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Generalized Trajectory Planning Using Polynomial Functions of Arbitrary Odd Degree †

Miroslava Ivanova * and Hristo Genev

Mathematical Analysis and Differential Equations–(MADE), Faculty of Applied Mathematics and Informatics, Technical University of Sofia, 1000 Sofia, Bulgaria; h_genev@tu-sofia.bg

* Correspondence: mj_ivanova@tu-sofia.bg

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Abstract

In this paper, we explore the synthesis of a trajectory with preassigned maximum velocity and acceleration for planning longitudinal horizontal motion. The trajectory includes polynomials of arbitrary odd degree for the acceleration and deceleration phases, and an intermediate segment of uniform motion.

Keywords: trajectory planning; arbitrary odd degree polynomials; uniform motion segment

1. Introduction

The fundamental task in the automation and robotization of machines is to plan their motion in a way that minimizes dynamic structural loads and vibrations. Numerical studies [1–3] have demonstrated that using motion laws with smooth increases and decreases in displacement can significantly improve the dynamic behavior of machines. High-degree polynomials are particularly suitable for motion planning, as they allow for the specification of the kinematic characteristics of motion at both the beginning and the end of the trajectory [4–9]. The motion law should meet various criteria [6–8], with particular importance placed on ensuring that specified values of kinematic characteristics, such as velocity and acceleration, are achieved to enhance machine functionality.

In [8,10], an algorithm for trajectory planning based on a fifth-degree polynomial law, including an intermediate segment of uniform motion, was developed. It was shown that motion based on a polynomial law improves machine performance by shortening the motion duration, achieving the preassigned maximum velocity and acceleration, and minimizing structural oscillations. In [11], an algorithm based on a seventh-degree polynomial motion law was developed, allowing the inclusion of the derivative of acceleration to achieve even smoother motion and to enhance the machine's dynamic response. In this study, the seventh-degree polynomials governing the acceleration and deceleration phases are explicitly derived, and the conditions under which a segment of uniform motion can be included are established.

Figure 1 shows the structure of a polynomial-based motion profile, as discussed in detail in [10,11].

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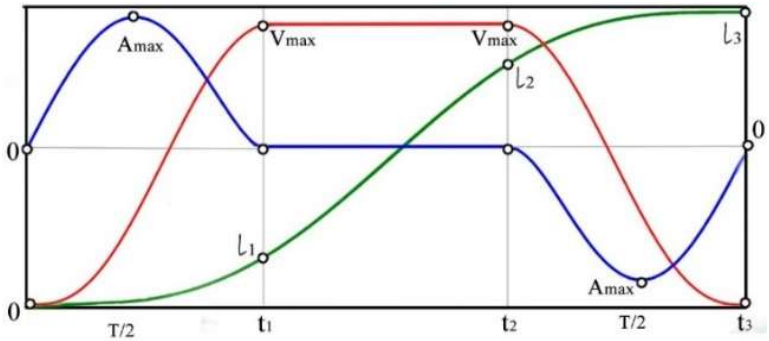


Figure 1. Polynomial-Based Motion Structure [11]. The blue, red, and green curves represent acceleration, velocity, and displacement, respectively.

In brief, the motion is longitudinal, with predefined values for the maximum velocity and acceleration. The motion is divided into three main phases: acceleration, uniform motion, and deceleration. Each phase is smooth and continuous. Now, we set the goal to generalize the problem using polynomial functions of an arbitrary odd degree and developing an algorithm for trajectory planning.

2. Statement 1

Let n be an odd natural number and $n \geq 5$, $n = 2m + 1$ where $m \in \mathbb{N}$ and $m \geq 2$.

Then in the set of all polynomials in an odd degree there exists only one satisfying the following conditions:

$$\begin{vmatrix} s(0) = 0 \\ s'(0) = 0 \\ \ldots\ldots\ldots\ldots \\ s^{(m)}(0) = 0 \end{vmatrix} \tag{1}$$

and

$$\begin{vmatrix} s(T) = l \\ s'(T) = 0 \\ \ldots\ldots\ldots\ldots \\ s^{(m)}(T) = 0, \end{vmatrix} \tag{2}$$

where $l = const > 0$, $T = const > 0$.

The searched polynomial is:

$$s(t) = \frac{n!}{m!} l \left(\frac{t}{T}\right)^{m+1} \sum_{k=0}^{m} \frac{(-1)^k}{k!(m-k)!(k+m+1)} \left(\frac{t}{T}\right)^k. \tag{3}$$

The first, second, and third derivatives are as follows:

$$v(t) = s'^{(t)} = \frac{n!}{(m!)^2} \frac{l}{T} \left(\frac{t}{T}\right)^m \left(1 - \frac{t}{T}\right)^m; \tag{4}$$

$$a(t) = s''^{(t)} = \frac{n!m}{(m!)^2} \frac{l}{T^2} \left(\frac{t}{T}\right)^{m-1} \left(1 - \frac{t}{T}\right)^{m-1} \left(1 - 2\frac{t}{T}\right); \tag{5}$$

$$a'(t) = s'''(t) = \frac{n!m}{(m!)^2} \frac{l}{T^3} \left(\frac{t}{T}\right)^{m-2} \left(2(2m-1)\left(\frac{t}{T}\right)^2 - 2(2m-1)\frac{t}{T} + m - 1\right). \tag{6}$$

Proof. For large values of n solving Problems (1)–(2) becomes nontrivial. Therefore, in the proof of the statement, we adopt an approach different from the one used in our previous work [11]. □

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From (1) and (2) it follows that:

$$\begin{vmatrix} v(0) = 0 \\ v'(0) = 0 \\ \ldots\ldots\ldots\ldots \\ v^{(m-1)}(0) = 0 \end{vmatrix} \tag{7}$$

and

$$\begin{vmatrix} v(T) = 0 \\ v'(T) = 0 \\ \ldots\ldots\ldots\ldots \\ v^{(m-1)}(T) = 0. \end{vmatrix} \tag{8}$$

As $s(t)$ is a polynomial of degree n, then $v(t) = s'(t)$ is a polynomial of $n - 1 = 2m$ degree.

As a result, from (7) we have: $v(t) = t^m q(t)$, where $q(t)$ is a polynomial of m degree.

From (8) we obtain: $q(t) = A(T - t)^m$, where $A = const \neq 0$ and therefore:

$$v(t) = At^m(T - t)^m.$$

Because $s(0) = 0$, then:

$$s(t) = \int_0^t v(\tau)d\tau = A\int_0^t \tau^m(T - \tau)^m d\tau = AT^{2m+1}\int_0^{\frac{t}{T}} \sigma^m(1 - \sigma)^m d\sigma.$$

If $I_m = \int_0^{\frac{t}{T}} \sigma^m(1 - \sigma)^m d\sigma$, we obtain: $l = s(T) = AT^{2m+1}I_m$.

By applying integration by parts, we derive:

$$I_m = \frac{(m!)^2}{(2m+1)!} = \frac{(m!)^2}{n!},\ A = \frac{n!}{(m!)^2}\frac{l}{T^n} \text{ and } v(t) = \frac{n!}{(m!)^2}\frac{l}{T}\left(\frac{t}{T}\right)^n\left(1 - \frac{t}{T}\right)^m.$$

Since $(1 - \sigma)^m = \sum_{k=0}^m \binom{m}{k}(-1)^k\sigma^k$, it follows that:

$$\int_0^{\frac{t}{T}} \sigma^m(1 - \sigma)^m d\sigma = \sum_{k=0}^m \binom{m}{k}\frac{(-1)^k}{k+m+1}\left(\frac{t}{T}\right)^{k+m+1} = m!\left(\frac{t}{T}\right)^{m+1}\sum_{k=0}^m \frac{(-1)^k}{k!(m-k)!(k+m+1)}\left(\frac{t}{T}\right)^k$$

and (3) is proved.

By differentiating (4), we obtain (5) and (6).

3. Statement 2

Let $s(t), v(t)$ and $a(t)$ are the functions obtained in Statement 1 and $t \in [0; T]$.

Then we have:

1. $s(t)$ is a strictly increasing function in the interval $[0; T]$ and

$$\min_{t\in[0;T]} s(t) = s(0) = 0,$$
$$\max_{t\in[0;T]} s(t) = s(T) = l.$$

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2. $v(t) > 0$ in the interval $(0; T)$ and

$$v_{min} = \min_{t\in[0;T]} v(t) = v(0) = v(T) = 0,$$
$$v_{max} = \max_{t\in[0;T]} v(t) = v\left(\frac{T}{2}\right) = \frac{n!}{4^m(m!)^2}\frac{l}{T}.$$

3. $a(t) > 0$ in the interval $\left(0; \frac{T}{2}\right)$, $a(t) < 0$ in the interval $\left(\frac{T}{2}; T\right)$, $a(0) = a\left(\frac{T}{2}\right) = a(T) = 0$ and

$$a_{max} = \max_{t\in[0;T]} a(t) = a(\sigma_1 T) = \frac{n!m}{(m!)^2 4^{m-1}}\left(\frac{n-3}{n-2}\right)^{m-1}\frac{1}{\sqrt{n-2}}\frac{l}{T^2}$$
$$a_{min} = \min_{t\in[0;T]} a(t) = a(\sigma_2 T) = -a_{max} \quad ,$$

where $\sigma_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{n-2}}\right) \in \left(0, \frac{1}{2}\right)$, $\sigma_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{n-2}}\right) \in \left(\frac{1}{2}; 1\right)$.

The proof of this statement is analogous to the one provided in our previous work [11].

As a direct consequence of statement 2, the following dependencies are obtained:

$$T = \frac{4m}{\sqrt{n-2}}\left(\frac{n-3}{n-2}\right)^{m-1}\frac{v_{max}}{a_{max}}, \tag{9}$$

$$l = \frac{4^{m+1}m(m!)}{n!}\left(\frac{n-3}{n-2}\right)^{m-1}\frac{1}{\sqrt{n-2}}\frac{v_{max}^2}{a_{max}}. \tag{10}$$

The derived expressions define the limit displacement and duration of motion, in which the set maximum values of velocity and acceleration can be simultaneously achieved. As seen from Equations (9) and (10), these parameters, which are fundamental to including a segment with uniform motion, now also depend on the degree of the polynomial.

4. Obtained Results

As was done in the previous paper [11] we define:

$$\widetilde{v}(t) = \begin{cases} v_1(t) = v(t), & 0 \le t \le t_1 \quad \text{acceleration section;} \\ v_2(t) = v_{max} = const & t_1 \le t \le t_2 \quad \text{uniform motion section;} \\ v_3(t) = v(t_3 - t) & t_2 \le t \le t_3 \quad \text{deceleration section.} \end{cases} \tag{11}$$

where $t_1 = \frac{T}{2}$, $t_3 - t_2 = \frac{T}{2}$.

The main idea in trajectory planning is to let the function $\widetilde{v}(t)$ describe the instantaneous velocity.

We define the functions $s_1(t), s_2(t)$ and $s_3(t)$ in the following way:

$$\begin{aligned} s_1(t) &= s(t), & 0 \le t \le t_1; \\ s_2(t) &= l_1 + v_{max}(t - t_1), & t_1 \le t \le t_2; \\ s_3(t) &= l_1 + l_2 - s(t_3 - t), & t_2 \le t \le t_3. \end{aligned}$$

where $l_1 = \frac{1}{2}l$, $l_2 = l_1 + v_{max}(t_2 - t_1)$.

By defining the function:

$$\widetilde{s}(t) = \begin{cases} s_1(t), & 0 \le t \le t_1 \\ s_2(t), & t_1 < t < t_2 \\ s_3(t), & t_2 \le t \le t_3 \end{cases}$$

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we obtain that this function is twice continuously differentiable in the interval $[0; t_3]$ as at all points, except $t = t_1$ and $t = t_2$ this function is infinitely differentiable.

Also, it can be proved that

$$\begin{vmatrix} \widetilde{s}'(t) = \widetilde{v}(t) \\ \widetilde{s}(0) = 0 \end{vmatrix} \quad \text{and} \quad \widetilde{s}''(t) = \widetilde{v}'(t) = \widetilde{a}(t) = \begin{cases} a_1(t) = a(t) \\ a_2(t) = 0 \\ a_3(t) = -a(t_3 - t) \end{cases}$$

so $\widetilde{s}(t)$ describes the displacement from $t = 0$ to the current moment t and $\widetilde{a}(t)$ describes the acceleration.

5. Conclusions

The developed trajectory planning algorithm, based on arbitrary odd-degree polynomials, ensures smooth transitions through the three motion phases: acceleration, uniform motion, and deceleration. In addition to satisfying the predefined maximum velocity and acceleration, the duration of the acceleration and deceleration phases can be adjusted by selecting an appropriate polynomial degree. However, regardless of the chosen degree, there is only second-order smoothness at the transitions between the phases and the uniform motion segment. The resulting polynomial trajectories are well suited for planning the longitudinal motion of machines or their components in applications where smooth motion and precise positioning are required.

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