) in Ω_1.

Proof. Let $\xi \in \Omega_1$ satisfy $F(\xi)=0$. Set $B=\int_0^1 F'(b+q(\xi-b))dq$. Then, by (H2) and (40), we obtain in turn that

$$
\begin{aligned}
\|F'(u_0)^{-1}(B-F'(u_0))\| &\leq L_0\int_0^1((1-q)\|u_0-b\|+q\|u_0-\xi\|)dq \\
&\leq \frac{L_0}{2}(r_0+r)<1.
\end{aligned}
$$

Therefore, $\xi=b$ follows from $B\neq 0$ and $B(\xi-b)=F(\xi)-F(b)=0-0=0$. □

3. Local Convergence

Set K_0, K, and K_1 to be positive parameters. Define function $g_1:[0,\frac{1}{K_0})\longrightarrow \mathbb{R}$ by

$$g_1(t)=\frac{Kt}{2(1-K_0t)}.$$

Notice that

$$\rho_0=\frac{2}{2K_0+K}<\frac{1}{K_0} \tag{41}$$

is a radius of convergence for Newton's method provided by us in [11–13]. This point ρ_0 also solves the equation

$$G_1(t)=g_1(t)-1=0.$$

Develop $q:[0,\frac{1}{K_0})\longrightarrow\mathbb{R}$, $Q:[0,\frac{1}{K_0})\longrightarrow\mathbb{R}$ by

$$q(t)=|\gamma-2|K_1g_1(t)+\frac{K}{2}t$$

and

$$Q(t)+1=q(t).$$

Then, we have $Q(0)=-1$ and $Q(\rho_0)=\frac{K}{2}\rho_0+|\gamma-2|K_1>0$. The intermediate value theorem assures Q has zeros in $(0,\rho_0)$. Let ρ_Q stand for the smallest zero in $(0,\rho_0)$. Define functions $g_2:[0,\rho_Q)\longrightarrow\mathbb{R}$ and $G_2:[0,\rho_Q)\longrightarrow\mathbb{R}$ by

$$g_2(t)=g_1(t)\left(1+\frac{K_1^2(1+|\gamma|g_1(t))}{(1-q(t))(1-K_0t)}\right)$$

and

$$G_2(t)=g_2(t)-1.$$

It follows $G_2(0) = -1$ and $G_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_Q^-$. Let ρ be the smallest such zero of G_2 on $(0, \rho_Q)$. Set $I = [0, \rho)$. Then, the definition of ρ implies that for all $t \in I$

$$0 \le g_1(t) < 1, \tag{42}$$

$$0 \le q(t) < 1 \tag{43}$$

and

$$0 \le g_2(t) < 1. \tag{44}$$

The local convergence of KM uses conditions (C). Suppose that there is

(C1) a solution $x^* \in \Omega$ of Equation (1) with $F'(x^*) \neq 0$;

(C2) $K_0 > 0$, so that

$$\|F'(x^*)^{-1}(F'(x^*) - F'(w))\| \le K_0 \|x^* - w\|$$

for all $w \in \Omega$. Define $\Omega_2 = U(x^*, \frac{1}{K_0}) \cap \Omega$;

(C3) There exist $K > 0, K_1 > 0$ such that

$$\|F'(x^*)^{-1}(F'(w) - F'(v))\| \le K\|w - v\|$$

and

$$\|F'(x^*)^{-1}F'(v)\| \le K_1\|x^* - v\|$$

for all $v, w \in \Omega_2$;

(C4) $U[x^*, \rho] \subset \Omega$.

Theorem 2. *Choose $u_0 \in U(x^*, \rho) - \{x^*\}$. Then, under conditions (C), sequence $\{u_n\}$ generated by KM converges to x^*, so that*

$$\|v_n - x^*\| \le g_1(d_n)d_n \le d_n < \rho \tag{45}$$

and

$$d_{n+1} \le g_2(d_n)d_n \le d_n, \tag{46}$$

where $d_n = \|u_n - x^*\|$, and the functions g_1, g_2 were previously defined.

Proof. Pick $z \in U(x^*, \rho) - \{x^*\}$. Then, by (C1) and (C2)

$$\|F'(x^*)^{-1}(F'(z) - F'(x^*))\| \le K_0\|z - x^*\| \le K_0\rho < 1. \tag{47}$$

So, we have $F'(z) \neq 0$ and

$$\|F'(z)^{-1}F'(x^*)\| \le \frac{1}{1 - K_0\|z - x^*\|}. \tag{48}$$

If $z = u_0$, we see that iterate v_0 is well-defined by KM for $n = 0$. Moreover, we can write

$$\begin{aligned} v_0 - x^* &= u_0 - x^* - F'(u_0)^{-1}F(u_0) \\ &= F'(u_0)^{-1}[\int_0^1 (F'(x^* + \theta(u_0 - x^*)) - F'(u_0))d\theta(u_0 - x^*)]. \end{aligned} \tag{49}$$

By (42), (48) (for $z = u_0$), (C3) and (46), we have in turn that

$$\begin{aligned} \|v_0 - x^*\| &\le \frac{K\|u_0 - x^*\|^2}{2(1 - K)\|u_0 - x^*\|} \\ &= g_1(\|u_0 - x^*\|)\|u_0 - x^*\| \\ &\le \|u_0 - x^*\| < \rho. \end{aligned} \tag{50}$$

Hence, iterate $v_0 \in U(x^*, \rho)$ and (42) holds if $n = 0$. Next, we show that $A_0 \neq 0$. If $u_0 \neq x^*$, we obtain by (C1), (C2), and (46)

$$\begin{aligned}
& \|(F'(x^*)(u_0 - x^*)^{-1}[A_0 - F'(x^*)(u_0 - x^*)]\| \\
\leq \; & \frac{1}{\|u_0 - x^*\|}[\|F'(x^*)^{-1}(F(u_0) - F(x^*) - F'(x^*)(u_0 - x^*))\| \\
& +|\gamma - 2|\|F'(x^*)^{-1}F(v_0)\|] \\
\leq \; & \frac{1}{\|u_0 - x^*\|}[\frac{K}{2}\|u_0 - x^*\|^2 + |\gamma - 2|K_1\|v_0 - x^*\|] \\
\leq \; & \frac{K}{2}\|u_0 - x^*\| + |\gamma - 2|K_1 g_1(\|u_0 - x^*\|) \\
= \; & q(\|u_0 - x^*\|) \leq q(\rho) < 1.
\end{aligned}$$

It follows that $A_0 \neq 0$, and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{\|u_0 - x^*\|(1 - q(\|u_0 - x^*\|))}. \tag{51}$$

Then, using (44), (C3), (48), (50), and (51)

$$\begin{aligned}
\|u_1 - x^*\| \leq \; & \|v_0 - x^*\| + \|A_0^{-1}F'(x^*)\|(\|F'(x^*)^{-1}F(u_0)\| \\
& +|\gamma|\|F'(x^*)^{-1}F(v_0)\|)\|F'(u_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(v_0)\| \\
\leq \; & \left(1 + \frac{K_1^2(\|u_0 - x^*\|) + |\gamma|\|v_0 - x^*\|}{\|u_0 - x^*\|(1 - q(\|u_0 - x^*\|))(1 - K_0\|u_0 - x^*\|)}\right)\|v_0 - x^*\| \\
\leq \; & \left(1 + \frac{K_1^2(\|u_0 - x^*\|) + |\gamma|g_1(\|u_0 - x^*\|)}{(1 - q(\|u_0 - x^*\|))(1 - K_0\|u_0 - x^*\|)}\right)g_1(\|u_0 - x^*\|)\|u_0 - x^*\| \\
= \; & g_2(\|u_0 - x^*\|)\|u_0 - x^*\| \leq \|u_0 - x^*\| < \rho.
\end{aligned} \tag{52}$$

That is iterate $u_1 \in U(u_0, x^*)$ and (43) holds for $n = 0$. Simply switch u_0, v_0, u_1 by u_k, v_k, u_{k+1} in the above calculations to terminate the induction for (42) and (43). Then, it follows from the estimate

$$\|u_{k+1} - x^*\| \leq \lambda\|u_k - x^*\| < \rho, \tag{53}$$

where $\lambda = g_2(\|u_0 - x^*\|) \in [0, 1)$. We conclude $\lim_{k \longrightarrow \infty} u_k = x^*$ and $u_{k+1} \in U(x^*, \rho)$. □

A uniqueness of the solution result follows next.

Proposition 2. *Suppose*

(1) *Element $\lambda \in U(x^*, \rho_0) \subseteq \Omega$ solves Equation (1), $F(\lambda) = 0$, and (C2) holds;*
(2) *There exists $\rho^* \geq \rho_0$ such that*

$$K_0\rho^* < 2. \tag{54}$$

Set $\Omega_3 = U[\lambda, \rho^*] \cap \Omega$. Then, element λ uniquely solves Equation (1) in Ω_3.

Proof. Let $\bar{x} \in \Omega_3$ with $F(\bar{x}) = 0$. Set $E = \int_0^1 F'(\lambda + \tau(\bar{x} - \lambda))d\tau$. Then, using (C2) and (54), we get in turn that

$$\begin{aligned}
\|F'(\lambda)^{-1}(E - F'(\lambda))\| \leq \; & K_0 \int_0^1 (1 - \tau)\|\lambda - \bar{x}\|d\tau \\
\leq \; & \frac{K_0}{2}\rho^* < 1.
\end{aligned}$$

Hence, $\bar{x} = \lambda$ follows from $E \neq 0$ and $E(\lambda - \bar{x}) = F(\lambda) - F(\bar{x}) = 0 - 0 = 0$. □

Next, the fourth-order convergence is shown using only the first derivative. Suppose:

$$\|A_n^{-1}F'(z)\| \leq \omega \tag{55}$$

and

$$\|F'(x)^{-1}(F'(x) - F'(y))\| \leq \omega_0\|x - y\| \tag{56}$$

hold for all $x, y, z \in \Omega$, for some constants $\omega > 0$ and $\omega_0 > 0$. Further, suppose

$$\frac{\theta\omega_0^2}{2}(\frac{3}{2} + \frac{\omega_0}{4} + \frac{|\gamma|\omega_0}{2}(1 + \frac{\omega_0}{4})) - 1 > 0. \tag{57}$$

Let $\psi(t) = \varphi(t) - 1 = 0$, where $\varphi(t) = \frac{\omega\omega_0^2}{2}(\frac{3}{2} + \frac{\omega_0}{4}t + \frac{|\gamma|\omega_0}{2}(t + \frac{\omega_0}{4}t^2)t^3$. Then, $\psi(0) = -1 < 0$ and $\psi(1) = \frac{\theta\omega_0^2}{2}(\frac{3}{2} + \frac{\omega_0}{4} + \frac{|\gamma|\omega_0}{2}(1 + \frac{\omega_0}{4})) - 1 > 0$. Hence, by the intermediate value theorem, $\psi(t) = 0$ has positive solutions. Let r_o be the smallest such solution.

Theorem 3. *Suppose conditions (55)–(57) hold. Then, sequence $\{u_n\}$ given in (2) is convergent to x^* with order four, i.e.,*

$$\|u_{n+1} - x^*\| \leq \varrho(r_o)d_n^4,$$

where $\varrho(r_o) = \frac{\theta\omega_0^2}{2}(\frac{3}{2} + \frac{\omega_0}{4}r_0 + \frac{|\gamma|\omega_0}{2}(r_o + \frac{\omega_0}{4}r_o^2)$.

Proof. The first substep of (2) and (56) gives

$$\begin{aligned}\|v_n - x^*\| &\leq \|F'(u_n)^{-1}\int_0^1[F'(u_n) - F'(x^* + \theta(u_n - x^*))]d\theta(u_n - x^*)\| \\ &\leq \frac{\omega_0}{2}d_n^2.\end{aligned}$$

Note

$$\begin{aligned}u_{n+1} - x^* &= v_n - x^* - A_n^{-1}(F(u_n) + \gamma F(v_n))F'(u_n)^{-1}F(v_n) \\ &= A_n^{-1}[A_n - (F(u_n) + \gamma F(v_n))F'(u_n)^{-1}\int_0^1 F'(x^* + \theta(v_n - x^*))d\theta](v_n - x^*) \\ &= A_n^{-1}F(u_n)F'(u_n)^{-1}[F'(u_n) - \int_0^1 F'(x^* + \theta(v_n - x^*))d\theta](v_n - x^*) \\ &\quad + A_n^{-1}F(v_n)F'(u_n)^{-1}\gamma[F'(u_n) - \int_0^1 F'(x^* + \theta(v_n - x^*))d\theta](v_n - x^*) \\ &\quad - 2A_n^{-1}F(v_n)(v_n - x^*),\end{aligned}$$

so, since $F(u_n) = \int_0^1 F'(x^* + u(u_n - x^*))du(u_n - x^*)$ and $F(v_n) = \int_0^1 F'(x^* + u(v_n - x^*))du(v_n - x^*)$

$$\begin{aligned}d_{n+1} &\leq \|A_n^{-1}\int_0^1 F'(x^* + u(u_n - x^*))duF'(u_n)^{-1}[F'(u_n) \\ &\quad - \int_0^1 F'(x^* + \theta(v_n - x^*))d\theta](u_n - x^*)(v_n - x^*)\| \\ &\quad + \|A_n^{-1}\int_0^1 F'(x^* + u(v_n - x^*))duF'(u_n)^{-1}\gamma[F'(u_n) \\ &\quad - \int_0^1 F'(x^* + \theta(v_n - x^*))d\theta](v_n - x^*)^2\| \\ &\quad + 2\|A_n^{-1}\int_0^1 F'(x^* + u(v_n - x^*))du(v_n - x^*)^2\|\end{aligned}$$

$$\begin{aligned}
&\leq \| \int_0^1 A_n^{-1} F'(x^* + u(u_n - x^*))du \\
&\int_0^1 F'(u_n)^{-1}[F'(u_n) - F'(x^* + \theta(v_n - x^*))d\theta](u_n - x^*)(v_n - x^*)\| \\
&+|\gamma|\| \int_0^1 A_n^{-1} F'(x^* + u(v_n - x^*))du \\
&+ \int_0^1 F'(u_n)^{-1}[F'(u_n) - F'(x^* + \theta(v_n - x^*))d\theta](v_n - x^*)^2\| \\
&+2\| \int_0^1 A_n^{-1} F'(x^* + u(v_n - x^*))du(v_n - x^*)^2\|.
\end{aligned}$$

Therefore, (55) and (56) give

$$\begin{aligned}
d_{n+1} &\leq \omega\omega_0 \left[d_n + \frac{\|v_n - x^*\|}{2} \right] d_n \|v_n - x^*\| \\
&\quad +|\gamma|\omega\omega_0 \left[d_n + \frac{\|v_n - x^*\|}{2} \right] \|v_n - x^*\|^2 \\
&\quad +2\omega\|v_n - x^*\|^2 \\
&\leq \omega\frac{\omega_0^2}{2}\left[1 + \frac{\omega_0}{4} d_n\right] d_n^4 \\
&\quad +|\gamma|\omega\frac{\omega_0^3}{4}\left[d_n + \frac{\omega_0}{4} d_n^2\right] d_n^4 \\
&\quad +\frac{\omega_0^2}{4}\omega d_n^4 \\
&\leq \varphi(d_n) d_n \\
&\leq \varrho(r_o) d_n^4.
\end{aligned}$$

□

4. Numerical Example

We verify convergence criteria using KM.

Example 1. *Let us consider a scalar function F defined on the set $\Omega = U[u_0, 1-s]$ for $s \in (0, \frac{1}{2})$ by*

$$F(x) = x^3 - s.$$

Choose $\gamma = 2$ and $u_0 = 1$. Then, we obtain the estimates $\eta = \frac{1-s}{3}$,

$$\begin{aligned}
|F'(u_0)^{-1}(F'(x) - F'(u_0))| &= |x^2 - u_0^2| \\
&\leq |x + u_0||x - u_0| \leq (|x - u_0| + 2|u_0|)|x - u_0| \\
&= (1 - s + 2)|x - u_0| = (3 - s)|x - u_0|,
\end{aligned}$$

for each $x \in \Omega$, so $L_0 = 3 - s$, $\Omega_0 = U(u_0, \frac{1}{L_0}) \cap \Omega = U(u_0, \frac{1}{L_0})$,

$$\begin{aligned}
|F'(u_0)^{-1}(F'(y) - F'(x)| &= |y^2 - x^2| \\
&\leq |y + x||y - x| \leq (|y - u_0 + x - u_0 + 2u_0)||y - x| \\
&= (|y - u_0| + |x - u_0| + 2|u_0|)|y - x| \\
&\leq (\frac{1}{L_0} + \frac{1}{L_0} + 2)|y - x| = 2(1 + \frac{1}{L_0})|y - x|,
\end{aligned}$$

for each $x, y \in \Omega$, and so $L = 2(1 + \frac{1}{L_0})$,

$$\begin{aligned} |F'(u_0)^{-1}(F'(y)-F'(x)| &= (|y-u_0|+|x-u_0|+2|u_0|)|y-x| \\ &\leq (1-s+1-s+2)|y-x| = 2(2-s)|y-x|, \end{aligned}$$

for each $x, y \in \Omega$, *so* $L_1 = (2-s)^2$ *and* $L_2 = \frac{3(2-s)^2}{F(1)+|\gamma-2|(1-\frac{F(1)}{F'(1)})^2-s}$.

Then, for $s = 0.95$, $\gamma = 0.5$, *we have* $\frac{1}{L_0} = 0.4878$.

According to the information taken from Table 1, the conditions of Lemma 1 hold. Consequently, the sequence converges and the interval of initial points has been further extended.

Table 1. Sequence (3) and condition (4).

n	1	2	3	4	5	6
p_n	0	0.1004	0.1033	0.1033	0.1033	0.1033
t_n	0.0167	0.0172	0.0172	0.0172	0.0172	0.0172
s_n	0.0167	0.0172	0.0172	0.0172	0.0172	0.0172

Example 2. *Set function* $F : I = [-1, 1] \longrightarrow \mathbb{R}$ *as*

$$F(x) = e^x - 1.$$

Notice that $x^* = 0$ *solves equation* $F(x) = 0$. *Choose* $\gamma = 2$. *Then, conditions of Theorem 3 hold for* $\omega = \omega_0 = e^2$. *Then, the radius is* $r_o = 0.1381$.

Example 3. *The example used in the introduction gives* $\omega = \omega_0 = 96.6629073$. *Then, for* $\gamma = 2$, *the radius is*

$$r_o = 0.0092.$$

Recall that it was shown in the Introduction that earlier articles cannot be used to solve this problem. The method used is a specialization of KM for $\gamma = 2$.

5. Conclusions

In this article, the extension of KM is presented. The convergence of KM has been shown by assuming the existence of a fifth derivative which was not considered before. This observation holds true for other high-convergence order methods such as Traub's and Jarratt's method. Other such methods can be found in [1–8] and the references therein. Therefore, these results cannot assure convergence. However, these methods may converge. Other concerns involve the absence of error estimates or uniqueness results that can be computed. This is our motivation for presenting a convergence analysis based on the first derivative used in KM. The generality of the technique allows its usage in other methods mentioned previously. This can be a fruitful direction of future research.

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