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Green's Function for Non-Homogeneous Magneto-Electro-Elastic Half-Plane †

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Abstract

An exponentially graded magneto-electro-elastic (MEE) half-plane subjected to anti-plane mechanical and in-plane electric and magnetic time-harmonic external load is considered. Green's function in the MEE half-plane is analytically derived using the fundamental solution in the MEE plane and the integral representation of the Hankel functions. The solutions presented here can be further used in computational schemes based on the boundary elements method (BEM) to obtain numerical results in a cracked homogeneous and non-homogeneous half-plane.

Keywords: exponentially graded MEE half-plane; Green's function; fundamental solution

1. Introduction

Magneto-electro-elastic materials (MEEM) are widely used in modern industry as sensors, actuators, transducers, filters, memory elements, batteries, etc. Single-phase MEEM-like Cr_2O_3, $BiFeO_3$, $BiMnO_3$ etc. are not suitable for practical use because of their weak magneto-electric effect and low Curie temperature. Unlike them, magneto-electro-elastic composite materials, for example $BaTiO_3/CoFe_2O_4$, laminate composite $PZT/Terfenol-D$ possess a giant magneto-electric effect at room temperature that makes them very suitable for applications, see Nan et al. [1].

The main drawback of these materials is delamination. To overcome this problem, functionally graded materials (FGM) have been developed. The properties of FGM vary gradually in space, which is a significant advantage over the discretely layered composites. These materials are brittle, and fracture analysis is very important for their structural integrity and reliable service life.

Fundamental solutions play a key role in finding numerical solutions for crack problems in the whole plane by BIEM. For half-plane problems, it is essential to know the Green's function that satisfies the additional boundary condition along the half-plane boundary. If we have the Green's function, we can reduce the boundary value problem to a system of boundary integral equations by using the Gauss theorem. This leads to increased accuracy of the numerical results, reduced execution time for the numerical procedure, and reduced memory requirements.

There is a limited number of works for Green's functions of a half-plane under time-harmonic external loading. Green's function for an orthotropic elastic half-plane can be seen in Rajapakse and Wang [2], and for graded elastic materials, see Rangelov and Manolis [3], Manolis et al. [4]. For a graded piezoelectric half-plane, see Rangelov and Dineva [5], and

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for an anisotropic magneto-electro-elastic plane in 2D and 3D case by Radon transform, see Rojas-Díaz et al. [6].

The aim of this study is to present Green's function in an exponentially graded magneto-electro-elastic half-plane under an anti-plane time-harmonic external load. The obtained results can be used to find numerical solutions for various crack problems in homogeneous and functionally graded magneto-electro-elastic half-planes by the boundary integral equation method.

2. Problem Statement

In a rectangular coordinate system $Ox_1x_2x_3$, we consider a transversely isotropic magneto-electro-elastic material, poled in the Ox_3 direction and MEE half-plane $R^2_- = \{(x_1, x_2), x_2 < 0\}$. The material is subjected to an incident time-harmonic with frequency ω and anti-plane mechanical, in-plane electrical, and magnetic load with respect to the isotropic plane Ox_1x_2. The problem is two-dimensional, i.e., the material properties are the same in all planes parallel to the isotropic plane and do not depend on x_3.

A generalized elasticity tensor C_{iJKl} is introduced as follows:

$$C_{iJKl}(x) = 0, for\, i \neq l, C_{i33l}(x) = \begin{cases} c_{44}(x),\, i = l \\ 0,\, i \neq l \end{cases}, C_{i34l}(x) = C_{i43l}(x) = \begin{cases} e_{15}(x),\, i = l \\ 0,\, i \neq l \end{cases}, C_{i35l}(x) = C_{i53l}(x) = \begin{cases} q_{15}(x),\, i = l \\ 0,\, i \neq l \end{cases},$$
$$C_{i44l}(x) = \begin{cases} -\varepsilon_{11}(x),\, i = l \\ 0,\, i \neq l \end{cases} C_{i45l}(x) = C_{i54l}(x) = \begin{cases} -d_{11}(x),\, i = l \\ 0,\, i \neq l \end{cases}, C_{i55l}(x) = \begin{cases} -\mu_{11}(x),\, i = l \\ 0,\, i \neq l \end{cases},$$

where c_{44} is the elastic module, e_{15} is the piezoelectric coefficient, q_{15} is the piezomagnetic coefficient, d_{11} is the magneto-electric coefficient, ε_{11} is the dielectric permittivity, and μ_{11} is the magnetic permeability. There is a generalized displacement, $u_J = (u_3, \phi, \varphi)$, where u_3 is the out-of-plane displacement, ϕ is the electric potential, and φ is the magnetic potential. There is a generalized stress, $\sigma_{iJ} = (\sigma_{i3}, D_i, B_i)$, where σ_{i3} and $i = 1, 2$ are components of the stress tensor, D_i is the component of the vector of the electric induction along Ox_i, $i = 1, 2$, and B_i is the component of the vector of the magnetic induction along Ox_i, $i = 1, 2$. We assume that $J = 3, 4, 5$ and that, in this notation, $u_4 = \phi$, $u_5 = \varphi$, $\sigma_{i4} = D_i$ and $\sigma_{i5} = B_i$, $i = 1, 2$.

Using generalized notations, the constitutive equation for this type of medium can be written in the following way:

$$\sigma_{iJ} = C_{iJKl} u_{K,l}, \tag{1}$$

see Soh and Liu [7], where the Einstein summation convention for repeated indices is assumed, the comma means differentiation, $i, l = 1, 2$ and $J, K = 3, 4, 5$.

The material inhomogeneity is described as follows:

$$e_{15}(x) = h(x)e_{15}, c_{44}(x) = h(x)c_{44}, q_{15}(x) = h(x)q_{15},$$
$$\varepsilon_{11}(x) = h(x)\varepsilon_{11}, d_{11}(x) = h(x)d_{11}, \mu_{11}(x) = h(x)\mu_{11}, \rho(x) = h(x)\rho,$$

where $\rho(x)$ is the density and $h(x)$ is the inhomogeneous function.

Using the constitutive Equation (1), the equation of motion, and the Maxwell equations, we obtain the following system of partial differential equations:

$$\left(C_{iJKl}(x)u_{K,l}\right)_{,l} + \rho_{JK}(x)\omega^2 u_K = 0 \tag{2}$$

where $\rho_{JK} = \begin{cases} \rho,\, J = K = 3 \\ 0,\, J\, or\, K \neq 3 \end{cases}$ and ω is the frequency of the incident wave.

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When the incident wave reaches the half-plane boundary, it scatters and produces a scattered wave. The total field at any point can be found using the superposition principle as follows:

$$u_J = u_J^{in} + u_J^{sc}, t_J = t_J^{in} + t_J^{sc},$$

where $t_J = \sigma_{iJ} n_i$ is the traction, $n = (n_1, n_2)$ is the normal vector to the crack, u_J^{in}, t_J^{in} are the displacement and traction of the incident wave, and u_J^{sc} and t_J^{sc} are the displacement and traction of the scattered wave. We assume that

$$t_J \big|_{x_2=0} = 0.$$

3. Fundamental Solution

The fundamental solution of the system of partial differential Equation (2) is the solution of:

$$\sigma^*_{iJM,i} + \rho_{JK}(x)\omega^2 u^*_{KM} = \delta_{JM}\delta(x,\xi) \tag{3}$$

where $\sigma^*_{iJK} = C_{iJKl}(x)u^*_{KM,l}$ and $\delta(x,\xi)$ are the Dirac's delta function and δ_{JM} is the Kronecker's symbol. The system (3) is a system with variable coefficients. We make the transformation $u^*_{KM} = h^{-1/2}U^*_{KM}$, which was successfully applied by Manolis and Shaw [8] for elastic media. The function U^*_{KM} satisfies the equation:

$$C_{iJKl}U^*_{KM,ii} + [\rho_{JK}\omega^2 - C_{iJKl}h^{-1/2}(h^{1/2})_{,ii}]U^*_{KM} = h^{-1/2}\delta_{JM}\delta(x,\xi). \tag{4}$$

Equation (4) is with constant coefficients if $p_{JK} = C_{iJKi}h^{-1/2}(h^{1/2})_{,ii} = const$. Here, we will consider exponentially graded MEEM with inhomogeneity function $h(x) = e^{2ax_2}$. In this case $p_{JK} = C_{2JK2}a^2$.

The following coefficients are introduced:

$$\widetilde{c}_{44} = c_{44} + \frac{q_{15}^2}{\mu_{11}}, \widetilde{e}_{15} = e_{15} - \frac{d_{11}q_{15}}{\mu_{11}}, \widetilde{\varepsilon}_{11} = \varepsilon_{11} - \frac{d_{11}^2}{\mu_{11}}, \widetilde{c} = \widetilde{c}_{44} + \frac{\widetilde{e}_{15}^2}{\widetilde{\varepsilon}_{11}}, k = \sqrt{\frac{\rho\omega^2}{\widetilde{c}} - |a|^2}.$$

Here, we consider the case $k^2 > 0$. Using the solution of the following equation:

$$\Delta V + k^2 V = \frac{1}{\widetilde{c}}h^{-1/2}(\xi)\delta(x,\xi),$$

See Vladimirov [9], Gradsteyn and Ryzhik [10], we find the following fundamental solution:

$$u^*(x,\xi,\omega) = -\frac{ie^{-(x_2+\xi_2)}}{4\widetilde{c}}\begin{pmatrix} 1 & A & B \\ A & A^2 & AB \\ B & AB & B^2 \end{pmatrix} H_0^{(1)}(|k||x-\xi|) - \\ -\frac{ie^{-(x_2+\xi_2)}}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{\widetilde{\varepsilon}_{11}} & \frac{d_{11}}{\widetilde{\varepsilon}_{11}\mu_{11}} \\ 0 & \frac{d_{11}}{\widetilde{\varepsilon}_{11}\mu_{11}} & -(\frac{d_{11}^2}{\widetilde{\varepsilon}_{11}\mu_{11}^2} + \frac{1}{\mu_{11}}) \end{pmatrix} H_0^{(1)}(i|a||x-\xi|),$$

where $A = \frac{\mu_{11}e_{15} - q_{15}d_{11}}{\mu_{11}\varepsilon_{11} - d_{11}^2}$ and $B = \frac{q_{15}\varepsilon_{11} - e_{15}d_{11}}{\mu_{11}\varepsilon_{11} - d_{11}^2}$ and $H_0^{(1)}$ is the Henkel function of the first kind and 0-th order.

The Henkel's function of the first kind is defined as follows:

$$H_n^{(1)} = J_n(z) + iY_n(z),$$

where $J_n(z)$ and $Y_n(z)$ are Bessel's functions of the first and second kind.

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4. Green's Function

To derive the Green's function, we use the following integral representation of Henkel's functions (see Gradshteyn and Ryzhik [10]):

$$H_0^{(1)}(kr) = \frac{1}{i\pi}\int\limits_{i\alpha-\infty}^{i\alpha+\infty}\frac{1}{\beta}e^{-\beta\,|\,x_2-\xi_2\,|}\,e^{i\eta(x_1+\xi_1)}d\eta, x_2<0, \xi_2<0,$$

where

$$r=|x-\xi|,\ k=k_1+ik_2\ k_1=\frac{\rho\omega^2}{\gamma}-a^2, k_2>0,\ -k_2<\alpha<k_2\ \beta=\sqrt{\eta^2-k^2}$$

Note that we can pass $k_2\to 0$, then $\alpha=0$ and the integral is over the $\mathbf{R}$.

Following Rangelov and Manolis [3] for the elastic case and Rangelov and Dineva [5] for the piezoelectric media, we obtain the following representation of the fundamental solution:

$$u^*(x,\xi,\omega)=-e^{-a(x_2+\xi_2)}\frac{1}{4\pi\widetilde{c}}G\int\limits_R\frac{1}{\beta}e^{-\beta\,|\,x_2-\xi_2\,|}\,e^{i\eta(x_1+\xi_1)}d\eta\ +e^{-a(x_2+\xi_2)}\frac{1}{4\pi}H\int\limits_R\frac{1}{\beta_a}e^{-\beta_a\,|\,x_2-\xi_2\,|}\,e^{i\eta(x_1+\xi_1)}d\eta,$$

where

$$G=\begin{pmatrix}1 & A & B\\ A & A^2 & AB\\ B & AB & B^2\end{pmatrix},\ H=\begin{pmatrix}0 & 0 & 0\\ 0 & -\frac{1}{\widetilde{\varepsilon}_{11}} & \frac{d_{11}}{\widetilde{\varepsilon}_{11}\mu_{11}}\\ 0 & \frac{d_{11}}{\widetilde{\varepsilon}_{11}\mu_{11}} & -(\frac{d_{11}^2}{\widetilde{\varepsilon}_{11}\mu_{11}^2}+\frac{1}{\mu_{11}})\end{pmatrix}.$$

We ask for Green's function in the following form:

$$g(x,\xi,\omega)=u^*(x,\xi,\omega)+w(x,\xi,\omega),$$

where

$$w=\{w_{IJ}\}$$
$$w_{IJ}=e^{-ax_2}W_{IJ};\ W_{IJ}=e^{-a\xi_2}\int\limits_R(S_{IJ}\frac{1}{\beta}e^{\beta(x_2+\xi_2)}+D_{IJ}\frac{1}{\beta_a}e^{\beta_a(x_2+\xi_2)})e^{i\eta(x_1+\xi_1)}d\eta,$$
$$\beta=\sqrt{\eta^2-k^2},\ \beta_a=\sqrt{\eta^2+a^2}, I,J=3,4,5$$

and S_{IJ} and D_{IJ} do not depend on x_1, x_2.

If $J=3$ from the boundary conditions, we have:

$$\begin{aligned}&C_{2332}[-aW_{3M}+W_{3M,2}]+C_{2342}[-aW_{4M}+W_{4M,2}]+C_{2352}[-aW_{5M}+W_{5M,2}]|_{x_2=0}=\\&=C_{2332}[-aU^*_{3M}+U^*_{3M,2}]+C_{2342}[-aU^*_{4M}+U^*_{4M,2}]+C_{2352}[-aU^*_{5M}+U^*_{5M,2}]\,|_{x_2=0}.\end{aligned}\tag{5}$$

After substituting W_{IJ} and U^*_{IJ} in the above equality (5) and simplifying, we find the following system of equations:

$$\left|\begin{aligned}&(c_{44}S_{3M}+e_{15}S_{4M}+q_{15}S_{5M})(1-a\tfrac{1}{\beta})=-\tfrac{\widetilde{c}(\beta+a)}{4\pi\widetilde{c}\beta}G_{3M}\\&(c_{44}D_{3M}+e_{15}D_{4M}+q_{15}D_{5M})(1-a\tfrac{1}{\beta_a})=-(e_{15}H_{4M}+q_{15}H_{5M})(1+a\tfrac{1}{\beta_a})\tfrac{1}{4\pi}.\end{aligned}\right.\tag{6}$$

Similarly if $J=4$ and $J=5$, we have the following:

$$\left|\begin{aligned}&(e_{15}S_{3M}-\varepsilon_{11}S_{4M}-d_{11}S_{5M})(1-a\tfrac{1}{\beta})=0\\&(e_{15}D_{3M}-\varepsilon_{11}D_{4M}-d_{11}D_{5M})(1-a\tfrac{1}{\beta_a})=(\varepsilon_{11}H_{4M}+d_{11}H_{5M})(1+a\tfrac{1}{\beta_a})\tfrac{1}{4\pi}\end{aligned}\right.\tag{7}$$

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and

$$\begin{cases} (q_{15}S_{3M} - d_{11}S_{4M} - \mu_{11}S_{5M})(1 - a\frac{1}{\beta}) = 0 \\ (q_{15}D_{3M} - d_{11}D_{4M} - \mu_{11}D_{5M})(1 - a\frac{1}{\beta_a}) = (d_{11}H_{4M} + \mu_{11}H_{5M})(1 + a\frac{1}{\beta_a})\frac{1}{4\pi}. \end{cases} \quad (8)$$

From the first equations in the systems (6), (7), and (8), we obtain the following:

$$S_{IJ} = \frac{1}{4\pi\widetilde{c}}\frac{a+\beta}{a-\beta}G_{IJ}.$$

Similarly, from the second equations in (6), (7), and (8), we have the following:

$$D_{IJ} = \frac{1}{4\pi\widetilde{\varepsilon}_{11}}\frac{\beta_a + a}{\beta_a - a}H_{IJ}.$$

5. Plane–Wave Solution

To find the incident SH wave, we will use the transformation $u_K^{in} = h^{-1/2}U_K^{in}$ again. The vector function U_K^{in} is obtained by wave decomposition and has the form, see Rangelov et al. [11]:

$$U_K^{in} = \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} e^{ik(x_1\eta_1 + x_2\eta_2)},$$

where $\eta = (\eta_1, \eta_2)$ is the direction of the incident shear wave.

We can find the free-field solution in the half-plane in the form:

$$u_K^{ff} = h^{-1/2}(A_1U_K^{in} + A_2U_K^{sc}),$$

where $U_K^{sc} = \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} e^{ik(x_1\eta_1 - x_2\eta_2)}$ is the scattered wave. The coefficients A_1 and A_2 have to be determined so that the condition along the half-plane boundary is fulfilled. Using the boundary conditions, we obtain the following relation:

$$A_2 = -\frac{ik\eta_2 - a}{ik\eta_2 + a}A_1.$$

6. Conclusions

An exponentially graded magneto-electro-elastic half-plane subjected to anti-plane time-harmonic mechanical, in-plane electric, and magnetic external loads is considered here. Green's function is analytically derived using the fundamental solution in the MEE plane and the integral representation of the Hankel functions. The plane–wave solution is also presented. The obtained results can be further used in numerical schemes for solving crack problems in homogeneous and functionally graded MEE half-planes based on the BEM.

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