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Partial Differential Equations of Motion for a Single-Link Flexible Manipulator †

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Abstract: Robot manipulators have played an enormous role in the industry during the twenty-first century. Due to the advances in materials science, lightweight manipulators have emerged with low energy consumption and positive economic aspect regardless of their complex mechanical model and control techniques problems. This paper presents a dynamic model of a single link flexible robot manipulator with a payload at its free end based on the Euler–Bernoulli beam theory with a complete second-order deformation field that generates a complete second-order elastic rotation matrix. The beam experiences an axial stretching, horizontal and vertical deflections, and a torsional deformation ignoring the shear due to bending, warping due to torsion, and viscous air friction. The deformation and its derivatives are assumed to be small. The application of the extended Hamilton principle while taking into account the viscoelastic internal damping based on the Kelvin–Voigt model expressed by the Rayleigh dissipation function yields both the boundary conditions and the coupled partial differential equations of motion that can be decoupled when the manipulator rotates with a constant angular velocity. Equations of motion solutions are still under research, as it is required to study the behavior of flexible manipulators and develop novel ways and methods for controlling their complex movements.

Keywords: flexible manipulator; Euler-Bernoulli beam; Viscoelasticity; Kelvin-Voigt model; Rayleigh dissipation function; extended Hamilton principle; partial differential equations

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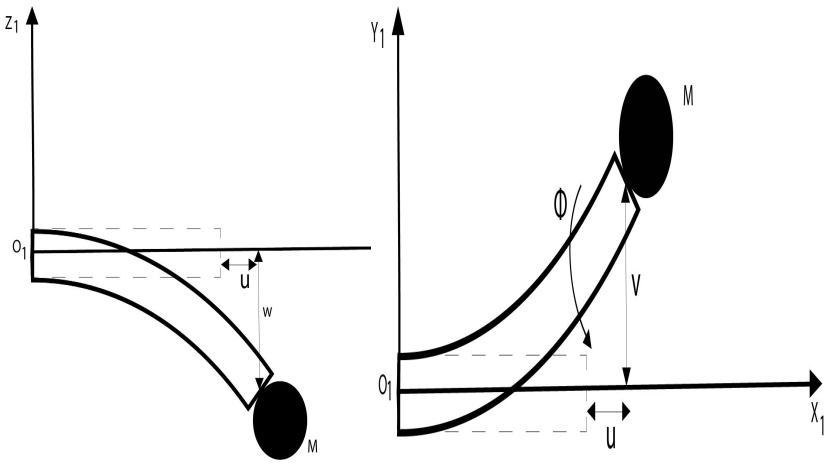
1. Introduction

The focus of robotics research in the last decade has been on building lightweight manipulators due to their low energy consumption despite their complex mechanical models and control systems. Lightweight manipulators are considered flexible manipulators that suffer from flexural effects, which leads to growing interest toward modeling and control architecture of such systems. In general, the research is restricted to single-link flexible manipulator [1] due to the intricacy of serial link flexible manipulators. In the literature, the single link is usually modeled by one deformation parameter [2], and the kinematics of the Euler–Bernoulli beam is usually approached by the assumed traditional deformation field that cannot allow having an orthogonal elastic rotation matrix to the second-order. For this article, the deformations and their partial derivatives are assumed to be small. The kinematic model described in Section 2.1 is based on the complete second-order deformation field [3]. Section 2.2 presents the dynamics model that includes the kinetic energy and potential energy of the system that is composed of gravitational and strain potential energies due to gravity and elasticity. Section 2.3 takes into account the Rayleigh dissipation function due to motor friction and the viscoelastic internal damping based on the Kelvin–Voigt model. Section 2.4 gives the motion equations using the extended Hamilton principle that

yields four partial differential equations satisfied by the deformation variables and seven boundary conditions. Lastly, Section 3 deals with the decoupling of partial differential equations in a particular case which allows small simplifications of the equations.

2. Mechanical Modeling

The system consists of a base subjected to an applied torque T_{mot} by a motor, a flexible link modeled as an Euler–Bernoulli beam with a circular cross-section with radius R and length L, and a payload with mass m_p and inertia matrix I_p at the free end of the link. The beam is subjected to an axial stretching $u(x,t)$, a horizontal deflection $v(x,t)$, a vertical deflection $w(x,t)$ and a torsional deformation $\phi(x,t)$, as shown in (Figure 1a,b) where the axis Z_1 is perpendicular to the ground. The beam deformations and their partial derivatives are assumed to be small, and shear due to bending, warping due to torsion, and air viscous friction are neglected. To simplify the notation, $u(x,t)$, $v(x,t)$, $w(x,t)$, $\phi(x,t)$, $\frac{d}{dt}(.)$, $\frac{d}{dx}(.)$ are denoted by u, v, w, ϕ, $\dot{(.)}$ and $(.)'$ respectively.



(**a**) Front view (**b**) Top view

Figure 1. Flexible beam with payload.

2.1. Kinematics

Let $\mathcal{R}_0$ be an inertial frame with origin O_0, $\mathcal{R}_1$ a frame attached to the motor with origin O_1 that coincides with O_0, and $\mathcal{R}_{dm}$ a frame attached to the cross-section of mass dm whose axes are parallel to those of $\mathcal{R}_1$ before deformation and whose origin O_{dm} is the center of the cross-section that is at a distance x from O_1 along the neutral axis of the link before deformation. The rotation matrix of $\mathcal{R}_1$ relative to $\mathcal{R}_0$ [4] is ${}^0R_1 = R_{Z_0,\theta}$ which means the frame R_1 rotates by an angle θ of about Z_0.

The position of O_{dm} relative to $\mathcal{R}_1$ expressed in $\mathcal{R}_1$ after deformation [3] expressed by:

$$ {}^1\overrightarrow{O_1O_{dm}} = [x + u - \frac{1}{2}\int_0^x (v'^2 + w'^2)ds, v, w]^T \tag{1} $$

The rotation matrix of $\mathcal{R}_{dm}$ relative to $\mathcal{R}_1$ after deformation [3] is:

$$ {}^1R_{dm} = \begin{bmatrix} 1 - \frac{1}{2}(v'^2 + w'^2) & -v' + u'v' - w'\phi & -w' + u'w' + v'\phi \\ v' - u'v' & 1 - \frac{1}{2}(v'^2 + \phi^2) & -\phi - \frac{1}{2}v'w' \\ w' - u'w' & \phi - \frac{1}{2}v'w' & 1 - \frac{1}{2}(w'^2 + \phi^2) \end{bmatrix} \tag{2} $$

${}^1R_{dm}$ is verified to be orthogonal to the second-order of Taylor expansion in the deformation variables. Let P be a point of the cross-section with (x, y, z) its coordinates relative to $\mathcal{R}_1$ before deformation. The position of P relative to $\mathcal{R}_1$ expressed in $\mathcal{R}_1$ after deformation [4] is

$$ {}^1\overrightarrow{O_1P} = {}^1\overrightarrow{O_1O_{dm}} + {}^1R_{dm}\,{}^{dm}\overrightarrow{O_{dm}P} $$

where ${}^{dm}\overrightarrow{O_{dm}P} = [0, y, z]^T$ and ${}^0\overrightarrow{O_0P} = {}^0R_1\,{}^1\overrightarrow{O_1P}$.

Let $\mathcal{R}_2$ be a frame attached to the free end of the link whose origin is O_2 and obtained from $\mathcal{R}_{dm}$ by replacing x by L (for example v(x,t) at x=L becomes $v(L,t)$, shortened v_L). If the position of the center of mass C of the payload relative to $\mathcal{R}_2$ expressed in $\mathcal{R}_2$ is ${}^2\overrightarrow{O_2C} = [c, 0, 0]^T$, then the position of C relative to $\mathcal{R}_1$ expressed in $\mathcal{R}_1$ is given by:

$$ {}^1\overrightarrow{O_1C} = [L + u_L - \frac{1}{2}\int_0^L (v'^2 + w'^2)ds + c(1 - \frac{1}{2}(v'^2_L + w'^2_L)), v_L + c(v'_L - u'_L v'_L), w_L + c(w'_L - u'_L w'_L)]^T \quad (3) $$

Since ${}^1\overrightarrow{O_1C} = {}^1\overrightarrow{O_1O_2} + {}^1R_2\,{}^2\overrightarrow{O_2C}$, and ${}^1R_2$ is deduced from ${}^1R_{dm}$ by replacing x by L; hence, ${}^0\overrightarrow{O_0C} = {}^0R_1\,{}^1\overrightarrow{O_1C}$. The angular velocity of $\mathcal{R}_1$ relative to $\mathcal{R}_0$ expressed in $\mathcal{R}_0$ is ${}^0\overrightarrow{\Omega_{1/0}} = [0, 0, \dot{\theta}]^T$. The angular velocity of $\mathcal{R}_{dm}$ relative to $\mathcal{R}_1$ expressed in $\mathcal{R}_1$ [4] is found from the following matrix

$$ S = {}^1\dot{R}_{dm}\,{}^1R^T_{dm} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} $$

Hence,

$$ {}^1\overrightarrow{\Omega_{dm/1}} = [\omega_x, \omega_y, \omega_z]^T $$

The Taylor expansion of ${}^1\overrightarrow{\Omega_{dm/1}}$ to the second-order in the deformation variables and after simplification gives:

$$ \omega_x \approx \dot{\phi} + \frac{1}{2}(v'\dot{w}' - \dot{v}'w') \qquad \omega_y \approx -\dot{w}' + \dot{u}'w' + u'\dot{w}' + v'\dot{\phi} \qquad \omega_z \approx \dot{v}' - \dot{u}'v' - u'\dot{v}' + \dot{\phi}w' \quad (4) $$

Hence, the angular velocity of $\mathcal{R}_{dm}$ relative to $\mathcal{R}_0$ expressed in $\mathcal{R}_0$ is given by:

$$ {}^0\overrightarrow{\Omega_{dm/0}} = {}^0\overrightarrow{\Omega_{1/0}} + {}^0R_1\,{}^1\overrightarrow{\Omega_{dm/1}} $$

The gravity vector is represented in $\mathcal{R}_0$ by: ${}^0\overrightarrow{g} = [0, 0, -g]^T$.

2.2. Dynamics

2.2.1. Kinetic Energy

The kinetic energy T of the system is the sum of kinetic energies: T_B of the base, T_l of the flexible link and T_p of the payload. Where $T_B = \frac{1}{2}I_B\dot{\theta}^2$, with I_B is the base inertia about the Z_0 axis. The kinetic energy of the link [5] is given by:

$$ T_l = \frac{1}{2}\iiint_V v(P/0)^2 dm = \frac{1}{2}\int_{z=-R}^{R}\int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}}\int_{x=0}^{L} \rho\, v(P/0)^2 dxdydz \quad (5) $$

Since the beam cross-section is circular, $y^2 + z^2 = r^2$, $r \in [0, R]$ and the last triple integral is written [6] as:

$$T_l = \frac{1}{2}\int_{r=0}^{R}\int_{\gamma=0}^{2\pi}\int_{x=0}^{L}\rho v(P/0)^2 r dr d\gamma dx \tag{6}$$

where $y = rcos(\gamma)$, $z = rsin(\gamma)$. Therefore, the kinetic energy of the link linearized to the second-order and after simplifications is given by:

$$\begin{aligned} T_l = \frac{\rho}{2}\Bigg\{ & \pi R^2\int_0^L(\dot{u}^2+\dot{v}^2+\dot{w}^2)dx + \frac{1}{4}\pi R^4\int_0^L(\dot{v}'^2+\dot{w}'^2+2\dot{\phi}^2)dx + \dot{\theta}^2\Bigg[\frac{1}{3}\pi R^2L^3 + \frac{1}{4}\pi R^4 L + \pi R^2\int_0^L(u^2+v^2)dx + \\ & \frac{1}{4}\pi R^4\int_0^L w'^2dx + 2\pi R^2\int_0^L xudx - \frac{1}{2}\pi R^2\int_0^L(L^2-x^2)(v'^2+w'^2)dx\Bigg] + 2\dot{\theta}\Bigg[\pi R^2\int_0^L x\dot{v}dx - \\ & \frac{1}{4}\pi R^4\int_0^L(-\dot{v}'+\dot{u}'v'+u'\dot{v}'-2w'\dot{\phi})dx + \pi R^2\int_0^L(u\dot{v}-\dot{u}v)dx\Bigg]\Bigg\} \end{aligned} \tag{7}$$

The kinetic energy of the payload [7] is expressed by :

$$T_p = \frac{1}{2}\overrightarrow{\Omega_{p/0}}.I_p\overrightarrow{\Omega_{p/0}} + \frac{1}{2}m_p v(C/0)^2 \tag{8}$$

where $I_p = \begin{bmatrix} I_1 & I_4 & I_5 \\ I_4 & I_2 & I_6 \\ I_5 & I_6 & I_3 \end{bmatrix}$ and $\overrightarrow{\Omega_{p/0}}$ is obtained from $\overrightarrow{\Omega_{dm/0}}$ by replacing x by L; hence, the expression of T_p linearized to the second-order in the deformation variables is given by:

$$\begin{aligned} T_p = \frac{1}{2}\Bigg[& I_1\Big(\dot{\phi}_L^2 cos(\theta)^2 + \dot{w}_L'^2 sin(\theta)^2 + 2\dot{\phi}_L\dot{w}_L' cos(\theta)sin(\theta)\Big) + I_2\Big(\dot{\phi}_L^2 sin(\theta)^2 + \dot{w}_L'^2 cos(\theta)^2 - 2\dot{\phi}_L\dot{w}_L' cos(\theta)sin(\theta)\Big) \\ & + I_3\Big(\dot{\theta}^2 + \dot{v}_L'^2 + 2\dot{\theta}(\dot{v}_L' - \dot{u}_L'v_L' - u_L'\dot{v}_L' + \dot{\phi}_L w_L')\Big) + 2I_4\Big((\dot{\phi}_L^2 - \dot{w}_L'^2)cos(\theta)sin(\theta) - \dot{\phi}_L\dot{w}_L'(2cos(\theta)^2-1)\Big) + \\ & 2I_5\Big(\dot{\theta}\Big((\dot{\phi}_L + \frac{1}{2}(v_L'\dot{w}_L' - \dot{v}_L'w_L'))cos(\theta) - (-\dot{w}_L' + \dot{u}_L'w_L' + u_L'\dot{w}_L' + v_L'\dot{\phi}_L)\sin(\theta)\Big) + \dot{v}_L'\dot{\phi}_L cos(\theta) + \dot{v}_L'\dot{w}_L' sin(\theta)\Big) \\ & + 2I_6\Big(\dot{\theta}\Big((\dot{\phi}_L + \frac{1}{2}(v_L'\dot{w}_L' - \dot{v}_L'w_L'))sin(\theta) + (-\dot{w}_L' + \dot{u}_L'w_L' + u_L'\dot{w}_L' + v_L'\dot{\phi}_L)\cos(\theta)\Big) + \dot{v}_L'\dot{\phi}_L sin(\theta) - \dot{v}_L'\dot{w}_L' cos(\theta)\Big)\Bigg] + \\ & \frac{1}{2}m_p\Bigg[\dot{u}_L^2 + \dot{v}_L^2 + \dot{w}_L^2 + c^2(\dot{v}_L'^2 + \dot{w}_L'^2) + 2c(\dot{v}_L\dot{v}_L' + \dot{w}_L\dot{w}_L') + \dot{\theta}^2\Big(L^2 + u_L^2 + v_L^2 + c^2(1-w_L'^2) + 2L\big[u_L - \frac{1}{2}\int_0^L(v'^2+w'^2)ds + \\ & c(1-\frac{1}{2}(v_L'^2+w_L'^2))\big] + 2c(u_L+v_Lv_L') - c\int_0^L(v'^2+w'^2)ds\Big) + 2\dot{\theta}\Big((L+c)(\dot{v}_L + c(\dot{v}_L' - \dot{u}_L'v_L' - u_L'\dot{v}_L')) + u_L(\dot{v}_L + c\dot{v}_L') - \\ & \dot{u}_L(v_L + cv_L')\Big)\Bigg] \end{aligned} \tag{9}$$

2.2.2. Potential Energy

The potential energy V of the system is the sum of potential energies:V_B of the base , V_l of the flexible link and V_p of the payload. The potential energy V_B of the base which is its gravitational potential energy equals a constant C_B because its mass center is fixed in the inertial frame $\mathcal{R}_0$ whose origin level is taken as reference $V_B = C_B$.The potential energy of the link is the sum of its gravitational potential energy and its strain energy:

$$V_l = V_{gravit} + V_{str} \tag{10}$$

V_{gravit} is the gravitational potential energy of the link [7] that equals:

$$V_{gravit} = -\int_{r=0}^{R}\int_{\gamma=0}^{2\pi}\int_{x=0}^{L}\vec{g}\,\overrightarrow{O_0P}\rho r dr d\gamma dx = \rho g\pi R^2\int_{x=0}^{L}w dx \tag{11}$$

V_{str} is the strain energy of the link [8] and it is the sum of strain energies due to different strains:

$$V_{str} = V_u + V_v + V_w + V_\phi \tag{12}$$

The expressions of different strain energies [9] are:

$$V_u = \frac{1}{2}\iiint_V Eu'^2 dV = \frac{1}{2}\pi R^2 E\int_0^L u'^2 dx \qquad V_v = \frac{1}{2}\iiint_V Ev''^2 y^2 dV = \frac{1}{8}\pi R^4 E\int_0^L v''^2 dx \qquad V_w = \frac{1}{2}\iiint_V Ew''^2 z^2 dV = \frac{1}{8}\pi R^4 E\int_0^L w''^2 dx$$
$$V_\phi = \frac{1}{2}\iiint_V Gr^2\phi'^2 dV = \frac{1}{4}\pi R^4 G\int_0^L \phi'^2 dx$$

where E and G are the young modulus and the shear modulus of the beam material respectively. The potential energy of the payload is its gravitational potential energy that equals:

$$V_p = -m_p\,\vec{g}\,\overrightarrow{O_0C} = m_p g\Big(w_L + c(w'_L - u'_L w'_L)\Big) \tag{13}$$

2.3. Rayleigh Dissipation Function

Rayleigh dissipation function $\mathscr{R}$ expresses the energy dissipated due to motor friction and internal damping effect of each deformation (u, v, w, ϕ), the dissipation is based on the Kelvin–Voigt model [10], and can be expressed [11] as follows:

$$\mathscr{R} = \mathscr{R}_{mot} + \mathscr{R}_u + \mathscr{R}_v + \mathscr{R}_w + \mathscr{R}_\phi \tag{14}$$

where

$$\mathscr{R}_u = \frac{1}{2}\iiint_V \sigma_u^d \dot{\epsilon}_u dV = \frac{1}{2}\pi R^2 C_X\int_{x=0}^L \dot{u}'^2 dx \qquad \mathscr{R}_v = \frac{1}{2}\iiint_V \sigma_v^d \dot{\epsilon}_v dV = \frac{1}{8}\pi R^4 C_Y\int_{x=0}^L \dot{v}''^2 dx \qquad \mathscr{R}_w = \frac{1}{2}\iiint_V \sigma_w^d \dot{\epsilon}_w dV = \frac{1}{8}\pi R^4 C_Z\int_{x=0}^L \dot{w}''^2 dx$$
$$\mathscr{R}_\phi = \frac{1}{2}\iiint_V \tau_\phi^d \dot{\gamma}_\phi dV = \frac{1}{4}\pi R^4 C_\Phi\int_{x=0}^L \dot{\phi}'^2 dx \qquad \mathscr{R}_{mot} = \frac{1}{2}b_m\dot{\theta}^2$$

Since

$$|\epsilon_u| = |u'| \;,\; \sigma_u^d = C_X\dot{\epsilon}_u, |\epsilon_v| = |yv''| = |rcos(\gamma)v''| \quad,\quad \sigma_v^d = C_Y\dot{\epsilon}_v,$$
$$|\epsilon_w| = |zw''| = |rsin(\gamma)w''| \quad,\quad \sigma_w^d = C_Z\dot{\epsilon}_w, |\gamma_\phi| = |r\phi'| \quad and \quad \tau_\phi^d = C_\Phi\dot{\gamma}_\phi.$$

2.4. Motion Equations

The extended Hamilton principle [12] is used to get motion equations and boundary conditions: $0 = \int_{t_1}^{t_2}(\delta T - \delta V + T_{mot}\,\delta\theta + \delta\zeta)dt$ where $\delta\zeta$ is the variation of work done by the dissipative forces, its expression is derived from Rayleigh dissipation function as follows: If the expression of Rayleigh dissipation function is given by: $\mathscr{R} = \frac{1}{2}\iiint_V \sigma^d\,\dot{\epsilon}\,dV$, then the expression of work variation $\delta\zeta$ done by dissipative forces is: $\delta\zeta = -\iiint_V \sigma^d\,\delta\epsilon\,dV$. Hence, using the fact that the beam is clamped at the joint i.e., $u(0,t) = v(0,t) = w(0,t) = \phi(0,t) = 0, v'(0,t) = w'(0,t) = 0$

The dynamic equation associated with θ is given by:

$$
\begin{aligned}
T_{mot} = {} & b_m\dot{\theta} + \frac{1}{2}I_B\ddot{\theta} - \Bigg[I_1\Big(cos(\theta)sin(\theta)(\dot{w}_L'^2 - \dot{\phi}_L^2) + \dot{\phi}_L\dot{w}_L'(2cos(\theta)^2 - 1)\Big) + I_2\Big(cos(\theta)sin(\theta)(\dot{\phi}_L^2 - \dot{w}_L'^2) - \dot{\phi}_L\dot{w}_L'(2cos(\theta)^2 - 1)\Big) \\
& + I_4\Big((\dot{\phi}_L^2 - \dot{w}_L'^2)(2cos(\theta)^2 - 1) + 4\dot{\phi}_L\dot{w}_L'cos(\theta)sin(\theta)\Big) + \dot{\theta}\big[(\dot{\phi}_L + \frac{1}{2}(v_L'\dot{w}_L' - \dot{v}_L'w_L'))(-I_5sin(\theta) + I_6cos(\theta)) + \\
& (-\dot{w}_L' + \dot{u}_L'w_L' + u_L'\dot{w}_L' + v_L'\dot{\phi}_L)(-I_5cos(\theta) - I_6sin(\theta))\big] + \dot{v}_L'\big[\dot{\phi}_L\big(-I_5sin(\theta) + I_6cos(\theta)\big) + \dot{w}_L'\big(I_5cos(\theta) + I_6sin(\theta)\big)\big] - \\
& \frac{\partial}{\partial t}\Big(I_3\big(\dot{\theta} + \dot{v}_L' - \dot{u}_L'v_L' - u_L'\dot{v}_L' + \dot{\phi}_Lw_L'\big) + (\dot{\phi}_L + \frac{1}{2}(v_L'\dot{w}_L' - \dot{v}_L'w_L'))(I_5cos(\theta) + I_6sin(\theta)) + \\
& (-\dot{w}_L' + \dot{u}_L'w_L' + u_L'\dot{w}_L' + v_L'\dot{\phi}_L)(-I_5sin(\theta) + I_6cos(\theta)) + m_p\Big\{\dot{\theta}\Big(L^2 + u_L^2 + v_L^2 + c^2(1 - w_L'^2) + 2L\big[u_L - \frac{1}{2}\int_0^L(v'^2 + w'^2)ds + \\
& c\big(1 - \frac{1}{2}(v_L'^2 + w_L'^2)\big)\big] + 2c(u_L + v_Lv_L') - c\int_0^L(v'^2 + w'^2)ds\Big) + \big(L + c\big)\big(\dot{v}_L + c(\dot{v}_L' - \dot{u}_L'v_L' - u_L'\dot{v}_L')\big) + u_L(\dot{v}_L + c\dot{v}_L') - \\
& \dot{u}_L(v_L + cv_L')\Big\}\Big)\Big] + \frac{\rho}{2}\Big\{\frac{\partial}{\partial t}\Big(2\dot{\theta}\Big[\frac{1}{3}\pi R^2L^3 + \frac{1}{4}\pi R^4L + \pi R^2\int_0^L(u^2 + v^2)dx + \frac{1}{4}\pi R^4\int_0^L w'^2dx + 2\pi R^2\int_0^L xudx - \\
& \frac{1}{2}\pi R^2\int_0^L(L^2 - x^2)(v'^2 + w'^2)dx\Big]\Big) + \\
& 2\frac{\partial}{\partial t}\Big(\pi R^2\int_0^L x\dot{v}dx - \frac{1}{4}\pi R^4\int_0^L(-\dot{v}' + \dot{u}'v' + u'\dot{v}' - 2w'\dot{\phi})dx + \pi R^2\int_0^L(u\dot{v} - \dot{u}v)dx\Big)\Big\}
\end{aligned} \tag{15}
$$

⋆ The equation satisfied by u:

$$
0 = \frac{\rho}{2}\Big(-2\pi R^2\ddot{u} + 2\pi R^2\dot{\theta}^2u + 2\pi R^2\dot{\theta}^2x + \pi R^2(4\dot{\theta}\dot{v} + 2\ddot{\theta}v) - \frac{1}{2}\pi R^4\ddot{\theta}v''\Big) + \pi R^2C_X\dot{u}'' + \pi R^2Eu'' \tag{16}
$$

⋆ The equation satisfied by v:

$$
\begin{aligned}
0 = {} & \frac{\rho}{2}\Big(-2\pi R^2\ddot{v} + \frac{1}{2}\pi R^4\ddot{v}'' + 2\pi R^2\dot{\theta}^2v - \pi R^2\dot{\theta}^2(2xv' + (x^2 - L^2)v'') - \pi R^2(4\dot{\theta}\dot{u} + 2\ddot{\theta}u + 2x\ddot{\theta}) - \frac{1}{2}\pi R^4\ddot{\theta}u''\Big) \\
& - \frac{1}{4}\pi R^4C_Y\dot{v}'''' - \frac{1}{4}\pi R^4Ev'''' + m_p\dot{\theta}^2(L + c)v''
\end{aligned} \tag{17}
$$

⋆ The equation satisfied by w:

$$
\begin{aligned}
0 = {} & \frac{\rho}{2}\Big(-2\pi R^2\ddot{w} + \frac{1}{2}\pi R^4\ddot{w}'' - \frac{1}{2}\pi R^4\dot{\theta}^2w'' - \pi R^2\dot{\theta}^2(2xw' + (x^2 - L^2)w'') - \pi R^4\dot{\theta}\dot{\phi}'\Big) - \frac{1}{4}\pi R^4C_Z\dot{w}'''' + m_p\dot{\theta}^2(L + c)w'' \\
& - \frac{1}{4}\pi R^4Ew'''' - \rho g\pi R^2
\end{aligned} \tag{18}
$$

⋆ The equation satisfied by ϕ:

$$
0 = \frac{\rho}{2}\Big(-\pi R^4\ddot{\phi} - \pi R^4(\ddot{\theta}w' + \dot{\theta}\dot{w}')\Big) + \frac{1}{2}\pi R^4C_\Phi\dot{\phi}'' + \frac{1}{2}\pi R^4G\phi'' \tag{19}
$$

⋆ Since the free end of the beam is at $x = L$, the following quantities δu_L, $\delta u_L'$, δv_L, $\delta v_L'$, δw_L, $\delta w_L'$, and $\delta\phi_L$ are arbitrary, therefore the final equations of boundary conditions are:

$$
0 = -\pi R^2C_X\dot{u}_L' + \frac{\rho}{4}\pi R\ddot{\theta}v_L' - \pi R^2Eu_L' - \frac{\partial}{\partial t}\Big[m_p\Big(\dot{u}_L - \dot{\theta}(v_L + cv_L')\Big)\Big] + m_p\big[\dot{\theta}^2(u_L + L + c) + \dot{\theta}(\dot{v}_L + c\dot{v}_L')\big] \tag{20}
$$

$$
\begin{aligned}
0 = {} & -\frac{\partial}{\partial t}\Big[-I_3\dot{\theta}v_L' + w_L'\big(-I_5\dot{\theta}sin(\theta) + I_6\dot{\theta}cos(\theta)\big) - m_p\dot{\theta}c(L + c)v_L'\Big] - I_3\dot{\theta}\dot{v}_L' - I_5\dot{\theta}\dot{w}_L'sin(\theta) + I_6\dot{\theta}cos(\theta)\dot{w}_L' - m_p\dot{\theta}c(L + c)\dot{v}_L' \\
& + m_pgcw_L'
\end{aligned} \tag{21}
$$

$$0 = \frac{1}{4}\pi R^4 C_Y \dot{v}_L''' - \frac{\rho}{4}\pi R^4 \ddot{v}_L' + \frac{\rho}{4}\pi R^4 \ddot{\theta}(u_L' - 1) + \frac{1}{4}\pi R^4 E v_L''' - \frac{\partial}{\partial t}\left[m_p\left(\dot{v}_L + c\dot{v}_L' + \dot{\theta}(L + c + u_L)\right)\right] + m_p\left(\dot{\theta}^2(v_L + cv_L') - \dot{\theta}\dot{u}_L - \dot{\theta}^2(L + c)v_L'\right) \tag{22}$$

$$0 = -\frac{1}{4}\pi R^4 C_Y \dot{v}_L'' - \frac{1}{4}\pi R^4 E v_L'' - \frac{\partial}{\partial t}\left[I_3\left(\dot{v}_L' + \dot{\theta}(1 - u_L')\right) + I_5\left(-\frac{1}{2}\dot{\theta}w_L' cos(\theta) + \dot{\phi}_L cos(\theta) + \dot{w}_L' sin(\theta)\right) + I_6\left(-\frac{1}{2}\dot{\theta}w_L' sin(\theta) + \dot{\phi}_L sin(\theta) - \dot{w}_L' cos(\theta)\right) + m_p\left(c^2\dot{v}_L' + c\dot{v}_L + \dot{\theta}c(L + c)(1 - u_L') + \dot{\theta}cu_L\right)\right] - I_3\dot{\theta}\dot{u}_L' + I_5\left(\frac{1}{2}\dot{\theta}\dot{w}_L' cos(\theta) - \dot{\theta}\dot{\phi}_L sin(\theta)\right) + I_6\left(\frac{1}{2}\dot{\theta}\dot{w}_L' sin(\theta) + \dot{\theta}\dot{\phi}_L cos(\theta)\right) + m_p\left(\dot{\theta}^2(-Lcv_L' + cv_L) - \dot{\theta}c(L + c)\dot{u}_L' - \dot{\theta}c\dot{u}_L\right) \tag{23}$$

$$0 = \frac{1}{4}\pi R^4 C_Z \dot{w}_L''' + \frac{1}{4}\pi R^4 E w_L''' - m_p g - \frac{\rho}{4}\pi R^4 \ddot{w}_L' + \frac{\rho}{4}\pi R^4 \dot{\theta}^2 w_L' + \frac{\rho}{2}\pi R^4 \dot{\theta}\dot{\phi}_L - m_p(\ddot{w}_L + c\ddot{w}_L') - m_p\dot{\theta}^2(L + c)w_L' \tag{24}$$

$$0 = -\frac{1}{4}\pi R^4 C_Z \dot{w}_L'' - \frac{1}{4}\pi R^4 E w_L'' - m_p g c(1 - u_L') - \frac{\partial}{\partial t}\left[I_1\left(sin(\theta)^2\dot{w}_L' + cos(\theta)sin(\theta)\dot{\phi}_L\right) + I_2\left(cos(\theta)^2\dot{w}_L' - cos(\theta)sin(\theta)\dot{\phi}_L\right) + I_4\left(-2\dot{w}_L' cos(\theta)sin(\theta) - \dot{\phi}_L(2cos(\theta)^2 - 1)\right) + I_5\left(\frac{1}{2}\dot{\theta}v_L' cos(\theta) + sin(\theta)(\dot{v}_L' - \dot{\theta}(u_L' - 1))\right) + I_6\left(\frac{1}{2}\dot{\theta}v_L' sin(\theta) + cos(\theta)(-\dot{v}_L' + \dot{\theta}(u_L' - 1))\right) + m_p\left(c^2\dot{w}_L' + c\dot{w}_L\right)\right] + I_3\dot{\theta}\dot{\phi}_L + I_5\left(-\frac{1}{2}\dot{\theta}\dot{v}_L' cos(\theta) - \dot{\theta}\dot{u}_L' sin(\theta)\right) + I_6\left(-\frac{1}{2}\dot{\theta}\dot{v}_L' sin(\theta) + \dot{\theta}\dot{u}_L' cos(\theta)\right) - m_p\dot{\theta}^2 c(L + c)w_L' \tag{25}$$

$$0 = -\frac{1}{2}\pi R^4 C_\Phi \dot{\phi}_L' - \frac{1}{2}\pi R^4 G\phi_L' - \frac{\partial}{\partial t}\left[I_1\left(cos(\theta)^2\dot{\phi}_L + cos(\theta)sin(\theta)\dot{w}_L'\right) + I_2\left(sin(\theta)^2\dot{\phi}_L - cos(\theta)sin(\theta)\dot{w}_L'\right) + I_4\left(2cos(\theta)sin(\theta)\dot{\phi}_L - \dot{w}_L'(2cos(\theta)^2 - 1)\right) + I_5\left((\dot{\theta} + \dot{v}_L')cos(\theta) - \dot{\theta}v_L' sin(\theta)\right) + I_6\left((\dot{\theta} + \dot{v}_L')sin(\theta) + \dot{\theta}v_L' cos(\theta)\right)\right] \tag{26}$$

u, v, ϕ must also satisfy these conditions: $u(x,0) = \lim_{t\to\infty} u(x,t) = 0$, $v(x,0) = \lim_{t\to\infty} v(x,t) = 0, \phi(x,0) = \lim_{t\to\infty} \phi(x,t) = 0$ and w must satisfy $w(x,0) = \lim_{t\to\infty} w(x,t) = \tilde{w}(x)$ whose expression [13] is given by: $\tilde{w}'(x) = tan\left(\frac{x(2a-x)}{2b}\right)$, since $\tilde{w}(0) = 0$, then $\tilde{w}(x) = \int_0^x tan\left(\frac{l(2a-l)}{2b}\right)dl$, where $a = L - \delta$, $b = \frac{EI}{F}$, the expression of the foreshortening δ term due to beam bending [14] is given by: $\delta = -\frac{1}{2}\int_0^L \tilde{w}'^2(x)dx$, where payload weight F equals $m_p g$, and beam area second moment I equals:$I = \iint y^2 dydz = \frac{\pi R^4}{4}$.

3. Discussion

Considering the reference of angle θ is zero when the manipulator is at rest ($t = 0$) and the angular velocity is constant ($\dot{\theta} = \Omega$), then θ and $\dot{\theta}$ are replaced by Ωt and Ω respectively in the equations of the previous section. Equation (16) yields $\dot{u} = L_1(v)$, taking the time derivative of Equation (15) and using the last expression yields $L_2(v) = 0$. Equation (17) yields $\dot{\phi}' = L_3(w) + c$, taking both time and spatial derivatives of Equation (18) and using the last expression yields $L_4(w) = 0$, where c is a constant and L_1, L_2, L_3, L_4 are linear operators. Hence, the motions equations are decoupled but the boundary conditions are still coupled. The goal of future work is to develop a numerical method for solving previous partial differential equations with coupled boundary conditions while ensuring the stability

of the solutions. Once the solutions are found, the mechanical modeling will be generalized to flexible manipulators with serial links where the payload attached to each link is the rest of the chain.

4. Conclusions

Modeling the single-link flexible manipulator as an Euler–Bernoulli beam with a payload at its free end subjected to small deformations, and using a rotation matrix orthogonal to the second-order of Taylor expansion in the deformations variables, the extended Hamilton principle is applied to get both the motion equations and boundary conditions. The motion partial differential equations are decoupled when the angular velocity is constant. Once the solutions are available, it will help to study more accurately the movements of flexible manipulators and to find new techniques for robust control of such systems.

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Nomenclature

T_{mot}	Torque applied by the motor
R	Beam cross section radius
L	Beam length
m_p	Payload mass
I_p	Payload inertia matrix
$u(x,t)$	Axial stretching
$v(x,t)$	Horizontal deflection
$w(x,t)$	Vertical deflection
$\phi(x,t)$	Torsional deformation
R_0	Inertial frame
R_1	Motor frame
R_{dm}	Cross section frame
R_2	Beam free end frame
O_0,O_1,O_{dm},O_2	Origins of frames R_0,R_1,R_{dm},R_2 respectively
C	Payload center of mass
${}^i\overrightarrow{v}$	Vector v expressed in frame R_i
${}^0\overrightarrow{\Omega_{1/0}}$	Rotation vector of frame R_1 relative to R_0 expressed in R_0
${}^0\overrightarrow{\Omega_{dm/0}}$	Rotation vector of frame R_{dm} relative to R_0 expressed in R_0
${}^1\overrightarrow{\Omega_{dm/1}}$	Rotation vector of frame R_{dm} relative to R_1 expressed in R_1
w_x,w_y,w_z	${}^1\overrightarrow{\Omega_{dm/1}}$ coordinates in frame R_1
T,T_B,T_l,T_p	Kinetic energy of the system,base,link,payload
I_1,I_2,I_3,I_4,I_5,I_6	Inertia matrix coefficients
ρ	Beam mass density
θ	Rotation angle of frame R_1 relative R_0
V,V_B,V_l,V_p	Potential energy of the system,base,link,payload
V_{gravit},V_{str}	link gravitational and strain potential energy
V_u,V_v,V_w,V_ϕ	Strain energies due to u, v, w, ϕ
$\mathscr{R}_{mot}$,$\mathscr{R}_u$,$\mathscr{R}_v$,$\mathscr{R}_w$,$\mathscr{R}_\phi$	Rayleigh dissipation function due to motor,u,v,w,ϕ
E, G	Beam young moduls and shear modulus
$\epsilon_u, \epsilon_v, \epsilon_w$	strains due to u, v, w

$\sigma_u^d, \sigma_v^d, \sigma_w^d, \tau_\phi^d$	stress due to internal damping caused by u,v,w and
C_X, C_Y, C_Z	Internal damping coefficient along x_1,y_1,z_1 axis
γ_ϕ,τ_ϕ^d	Shear strain and shear stress due to torsional damping
ζ, F	Work done by dissipative forces and payload weight
δ, b_m	Foreshortening term,motor viscous friction coefficient
C_ϕ, I	Torsional deformation coefficient, area second moment

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