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On the Semi-Local Convergence of a Traub-Type Method for Solving Equations

Samundra Regmi ¹ , Christopher I. Argyros ², Ioannis K. Argyros ^{3,*} and Santhosh George ⁴

¹ Learning Commons, University of North Texas at Dallas, Dallas, TX 75201, USA; samundra.regmi@untDallas.edu

² Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA; christopher.Argyros@cameron.edu

³ Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA

⁴ Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangaluru 575025, India; sgeorge@nitk.edu.in

* Correspondence: iargyros@cameron.edu

Abstract: The celebrated Traub's method involving Banach space-defined operators is extended. The main feature in this study involves the determination of a subset of the original domain that also contains the Traub iterates. In the smaller domain, the Lipschitz constants are smaller too. Hence, a finer analysis is developed without the usage of additional conditions. This methodology applies to other methods. The examples justify the theoretical results.

Keywords: Traub's method; Banach space; convergence criterion

MSC: 49M15; 47H17; 65J15; 65G99; 41A25

Citation: Regmi, S.; Argyros, C.I.; Argyros, I.K.; George, S. On the Semi-Local Convergence of a Traub-Type Method for Solving Equations. *Foundations* **2022**, *2*, 114–127. <https://doi.org/10.3390/foundations2010006>

Academic Editor: Vladimir Iosifovich Semenov

Received: 25 December 2021

Accepted: 12 January 2022

Published: 14 January 2022

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1. Introduction

The purpose of this article is to locate a solution x^* of equation

$$F(x) = 0, \quad (1)$$

provided that $F : \Omega \subset E_1 \longrightarrow E_2$ is derivable according to Fréchet. Moreover, E_1, E_2 stand for Banach spaces, whereas Ω is nonempty and open.

The famous quadratically convergent Newton–Kantorovich method is defined for all $j = 0, 1, 2, \dots$ as

$$x_0 \in \Omega, x_{j+1} = x_j - F'(x_j)^{-1}F(x_j) \quad (2)$$

has been used extensively to produce sequence $\{x_j\}$ such that $\lim_{j \rightarrow \infty} x_j = x^*$ [1–8]. Although there is a plethora on convergence results for (2) there exist some problems. In particular, the convergence ball is in general small [1–18]. Hence, it is important to extend this ball but with no additional conditions. Other defects relate to the accuracy of bounds on $\|x_{j+1} - x_j\|$ or $\|x_j - x^*\|$, as well as the results on the location uniqueness of x^* . The same defects appear in the study of high convergence order methods [19–22]. We have developed a technique that helps determine some $D \subseteq \Omega$, where iterates can also be found. This way, using D instead of Ω , a finer analysis is possible with no additional conditions.

We demonstrate our techniques for a certain high convergence order, although it can similarly be used on other methods [12,15–17].

We extend the two step Traub method [21] (see also [18,22]) to the following three step fifth order method

$$\begin{aligned}y_j &= x_j - F'(x_j)^{-1}F(x_j) \\z_j &= y_j - F'(x_j)^{-1}F(y_j) \\x_{j+1} &= z_j - F'(x_j)^{-1}F(z_j).\end{aligned}\tag{3}$$

Traub's two-step method requires less computational effort than any third-order method utilizing the second derivative [2,4,5,14].

Let us provide the earlier results.

- (i) Convergence has been shown in Potra and Pták in [14] using

$$\begin{aligned}\|F'(x_0)^{-1}F(x_0)\| &\leq \mu \\ \|F'(x_0)^{-1}(F'(w) - F'(v))\| &\leq L_1\|w - v\| \text{ for all } w, v \in \Omega\end{aligned}\tag{4}$$

$$T_1 = 2L_1\mu \leq 1\tag{5}$$

and

$$U[x_0, \rho_0] \subset \Omega,$$

and

$$\rho_0 = \frac{1 - \sqrt{1 - T_1}}{L_1},\tag{6}$$

where $U[x_0, \rho_0]$ is a closed ball with radius ρ_0 and center at $x_0 \in \Omega$. The center-Lipschitz condition is introduced by us as

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq L_0\|w - x_0\| \text{ for all } w \in \Omega.\tag{7}$$

Define the set

$$D = U(x_0, \frac{1}{L_0}) \cap \Omega.\tag{8}$$

Moreover, we introduced the restricted-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(w) - F'(v))\| \leq L\|w - v\| \text{ for all } w, v \in D.\tag{9}$$

However, then we notice that

$$L_0 \leq L_1\tag{10}$$

and

$$L \leq L_1\tag{11}$$

hold, since

$$D \subseteq \Omega.\tag{12}$$

Suppose

$$T = 2L\mu \leq 1.\tag{13}$$

It follows that (9), (13), and

$$\bar{\rho}_0 = \frac{1 - \sqrt{1 - T}}{L}\tag{14}$$

can be used for (4), (5), and ρ_0 , respectively, given in [14] (Theorem 5.2, p. 79). Hence,

$$T_1 \leq 1 \Rightarrow T \leq 1\tag{15}$$

and

$$\bar{\rho}_0 \leq \rho_0\tag{16}$$

hold. So, the applicability of Traub's method is extended. The parameters L_0 and L are special cases of L_1 , so no additional effort is used. It is also worth to mention that $L_1 = L_1(\Omega)$, $L_0 = L_0(\Omega)$ but $L = L(\Omega, L_0)$. The proof in [14] (Theorem 5.2) utilized

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq L_1\|w - x_0\| < 1$$

leading to (by the Banach lemma [12] on linear invertible operators)

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_1\|w - x_0\|}.$$

However, we get

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq L_0\|w - x_0\| < 1$$

leading to tighter

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|w - x_0\|}.$$

This modification in the proof brings the aforementioned advantages. In the numerical section one can find cases when (10)–(12) are strict.

(ii) In [12], they used

$$\|F'(x_0)^{-1}F(x_0)\| \leq \mu,$$

$$\|F'(x_0)^{-1}F''(x_0)\| \leq \tau,$$

$$\|F'(x_0)^{-1}(F''(w) - F''(v))\| \leq K_1\|w - v\| \text{ for all } w, v \in \Omega,$$

$$\mu \leq \frac{(\tau^2 + 2K_1)^{\frac{3}{2}} - \tau(\tau^2 + 3K_1)}{3K_1}$$

and

$$U[x_0, \rho_1] \subset \Omega,$$

with ρ_1 denoting the minimal positive solution of equation

$$\frac{K_1}{6}t^3 + \frac{\tau}{2}t^2 - t + \mu = 0.$$

In our case we use

$$\|F'(x_0)^{-1}(F''(w) - F''(v))\| \leq K\|w - v\| \text{ for all } w, v \in D_0,$$

where

$$D_0 = U(x_0, \frac{1}{L_0}) \cap \Omega$$

or

$$D_0 = U(x_0, \rho_2) \cap \Omega,$$

if

$$\|F'(x_0)^{-1}(F''(w) - F''(x_0))\| \leq K_0\|w - x_0\| \text{ for all } w \in \Omega,$$

is used, instead, where ρ_2 is the minimal positive solution of equation

$$\frac{K}{6}t^3 + \frac{\tau}{2}t^2 - t + \mu = 0.$$

Then, condition

$$\mu \leq \frac{(\tau^2 + 2K)^{\frac{3}{2}} - \tau(\tau^2 + 3K)}{3K}$$

is the corresponding and weaker sufficient convergence criterion. Notice again that

$$K \leq K_1,$$

$$\rho_2 \leq \rho_1,$$

$K_1 = K_1(\Omega)$, $K_0 = K_0(\Omega)$, and $K = K(\Omega, K_0)$. The old estimate in [12] involving the bounds on $\|F'(w)^{-1}F'(x_0)\|$ is

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1 - (\tau\|w - x_0\| + \frac{K_1}{2}\|w - x_0\|^2)},$$

whereas, we use

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1 - (\tau\|w - x_0\| + \frac{K}{2}\|w - x_0\|^2)},$$

which is more precise.

The rest of the paper is organized as follows: Next, Section 2 contains the convergence analysis, whereas the numerical examples and conclusions can be found in Sections 3 and 4, respectively.

2. Majorizing Sequences

We introduce some auxiliary results on scalar majorizing sequences.

Definition 1. Let $\{p_j\}$ be a Banach space valued sequence. Then, a nondecreasing scalar sequence $\{q_j\}$ is majorizing for $\{p_j\}$, if

$$\|p_{j+1} - p_j\| \leq q_{j+1} - q_j \text{ for each } j = 0, 1, 2, \dots$$

So, the convergence of sequence $\{p_j\}$ reduces to studying that of $\{q_j\}$ [14]. Set $H = [0, \infty)$ and $H_0 = [0, b)$ for some $b > 0$.

Let $\mu \geq 0$ be a parameter and $\varphi_0 : H \rightarrow H$, $\varphi : H_0 \rightarrow H$ be continuous and non-decreasing function. We shall use scalar sequences $\{t_j\}$, $\{s_j\}$ and $\{u_j\}$ defined for each $j = 0, 1, 2, \dots$ by $t_0 = 0, s_0 = \mu$

$$\begin{aligned} u_j &= s_j + \frac{\int_0^1 \bar{\varphi}((1-\theta)(s_j - t_j))d\theta(s_j - t_j)}{1 - \varphi_0(t_j)} \\ t_{j+1} &= u_j + \frac{\int_0^1 \bar{\varphi}(s_j - t_j + \theta(u_j - s_j))d\theta(u_j - s_j)}{1 - \varphi_0(t_j)} \\ s_{j+1} &= t_{j+1} + \frac{\int_0^1 \bar{\varphi}(u_j - t_j + \theta(t_{j+1} - u_j))d\theta(t_{j+1} - u_j)}{1 - \varphi_0(t_{j+1})}, \end{aligned} \quad (17)$$

where $\bar{\varphi} = \begin{cases} \varphi_0, & j = 0 \\ \varphi, & j = 1, 2, \dots \end{cases}$

Next, we present a convergence result for a sequence $\{t_j\}$ under very general conditions

Lemma 1. Suppose:

(a) for all $j = 0, 1, 2, \dots$

$$\varphi_0(t_j) < 1 \quad (18)$$

and

$$t_j \leq a \text{ for some } a > 0. \quad (19)$$

(b) Function $\varphi_0 : H \longrightarrow H$ is continuous, increasing and

$$t_j \leq \varphi_0^{-1}(1) \quad (20)$$

for each $j = 0, 1, 2, \dots$. Then, sequences $\{t_j\}, \{s_j\}, \{u_j\}$ converge monotonically to s_{**} , which is their unique upper bound (least).

Proof. (a) By (17)–(19): $0 \leq t_j \leq s_j \leq u_j \leq t_{j+1}$, are bounded by a so, they converge to $s_* \in [\mu, a]$.

(b) By (17) and (20) one has again

$$\lim_{j \rightarrow \infty} t_j = \lim_{j \rightarrow \infty} s_j = \lim_{j \rightarrow \infty} u_j = s_*.$$

□

Remark 1. Conditions (18)–(20) can be replaced by stronger, which however are easier to satisfy. This is why we give alternative criteria (but stronger) that can easier be verified.

We introduce, sequences functions and sequences of functions as follows

$$\begin{aligned} a_j &= \frac{\int_0^1 \bar{\varphi}((1-\theta)(s_j - t_j)) d\theta}{1 - \varphi_0(t_j)}, \\ b_j &= \frac{\int_0^1 \bar{\varphi}(u_j - t_j + \theta(u_j - s_j)) d\theta}{1 - \varphi_0(t_j)}, \\ c_j &= \frac{\int_0^1 \bar{\varphi}(u_j - t_j + \theta(t_{j+1} - u_j)) d\theta}{1 - \varphi_0(t_{j+1})}, \\ \bar{h}_j^{(1)}(t) &= \int_0^1 \varphi((1-\theta)(s_j - t_j)) d\theta + t\varphi_0(t_j) - t, \\ h_j^{(1)}(t) &= \int_0^1 \varphi((1-\theta)t^j \mu) d\theta \\ &\quad + t\varphi_0\left(\frac{1-t^{j+1}}{1-t} \mu\right) - t, \\ h^{(1)}(t) &= \int_0^1 \varphi((1-\theta)\mu) d\theta + t\varphi_0\left(\frac{\mu}{1-t}\right) - t, \\ \bar{h}_j^{(2)}(t) &= \int_0^1 \varphi(s_j - t_j + \theta(u_j - s_j)) d\theta + t\varphi_0(t_j) - t, \\ h_j^{(2)}(t) &= \int_0^1 \varphi((1+\theta t)t^{2j} \mu) d\theta \\ &\quad + t\varphi_0\left(\frac{1-t^{2j+1}}{1-t} \mu\right) - t, \\ h^{(2)}(t) &= \int_0^1 \varphi((1+\theta t)\mu) d\theta + t\varphi_0\left(\frac{\mu}{1-t}\right) - t, \\ \bar{h}_j^{(3)}(t) &= \int_0^1 \varphi(u_j - t_j + \theta(t_{j+1} - u_j)) d\theta + t\varphi_0(t_{j+1}) - t, \\ h_j^{(3)}(t) &= \int_0^1 \varphi((1+t+\theta t^2)t^{2j} \mu) d\theta + t\varphi_0\left(\frac{1-t^{2j+3}}{1-t} \mu\right) - t \end{aligned}$$

and

$$h^{(3)}(t) = \int_0^1 \varphi((1+t+\theta t^2)\mu) d\theta + t\varphi_0\left(\frac{\mu}{1-t}\right) - t.$$

Next, we present a second convergence result for $\{t_j\}$.

Lemma 2. *Suppose:*

There exists parameter $\gamma \in [0, 1)$ such that

$$0 \leq a_0 \leq \gamma, 0 \leq b_0 \leq \gamma, 0 \leq c_0 \leq \gamma, \quad (21)$$

$$0 \leq \varphi_0(t_1) < 1, \quad (22)$$

$$h^{(1)}(\gamma) \leq 0, h^{(2)}(\gamma) \leq 0 \text{ and } h^{(3)}(\gamma) \leq 0. \quad (23)$$

Then, sequences $\{t_j\}$, $\{s_j\}$, $\{u_j\}$ converge to $s_ \in [\mu, s_{**}]$, where $s_{**} = \frac{1}{1-\gamma}\mu$. Moreover, the following estimates hold*

$$0 \leq s_j - t_j \leq \gamma^j(t_j - s_{j-1}) \leq \gamma^{2j}\mu, \quad (24)$$

$$0 \leq u_j - s_j \leq \gamma(s_j - t_j) \leq \gamma^{2j+1}\mu, \quad (25)$$

$$0 \leq t_{j+1} - u_j \leq \gamma(u_j - s_j) \leq \gamma^{2j+2}\mu, \quad (26)$$

$$t_{j+1} \leq \frac{1 - \gamma^{2j+3}}{1 - \gamma}\mu, \quad (27)$$

and

$$t_j \leq s_j \leq u_j \leq t_{j+1} \leq s_{**}. \quad (28)$$

Furthermore, we have $\varphi_0(t_j) < 1$ for each $j = 0, 1, 2, \dots$

Proof. Estimates (24)–(28) hold if

$$0 \leq a_m \leq \gamma, \quad (29)$$

$$0 \leq b_m \leq \gamma, \quad (30)$$

$$0 \leq c_m \leq \gamma, \quad (31)$$

$$\varphi_0(t_{m+1}) < 1 \quad (32)$$

and

$$t_m \leq s_m \leq u_m \leq t_{m+1} \quad (33)$$

hold for each $m = 0, 1, 2, \dots$. However, they are true for $m = 0$, by (21)–(23). Notice that we have by the definition (17) and these conditions that

$$\begin{aligned} t_{m+1} &\leq u_m + \gamma^{2m+2}\mu \leq s_m + \gamma^{2m+1}\mu + \gamma^{2m+2}\mu \\ &\leq t_m + \gamma^{2m}\mu + \gamma^{2m+1}\mu + \gamma^{2m+2}\mu \\ &\dots \mu + \gamma\mu + \dots + \gamma^{2m+2}\mu \\ &= \frac{1 - \gamma^{2m+3}}{1 - \gamma}\mu \leq \frac{\mu}{1 - \gamma} = s_{**}. \end{aligned}$$

Suppose, estimates (24)–(28) hold for all integers smaller or equal to k . Hence, by replacing t_0, s_0, u_0, t_1 by t_m, s_m, u_m, t_{m+1} and using the induction hypotheses we see that (29)–(33) shall be true if

$$\bar{h}_j(\gamma)^{(i)} \leq 0$$

or

$$h_j(\gamma)^{(i)} \leq 0$$

or

$$h(\gamma)^{(i)} \leq 0,$$

for $i = 1, 2, 3$ and $j = 0, 1, 2, \dots, m$, which holds true by (23). The induction for (24)–(28) is terminated. The remaining of the proof can be found in Lemma 1. $\square$

Remark 2. (a) The conditions of Lemma 2 imply those of Lemma 1 but not necessarily vice versa.
(b) Consider functions “ φ ” to be given by φ_0 and φ in the interesting case $\varphi_0(t) = L_0 t$ and $\varphi(t) = Lt$ for $L_0 > 0$, and $L > 0$.

Then, consider functions $f_j^{(i)}$ on $[0, 1)$ given by

$$\begin{aligned} f_j^{(1)}(t) &= \frac{1}{2}t^{2j-1}\mu \\ &\quad + L_0(1 + t + \dots + t^{2j})\mu - 1, \end{aligned}$$

$$\begin{aligned} f_j^{(2)}(t) &= L(1 + \frac{1}{2}t)t^{2j-1}\mu \\ &\quad + L_0(1 + t + \dots + t^{2j})\mu - 1, \end{aligned}$$

$$\begin{aligned} f_j^{(3)}(t) &= L(1 + t + \frac{t^2}{2})t^{2j-1}\mu \\ &\quad + L_0(1 + t + \dots + t^{2j+2})\mu - 1, \end{aligned}$$

$$g_1(t) = L_0 t^3 + (L_0 + \frac{L}{2})t^2 - \frac{L}{2},$$

$$g_2(t) = (L_0 + \frac{L}{2})t^3 + (L_0 + L)t^2 - \frac{L}{2}t - L$$

and

$$g_3(t) = L_0 t^5 + (L_0 + \frac{L}{2})t^4 + Lt^3 + \frac{L}{2}t^2 - Lt - L.$$

By these definitions, we have

$$g_1(0) = -\frac{L}{2} < 0, \quad g_1(1) = 2L_0 > 0,$$

$$g_2(0) = -L, \quad g_2(1) = 2L_0$$

$$g_3(0) = -L \text{ and } g_3(1) = 2L_0.$$

It follows from the intermediate value theorem (IVT) that functions g_i have zeros in $(0, 1)$. Denote the minimal such zeros by γ_i , respectively.

Define parameters

$$\lambda_0 = \max\{a_0, b_0, c_0\}, \quad \lambda_1 = \min\{\gamma_1, \gamma_2, \gamma_3\} \quad (34)$$

and

$$\lambda_2 = \max\{\gamma_1, \gamma_2, \gamma_3\}.$$

Then, we can show a third result on the convergence of sequence $\{t_j\}$.

Lemma 3. Suppose:

There exists $\gamma \in [0, 1)$ satisfying

$$\lambda_0 \leq \lambda_1 \leq \gamma \leq \lambda_2 < 1 - L_0\mu. \quad (35)$$

Then, the conclusions of Lemma 2 for a sequence $\{t_j\}$ follow.

Proof. We must show by Lemma 2

$$h_m^{(i)}(\gamma) \leq 0. \quad (36)$$

But by the preceding definitions, we can show instead

$$f_m^{(i)}(\gamma) \leq 0. \quad (37)$$

We must relate $f_{m+1}^{(i)}(t)$ to $f_m^{(i)}$. We can write

$$\begin{aligned} f_{m+1}^{(i)}(t) &= \frac{L}{2}t^{2m+1} + L_0(1+t+\dots+t^{2m+2})\mu - 1 \\ &\quad - \frac{L}{2}t^{2m-1}\mu - L_0(1+t+\dots+t^{2m})\mu + 1 + f_m^{(1)}(t) \\ &= f_m^{(1)}(t) + \left(\frac{L}{2}t^2 + L_0(t^2+t^3) - \frac{L}{2}\right)t^{2m-1}\mu \\ &= f_m^{(1)}(t) + g_1(t)t^{2m-1}\mu, \end{aligned}$$

so

$$f_{m+1}^{(1)}(t) = f_m^{(1)}(t) + g_1(t)t^{2m-1}\mu. \quad (38)$$

In particular, by (38) and the definition of γ

$$f_{m+1}^{(1)}(\gamma) \leq f_m^{(1)}(\gamma). \quad (39)$$

Define function $f_\infty^{(1)}$ by

$$f_\infty^{(1)}(t) = \lim_{m \rightarrow \infty} f_m^{(1)}(t). \quad (40)$$

Then, we have by (40)

$$f_\infty^{(1)}(t) = \frac{L_0\mu}{1-t} - 1. \quad (41)$$

So, we can show instead of (37) (for $i = 1$) that

$$f_\infty^{(1)}(\gamma) \leq 0, \quad (42)$$

which is true by (35). Similarly, we get

$$\begin{aligned} f_{m+1}^{(2)}(t) &= L\left(1+\frac{t}{2}\right)t^{2m+1}\mu + L_0(1+t+\dots+t^{2m+2})\mu - 1 \\ &\quad - L\left(1+\frac{t}{2}\right)\left(1+\frac{t}{2}\right)t^{2m-1}\mu - L_0(1+t+\dots+t^{2m})\mu + 1 + f_m^{(2)}(t) \\ &= f_m^{(2)}(t) + \left[\left(1+\frac{t}{2}\right)Lt^2 - \left(1+\frac{t}{2}\right)L + L_0(t^2+t^3)\right]t^{2m-1}\mu \\ &= f_m^{(2)}(t) + g_2(t)t^{2m-1}\mu, \end{aligned}$$

so

$$f_{m+1}^{(2)}(t) = f_m^{(2)}(t) + g_2(t)t^{2m-1}\mu.$$

In particular, we have

$$f_{m+1}^{(2)}(\gamma) \leq f_m^{(2)}(\gamma),$$

and again

$$f_\infty^{(2)}(t) = \frac{L_0\mu}{1-t} - 1.$$

Therefore, (37) (for $i = 2$) reduces to showing

$$f_{\infty}^{(2)}(\gamma) \leq 0,$$

which is true by (35). Moreover, we have analogously

$$\begin{aligned} f_{m+1}^{(3)}(t) &= L(1+t+\frac{t^2}{2})t^{2m+1}\mu + L_0(1+t+\dots+t^{2m+1})\mu - 1 \\ &\quad - L(1+t+\frac{t^2}{2})t^{2m-1}\mu - L_0(1+t+\dots+t^{2m+2})\mu + 1 + f_m^{(3)}(t) \\ &= f_m^{(3)}(t) + [L(1+t+\frac{t^2}{2})t^2 \\ &\quad + L_0(t^4+t^5) - L(1+t+\frac{t^2}{2})]t^{2m-1}\mu \\ &= f_m^{(3)}(t) + g_3(t)t^{2m-1}\mu, \end{aligned}$$

so

$$f_{m+1}^{(3)}(\gamma) \leq f_m^{(3)}(\gamma),$$

Hence, we can show instead of (37) (for $i = 3$) that

$$f_{\infty}^{(3)}(\gamma) \leq 0,$$

which is true by (34), where

$$f_{\infty}^{(3)}(t) = \lim_{m \rightarrow \infty} f_m^{(3)}(t) = \frac{L_0\mu}{1-t} - 1.$$

Therefore, sequence $\{t_j\}$ is nondecreasing and bounded from above by $s_{**} = \frac{1}{1-\gamma}\mu$, so it converges to s_* . $\square$

Next, we connect Lemmas 1 and 2 to method (3). We first consider conditions (A):
Suppose

(A1) There exists $x_0 \in \Omega$, $\mu \geq 0$ such that $F'(x_0)^{-1} \in L(E_2, E_1)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \mu.$$

(A2) For all $w \in \Omega$

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \leq \varphi_0(\|w - x_0\|).$$

(A3) Function $\varphi_0(t) - 1$ has a smallest positive solution ρ . Set

$$\Omega_0 = U(x_0, \rho) \cap \Omega.$$

(A4) For each $w, v \in \Omega_0$

$$\|F'(x_0)^{-1}(F'(w) - F'(v))\| \leq \varphi(\|w - v\|)$$

(A5) Hypotheses of Lemma 1 or Lemma 2 or Lemma 3 hold and

(A6) $U[w_0, s_*] \subset \Omega$ (or $U[w_0, s_{**}] \subset \Omega$).

Next, we prove the first semi-local convergence theorem for sequence $\{w_j\}$.

Theorem 1. Suppose hypotheses (A) hold. Then, sequences $\{w_j\}$ produced by method (3) is well defined in $U[w_0, s_*]$, remains in $U[w_0, s_*]$ for each $j = 0, 1, 2, \dots$ and converges to a solution $w_* \in U[w_0, s_*]$ (or $w_* \in U[w_0, s_{**}]$) of equation $F(w) = 0$.

Proof. Using condition (A1) and the first substep of method (3) for $j = 0$, we see that y_0 is well defined and

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \mu = s_0 - t_0 = s_0 \leq s_*,$$

so $y_0 \in U(x_0, s_*)$. Iterate z_0 exists by (A1) and (3) for $j = 0$. So, by (3) and (A3) one has

$$\begin{aligned} \|z_0 - y_0\| &= \|F'(x_0)^{-1}F(y_0)\| \\ &= \|F'(x_0)^{-1}(F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0))\| \\ &= \left\| \int_0^1 F'(x_0)^{-1}(F'(x) - F'(x_0 + \theta(y_0 - x_0)))d\theta(y_0 - x_0) \right\| \\ &\leq \int_0^1 \bar{\varphi}((1 - \theta)\|y_0 - x_0\|)d\theta\|y_0 - x_0\| \\ &\leq \int_0^1 \bar{\varphi}((1 - \theta)(s_0 - t_0))d\theta(s_0 - t_0) \leq u_0 - s_0. \end{aligned}$$

We also have $\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq u_0 - s_0 + s_0 - t_0 = u_0 - t_0 < s_*$, so $z_0 \in U(x_0, s_*)$. By condition (A1) and (3) for $j = 0$, x_1 exists, and we can write

$$\begin{aligned} x_1 - y_0 &= z_0 - x_0 - F'(x_0)^{-1}(F(z_0) - F(x_0)) \\ &= F'(x_0)^{-1} \int_0^1 [F'(x_0) - F'(x_0 + \theta(z_0 - x_0))]d\theta(z_0 - x_0). \end{aligned}$$

The condition (A2) and (17) give in turn that

$$\begin{aligned} \|x_1 - y_0\| &\leq \int_0^1 \varphi_0((1 - \theta)\|z_0 - x_0\|)d\theta\|z_0 - x_0\| \\ &\leq \frac{\int_0^1 \varphi_0((1 - \theta)\|z_0 - x_0\|)d\theta\|z_0 - x_0\|}{1 - \varphi_0(\|x_0 - x_0\|)} \\ &\leq \frac{\int_0^1 \varphi_0((1 - \theta)(u_0 - t_0))d\theta(u_0 - t_0)}{1 - \varphi_0(t_0)} \\ &= t_1 - s_0. \end{aligned} \tag{43}$$

We also have

$$\begin{aligned} \|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \\ &\leq t_1 - s_0 + s_0 - t_0 = t_1 \leq s_*, \end{aligned}$$

so $w_1 \in U(w_0, s_*)$. Let $w \in U(w_0, s_*)$. Using (A2) one obtains

$$\|F'(w_0)^{-1}(F'(w) - F'(w_0))\| \leq \varphi_0(\|w - w_0\|) \leq \varphi_0(s_*) < 1,$$

so the Banach lemma for linear invertible operators [5] assures the existence of $F'(w)^{-1}$ and

$$\|F'(w)^{-1}F'(w_0)\| \leq \frac{1}{1 - \varphi_0(\|w - w_0\|)}. \tag{44}$$

In particular for $w = w_1$, $F'(w_1)^{-1}$ exists, so does iterate y_1 . Then, we can write by the first substep of method (3) for $k = 1$

$$\begin{aligned}\|y_1 - x_1\| &= \|F'(x_1)^{-1}F(x_1)\| \\ &\leq \|F'(x_1)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_1)\| \\ &\leq \frac{\|\int_0^1 F'(x_0)^{-1}(F'(z_0 + \theta(x_1 - z_0)) - F'(x_0))d\theta(x_1 - z_0)\|}{1 - \varphi_0(\|x_1 - x_0\|)} \\ &\leq \frac{\int_0^1 \varphi_0(\|z_0 - x_0\| + \theta\|x_1 - z_0\|)d\theta\|x_1 - z_0\|}{1 - \varphi_0(\|x_1 - x_0\|)} \\ &\leq \frac{\int_0^1 \bar{\varphi}(u_0 - t_0 + \theta(t_1 - u_0))d\theta(t_1 - u_0)}{1 - \varphi_0(t_1)},\end{aligned}$$

where we also used by the definition of the method

$$\begin{aligned}\|F'(x_0)^{-1}F(x_1)\| &= \|F'(x_0)^{-1}(F(x_1) - F(z_0) + F(z_0))\| \\ &\leq \int_0^1 \|F'(x_0)^{-1}(F'(z_0 + \theta(x_1 - z_0)) - F'(x_0))d\theta(x_1 - z_0)\| \\ &\leq \int_0^1 \varphi_0(\|z_0 - x_0\| + \theta\|x_1 - z_0\|)d\theta\|x_1 - z_0\| \\ &\leq \int_0^1 \bar{\varphi}(u_0 - t_0 + \theta(t_1 - u_0))d\theta(t_1 - u_0) \\ &= s_1 - t_1,\end{aligned}\tag{45}$$

and

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 - t_0.$$

Hence, we showed so far

$$\|y_m - x_m\| \leq s_m - t_m, \quad m = 0, 1,\tag{46}$$

$$\|z_m - y_m\| \leq u_m - s_m, \quad m = 0,\tag{47}$$

$$\|x_{m+1} - z_m\| \leq t_{m+1} - u_m, \quad m = 0,\tag{48}$$

and

$$x_m, y_m, z_m, x_{m+1} \in U(x_0, s_*)\tag{49}$$

for $m = 0$. Consider these estimates are true for all $m \leq j - 1$. Then, simply replace x_0, y_0, z_0, x_1 by x_m, y_m, z_m, x_{m+1} , to terminate the induction for items (46)–(49). So, $\{x_j\}$ is fundamental in a Banach space E_1 . Hence, $\lim_{j \rightarrow \infty} x_j = x^* \in U[x_0, s_*]$. Then, by letting $j \rightarrow \infty$ in the estimate (see also (46))

$$\|F'(x_0)^{-1}F(x_{j+1})\| \leq \int_0^1 \varphi(u_j - t_j + \theta(t_{j+1} - u_j))d\theta(t_{j+1} - u_j),$$

and the continuity of F , we conclude $F(x_*) = 0$. $\square$

A uniqueness of the solution result follows.

Proposition 1. Suppose:

- (i) There exists a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.

(ii) There exists $\alpha \geq s_*$ such that

$$\int_0^1 \varphi_0((1-\theta)\alpha + \theta s_*) d\theta < 1.$$

Set $\Omega_1 = U[x_0, \alpha] \cap \Omega$. Then, x^* is unique in Ω_1 .

Proof. Let $w^* \in \Omega_1$ with $F(w^*) = 0$. Set $M = \int_0^1 F'(w^* + \theta(x^* - w^*)) d\theta$. Then, in view of (A2), we obtain in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(M - F'(x_0))\| &\leq \int_0^1 \varphi_0((1-\theta)\alpha + \theta s_*) d\theta \\ &\leq \int_0^1 \varphi_0((1-\theta)\alpha + \theta s_*) d\theta < 1, \end{aligned}$$

so $w^* = x_*$ follows from the invertability of M and the estimate $M(x_* - w^*) = F(x_*) - F(w^*) = 0 - 0 = 0$. $\square$

3. Examples

We present examples to further justify the theoretical results.

Example 1. Consider

$$\psi(t) = b_0 t + b_1 + b_2 \sin b_3 t, \quad t_0 = 0,$$

where $b_j, j = 0, 1, 2, 3$ are parameters. Then, clearly for b_3 large and b_2 small, $\frac{L_0}{L_1}$ can be small (arbitrarily). Notice that as $\frac{L_0}{L_1} \rightarrow 0, \frac{T}{T_1} \rightarrow 0$.

Example 2. If $E_1 = E_2 = \mathbb{R}, x_0 = 1$ and $\Omega = U[1, 1 - p]$ for $p \in (0, \frac{2}{3})$, define polynomial ψ on Ω as

$$\psi(t) = t^3 - p.$$

If we consider case 1 of Newton's method, then, we obtain $L_0 = 3 - p, L = L_1 = 2(2 - p)$ and $\mu = \frac{1}{3}(1 - p)$. But then, $T_1 > \frac{1}{2}$ for all $p \in (0, \frac{1}{2})$. So, Theorem 5.2 in [14] cannot assure convergence. However, we have $T \leq \frac{1}{2}$ for all $p \in I = [0.4271907643, \frac{1}{2})$. Hence, our result guarantees convergence to $x^* = \sqrt[3]{p}$ as long as $p \in I$.

Example 3. Let $E_1 = E_2 = H([0, 1])$ the domain of functions given on $[0, 1]$ which are continuous. We consider the max-norm. Choose $\Omega = U(0, d), d > 1$. Define G on Ω be

$$G(x)(s) = x(s) - w(s) - \delta \int_0^1 N(s, t) x^3(t) dt, \quad (50)$$

$x \in E_1, s \in [0, 1], w \in E_1$ is given, δ is a parameter and N is the Green's kernel given by

$$N(b_2, b_1) = \begin{cases} (1 - b_2)b_1, & b_1 \leq b_2 \\ b_2(1 - b_1), & b_2 \leq b_1. \end{cases}$$

By (50), we have

$$(G'(x)(z))(s) = z(s) - 3\delta \int_0^1 N(s, t) x^2(t) z(t) dt,$$

$t \in E_1, s \in [0, 1]$. Consider $x_0(s) = w(s) = 1$ and $|\delta| < \frac{8}{3}$. We get

$$\|I - G'(x_0)\| < \frac{3}{8}|\delta|, \quad G'(x_0)^{-1} \in L(E_2, E_1),$$

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\delta|}, \quad \mu = \frac{|\delta|}{8 - 3|\delta|}, \quad L_0 = \frac{12|\delta|}{8 - 3|\delta|},$$

and $L_1 = L = \frac{6\mu|\delta|}{8-3|\delta|}$.

In Table 1 that follows we have listed the results on the convergence criteria for various values of the parameter involved.

Table 1. Comparison table of criteria.

μ	δ^*	T_1	T
2.09899	0.9976613778	1.007515200	0.9639223786
2.19897	0.9831766058	1.055505600	0.9678118280
2.29597	0.9698185659	1.102065600	0.9715205068
3.095467	0.87963113211	1.485824160	1.000082409

Example 4. Let E_1, E_2 , and Ω be as in the Example 3. It is well known that the boundary value problem [16]

$$\begin{aligned}\psi(0) &= 0, (1) = 1, \\ \psi'' &= -\psi - \ell\psi^2\end{aligned}$$

can be presented as a Hammerstein-like nonlinear integral equation [12]

$$\psi(s) = s + \int_0^1 K(s, t)(\psi^3(t) + \ell\psi^2(t))dt$$

for ℓ being a parameter, consider $F : \Omega \rightarrow E_2$ given by

$$[F(x)](s) = x(s) - s - \int_0^1 K(s, t)(x^3(t) + \ell x^2(t))dt.$$

Choose $\psi_0(s) = s$ and $\Omega = U(\psi_0, \rho_0)$. Then, clearly $U(\psi_0, \rho_0) \subset U(0, \rho_0 + 1)$, since $\|\psi_0\| = 1$. Suppose $2\ell < 5$. Then, conditions (A) are satisfied for

$$L_0 = \frac{2\ell + 3\rho_0 + 6}{8}, L_1 = L = \frac{\ell + 6\rho_0 + 3}{4}$$

and $\mu = \frac{1+\ell}{5-2\ell}$. Notice that $L_0 < L_1$.

4. Conclusions

Two different techniques and a new domain D included in the original one are introduced. This change in the analysis gives a finer convergence with no additional conditions.

Author Contributions: Conceptualization, S.R., C.I.A., I.K.A. and S.G.; methodology, S.R., C.I.A., I.K.A. and S.G.; software, S.R., C.I.A., I.K.A. and S.G.; validation, S.R., C.I.A., I.K.A. and S.G.; formal analysis, S.R., C.I.A., I.K.A. and S.G.; investigation, S.R., C.I.A., I.K.A. and S.G.; resources, S.R., C.I.A., I.K.A. and S.G.; data curation, S.R., C.I.A., I.K.A. and S.G.; writing—original draft preparation, S.R., C.I.A., I.K.A. and S.G.; writing—review and editing, S.R., C.I.A., I.K.A. and S.G.; visualization, S.R., C.I.A., I.K.A. and S.G.; supervision, S.R., C.I.A., I.K.A. and S.G. project administration, S.R., C.I.A., I.K.A. and S.G.; funding acquisition, S.R., C.I.A., I.K.A. and S.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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