

Article

Modeling the Thermomechanical Characteristics of a Heat-Insulated Rod with a Variable Cross-Section

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Abstract

In this study, the thermomechanical behavior of a variable cross-section rod with fixed ends is studied using an analytical method. A rod with a radius that varies quadratically with length is considered, and it is thermally insulated on its side surface. Heat flow is applied to the left end of the rod, and heat exchange with the environment occurs at the right end. Based on the obtained temperature distribution, the thermal strains, stresses, displacements, and total elongation are determined. The results highlight the influence of boundary thermal conditions on the thermomechanical response of the rod, which is of interest for the design and analysis of structural elements operating under conditions of uneven thermal stress.

Keywords: variable cross-section rod; thermal insulation; heat exchange; analytical solution; finite element method; elongation; strain; stress

1. Introduction

Many mechanical engineering objects are rods of variable cross-section that are exposed to various types of heat sources. To ensure the reliable operation of such objects, it is necessary to know the laws of temperature distribution along the length of the rods used and the values of other thermomechanical characteristics, such as elongation, axial force, stress, strain, and displacement.

Due to increased stresses in narrowed areas, a rod with a variable cross-section—with the same length and weight and under the influence of temperature—elongates more than a rod with a constant cross-section [1–4]. Therefore, when creating devices that use rod elongation upon heating, it is economically advantageous to use a rod with a variable cross-section, as the desired result can be achieved using a smaller amount of material. Such devices include dilatometric and bimetallic thermometers, thermomechanical sensors, thermal expansion compensators, etc. Nuclear power plant (NPP) tubes can also be considered as a variable cross-section rod exposed to high temperature under the condition that the length-to-diameter ratio is >10 [5–7]. Under the same conditions, the pipes of thermal power plants (TPPs) and chimneys can be considered as rods of variable cross-section [8] with internal heat sources.

A review of published research shows that a sufficient number of studies have been devoted to studying the thermomechanical characteristics of rods using various methods. It is relatively easy to arrive at an analytical decision in the case of a rod with a constant cross-section. Thus, the authors of [9,10] considered analytical thermal conductivity solutions



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for a rod with a constant cross-section and finite length. Heat conduction problems are analyzed when heat flux is supplied to the lateral surface of the rod or when heat exchange with the environment occurs. A power series expansion method is used to solve the same problem [11]. An analytical solution to the thermal conductivity problem of a thin, non-insulated rod embedded in a heated wall was considered by Eckert and Drake [12].

Some authors prefer using analytical and numerical methods in parallel. For example, the authors of [13] utilize a Fourier series analytical solution, while the finite element method was applied as the numerical method. It has been shown that the method of separating variables can be successfully applied in one-dimensional heat transfer problems in a rod [14].

As expected, various numerical methods can be successfully applied to heat transfer problems in rods with both constant and variable cross-sections. The effective application of variational methods, the finite difference method, and the finite element method in such problems has been demonstrated by the authors of [14–16]. For example, the law of temperature distribution along the length of a rod was determined by Arshidinova et al. using a variational solution [10]. The elongation (when one end of the rod is rigidly fixed and the other is free) and the axial compressive force (in the case of both ends of the rod being clamped) were also calculated. Furthermore, the strain, stress, and displacement fields were determined. In the studies of the authors of [17,18], the thermal stress–strain state of a rod with a lateral surface that is completely thermally insulated was numerically investigated. The radius of the rod varies nonlinearly along its length and decreases quadratically along the length of the rod. Heat flow with constant intensity is applied to the left end of the rod, while heat exchange with the environment occurs at the right end. Changes in thermomechanical characteristics depending on changes in heat flow [17] and environmental temperature [18] were also studied.

A review of the literature shows that existing sources do not yet provide an analytical solution for thermal conductivity problems in a heat-insulated rod with a quadratic envelope, where heat flux is applied at the left end, and heat exchange with the environment occurs at the right end.

In this study, an analytical solution to this problem is obtained. Thermomechanical properties of the rod, such as elongation, stress, strain, and displacement, are determined. The results of the analytical solution are compared with the finite element method.

2. Problem Formulation

2.1. Differential Equation of Thermal Conductivity

Let us consider a rod with a variable cross-section that is thermally insulated on the sides (the radius changes quadratically along the length of the rod $ax^2 + bx + c$), the left end of which is under the influence of heat flow, and heat exchange occurs with the environment at the right end—the temperature of which is T_{env} (Figure 1). It is necessary to determine the thermomechanical characteristics of the rod.

First, we obtain the heat equation for the problem under consideration. To carry this out, we calculate the amount of heat passing through a certain cross-section of the rod—located at a distance x from the left end of the rod—in time $d\tau$. According to Fourier's law, it is equal to the following:

$$-K_{xx} \cdot F \cdot \frac{\partial T}{\partial x} \cdot d\tau, \quad (1)$$

where F denotes the cross-sectional area of the rod at a distance x from its left end;

T denotes temperature;

K_{xx} denotes the thermal conductivity coefficient;

$d\tau$ denotes an infinitely small time interval.

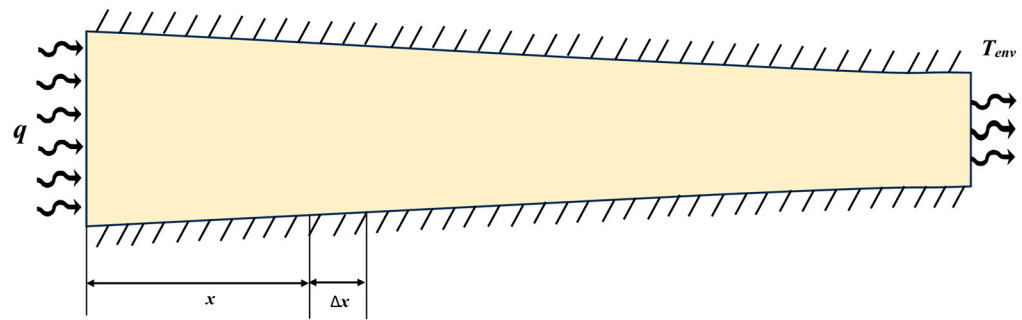


Figure 1. Thermally insulated rod with a variable cross-section.

During time $d\tau$, the amount of heat that passes through the section located at a distance $x + \Delta x$ from the left end of the rod will be equal to the following:

$$-K_{xx}F(x + \Delta x) \frac{\partial T(x + \Delta x)}{\partial x} d\tau. \quad (2)$$

Adding and subtracting $K_{xx}F(x) \frac{\partial T}{\partial x} d\tau$ to this expression, we obtain

$$\begin{aligned} & \left[-K_{xx}F(x + \Delta x) \frac{\partial T(x + \Delta x)}{\partial x} + K_{xx}F(x) \frac{\partial T}{\partial x} - K_{xx}F(x) \frac{\partial T}{\partial x} \right] d\tau \\ &= - \left\{ \left[K_{xx}F(x + \Delta x) \frac{\partial T(x + \Delta x)}{\partial x} - K_{xx}F(x) \frac{\partial T}{\partial x} \right] + K_{xx}F(x) \frac{\partial T}{\partial x} \right\} d\tau \\ &= - \left\{ K_{xx} \frac{\partial}{\partial x} \left[F(x) \frac{\partial T}{\partial x} \right] dx + K_{xx}F(x) \frac{\partial T}{\partial x} d\tau \right\} \end{aligned} \quad (3)$$

Then, the section of the rod enclosed between the sections located at distances x and $x + \Delta x$ from the left end of the rod acquires, over time $d\tau$, an amount of heat equal to the difference between expressions (1) and (3) due to heat conduction:

$$\begin{aligned} & -K_{xx}F(x) \frac{\partial T}{\partial x} d\tau + K_{xx} \frac{\partial}{\partial x} \left[F(x) \frac{\partial T}{\partial x} \right] dx d\tau + K_{xx}F(x) \frac{\partial T}{\partial x} d\tau \\ &= K_{xx} \frac{\partial}{\partial x} \left[F(x) \frac{\partial T}{\partial x} \right] dx d\tau = K_{xx} \left[\frac{dF}{dx} \frac{\partial T}{\partial x} + F(x) \frac{\partial^2 T}{\partial x^2} \right] dx d\tau. \end{aligned} \quad (4)$$

On the other hand, the amount of heat that must be supplied to a homogeneous body to increase its temperature by ΔT is proportional to the volume V and the density ρ :

$$\Delta Q = c\rho V \Delta T = c\rho F(x) \Delta T \Delta x, \quad (5)$$

where c denotes the specific heat capacity of a body.

Equating expressions (4) and (5), we obtain the following:

$$K_{xx} \left[\frac{dS}{dx} \frac{\partial T}{\partial x} + F(x) \frac{\partial^2 T}{\partial x^2} \right] dx dt = c\rho F(x) \frac{\partial T}{\partial t}$$

or

$$K_{xx} \left[\frac{dS}{dx} \frac{\partial T}{\partial x} + F(x) \frac{\partial^2 T}{\partial x^2} \right] dx dt = c\rho F(x) \frac{\partial T}{\partial t}, \quad a = \frac{K_{xx}}{c\rho}.$$

When the process is stationary, this equation takes the following form:

$$\frac{dS}{dx} \frac{dT}{dx} + F(x) \frac{d^2 T}{dx^2} = 0.$$

Multiplying both sides of this equation by $\frac{1}{F(x)}$, the differential equation for steady-state heat conduction can be written as follows:

$$\frac{d^2T}{dx^2} + \frac{dT}{dx} \left(\frac{1}{F(x)} \frac{dF(x)}{dx} \right) = 0. \quad (6)$$

2.2. Boundary Conditions

The boundary conditions for our problem are as follows:

$$K_{xx} \frac{dT}{dx} \Big|_0 + q = 0, \quad (7)$$

$$K_{xx} \frac{dT}{dx} \Big|_L + h(T - T_{env}) = 0. \quad (8)$$

3. Analytical Solution of the Problem

3.1. Temperature Distribution

To analytically solve the differential Equation (6) with boundary conditions (7) and (8), we change some variables: $y = \frac{dT}{dx}$. Then, Equation (6) takes the following form:

$$\frac{dy}{dx} + y \left(\frac{1}{F(x)} \frac{dF(x)}{dx} \right) = 0. \quad (9)$$

Since the cross-sectional radius of the rod changes according to the quadratic equation $ax^2 + bx + c$, the cross-sectional area of the rod changes according to the following formula:

$$F(x) = \pi(ax^2 + bx + c)^2.$$

Then, the derivative of the cross-sectional area of the rod with respect to the length of the rod x is as follows:

$$\frac{dF(x)}{dx} = 2\pi(2ax + b) (ax^2 + bx + c).$$

Substituting the obtained value of the derivative into Equation (9), we obtain the following differential equation:

$$\frac{dy}{dx} + \left(\frac{2(2ax + b)}{ax^2 + bx + c} \right) y = 0$$

or

$$\frac{dy}{y} = - \left(\frac{2(2ax + b)}{ax^2 + bx + c} \right) dx.$$

Integrating this equation, we obtain the following:

$$\int \frac{\partial y}{y} = -2 \int \frac{(2ax + b)}{ax^2 + bx + c} dx = -2 \int \frac{d(ax^2 + bx + c)}{ax^2 + bx + c}.$$

From the last integral, we find

$$\ln(y) = -2\ln(ax^2 + bx + c) + \ln(c_1)$$

or

$$\ln(y) = \ln(c_1(ax^2 + bx + c)^{-2}).$$

From here, we determine y :

$$y = c_1(ax^2 + bx + c)^{-2}.$$

Carrying out the reverse substitution of variables ($y = \frac{dT}{dx}$), we return to variable T :

$$\frac{dT}{dx} = y = c_1(ax^2 + bx + c)^{-2}. \quad (10)$$

Integrating this equation with respect to x , we obtain the following:

$$T(x) = \int y dx = c_1 \int (ax^2 + bx + c)^{-2} dx + c_2. \quad (11)$$

Let us calculate the integral as follows:

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{a^2} \int \frac{1}{\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]^2} dx = \frac{1}{a^2} \int \frac{1}{\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]^2} dx.$$

Changing some variables, $t = x + \frac{b}{2a}$, and taking into account $dt = dx$, we have the following:

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{a^2} \int \frac{1}{\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]^2} dx = \frac{1}{a^2} \int \frac{1}{(t^2 + D)^2} dt, \quad D = -\frac{b^2}{4a^2} + \frac{c}{a}.$$

Let us consider three cases: $D > 0$, $D = 0$, and $D < 0$.

1. $D > 0$: Let us denote $d^2 = \left(-\frac{b^2}{4a^2} + \frac{c}{a}\right)$. Then,

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{a^2} \int \frac{1}{(t^2 + d^2)^2} dt.$$

We perform a change of variables, setting $t = d \cdot z$ and taking into account $dt = d \cdot dz$; thus, we have the following:

$$\frac{1}{a^2} \int \frac{1}{(t^2 + d^2)^2} dt = \frac{1}{d^3 a^2} \int \frac{1}{(z^2 + 1)^2} dz.$$

Because

$$\int \frac{1}{(z^2 + 1)^2} dz = \frac{1}{2} \left[\arctg(z) + \frac{z}{1 + z^2} \right] + c_2,$$

we obtain

$$\frac{1}{a^2} \int \frac{1}{(t^2 + d^2)^2} dt = \frac{1}{d^3 a^2} \int \frac{1}{(z^2 + 1)^2} dz = \frac{1}{2d^3 a^2} \left[\arctg(z) + \frac{z}{1 + z^2} \right] + c_2.$$

Carrying out the reverse change of variables, that is, moving from t to x , we have

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{2d^3 a^2} \left[\arctg\left(\frac{x + \frac{b}{2a}}{d}\right) + \frac{\left(x + \frac{b}{2a}\right)d}{d^2 + \left(x + \frac{b}{2a}\right)^2} \right]. \quad (12)$$

Taking into account expression (12), the temperature Equation (11) can be rewritten as follows:

$$T_1(x) = \int c_1(ax^2 + bx + c)^{-2} dx + c_2 = \frac{c_1}{2d^3a^2} \left[\operatorname{arctg}\left(\frac{x + \frac{b}{2a}}{d}\right) + \frac{\left(x + \frac{b}{2a}\right)d}{d^2 + \left(x + \frac{b}{2a}\right)^2} \right] + c_2.$$

By introducing the notation

$$\operatorname{arc}(x) = \left[\operatorname{arctg}\left(\frac{x + \frac{b}{2a}}{d}\right) + \frac{\left(x + \frac{b}{2a}\right)d}{d^2 + \left(x + \frac{b}{2a}\right)^2} \right],$$

the temperature can be represented as follows:

$$T_1(x) = \frac{c_1}{2d^3a^2} \operatorname{arc}(x) + c_2. \quad (13)$$

Using boundary condition (7) and taking into account expression (10) for $x = 0$, we have

$$K_{xx} \frac{\partial T}{\partial x} \Big|_0 = K_{xx} \frac{c_1}{c^2} = -.$$

From this equation, we find c_1 :

$$c_1 = -\frac{qc^2}{K_{xx}}.$$

Taking into account the second boundary condition (8), we have the following:

$$K_{xx} \frac{\partial T}{\partial x} \Big|_L = K_{xx} \frac{c_1}{(aL^2 + bL + c)^2} + h \left[\frac{c_1}{2d^3a^2} \operatorname{arc}(L) + c_2 - T_{env} \right] = 0.$$

From this equation, we obtain c_2 :

$$c_2 = -K_{xx} \frac{c_1}{h(aL^2 + bL + c)^2} - \frac{c_1}{2d^3a^2} \operatorname{arc}(L) + T_{env}.$$

Substituting the obtained expressions for c_1 and c_2 into Equation (13), we have

$$\begin{aligned} T_1(x) &= \frac{c_1}{2d^3a^2} \operatorname{arc}(x) - \frac{c_1 K_{xx}}{h(aL^2 + bL + c)^2} - \frac{c_1 \operatorname{arc}(L)}{2d^3a^2} + T_{env} = \\ &= \frac{c_1}{2d^3a^2} [\operatorname{arc}(x) - \operatorname{arc}(L)] - \frac{c_1 K_{xx}}{h(aL^2 + bL + c)^2} + T_{env}. \end{aligned} \quad (14)$$

2. $D = 0$: Then, we have

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{a^2} \int \frac{1}{(t^2 + D)^2} dt = \frac{1}{a^2} \int \frac{1}{t^4} dt = -\frac{1}{3a^2} \left(x + \frac{b}{2a}\right)^{-3}.$$

That is why the following is the case:

$$T_2(x) = -\frac{c_1}{3a^2} \left(x + \frac{b}{2a}\right)^{-3} + c_2,$$

Since $\frac{\partial T(x)}{\partial x} \Big|_0 = \frac{c_1}{a^2} \left(x + \frac{b}{2a}\right)^{-4} \Big|_0 = \frac{c_1}{a^2} \left(\frac{b}{2a}\right)^{-4}$, from the boundary condition of the left end of the rod, we have

$$K_{xx} \frac{\partial T}{\partial x} \Big|_0 = K_{xx} \frac{c_1}{a^2} \left(\frac{b}{2a}\right)^{-4} = -q,$$

where

$$c_1 = -\frac{qa^2}{K_{xx}} \left(\frac{b}{2a}\right)^4.$$

From boundary condition (8) for the right end of the rod, we have

$$K_{xx} \frac{c_1}{a^2} \left(L + \frac{b}{2a}\right)^{-4} + h \left(-\frac{c_1}{3a^2} \left(L + \frac{b}{2a}\right)^{-3} + c_2 - T_{env}\right) = 0.$$

From here, we obtain c_2 :

$$c_2 = -K_{xx} \frac{c_1}{a^2 h} \left(L + \frac{b}{2a}\right)^{-4} + \frac{c_1}{3a^2} \left(L + \frac{b}{2a}\right)^{-3} + T_{env}.$$

3. $D < 0$: Let us denote $d = -\left(-\frac{b^2}{4a^2} + \frac{c}{a}\right) > 0$. Then,

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = \frac{1}{a^2} \int \frac{1}{(t^2 - d)^2} dt.$$

Let us calculate the integral $\int \frac{dx}{(t^2 - d)^2}$.

To carry this out, let us consider the two following functions:

$$F(x) = \frac{t}{2d(t^2 - d)} \quad \text{и} \quad G(x) = \frac{1}{4d^{\frac{3}{2}}} \ln \left| \frac{t - \sqrt{d}}{t + \sqrt{d}} \right|.$$

The derivatives of these functions are equal to

$$F'(x) = \frac{-t^2 - d}{2d(t^2 - d)^2}, \quad G'(x) = \frac{1}{2d(t^2 - d)}.$$

Moreover, the sum of the derivatives is equal to

$$F'(x) + G'(x) = -\frac{1}{(t^2 - d)^2}.$$

This means that

$$\int \frac{dx}{(t^2 - t)^2} = -F(x) - G(x) = -\frac{t}{2d(t^2 - d)} - \frac{1}{4d^{\frac{3}{2}}} \ln \left| \frac{t - \sqrt{d}}{t + \sqrt{d}} \right|.$$

Then, we have

$$\int \frac{1}{(ax^2 + bx + c)^2} dx = -\frac{t}{2a^2 d(t^2 - d)} - \frac{1}{4a^2 d^{\frac{3}{2}}} \ln \left| \frac{t - \sqrt{d}}{t + \sqrt{d}} \right|.$$

The temperature in this case is equal to

$$T_3(x) = -\frac{c_1 \left(x + \frac{b}{2a}\right)}{2a^2 d \left(\left(x + \frac{b}{2a}\right)^2 - d\right)} - \frac{c_1}{4a^2 d^{\frac{3}{2}}} \ln \left| \frac{\left(x + \frac{b}{2a}\right) - \sqrt{d}}{\left(x + \frac{b}{2a}\right) + \sqrt{d}} \right| + c_2. \quad (15)$$

The constants of integration c_1 and c_2 are found from boundary conditions (7) and (8). Using boundary condition (7), we have

$$K_{xx} \frac{\partial T}{\partial x} \Big|_0 = K_{xx} \frac{c_1}{c^2} = -q.$$

where

$$c_1 = -\frac{qc^2}{K_{xx}}.$$

Taking constraint (8) into account, we have the following:

$$K_{xx} \frac{c_1}{(aL^2 + bL + c)^2} + h \left\{ -\frac{c_1 \left(L + \frac{b}{2a}\right)}{2a^2 d \left(\left(L + \frac{b}{2a}\right)^2 - d\right)} - \frac{c_1}{4a^2 d^{\frac{3}{2}}} \ln \left| \frac{\left(L + \frac{b}{2a}\right) - \sqrt{d}}{\left(L + \frac{b}{2a}\right) + \sqrt{d}} \right| + c_2 - T_{env} \right\} = 0.$$

Moreover, we obtain the following:

$$c_2 = -K_{xx} \frac{c_1}{h(aL^2 + bL + c)^2} + \left\{ \frac{c_1 \left(L + \frac{b}{2a}\right)}{2a^2 d \left(\left(L + \frac{b}{2a}\right)^2 - d\right)} + \frac{c_1}{4a^2 d^{\frac{3}{2}}} \ln \left| \frac{\left(L + \frac{b}{2a}\right) - \sqrt{d}}{\left(L + \frac{b}{2a}\right) + \sqrt{d}} \right| + T_{env} \right\}.$$

3.2. Rod Extension

If one or both ends of a rod are free, the rod's length changes due to temperature changes. As the temperature increases compared to its initial state, the rod elongates. The magnitude of the rod's elongation, taking into account relationship (13), can be determined using the following expression:

(1) For the case when $D > 0$,

$$L_1 = \alpha \int_0^L T_1(x) dx = \int_0^L \left\{ \frac{c_1}{2d^3 a^2} \left[\operatorname{arctg} \left(\frac{x + \frac{b}{2a}}{d} \right) + \frac{\left(x + \frac{b}{2a}\right)d}{d^2 + \left(x + \frac{b}{2a}\right)^2} \right] + c_2 \right\} dx.$$

Here, α is the coefficient of thermal expansion, $^{\circ}\text{C}^{-1}$.

To calculate the integral of the first term, we use the following well-known formula [19]:

$$\int \operatorname{arctg}(x) dx = x \operatorname{arctg}(x) - \frac{1}{2} \ln(x^2 + 1) + c.$$

Taking this formula into account, we have

$$\begin{aligned} J_{10} &= \int_0^L \operatorname{arctg} \left(\frac{x + \frac{b}{2a}}{d} \right) dx = d \int_0^L \operatorname{arctg} \left(\frac{x + \frac{b}{2a}}{d} \right) d \left(\frac{x + \frac{b}{2a}}{d} \right) = \\ &= \left[\left(x + \frac{b}{2a} \right) \operatorname{arctg} \left(\frac{x + \frac{b}{2a}}{d} \right) - \frac{d}{2} \ln \left(\left(\frac{x + \frac{b}{2a}}{d} \right)^2 + 1 \right) \right]_0^L = \\ &= \left[\left(L + \frac{b}{2a} \right) \operatorname{arctg} \left(\frac{L + \frac{b}{2a}}{d} \right) - \frac{d}{2} \ln \left(\left(\frac{L + \frac{b}{2a}}{d} \right)^2 + 1 \right) \right] - \left[\left(\frac{b}{2a} \right) \operatorname{arctg} \left(\frac{b}{2ad} \right) - \frac{d}{2} \ln \left(\left(\frac{b}{2ad} \right)^2 + 1 \right) \right]. \end{aligned}$$

Let us calculate the integral of the second term ΔL :

$$J_{20} = \int_0^L \left[\frac{\left(x + \frac{b}{2a}\right)d}{d^2 + \left(x + \frac{b}{2a}\right)^2} \right] dx.$$

Let us make a change of variables: $t = x + \frac{b}{2a}$. Since $dx = dt$, we have

$$\begin{aligned} J_{20} &= d \int_{\frac{b}{2a}}^{L + \frac{b}{2a}} \left[\frac{t}{d^2 + t^2} \right] dt = \frac{d}{2} \int_{\frac{b}{2a}}^{L + \frac{b}{2a}} \left[\frac{1}{d^2 + t^2} \right] d(d^2 + t^2) = \frac{d}{2} \ln(d^2 + t^2) \Big|_{\frac{b}{2a}}^{L + \frac{b}{2a}} = \\ &= \frac{d}{2} \ln \left(d^2 + \left(x + \frac{b}{2a}\right)^2 \right) \Big|_0^L = \frac{d}{2} \left[\ln \left(d^2 + \left(\frac{L + \frac{b}{2a}}{d}\right)^2 \right) - \ln \left(d^2 + \left(\frac{b}{2a}\right)^2 \right) \right]. \end{aligned}$$

Finally, the expression for the thermal elongation of the rod can be written as follows:

$$L_1 = \frac{\alpha c_1}{2d^3 a^2} (J_{10} + J_{20}) + \alpha c_2 L;$$

(2) For the case of $D = 0$, we have

$$L_2 = \alpha \int_0^L T_2(x) dx = \frac{\alpha c_1}{6a^2} \left[\left(L + \frac{b}{2a}\right)^{-2} - \left(\frac{b}{2a}\right)^{-2} \right] + \alpha c_2 L.$$

(3) For the case of $D < 0$, we have

$$L_3 = \alpha \int_0^L T_3(x) dx = \alpha \int_0^L \left[-\frac{c_1 \left(x + \frac{b}{2a}\right)}{2a^2 d \left(\left(x + \frac{b}{2a}\right)^2 - d \right)} - \frac{c_1}{4a^2 d^{\frac{3}{2}}} \ln \left| \frac{\left(x + \frac{b}{2a}\right) - \sqrt{d}}{\left(x + \frac{b}{2a}\right) + \sqrt{d}} \right| + c_2 \right] dx$$

The first integral is equal to

$$I_1 = \alpha \int_0^L -\frac{c_1 \left(x + \frac{b}{2a}\right)}{2a^2 d \left(\left(x + \frac{b}{2a}\right)^2 - d \right)} dx = -\frac{\alpha c_1}{4a^2 d} \ln \left| \left(x + \frac{b}{2a}\right)^2 - d \right| \Big|_0^L = -\frac{\alpha c_1}{4a^2 d} \ln \left| \frac{\left(L + \frac{b}{2a}\right)^2 - d}{\left(\frac{b}{2a}\right)^2 - d} \right|$$

The second integral is equal to

$$\begin{aligned} I_2 &= -\frac{\alpha c_1}{4a^2 d^{3/2}} \int_0^L \ln \left| \frac{\left(x + \frac{b}{2a}\right) - \sqrt{d}}{\left(x + \frac{b}{2a}\right) + \sqrt{d}} \right| dx \\ &= -\frac{\alpha c_1}{4a^2 d^{3/2}} \left\{ \left(L + \frac{b}{2a}\right) \ln \left| \frac{\left(L + \frac{b}{2a}\right) - \sqrt{d}}{\left(L + \frac{b}{2a}\right) + \sqrt{d}} \right| - \sqrt{d} \ln \left| \left(L + \frac{b}{2a}\right)^2 - d \right| \right\} \\ &\quad + \frac{\alpha c_1}{4a^2 d^{3/2}} \left\{ \left(\frac{b}{2a}\right) \ln \left| \frac{\frac{b}{2a} - \sqrt{d}}{\frac{b}{2a} + \sqrt{d}} \right| - \sqrt{d} \ln \left| \left(\frac{b}{2a}\right)^2 - d \right| \right\} \end{aligned}$$

Finally, we have

$$\Delta L_3 = I_1 + I_2.$$

3.3. Thermal Stress

If both ends of the rod are not able to move in the longitudinal direction, then due to the increase in temperature compared to the initial state, thermal stress arises in the rod, which can be calculated using the following formula:

$$\sigma_{Ti} = \varepsilon_{Ti} \cdot E, \quad i = 1, 2, 3. \quad (16)$$

where σ_{Ti} denotes thermal stress;

ε_{Ti} denotes the constrained thermal strain;

E denotes the modulus of elasticity of the rod's material.

The constrained thermal strain is determined using the following formula:

$$\varepsilon_{Ti} = \alpha \cdot T_i(x), \quad (17)$$

where α denotes the coefficient of thermal expansion of the rod's material.

In Formulas (16) and (17), the index i takes the values 1, 2, and 3 depending on the parameter D : $D > 0$; $D = 0$; $D < 0$.

Thermal stress is determined using Formula (16).

4. Finite Element Method

In this section, the main (resolving) equation of the finite element method (FEM) for modeling temperatures in the rod is obtained via the variational method.

Let us show that the solution of problem (1) under constraints (2) and (3) is equivalent to minimizing the following functional:

$$J = \int_0^L F(x) \frac{K_{xx}}{2} \left(\frac{dT}{dx} \right)^2 dx - qTF(0) + \frac{F(L)}{2} h(T - T_{env})^2. \quad (18)$$

Let us apply a small increment to the temperature $\hat{T} = T + \varepsilon\eta(x)$, where $\eta(x)$ is some function with a first derivative. Then, the functional J has the following form:

$$J = \int_0^L \left[\frac{F(x)K_{xx}}{2} \left(\frac{d(T + \varepsilon\eta(x))}{dx} \right)^2 \right] dx - q(T + \varepsilon\eta(x))F(0) + \frac{F(L)}{2} h(T + \varepsilon(x)\eta(x) - T_{env})^2.$$

We take the variation of J with respect to ε at point 0. We have

$$\frac{\partial J}{\partial \varepsilon} = \int_0^L \left[F(x)K_{xx} \frac{dT}{dx} \eta'(x) \right] dx - q\eta(x)F(x) + F(x)h(T - T_{env})\eta(x). \quad (19)$$

We integrate the first term via the following parts:

$$\int_0^L \left[F(x)K_{xx} \frac{\partial T}{\partial x} \eta'(x) \right] dx = F(x)K_{xx} \frac{dT}{dx} \eta(x) \Big|_0^L - \int_0^L \left[\left(F(x)K_{xx} \frac{d^2T}{dx^2} + \frac{dF(x)}{dx} K_{xx} \frac{dT}{dx} \right) \eta(x) \right] dx.$$

Substituting the above into (19), we have

$$\begin{aligned} \frac{\partial J}{\partial \varepsilon} = & \int_0^L \left[-F(x)K_{xx} \frac{d^2T}{dx^2} - \frac{dF(x)}{dx} K_{xx} \frac{dT}{dx} \right] \eta(x) dx + \\ & + F(x)K_{xx} \frac{dT}{dx} \eta(x) \Big|_0^L - qF(x)\eta(x) \Big|_0 + F(x)h(T - T_{env})\eta(x) \Big|_L = 0. \end{aligned} \quad (20)$$

In order for (20) to be identically equal to zero, Equation (1) must be satisfied under constraints (2) and (3).

Next, we divide the rod into m elements of equal length and approximate the temperature in the i -th element with a quadratic polynomial [11]:

$$T_i = \left(\frac{2x^2 - 3lx + l^2}{l^2} \right) T_{2i} + \left(\frac{4lx - 4x^2}{l^2} \right) T_{2i+1} + \left(\frac{2x^2 - lx}{l^2} \right) T_{2i+2}, \quad (21)$$

where $l = L/m$.

Let us define the following:

$$\frac{dT_i}{dx} = \left(\frac{4x - 3l}{l^2} \right) T_{2i} + \left(\frac{4l - 8x}{l^2} \right) T_{2i+1} + \left(\frac{4x - l}{l^2} \right) T_{2i+2}, \quad i = \overline{1, m-1}.$$

The number of nodes in this case is equal to $n = 2m + 1$.

Then, functional (18) has the following form:

$$J = \frac{1}{2} K_{xx} \sum_{i=0}^{m-1} \int_0^l \left(\left(\frac{4x - 3l}{l^2} \right) T_{2i} + \left(\frac{4l - 8x}{l^2} \right) T_{2i+1} + \left(\frac{4x - l}{l^2} \right) T_{2i+2} \right)^2 F(x) dx - q T_0 F(0) + \frac{F(L)}{2} h (T_n - T_{env})^2.$$

The minimization of this functional with respect to the variable T_i , $i = \overline{1, n}$, $n = 2m + 1$, was carried out using the Python 3.11 programming tool [20]. The general algorithm for finding the law of temperature distribution along the length of the rod consists of the following steps:

- (1) Find the derivative of a functional:

$$\frac{\partial J}{\partial T_i} = 0, \quad i = \overline{1, n}.$$

This is carried out using the `diff()` function of the Sympy module [21]: `sympy.diff(J, T[i])`, $i = \overline{1, n}$. This results in a system of linear equations of the form

$$A\overline{T} = \overline{B};$$

- (2) We solve the resulting system of equations using the `linalg.solve()` function of the `scipy` module: `scipy.linalg.solve(A, B)`. As a result, we obtain the temperature values at the nodal points T_i^* , $i = \overline{1, n}$;
- (3) Substituting the values of T_i^* ($i = \overline{1, n}$) into the quadratic temperature approximation function, we obtain the law of temperature distribution along the length of the rod.

5. Results and Discussion

5.1. Thermal Verification

Let us consider a thermally insulated copper rod with a variable cross-section. The rod's material characteristics and boundary conditions used in the calculations are given in Tables 1 and 2.

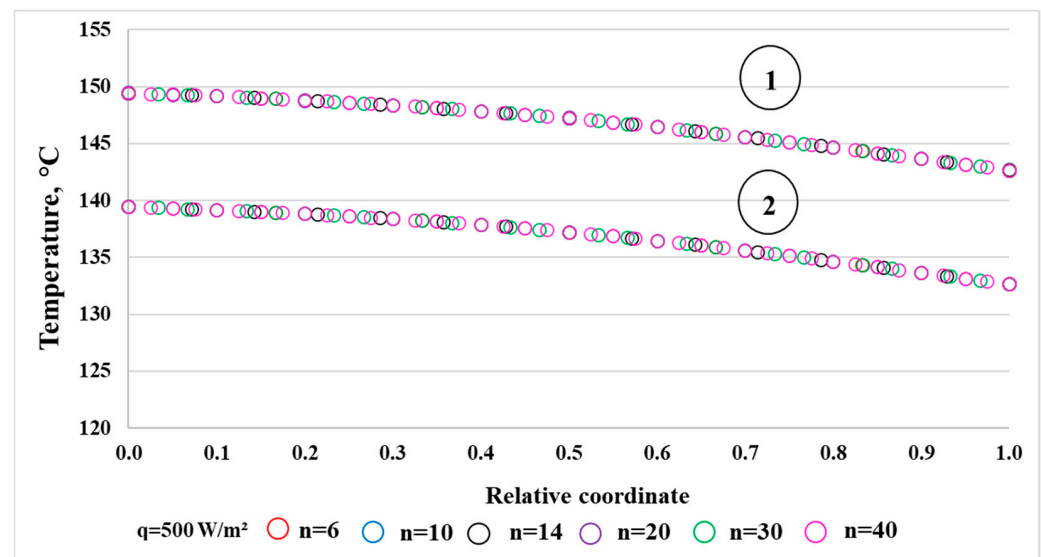
Table 1. Values of geometric characteristics and the modulus of elasticity of the rod material.

Length of Rod L , m	Radius Change Parameters			Modulus of Elasticity E , MPa
	a	b	c	
2.0	0.03	−0.11	0.2	120,000

Table 2. Values of thermophysical characteristics of the rod material and boundary conditions.

Thermophysical Characteristics of the Rod Material		Boundary Conditions Parameters		
K_{xx} , W/(m·°C)	α , 1/°C	q , W/m ²	h , W/(m ² ·°C)	T_{env} , °C
400	1.65×10^{-5}	3350	13.1	−10
400	1.65×10^{-5}	500	13.1	−10

It is well known that the accuracy of calculations using the finite element method also depends on the number of finite elements into which the object under consideration is discretized. To evaluate the accuracy of calculations in our cases, we performed calculations of the temperature distribution in the rod with the following finite element numbers: $n = 6, 10, 14, 20, 30$, and 40 . As Figure 2 shows, the convergence of the calculations is very high. All subsequent calculations were performed with 30 finite elements, although the use of 6 finite elements ensures high calculation accuracy.

**Figure 2.** Comparison of temperature distributions for a rod at $q = 500 \text{ W/m}^2$ obtained via FEM at different FE numbers (n): 1— $T_{env} = -10 \text{ °C}$; 2— $T_{env} = -20 \text{ °C}$.

Figures 3 and 4 show graphs of temperature changes along the rod for two heat flux q values, obtained analytically and via FEM.

As expected, for both high ($q = 3350 \text{ W/m}^2$) and moderate ($q = 500 \text{ W/m}^2$) heat flux values, the highest temperatures—equal to 1048.5 °C and 159.4 °C , respectively—occur at the left end of the rod. The temperature decreases monotonically and nonlinearly along the length of the rod. Moreover, the rate of temperature decrease in the first half of the rod is lower than in the second half. Despite the significant difference in heat flux ($3350/500 = 6.7$), the temperature decrease at the left end of the rod is approximately the same as at the right end (4.6 °C and 6.8 °C , respectively).

These figures also show that the agreement between the temperature values calculated analytically and those obtained using FEM is very high for both high and moderate heat flux values.

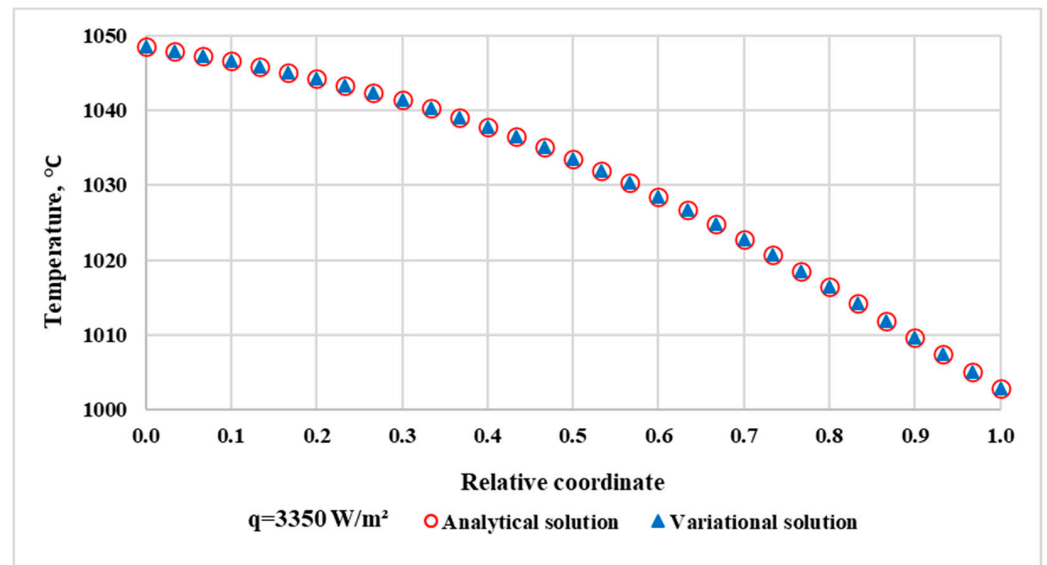


Figure 3. Comparison of temperature distributions for a rod at $q = 3350 \text{ W/m}^2$ obtained via the analytical solution and FEM.

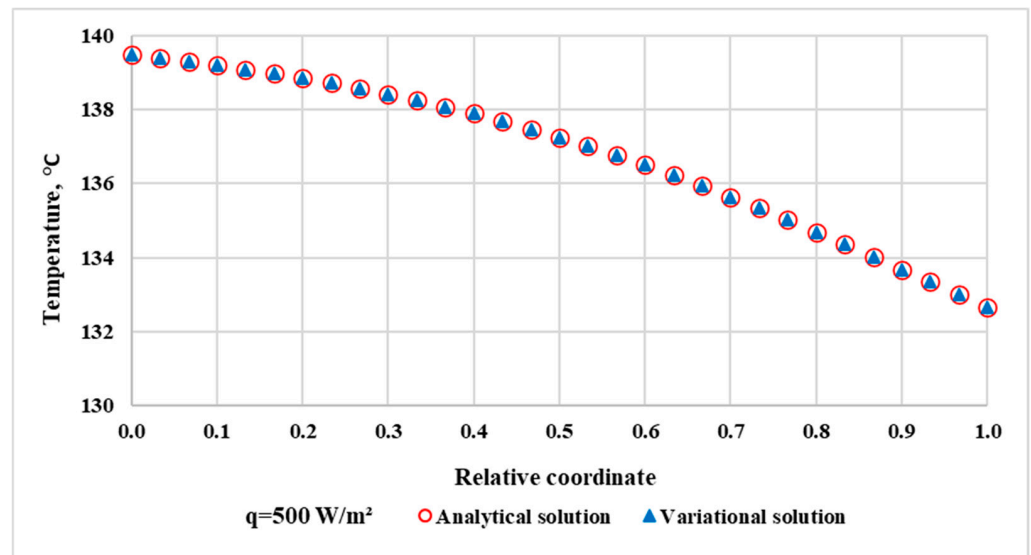


Figure 4. Comparison of temperature distributions for a rod at $q = 500 \text{ W/m}^2$ obtained via the analytical solution and FEM.

5.2. Thermo-Mechanical Response and Parametric Study

In real-world conditions, the environmental temperature at which the second (right) end of the rod is in contact can vary. Therefore, it is important to assess the influence of environmental temperature on the temperature distribution and other mechanical characteristics within the rod.

In further calculations, the mechanical and thermophysical characteristics of the rod were taken to be the same as in Tables 1 and 2, with the only exception that the heat flow supplied to the left end of the rod was taken to be $q = 500 \text{ W/m}^2$, and the environmental temperature values changed from -40°C to $+40^\circ\text{C}$ in 20°C increments.

Figures 5–8 show the graphs of temperature distribution, displacement values, constrained thermal strain, and thermal stress in the rod for the selected environmental temperature values.

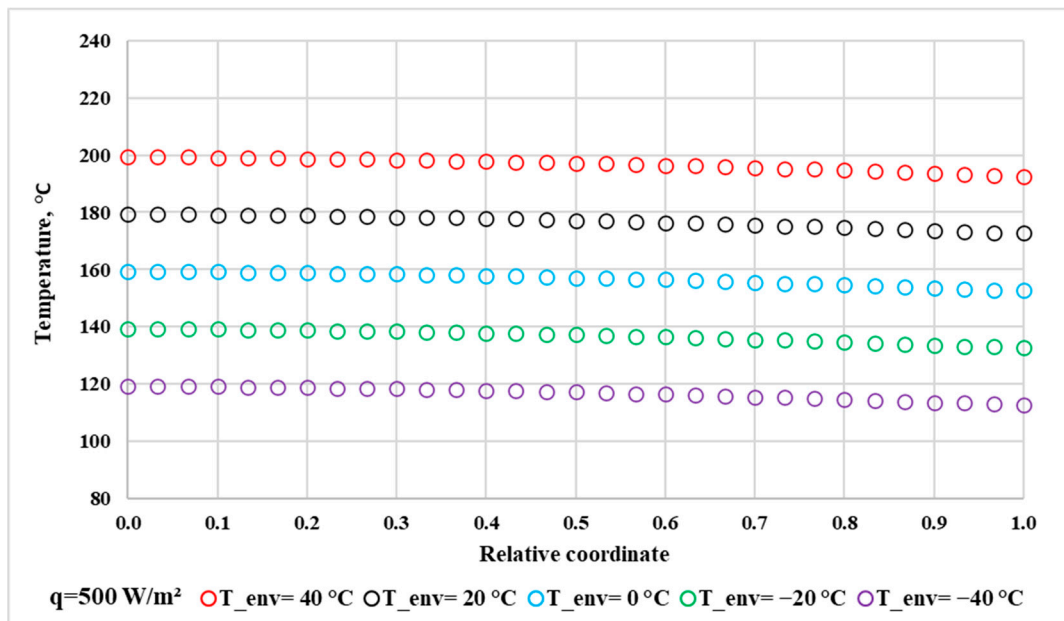


Figure 5. Temperature distribution in a rod at $q = 500\text{ W/m}^2$ and different environmental temperatures, T_{env} .

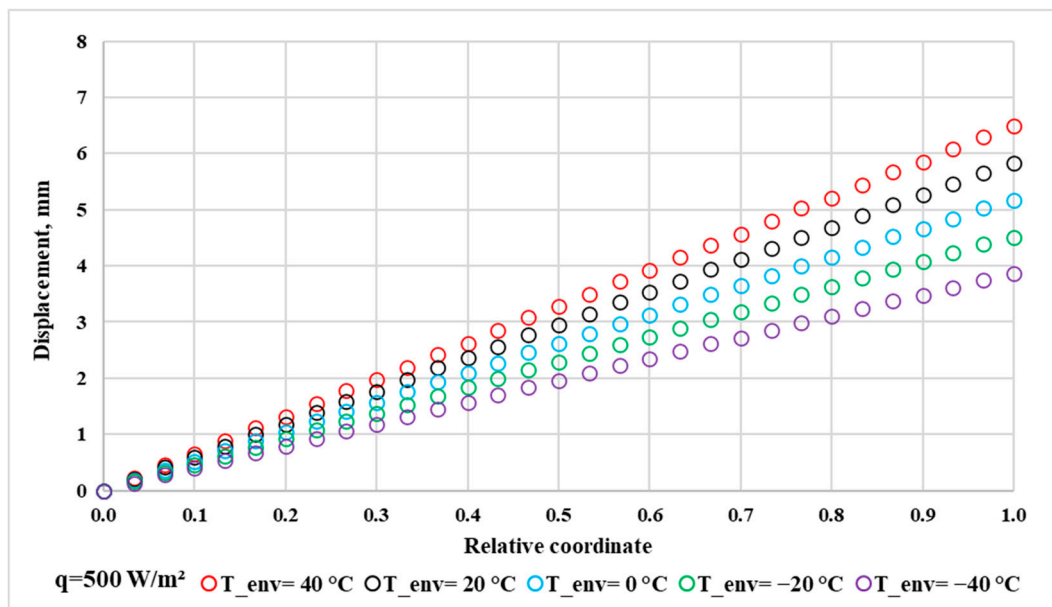


Figure 6. Values of displacement in the rod at $q = 500\text{ W/m}^2$ and different environmental temperatures, T_{env} .

As observed in Figure 5, the temperature distributions in the rod at different environmental temperatures are similar. That is, if the temperature distribution at any one environmental temperature is known, then the temperature distribution at other environmental temperature values can be obtained by simply shifting the known temperature distribution in the vertical direction (up or down). It was observed that the maximum temperature in the rod has a functional relationship with the environmental temperature (Figure 9, $R^2 = 1$). This relationship is linear, with a slope of 1. From this, it becomes clear that a change (increase or decrease) in the environmental temperature causes a similar change (increase or decrease) in the temperature at the left cross-section of the rod. The graph in Figure 9 is a function of the shear.

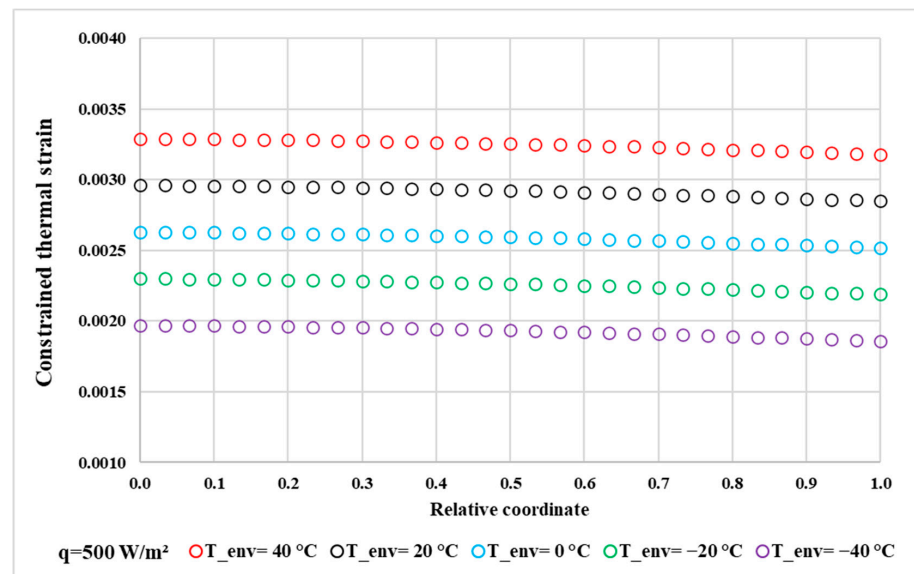


Figure 7. Constrained thermal strain in the rod at $q = 500 \text{ W/m}^2$ and different environmental temperatures, T_{env} .

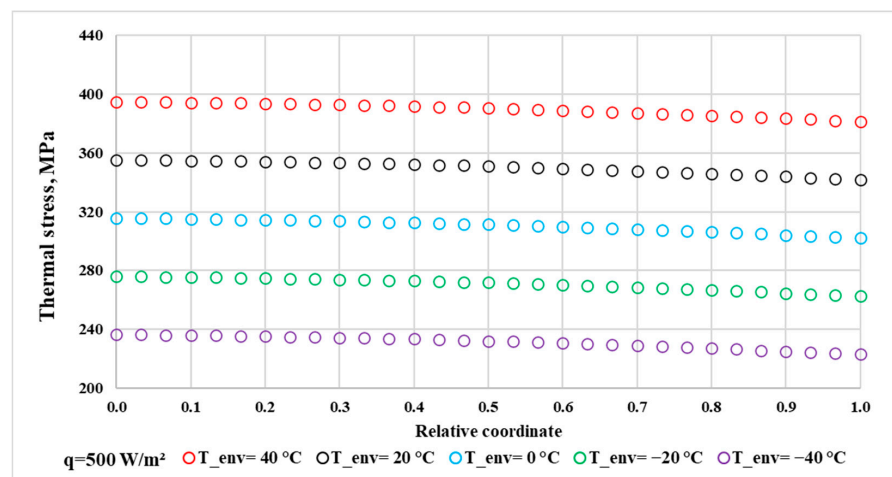


Figure 8. Thermal stress in the rod at $q = 500 \text{ W/m}^2$ and different environmental temperatures, T_{env} .

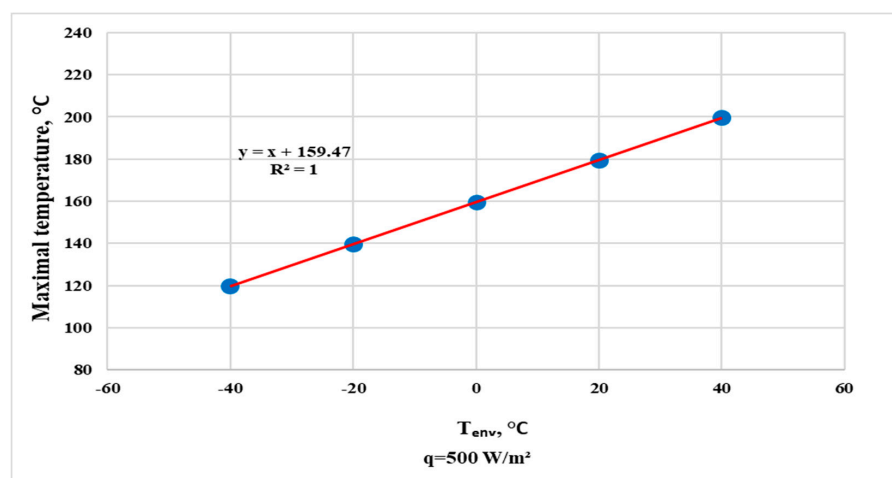


Figure 9. Relationship between the maximum temperature in the rod and the environmental temperature T_{env} at $q = 500 \text{ W/m}^2$.

Constrained thermal strain (Figure 7) and thermal stress (Figure 8) have similar properties. The shear functions for these strains are shown in Figures 10 and 11 respectively.

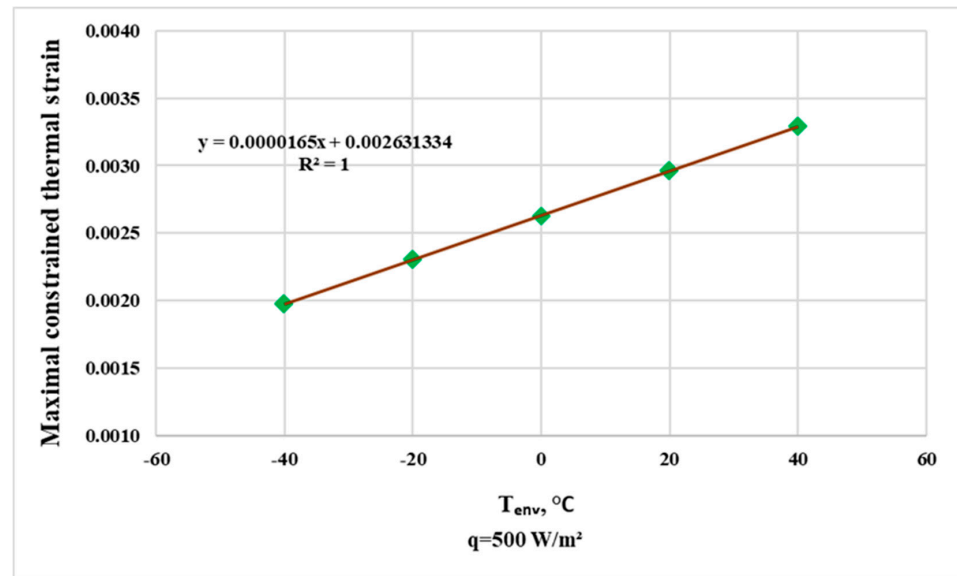


Figure 10. Relationship between the maximum constrained thermal strain in the rod and the environmental temperature T_{env} at $q = 500 \text{ W/m}^2$.

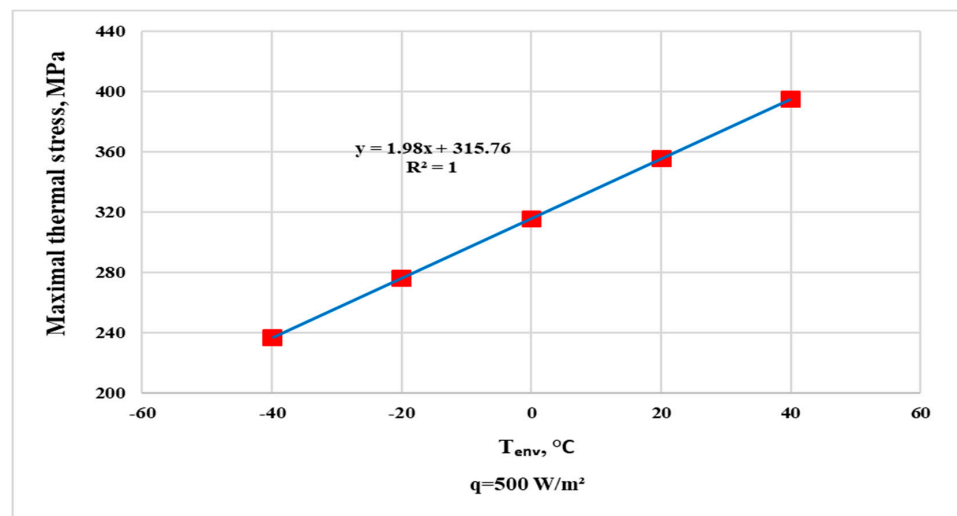


Figure 11. Relationship between the maximum thermal stress in the rod and the environmental temperature T_{env} at $q = 500 \text{ W/m}^2$.

It was observed that with a change in environmental temperature from -40°C to $+40^\circ\text{C}$, the maximum temperature, maximum constrained thermal strain, and maximum stress in the rod increase from 120°C to 200°C , from 0.0020 to 0.0033, and from 240 MPa to 400 MPa, respectively.

As expected, with an increase in rod temperature (due to increasing environmental temperature), the displacements at the rod points and the rod's elongation increase linearly. There is also a functional (linear) relationship between the elongation of the rod and the environmental temperature (Figure 12), which is also a function of the shift in the elongation of the rod at different environmental temperature values (Figure 6). Figures 6 and 12 show that when the environmental temperature increases from -40°C to $+40^\circ\text{C}$, the elongation of the rod ranges from 3.8 mm to 6.5 mm.

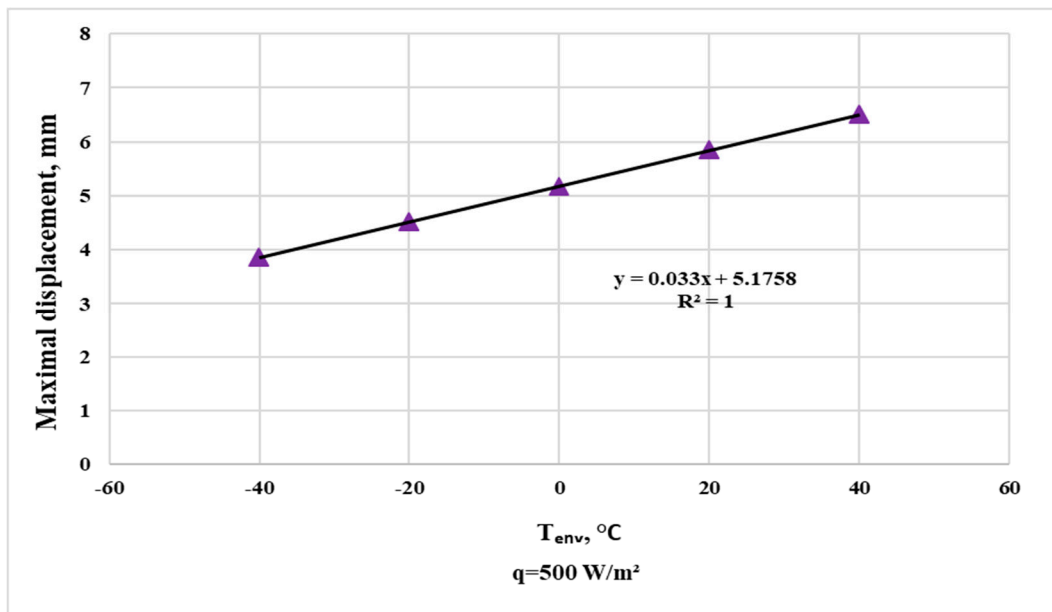


Figure 12. Relationship between the maximum displacement in the rod and the environmental temperature T_{env} at $q = 500 \text{ W/m}^2$.

In order to study the influence of material characteristics on the temperature field and mechanical behavior of the rod, temperature distributions (Figure 13), constrained thermal strains (Figure 14), and thermal stresses (Figure 15) were calculated in rods constructed with different materials (copper, aluminum, and iron). The thermophysical characteristic values of rod materials and the parameters of the boundary conditions are given in Table 3.

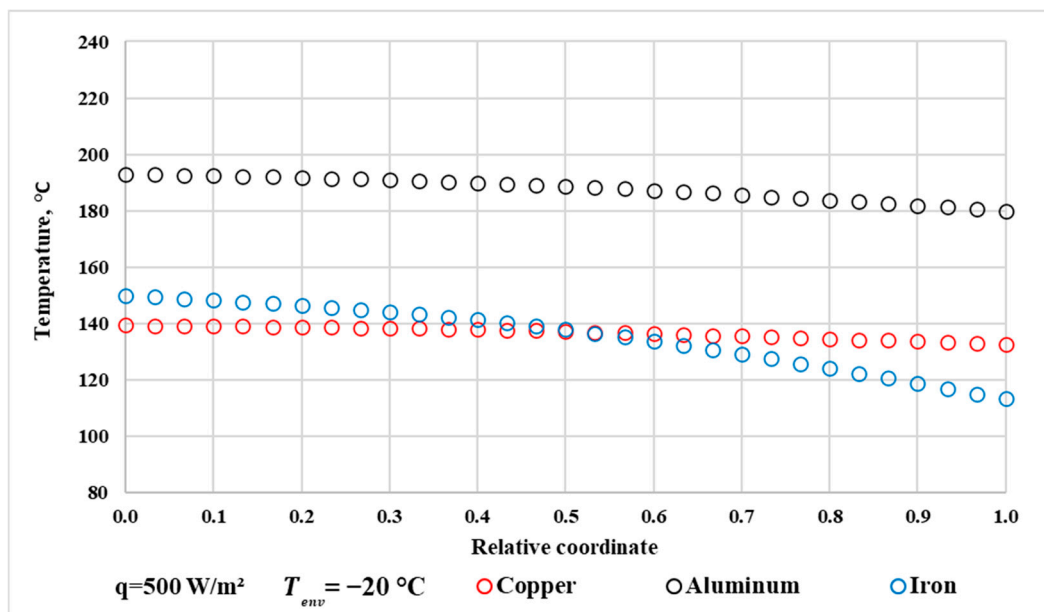


Figure 13. Temperature distribution in the rod at $q = 500 \text{ W/m}^2$ and environmental temperature $T_{env} = -20 ^\circ\text{C}$.

As observed, the type of material (its characteristics) greatly influences the temperature distribution and mechanical characteristics of a rod. Despite the fact that the thermal conductivity coefficient of aluminum has an intermediate value ($K_{xx} = 209.3 \text{ W/(m}\cdot^\circ\text{C)}$) compared to iron ($K_{xx} = 74.4 \text{ W/(m}\cdot^\circ\text{C)}$) and copper ($K_{xx} = 400.0 \text{ W/(m}\cdot^\circ\text{C)}$), the temperature value of the aluminum rod is significantly higher. This is explained by the fact that

the heat transfer coefficient of aluminum ($h = 10 \text{ W}/(\text{m}^2 \cdot ^\circ\text{C})$) is lower than that of iron ($h = 15 \text{ W}/(\text{m}^2 \cdot ^\circ\text{C})$) and copper ($h = 13.1 \text{ W}/(\text{m}^2 \cdot ^\circ\text{C})$).

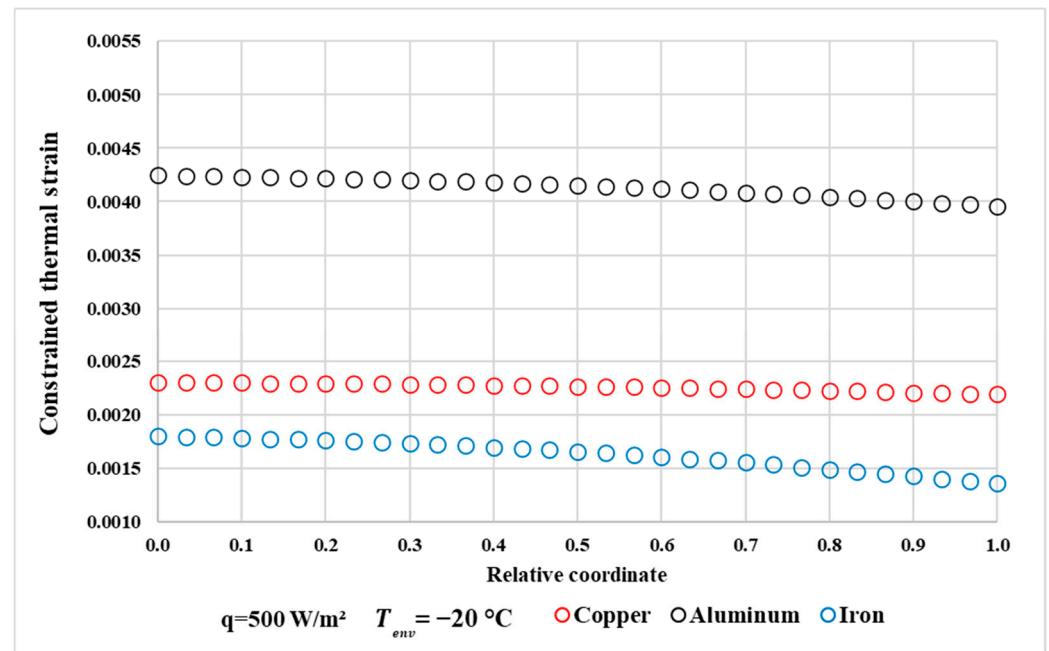


Figure 14. Constrained thermal strain in the rod at $q = 500 \text{ W}/\text{m}^2$ and environmental temperature $T_{env} = -20 \text{ }^\circ\text{C}$.

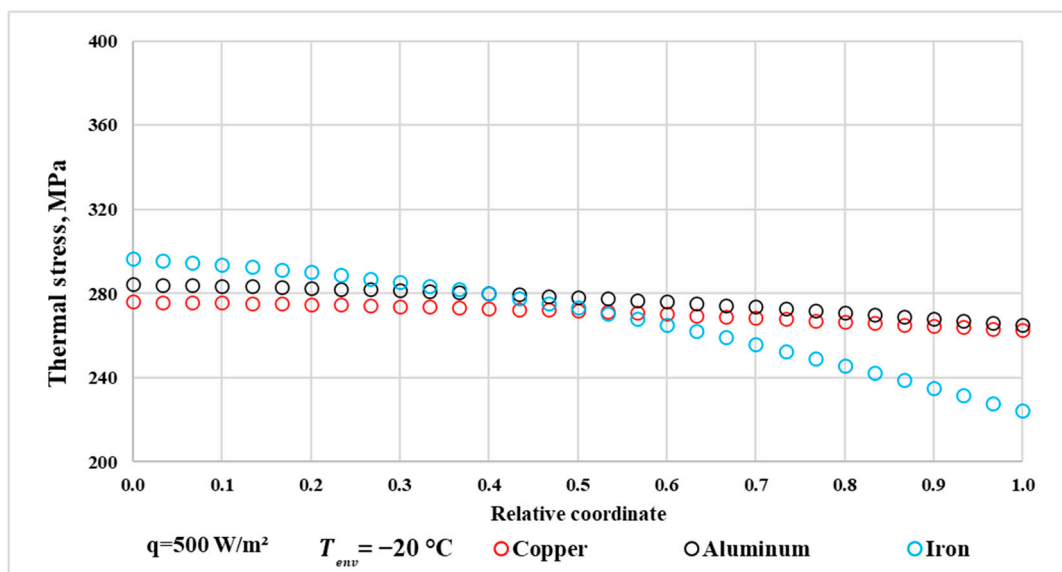


Figure 15. Thermal stress in the rod at $q = 500 \text{ W}/\text{m}^2$ and environmental temperature $T_{env} = -20 \text{ }^\circ\text{C}$.

The significantly high constrained thermal strain values of the aluminum rod are due to its large thermal expansion coefficient α .

The approximately equal thermal stress in most of the length (approximately 0.7 of the length) of all three rods can be explained by the fact that the elastic modulus of aluminum is almost two times less than that of copper, while the elastic moduli of iron and copper are of approximately the same order of magnitude.

Table 3. Values of the thermophysical characteristics of the rod's materials and boundary condition parameters.

Material of the Rod	Thermophysical Characteristics of the Rod's Material		Boundary Condition Parameters		
	K_{xx} , W/(m·°C)	α , 1/°C	q , W/m ²	h , W/(m ² ·°C)	T_{env} , °C
Aluminum	209.3	2.2×10^{-5}	500	10.0	−20
Iron	74.4	1.2×10^{-5}	500	15.0	−20

6. Failure Criteria

In order to ensure the safe and reliable operation of structural parts, etc., which are rods with a variable cross-section, the following failure criteria must be taken into account:

I. Mechanical criterion:

$$\sigma_{oper} \cdot k_{1m} < [\sigma], \quad (22)$$

where σ_{oper} denotes the value of maximum stress in the rod;

$[\sigma]$ denotes the permissible voltage value at temperature T ;

k_{1m} denotes the mechanical safety factor.

II. Thermal criterion:

$$T_{oper} \cdot k_T < T_m, \quad (23)$$

where T_{oper} denotes the maximum temperature value in the rod, °C;

T_m denotes the melting point of the rod's material;

k_T denotes the thermal safety factor.

7. Practical Applications

In the Introduction section of this study, it was stated that many mechanical engineering objects comprise rods with a variable cross-section. It was also noted that rods with variable cross-sections can be used as device components, such as dilatometers, bimetallic thermometers, thermomechanical sensors, thermal expansion compensators, etc.

In geotechnics, the following important problem often arises: It is necessary to construct the pile foundation of a new object in confined conditions (for example, between buildings), which during the spring, summer, and autumn periods can negatively affect the stability of existing neighboring objects. Therefore, this practical process is more reliably carried out in winter. In this case, it is effective to use a device that enables localized thawing of frozen ground to a certain depth. Such a device can be designed in the form of a rod with a variable cross-section, with heat supplied to one end, while the other end, embedded in frozen ground, exchanges heat with the environment (with frozen ground).

8. Conclusions

1. As the main result of our modeling, an analytical expression was obtained for the temperature distribution in a heat-insulated rod with a variable cross-section, where a constant-intensity heat flow is supplied to the left end, and heat exchange with the environment occurs at the right end. The analytical solution was obtained using the finite element method based on the variational solution.
2. Using analytical expressions for temperature, we obtained the thermophysical (temperature solution coefficient) and mechanical (elastic modulus) characteristics of the

rod material and expressions for the elongation, constrained thermal strain, and thermal stress of the rod.

3. The agreement between the temperature values obtained using the analytical method and the finite element method for both large ($q = 3350 \text{ W/m}^2$) and moderate ($q = 500 \text{ W/m}^2$) heat flux values is very high, which demonstrates the high accuracy of the obtained analytical solution.
4. The temperature field in the rod is nonlinear and decreases along its length, which is a result of one-sided heating and free heat exchange with the environment at the other end. The temperature distributions in the rod at different environmental temperatures are similar. There is a functional (linear) relationship between the maximum temperatures in the rod and the environmental temperatures. This relationship can serve as a shift function to obtain the temperature distribution in the rod at other environmental temperatures.
The constrained thermal strain and thermal stress in the rod have similar properties, for which the corresponding shear functions are also defined.
5. When the environmental temperature increases from -40°C to $+40^\circ\text{C}$, the maximum temperature, maximum constrained thermal strain, maximum thermal stress, and elongation of the rod increase from 120°C to 200°C , from 0.0020 to 0.0033, from 240 MPa to 400 MPa, and from 3.8 mm to 6.5 mm, respectively.

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Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Timoshenko, S.P. *Strength of Materials, Part I*, 2nd ed.; Van Nostrand: New York, NY, USA, 1947; 358p.
2. Feodosiev, V.I. *Strength of Materials: Textbook for Universities*, 9th ed.; revised; Nauka: Moscow, Russia, 1986; 512p. (In Russian)
3. Gere, J.M.; Goodno, B.J. *Mechanics of Materials*, 5th ed.; Brooks/Cole: Pacific Grove, CA, USA, 2001; 922p.
4. ASME. *ASME B31.1 Power Piping Code*; The American Society of Mechanical Engineers: New York, NY, USA, 2010.
5. Blevins, R.D. *Flow-Induced Vibration*; Van Nostrand Reinhold: New York, NY, USA, 1990; 472p.
6. Seong, S.H.; Kim, J.T.; Cheon, S.W.; Lee, C.K.; Lee, N.Y.; Hwang, I.S.; Lee, S.J. Vibration analysis for condition monitoring on piping elbow thinning. In Proceedings of the International Symposium on the Future I&C for Nuclear Power Plants (ISOFC 2005), Tongyeong, Republic of Korea, 1–4 November 2005; KNICS R and D Center: Changwon, Republic of Korea; pp. 372–378.
7. Kulikov, A.V. Raschet ustoychivosti truboprovodov AES s peremennym secheniem metodom konechnykh elementov [Stability analysis of NPP pipelines with variable cross-section using FEM]. *Teploenergetika* **2012**, *5*, 45–52. (In Russian)
8. Birger, I.A.; Mavlyutov, R.R. *Strength of Materials*; Nauka: Moscow, Russia, 1986; 554p. (In Russian)
9. Carslaw, H.S.; Jaeger, J.C. *Conduction of Heat in Solids*, 2nd ed.; Oxford University Press: Oxford, UK, 1959.
10. Arshidinova, M.; Tashev, A.; Kudaykulov, A. Developing a method of accounting for the existence of local surface heat exchange in rods of variable cross-section. *East.-Eur. J. Enterp. Technol.* **2023**, *3*, 53–64. [[CrossRef](#)]
11. Kanareykin, A. Distribution of the temperature field in the rod, taking into account the correction for heat loss through its side surface. *E3S Web Conf.* **2023**, *376*, 01078. [[CrossRef](#)]
12. Eckert, E.R.G.; Drake, R.M. *Heat and Mass Transfer*; McGraw-Hill: New York, NY, USA, 1959.
13. Węgrzyn-Skrzypczak, E.; Skrzypczak, T. Analytical and numerical solution of the heat conduction problem in the rod. *J. Appl. Math. Comput. Mech.* **2017**, *16*, 79–86. [[CrossRef](#)]

14. Eremin, A.V. Study on heat transfer between the rod and the environment under conditions of forced convection. *IOP Conf. Ser. Mater. Sci. Eng.* **2020**, *791*, 012008. [[CrossRef](#)]
15. Kudaykulov, A.; Tashev, A.A.; Askarova, A. A computational algorithm and the method of determining the temperature field along the length of the rod of variable cross section. *Open Eng.* **2018**, *8*, 170–175. [[CrossRef](#)]
16. Akhmetov, S.; Kudaykulov, A.; Bizhanov, D. Determination of thermally stressed state of rod elements of variable cross section under the impact of a lateral heat flow, heat exchange and surface thermal insulation. *E3S Web Conf.* **2020**, *216*, 01073. [[CrossRef](#)]
17. Arshidinova, M.; Kudaykulov, A.; Tashev, A.; Begaliyeva, K. Numerical study of the effect of local heat flow on the thermal stress–strain state of a rod of variable cross-section. In Proceedings of the IFToMM Asian MMS 2024, Almaty, Kazakhstan, 28–30 August 2024; pp. 34–43. [[CrossRef](#)]
18. Arshidinova, M.; Kudaykulov, A.; Tashev, A.; Begaliyeva, K.; Merkibayeva, B. Numerical study of the influence of ambient temperature on the thermo-stress–strain state of a rod with a curvilinear core. In Proceedings of the IFToMM Asian MMS 2024, Almaty, Kazakhstan, 28–30 August 2024; pp. 90–98. [[CrossRef](#)]
19. Dwight, H.B. *Tables of Integrals and Other Mathematical Data*, 4th ed.; Macmillan: New York, NY, USA, 1961.
20. Lutz, M. *Learning Python*, 5th ed.; O'Reilly Media: Sebastopol, CA, USA, 2013; ISBN 978-1-4493-5573-9.
21. Führer, C.; Verdier, O.; Solem, J.E. *Scientific Computing with Python*, 2nd ed.; Packt Publishing: Birmingham, UK, 2021; ISBN 978-1-83882-232-3.

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