

Investigation of Methods for Generating Smooth Continuous Curves Resulting from the Intersection of Conical Surfaces [†]

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Abstract

Studied extensively in the present paper is a geometric problem related to the intersection of two conical surfaces with crossed axes. The goal was to derive a spatial intersection curve between the surfaces that resembles the classical Viviani curve but is defined as the intersection of two rotational conical surfaces with crossed axes, forming an asymmetric spatial figure-of-eight shape. These unique spatial curves, as variations of the Viviani curve, possess intriguing geometric properties and are likely to have a wide range of potential applications in the fields of construction and engineering. Their presence can often be linked to the development of complex geometric shapes in architecture, particularly in the design of bridges with unconventional curves in their support structures or situations where inclined support beams intersect with other structural components. They are commonly employed in the construction of tunnels with conical and cylindrical shapes, in addition to various types of vaults and domes. These curves are critical in optimizing structures, such as roofs with intricate geometries or decorative façade elements, and can greatly enhance both their stability and aesthetic appeal.

Keywords: design; applied geometry; smooth continuous curves; intersection of cones

1. Introduction

When solving descriptive geometry problems related to the intersection of rotational surfaces, it is typical for the axes of these surfaces to intersect. This configuration is frequently encountered in architecture and engineering. In general, the solutions can be summarized in a theorem stating that when the axes of two second-degree rotational surfaces intersect, the rectilinear projection of the line of their mutual intersection onto a plane defined by their axes is a hyperbola. The solution, in specific cases, can be further simplified to two second-degree curves (intersecting ellipses) [1]. Occasionally, different arrangements emerge where the axes of intersecting rotational surfaces do not intersect but cross each other. Addressing these problems, in most cases, is quite feasible, as their solutions can usually be achieved by using the theorem mentioned earlier.

Despite employing an established method for solving the specific problem, such as intersecting surfaces with a bundle of planes intersecting along either a finite or infinite straight line, the solution to the particular problem of generating a smooth continuous curve in the shape of a spatially asymmetric intersection in the form of a lemniscate, resulting from the intersection of two conical surfaces with crossed axes, proves to be quite challenging, especially in terms of identifying the location of the node of the spatially



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asymmetric lemniscate. The node should be located at the intersection point of generators from each of the intersecting adjacent conical surfaces (Figure 1). The two intersecting generators define a plane that also contains the vertices of the conical surfaces. Establishing the exact location of this point through the use of conventional methods, like the Monges projection technique, however, is ineffective. An alternative approach to determining the node's precise location is to apply empirical optimization; however, this involves numerous labour-intensive constructions and likely a number of unsuccessful attempts. The objective of the present paper is to develop a method that enables the accurate positioning of the node on the smooth continuous curve at the very outset of the problem-solving process, as failure to do so may once again result in solutions resembling hyperbolic curves.

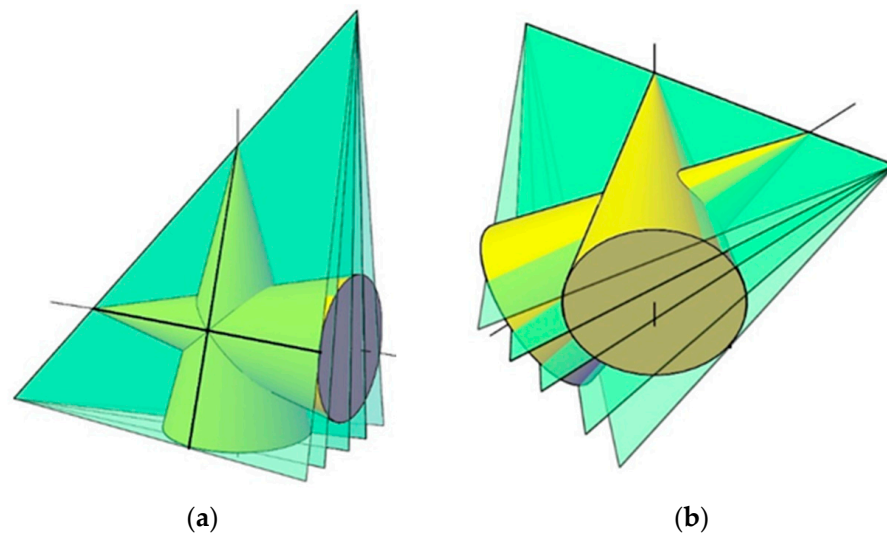


Figure 1. Method wherein a bundle of planes intersects along a finite line: (a) Front view; (b) Rear view.

2. Materials and Methods

All attempts to obtain a solution in the form of a smooth continuous lemniscate-like curve, resulting from the intersection of cones with crossed axes using Monge's projection method, were unsuccessful; instead, a hyperbolic solution was consistently produced.

Securing such a solution seems highly plausible through 3D modelling; however, the achievement of the desired spatial configuration with the Monge projection method relies on accurately defining the necessary conditions for the placement of the cones (Figure 2). Even though the cones share skew axes and intersect along one of their generators, the task of accurately identifying the intersecting generators for each cone remains unresolved.

The proper solution was actually achieved by adopting a theorem that stated that if two second-degree rotational surfaces with intersecting axes are described around a common sphere, their intersection breaks down into two second-degree curves that are symmetric with respect to the plane defined by the axes of the two intersecting surfaces. In an axonometric orthographic projection, the intersection of the cones results in two intersecting ellipses (Figure 3) [2,3]. It appears that the node we are searching for is positioned at the intersection point of these ellipses, where the intersecting generators of each of the two conic surfaces pass through. These two intersecting generators construct a plane that acts as a tangent to both conic surfaces (Figure 1) [4,5].

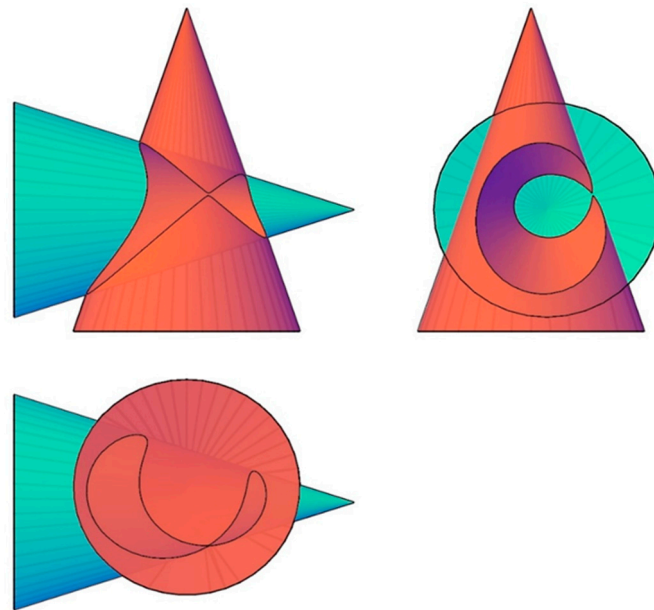


Figure 2. Intersection of conical surfaces with crossed axes and a distinctive spatial smooth continuous curve in the form of an asymmetric lemniscate, following the Monge method.

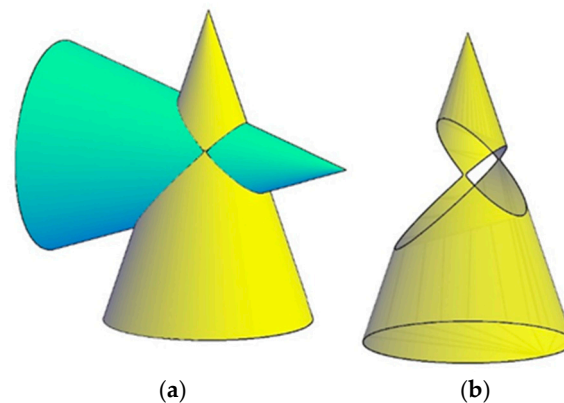


Figure 3. A curve that splits into two second-degree lines, symmetric with respect to the plane defined by the axes of the two intersecting surfaces: (a) two intersecting cones with coincident axes; (b) one cone is removed to expose the spatial curve of intersection.

3. Results and Discussions

Capitalizing on this situation, we could preserve the position of one of the previously acquired nodes by merely sliding one of the conical surfaces along its generator to a different location, where the axes no longer intersect but cross each other instead. This arrangement ensures the consistency of the angular divergence at the vertex of the conical surface (Figure 4b, depicted in red). This creates a configuration that enables the initiation of the problem-solving process aimed at obtaining the desired smooth continuous curve in the form of a spatially asymmetric lemniscate.

In Figure 4a, the frontal projection illustrates the sphere (in green) that is the tangent to the outlying generators of the two conical surfaces and has its centre at the intersection of their axes [6]. Easily distinguishable is also the sought-after intersection between the two conical surfaces, marking the spot where the two second-degree curves intersect. Flowing through the obtained node are the generators of each conical surface depicted in magenta. Displayed in Figures 4b and 5a is the conical surface that has shifted and one of the nodes that has been preserved.

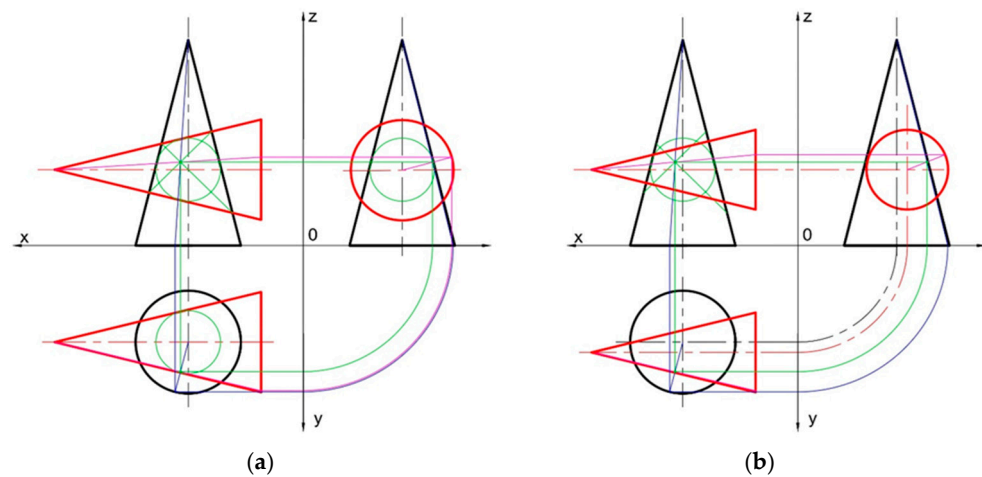


Figure 4. Determining the location of the node on the smooth continuous curve in the form of a spatially asymmetric lemniscate: (a) frontal projection; (b) shifted conical surface.

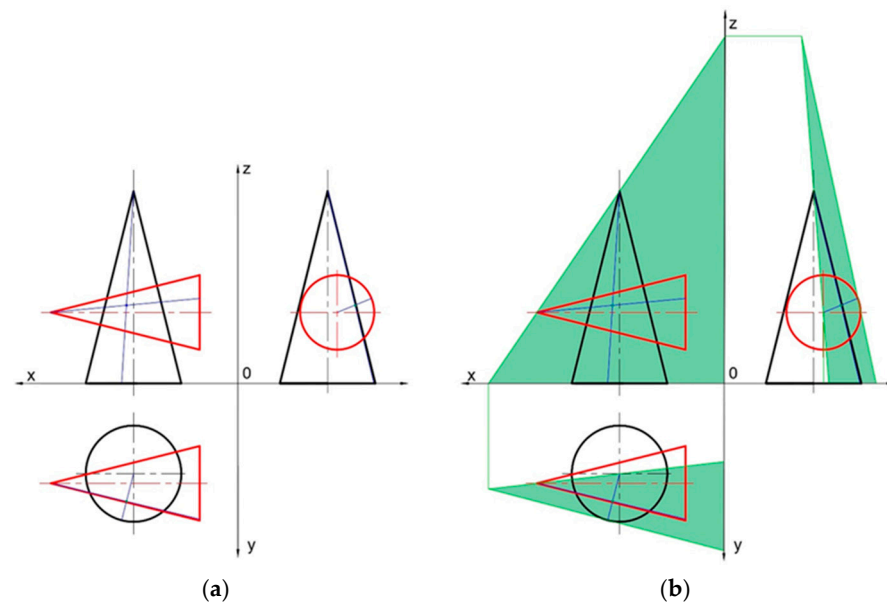


Figure 5. Determining the location of the two generators (a) and constructing the plane in which they lie (b).

The solution to the problem is found through the application of the method “a bundle of planes intersecting along a finite line”, running between the vertices of the two conical surfaces, as outlined above (Figure 5b). Emerging, subsequently, is the intersection of the two conical surfaces with a multitude of planes, each of which intercepts the cones to produce pairs of intersecting triangles (Figure 1). The intersection points between these pairs of triangles lie on the intersection curve of the conical surfaces, which takes the form of a smooth continuous spatial curve resembling an asymmetric lemniscate.

The problem could also be solved by applying the method of a bundle of planes intersecting along an infinite line; however, this approach would significantly complicate the solution, as the secant plane from one cone generates circular sections, while the sections from the other cone take the shape of a hyperbola.

The complete solution to the problem of constructing the smooth continuous spatial curve resembling an asymmetric lemniscate is illustrated in Figure 6.

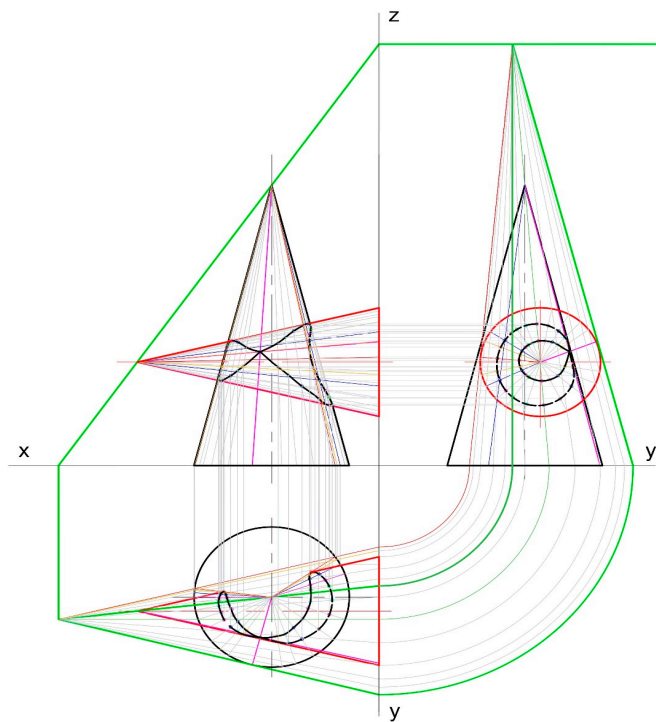


Figure 6. The final solution to the problem resulting in a smooth continuous spatial curve resembling an asymmetric lemniscate.

The solution, in the frontal (defined by XZ) and horizontal (defined by XY) projection planes, manifests itself as a distorted lemniscate.

The projection onto the profile (YZ) projection plane yields a closed, asymmetric curve in the shape of a cardioid.

Thus, in the provided example, the crossed axes of the conical surfaces form a right angle when projected using Monge's method. However, curves in the form of a spatially asymmetric lemniscate can also be obtained when the axes form an angle different from 90 degrees.

Depicted in Figure 7 are simulations of cones intersecting at crossed axes to illustrate the generation of the desired curve, visualizing the concept defined in the preceding statement. The method for identifying the position of the node along the smooth continuous curve in the form of a spatially asymmetric lemniscate during the initial stages of a similar problem-solving process, discussed above, proves to be efficient even if the angle between the two crossed axes of the conical surfaces is different from 90 degrees.

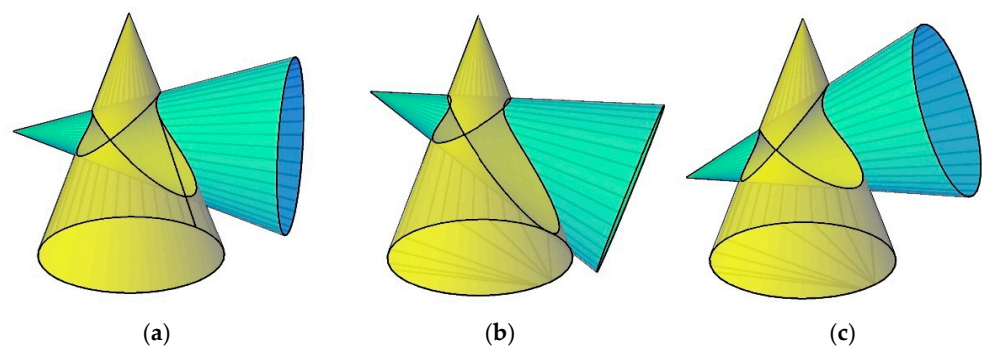


Figure 7. Variation in the angle between the crossed axes affects only the size of the loops of the spatially asymmetric lemniscate: (a) the axes of the cones intersect at a perpendicular angle; (b) the axes of the cones intersect at an angle exceeding 90 degrees; (c) the axes of the cones intersect at an angle less than 90 degrees.

4. Conclusions

Smooth continuous curves in the form of spatially asymmetric lemniscates appear in various fields such as architecture, mechanical engineering, optics, and other engineering sciences, where intersections between conical and/or cylindrical surfaces generate geometrically complex and practically valuable forms. Their applicability in the analysis and design of intricate structures and mechanisms makes them especially beneficial in projects that demand high precision and structural stability.

The present paper proposes an effective method for accurately determining the location of the node of a smooth continuous curve in the form of a spatially asymmetric lemniscate, obtained from the intersection of two conical surfaces with crossed axes, during the initial stages of the problem-solving process, by defining the geometric requirements for its construction.

The developed method for generating a smooth continuous curve in the form of a spatially asymmetric lemniscate through the intersection of conical surfaces with crossed axes represents an extension of classical geometric concepts. The study demonstrates that non-trivial configurations of the conical surface axes lead to a deformed curve that inherits key characteristics of the Viviani curve, such as closure and a dynamic structure resulting from the crossed axes. The implementation of analytical and numerical methods in synergy with descriptive geometry methods for constructing and representing intersection lines enables accurate modelling of intricate geometric shapes with potential applications in engineering, computer graphics, architecture, and other fields.

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