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Review

Remarks on Constitutive Modeling of Granular Materials

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Abstract: In this paper, we provide a brief overview of certain fundamental concepts which can be used to derive constitutive relations for the stress tensor of granular materials. These include concepts such as dilatancy, cohesion, yield criterion, shear banding, etc. The focus will be on the constitutive relations which are used in the so-called 'frictional flow' or 'slow flow' regime as opposed to the rapid flow regime; in the slow flow regime the material is about to yield or has just yielded and the flow has been initiated. This type of flow occurs in the storage of grains, etc., in silos and bins or hoppers after the valves/gates are opened. The techniques of continuum mechanics are used to discuss constitutive relations where the effects of non-linearities such as yield stress, dilatancy, density gradients, etc., are important.

Keywords: granular materials; dilatancy; yield stress; hypoplasticity; constitutive equations

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1. Introduction

Granular materials exist naturally, for example, sand, soil, ice, snow, etc. [1]. Each of these can be the subject of studies which can have a significant impact on our environment, for example, in cases of avalanches, mudflows, debris flow, etc. [2]. Granular materials also play a major role in agricultural and food industries, where rice and other kinds of grains have to be stored, processed, and transported.

Early studies of (flowing) granular materials were concerned with the engineering design of bins and silos. The inaccuracy of many of these theories for dynamic conditions of loading or emptying often resulted in failure of bins or silos. Despite wide interest and many decades of experimental and theoretical investigations, the behavior of granular materials is not well understood. Flowing granular materials, in some sense, are two-phase flows at high particle (solid) concentration and high solid-to-fluid density ratios. Confined flows of granular materials in bins and hoppers were the primary motivation to formulate quasi-static theories where the inertia effects were neglected. This helped in determining bin and hopper geometries which allow the bulk solid to flow freely and also allow for the ability to predict the stresses in the walls to avoid failure [3–5].

For industrial applications, such as fluidized beds, powder technology, slurry transport, etc., different theories have been developed to describe the behavior of flowing granular materials. Bagnold's studies [6,7] have led to many formulations where granular materials are considered to behave as non-Newtonian fluids [8–15]. Many researchers have developed rate-independent theories [16], plasticity theories [17], viscoelastic theories [18], and hypoplastic theories [19]. Theories with microstructure have also been proposed [20–22]. A general continuum theory with thermodynamical restrictions was proposed in [23,24]. This was subsequently modified, extended, and generalized by various researchers [25,26]. There are many excellent review articles [27–33] and books [34–37]. In this paper, we do not discuss or review the numerical or computational approaches used to study granular materials, nor do we talk about the experimental techniques used in characterizing granular materials. The emphasis here is on providing an overview of a few fundamental concepts which may be used in mathematical modeling of granular material using the techniques of continuum mechanics.

In this paper, for the sake of completeness and to show the necessity of constitutive relations, we first provide the basic governing equations. In Section 3, we discuss a few important physical concepts, such as dilatancy, yield criterion, etc., which can be used as guidelines to mathematically model granular materials, i.e., to derive constitutive relations for the stress tensor of granular materials which have just yielded and are flowing. This regime is usually known as the slow flow or the frictional flow, as opposed to the rapid flow regime. In the slow flow regime, there is a lot of contact between the particles, and friction plays a major role; the particles slide over each other and move slowly. In the rapid flow regime, there is a lot of collision between the particles, and the elasticity or inelasticity of collisions become important, and as a result concepts such as 'granular temperature' related to the fluctuation of the velocity may have to be considered. In this regime, techniques of turbulence modeling or the kinetic theory formulation could be used. In Section 4, we provide a short discussion of a few early constitutive models used in granular materials.

2. Governing Equations

For a densely packed system, the particles' response, in general, is governed by interparticle cohesion, friction, and possibly collisions. In some cases, we can ignore the presence of the fluid phase and consider the system as a single-phase system composed of particles only. The governing equations, when there are no chemical reactions, are the conservation of mass, conservation of linear momentum, conservation of angular momentum, and conservation of energy. Conservation of mass in the Lagrangian form is:

$$\rho_0 = \rho \det \boldsymbol{F} \tag{1}$$

where, ρ_0 is the reference density of the material, ρ is the current density, and $\boldsymbol{F}$ is the deformation gradient which is given by [38,39]:

$$\boldsymbol{F} = \frac{\partial \chi}{\partial \boldsymbol{X}} \tag{2}$$

where $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$, where χ is the mapping from the reference configuration to the present one. The conservation of mass in the Eulerian form is given by:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{3}$$

where $\partial/\partial t$ is the partial derivative with respect to time, *div* is the divergence operator, and $\mathbf{v}$ is the velocity vector. The conservation of linear momentum is given by

$$\rho \frac{d\mathbf{v}}{dt} = div\boldsymbol{T} + \rho \boldsymbol{b}, \tag{4}$$

where d/dt is the total time derivative given by $d(.)/dt = \partial(.)/\partial t + [grad(.)]\mathbf{v}$ and *grad* is the gradient operator, $\boldsymbol{b}$ is the body force vector, and $\boldsymbol{T}$ is the Cauchy stress tensor. The balance of angular momentum (when there are no couple stresses) indicates that the Cauchy stress is symmetric.

The energy equation is:

$$\rho \frac{d\varepsilon}{dt} = \boldsymbol{T}\cdot\boldsymbol{L} - div\boldsymbol{q} + \rho r \tag{5}$$

where ε is the specific internal energy, $\boldsymbol{q}$ is the heat flux vector, r is related to the radiation effects, and $\boldsymbol{L}$ is the velocity gradient. Thermodynamical considerations require the application of the second law of thermodynamics, where its local form (the entropy inequality) is given by ([39], p. 130):

$$\rho \dot{\eta} + div\boldsymbol{\varphi} - \rho s \geq 0 \tag{6}$$

where $\eta(\boldsymbol{x}, t)$ is the specific entropy density, $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ is the entropy flux, and s is the entropy supply density due to external sources, and the superposed dot denotes the material time derivative. If,

$$\boldsymbol{\varphi} = \frac{1}{\theta}\boldsymbol{q} \tag{7}$$

$$s = \frac{1}{\theta}r \tag{8}$$

where θ is the absolute temperature, then Equation (6) reduces to the Clausius–Duhem inequality,

$$\rho\dot{\eta} + div\frac{\boldsymbol{q}}{\theta} - \rho\frac{r}{\theta} \geq 0 \tag{9}$$

For a complete study of a thermo-mechanical problem, the second law of thermodynamics must be considered [40,41]. For a discussion of important concepts in constitutive equations of mechanics, we refer the reader to the following books [42–44]. In general, there seems to be no agreement on the functional form of the constitutive relation, and with the Helmholtz free energy as an unknown, a complete thermodynamical treatment of granular materials is lacking.

In the next section we will look at a few fundamental concepts, some of which can be used for modeling the stress tensor $\boldsymbol{T}$.

3. A Few Fundamental Concepts

Granular materials can sometimes behave like a solid material and sometimes like a fluid. Like liquids, they take the shape of the container, or they can be heaped, behaving like a solid. Bulk solids are composed of a variety of materials, and as a result it is difficult to characterize them, because small variations in primary properties such as size, shape, hardness, particle density, surface roughness, etc., can result in different behavior [45]. Other factors such as moisture, the severity of prior compaction, the temperature, etc., not directly associated with the particles, can significantly influence the behavior of the bulk solids. According to [46], a granular material covers any particulate material with particles ranging in size from about 10 μm up to 3 mm. A powder is composed of particles up to 100 μm (diameter) with further subdivision into ultrafine (0.1 to 1.0 μm), superfine (1 to 10 μm), or granular (10 to 100 μm) particles. A granular solid covers the range from about 100 to 3000 μm. This includes most of the materials used in laboratory experiments. They [46] define a bulk solid as:

> "*An assembly of discrete solid components dispersed in a fluid such that the constituents are substantially in contact with near neighbors. This definition excludes suspensions, fluidized beds, and materials embedded in a solid mixture.*"

3.1. Dilatancy

Early experimental studies of granular materials were conducted by [47]; he studied the flow of sand in tubes. Later, it was observed [48] that for a shearing motion to occur in a bed of densely packed particles, the bed must expand to increase the volume of the voids. Reynolds [49] termed this phenomena "**dilatancy**". He used this idea to describe the capillary action in wet sand. Reynolds provided an insightful description of dilatancy in sand:

> "*At one time the sand will be so firm and hard that you may walk with high heels without leaving a footprint; while at others, although the sand is not dry, one sinks in so as to make walking painful. Had you noticed, you would have found that the sand is firm as the tide falls and becomes soft again after it has been left dry for some hours. The tide leaves the sand, though apparently dry on the surface, with all its interstices perfectly full of water which is kept up to the surface of the sand by capillary attraction; at the same time the water is percolating through the sand from the sands above where the capillary action is not sufficient to hold the water. When the foot falls on this water-saturated sand, it*

> *tends to change its shape, but it cannot do this without enlarging the interstices—without drawing in more water. This is a work of time, so that the foot is gone again before the sand has yielded."*

The relationship between the stress in granular materials and the voidage was also mentioned [49]:

> *"Taking a small indiarubber bottle with a glass neck full of shot and water, so that the water stands well into the neck. If instead of shot the bag were full of water or had anything of the nature of a sponge in it, when the bag was squeezed, the water would be forced up the neck. With the shot the opposite result is obtained; as I squeeze the bag, the water decidedly shrinks in the neck... When we squeeze a sponge between two planes, water is squeezed out; when we squeeze sand, shot, or granular material, water is drawn in."*

The work of Reynolds was followed by the experimental studies in [50–53] to name a few. Many models include the effects of dilatancy in the theory [54–57]. In fact, Reiner [8] was one of the first who used a non-Newtonian model to describe 'dilatancy' in wet sand. This model, however, does not explain how the voidage (volume fraction) can influence the stress [58].

To measure the strength of granular materials with various packing densities, Casagrande [59] performed many experiments, where he observed:

> *"During the shearing test on dense sand, the shearing stress reaches a peak value and if the deformation continued, the shearing stress drops to a smaller value, at which value it remains constant for all further deformations. During the drop in shearing stress, the sand continues to expand, finally reaching a critical value at which continuous deformation is possible at constant sharing stress.*
>
> *When a loose sample of sand is subjected to shearing test under constant normal pressure, however, the shearing stress simply increases until it reaches the shearing strength and if the deformation is continued beyond this point the resistance remains unchanged. The volume of the sand in this state must correspond to the critical void ratio which is reached when performing a test on the same material in dense state therefore the curves representing the volume changes during shearing tests on material in the dense and the loose state must meet at the critical void ratio when stationary condition is established."*

The dilation or contraction of granular materials depend on the initial void ratio and the void ratio at the critical state, which is defined as the continued plastic shearing without change in volume or stress [60]. It is also possible to think that the dilatancy or contractancy of granular materials are due to the coupling of the deviatoric stresses and strains, i.e., the volumetric strain is affected by the deviatoric stress and vice versa [61]. In the course of tri-axial compression tests with dense granular materials, an initial contractancy is followed by dilatancy. The initial contractancy is attributed to compression since the volumetric decrease is accompanied by an increase in the hydrostatic stress. The dilatancy angle depends on factors such as the stress level, the density, and the friction coefficient, and it approaches zero when the friction coefficient approaches the critical value of the friction coefficient [62,63].

3.2. Cohesionless and Cohesive Materials

The response of granular materials such as soils ranges between those of plastic clay [64] and clean, perfectly dry sand. Different slopes, including riverbanks, seacoast bluffs, hills, mountains, etc., remain in place because of the shearing strength possessed by the soil or the rock [65]. If we dig a hole into a bed of dry (or completely immersed) sand, the material at the sides of the hole would slide toward the bottom. This indicates a complete absence of a bond between the individual particles [66]. This sliding continues until the angle of inclination becomes equal to the 'angle of repose.' Brown and Richards [46] define two angles of repose as:

> *"The angle to the horizontal assumed by the free surface of a heap at rest and obtained under stated conditions:*
>
> *(i) The poured angle of repose is formed by pouring the bulk solid to form a heap below the pour point.*
>
> *(ii) The drained angle of repose is formed by allowing a heap to emerge as superincumbent powder is allowed to drain away past the periphery of a horizontal flat platform previously buried in the powder."*

Different techniques to measure this angle are given by Wieghardt [67]. The internal angle of friction is related to the cohesion present in the material. The bond between the particles is influenced by a variety of forces including Van der Waals' forces, Coulomb forces, and capillary forces [68]. It is difficult to assign a definite angle of repose to a granular material with cohesion, since the steepest angle at which such a material can stand decreases with the increasing height of the slope [66]. The properties of granular materials are very complex, and a rigorous mathematical analysis of their behavior seems impossible. At times it has been necessary to study idealized cases such as ideal sands (cohesionless granular material) or ideal clays (ideally cohesive material, i.e., no internal friction) [16,69].

3.3. Yield Criterion

The Coulomb failure criterion [70,71], based on experiments, states that yielding occurs when

$$|S| = b_0 T + c \tag{10}$$

Here S and T are the shear stress and the normal stress, respectively, acting on a plane at a point; c is the coefficient of cohesion; and b_0 is the coefficient of static friction related to the internal angle of friction ϕ through

$$\mathrm{b}_0 = \tan \phi \tag{11}$$

When cohesion is absent (c = 0), a granular medium is considered an ideal one. One in which internal friction is absent (ϕ = 0) is called an ideally cohesive medium. For dry, coarse materials, the cohesion coefficient can be neglected. There are other yield criteria which can be used [72].

To give a more rigorous mathematical structure to the Mohr–Coulomb criterion, let $(\boldsymbol{s}, \boldsymbol{n})$ be a pair of orthonormal unit vectors in the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$-plane. The unit vector, $\boldsymbol{s}$, denotes the tangential direction of a surface element which is perpendicular to the $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$-plane, and the unit vector, $\boldsymbol{n}$, is in the direction normal to $\boldsymbol{s}$. The shear stress and the compressive normal stress on the surface element are given by

$$\boldsymbol{\tau} = \boldsymbol{s} \cdot \boldsymbol{\sigma} \boldsymbol{n}, \tag{12}$$

$$\sigma = -\boldsymbol{n} \cdot \boldsymbol{\sigma} \boldsymbol{n} \tag{13}$$

respectively. When a Mohr–Coulomb yield condition is assumed, it implies that the shearing resistance is overcome on those planes for which the yield function

$$f = \tau - c - \mu\sigma \tag{14}$$

is a maximum, where 'c' is the cohesion parameter and μ is the friction coefficient. For given values of c and μ, there are two directions for which the yield function f is a maximum. These two directions which are the potential slip lines are symmetrically situated about the maximum principal stress direction $\hat{\boldsymbol{e}}_1$, making an angle $\frac{\pi}{4} + \frac{\beta}{2}$ with respect to this direction, see Figure 1. The two potential slip lines are denoted by [73]:

$$\boldsymbol{s}^1 = cos\xi \hat{\boldsymbol{e}}_1 + sin\xi \hat{\boldsymbol{e}}_2, \tag{15}$$

$$\boldsymbol{s}^2 = cos\xi\hat{\boldsymbol{e}}_1 - sin\xi\hat{\boldsymbol{e}}_2, \tag{16}$$

$$\boldsymbol{n}^1 = sin\xi\hat{\boldsymbol{e}}_1 - cos\xi\hat{\boldsymbol{e}}_2, \tag{17}$$

$$\boldsymbol{n}^2 = sin\xi\hat{\boldsymbol{e}}_1 + cos\xi\hat{\boldsymbol{e}}_2 \tag{18}$$

where $\xi = \frac{\pi}{4} + \frac{\beta}{2}$.

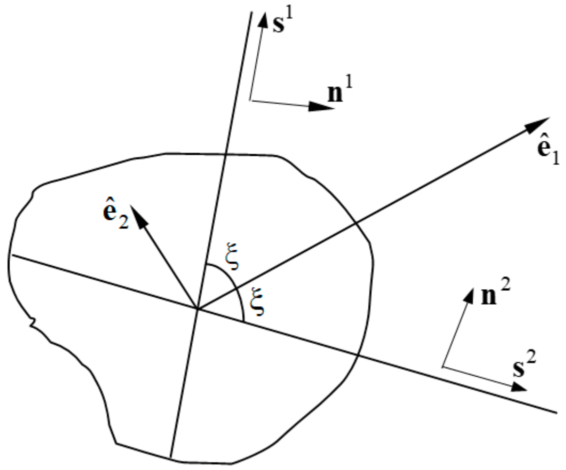

Figure 1. The two potential slip lines.

Thus, the shear stress and the compressive normal stress along the two slip lines can be expressed as:

$$\tau^\alpha = \boldsymbol{s}^\alpha \cdot \boldsymbol{\sigma}\boldsymbol{n}^\alpha, \tag{19}$$

$$\sigma^\alpha = -\boldsymbol{n}^\alpha \cdot \boldsymbol{\sigma}\boldsymbol{n}^\alpha, \ (\alpha = 1, 2). \tag{20}$$

Now, it can be shown that

$$\tau \equiv \tau^1 = \tau^2, \tag{21}$$

and $\sigma \equiv \sigma^1 = \sigma^2$.

3.4. Void Ratio

The behavior of granular materials depends strongly on the void ratio. Let v_s be the volume of solid particles and let v_v be the total volume of the voids. Then the void ratio, e, is defined as

$$e = \frac{v_v}{v_s} \tag{22}$$

The void ratio, e, is related to the solid volume fraction, φ, through

$$\varphi = \frac{v_s}{v_s + v_v} = \frac{1}{1+e} \tag{23}$$

The effects of void ratio are seen in various tests. In a uniaxial test, the compressibility of a loose packing is much higher than that of a dense packing. In a triaxial test, the stress ratio and the dilatancy at failure depend strongly on the initial void ratio. Based on experimental observations [74], some features related to void ratio are summarized below:

- For a given initial void ratio, the dilatancy decreases as the stress level increases.
- For low confining pressure, the stress–strain curve of dense granular materials shows a peak. After the peak, the dilatant volume change becomes less pronounced.
- Dense granular materials under high confining pressures and loose granular materials both show initially contractant volume change.

- A critical state, characterized by the critical stress state and the critical void ratio, will be reached asymptotically with continued deformation. The void ratio at a critical state depends on the stress level; it decreases with an increasing stress level.

At a given stress level, the void ratio of granular materials may vary widely depending mainly on the history of deformation. In contrast, the density of many fluids and solids depends only on the stress level.

3.5. Hardening/Softening

Hardening and softening are observed in certain applications of granular materials, where most of the observed softening is due to the inhomogeneous deformation of the sample. However, material softening can be exhibited by an ideal sample undergoing homogeneous deformation [75]. Dense granular materials have a higher strength than the loose ones, thus during the course of continued deformation, dilatancy transforms a sample of granular materials from a dense to a less dense arrangement which will lead to material softening. Strain softening granular materials can collapse dramatically if subjected to stresses that trigger their strengths to go over the peak. Granular materials which exhibit strain hardening or limited strain softening behavior are not prone to these dramatic collapses unless the materials become loose. The effect of hydrostatic stress on hardening/softening is an important point and requires further consideration.

3.6. Anisotropy

Probably three types of anisotropy can be identified for granular materials: the first one is the initial anisotropy, linked to the usual way in which granular materials are deposited under the action of gravity; the second one is called induced anisotropy, related to the loading path. Both anisotropic behaviors are due to the distribution of contact normals, the shape of particles and the non-linear local contact law. The distribution of contact normals plays an important role in giving rise to a global anisotropic behavior. The distribution of contact normals is usually characterized by approximate Fourier series function or the fabric tensor [76,77]. A fabric-dependent elasticity tensor with orthotropic symmetry is proposed in [78]. Finally, there could also be an anisotropy due to the directionality of rod-like or fiber-type granular materials [79].

3.7. Cyclic Loading

A considerable displacement of soil structures may be caused by irreversible cyclic loading. For undrained conditions a similar phenomenon may lead to an accumulation of pore water pressure (soil liquefaction) which may eventually lead to a loss of overall stability. The classical elasto-plasticity model bears some inherent deficiencies for cyclic loading [75]. From a physical point of view, some of the mechanical properties of granular materials depend strongly on the distribution of grain contact normals, the arrangement of grains, etc., which cannot be expressed by the customary state variables such as stress and void ratio, used in classical theories in the geotechnical engineering field. In the course of continued deformation, the microstructure (fabric) of the granular materials changes so as to increase its resistance to the continued deformation, resulting in hardening. As the microstructure gets rearranged, the granular materials dilate, and the potential for the reversed loading and the collapse of voids upon unloading increases. Similar considerations have been reported in [80–82].

3.8. Shear Banding

A typical pattern of inhomogeneous deformation is the localization of deformation in a narrow zone called the shear band; this is a well-known failure mode and occurs when a granular material is subjected to large shearing deformation. Shear localization induces intense inter-particle slip. Increasing the void ratio naturally reduces the coordination number (number of contacts per grain) of the granular assembly, causing material softening inside the shear band. The theoretical analysis of the initiation of localization of deformation dates to the work of [83–85]. A wide range of experimental data is also available in the literature [86–89].

The mechanism of shear banding in granular materials can be thought of as the instability triggered by initial imperfections. Within the shear band, the motion, deformation, and the rearrangement of granules soften the resistance of the granular assembly, as these mechanisms create additional geometric imperfections.

3.9. Rate Independence/Dependence

In a dense, dry granular material in the quasi-static and slow flow regime, dissipation is mainly due to the friction between sliding particles; the momentum transfer due to colliding particles is assumed to be negligible. Thus, in the frictional or slow flow regime, granular materials can be treated as rate-independent materials. However, some granular materials such as soils are in general rate dependent, that is, loading of undrained soil leads to the generation of excess pore pressures. As the pore pressures change, the effective stresses change, and the soil deforms. These deformations could be described as time-dependent deformations, which arise from the finite permeability of the soil, and not from any constitutive properties of the soil skeleton. However, in soils, rate dependence is more pronounced for clays, known as creep (secondary consolidation). The term creep usually refers to deformations continuing with time at constant effective stress, which is a result of the readjustment of particle contacts [90].

4. Constitutive Modeling of Granular Materials: The Frictional Flow Regime

In general, there are different ways of modeling complex materials such as granular materials [91]. We can, for example, use:

- Physical and experimental models
- Numerical simulations
- Statistical mechanics approaches (e.g., extension of kinetic theory of gases)
- Standard continuum mechanics
- Ad-hoc approaches

It is known that granular materials have certain 'structures' and as a result 'higher' order models or more advanced theories such as micro-mechanics, micropolar, Cosserat theories, non-Newtonian models, hypoplastic or hypoelastic models, viscoelastic, turbulence models, etc., are perhaps needed [92,93]. A different scheme used in constitutive modeling is:

- Explicit constitutive relations
- Implicit constitutive relations
- Ad hoc relations

An explicit constitutive equation is one where the constitutive variable being modeled is represented 'explicitly' in terms of other kinematical, thermal, or chemical variables, whereas an implicit constitutive relation, in addition to the above list, also depends on its own (time) derivatives, for example. This brings the concept of rate dependence or rate independence into the formulation. However, since, in general, terms such as 'rate dependent' are not clear, we prefer the explicit and implicit categorization [94–96].

Here, we provide a selection of constitutive relations which have been used. The focus is on the early attempts, and this work is not a thorough review article. We will not mention the micropolar or the kinetic theories.

4.1. Density (Volume Fraction) Gradient Theories

Goodman and Cowin [23,24] developed a continuum theory for the stresses that occur during the flow of granular materials. They assumed that the material properties are continuous functions of position and times. A distributed volume $V_t = \int \varphi dV$ and a distributed mass $M = \int \rho_s \varphi dV$ were defined, where the function φ is an independent kinematical variable called the volume distribution function (volume fraction), where

$$0 \leq \varphi(x,t) < 1 \tag{24}$$

In reality, φ is either one or zero at any position and time, depending on whether there is a granule or a void at that position. The classical mass density or the bulk density, ρ, is related to the particle density ρ_s and φ through

$$\rho = \rho_s \varphi \tag{25}$$

They derived a constitutive relation for the Cauchy stress tensor defining a Coulomb granular material as:

$$\boldsymbol{T} = (\beta_0 - \beta\varphi^2 + \alpha\nabla\varphi\cdot\nabla\varphi + 2\alpha\varphi\Delta\varphi)\mathbf{1} - 2\alpha\nabla\varphi \otimes \nabla\varphi + \lambda(tr\boldsymbol{D})\mathbf{1} + 2\mu\boldsymbol{D} \tag{26}$$

where ∇ is the gradient operator, Δ is the Laplacian operator and $\otimes$ represents the outer (dyadic) product of two vectors. The coefficients β_0, β, and α are material parameters; λ and μ are, in general, functions of ρ_s and φ; and a comma denotes differentiation with respect to $\boldsymbol{x}$. They assumed that the stress tensor is obtained by the linear superposition of two parts: $\boldsymbol{T}_0$, a rate-independent (also referred to as equilibrium or non-dissipative) part, which depends on the solids fraction φ and its gradients, and $\boldsymbol{T}_*$, a rate-dependent (viscous) part. Thus [97,98],

$$\boldsymbol{T} = \boldsymbol{T}_0 + \boldsymbol{T}_* \tag{27}$$

Later, Savage [10] assumed that the stress is an isotropic function of φ_0, φ, $\nabla\varphi$, and the symmetric part of the velocity gradient, $\boldsymbol{D} = 1/2[\boldsymbol{grad}\ \boldsymbol{v} + (\boldsymbol{grad}\ \boldsymbol{v})^{\boldsymbol{T}}]$, where φ_0 is a reference value of the volume fraction, φ. That is

$$\boldsymbol{T} = \boldsymbol{T}(\varphi_0, \varphi, \nabla\varphi, \boldsymbol{D}) \tag{28}$$

An early example of such an approach was proposed by Korteweg [99] to describe the structure of capillarity, who generalized the Navier–Stokes relation and assumed that the stress tensor was a function of the gradient of density, the second gradient of density, and the rate of the deformation tensor [42]. By introducing a tensor, $\boldsymbol{M} = \nabla\varphi \otimes \nabla\varphi$, and observing that $tr\boldsymbol{M} = |\nabla\varphi|^2$ and $\boldsymbol{M} = |\nabla\varphi|^2\boldsymbol{M}$ an isotropic form for $\boldsymbol{T}$ can be obtained:

$$\begin{aligned}\mathbf{T} = {} & \mathrm{a}_0\mathbf{1} + \mathrm{a}_1\mathbf{D} + \mathrm{a}_2\mathbf{M} + \mathrm{a}_3\mathbf{D}^2 + \mathrm{a}_4\mathbf{M}^2 + \mathrm{a}_5(\mathbf{DM} + \mathbf{MD}) + \mathrm{a}_6(\mathbf{D}^2\mathbf{M} + \mathbf{MD}^2) \\ & + \mathrm{a}_7(\mathbf{DM}^2 + \mathbf{M}^2\mathbf{D}) + \mathrm{a}_8(\mathbf{D}^2\mathbf{M}^2 + \mathbf{M}^2\mathbf{D}^2)\end{aligned} \tag{29}$$

where a_0 through a_8 are polynomials in the 10 basic invariants

$$\mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}\mathbf{M}, \mathrm{tr}\mathbf{M}^2, \mathrm{tr}\mathbf{M}^3, \mathrm{tr}(\mathbf{DM}), \mathrm{tr}(\mathbf{DM}^2), \mathrm{tr}(\mathbf{MD}^2), \mathrm{tr}(\mathbf{D}^2\mathbf{M}^2) \tag{30}$$

This is a very general representation for $\boldsymbol{T}$. It was shown [10,97,98] that by giving a special structure to the stress $\mathbf{T}_0$, one can satisfy the Coulomb failure criterion [100,101].

4.2. Non-Newtonian Fluid Models

Non-Newtonian fluid models can generally be classified as complex or non-linear materials, with the following caveats: (1) the ability to shear thin or shear thicken; (2) the ability to creep; (3) the ability to relax stresses; (4) the presence of normal stress differences in simple shear flows; (5) the presence of yield stress; (6) memory effects, etc. [102]. The models which are used for granular materials usually incorporate one of more of the above-mentioned effects.

Reiner [8,103] derived a constitutive relation for wet sand. This model does not take into account how the voidage (volume fraction) affects the stress [58]. Reiner showed that a non-zero shear stress produces a change in volume. The constitutive relation of the type

$$S_{lm} = F_0\delta_{lm} + 2\eta D_{lm} + 4\eta_c D_{lj} D_{jm} \tag{31}$$

describing the rheological behavior of a non-linear fluid was named by Truesdell [104] as Reiner–Rivlin fluid [105], where $\boldsymbol{T}$ is related to $\boldsymbol{D}$ (see [38], p. 221):

$$\boldsymbol{T} = -p(\rho)\mathbf{1} + f_1\boldsymbol{D} + f_2\boldsymbol{D}^2 \tag{32}$$

where f_1 and f_2 are functions of $\rho,\ tr\boldsymbol{D},$ and $tr\boldsymbol{D}^2$. Bingham [106] proposed a constitutive relation for a visco-plastic material in a simple shear flow where the relationship between the shear stress (or stress $\boldsymbol{T}$ in general) and the rate of shear (or the symmetric part of the velocity gradient $\boldsymbol{D}$) is given by (see [107], p. 137)

$$2\mu D_{ij} = \left\{ \begin{array}{ll} 0 & for\ F < 0 \\ FT'_{ij} & for\ F \geq 0 \end{array} \right\} \tag{33}$$

where T'_{ij} denotes the stress deviator and F, called the yield function, is given by

$$F = 1 - \frac{K}{II_2'^{1/2}} \tag{34}$$

where II_2' is the second invariant of the stress deviator, and in simple shear flows it is equal to the square of the shearing stress. K is called the yield stress (a constant). For one-dimensional flow, these relationships reduce to [106], i.e.,

$$F = 1 - \frac{K}{|T_{12}|} \tag{35}$$

and

$$2\mu D_{12} = \left\{ \begin{array}{ll} 0 & for\ F < 0 \\ FT_{12} & for\ F \geq 0 \end{array} \right\} \tag{36}$$

The formulation is known as the Bingham model (see [41], p. 170).

Bagnold [6] experimentally studied the behavior of neutrally buoyant, spherical particles suspended in Newtonian fluids in coaxial rotating cylinders. He measured the torque and normal stress in the radial direction for various concentrations of the grains.

Schaeffer [108] used a different yield criterion, namely, the von Mises yield criterion where

$$\sum_{i=1}^{3} (\sigma_i - \sigma)^2 \leq k^2\sigma^2, \tag{37}$$

where $\sigma = \frac{1}{3}tr\boldsymbol{T}$ and k is a constant characteristic of the material, and σ_i are the eigenvectors of T_{ij}. For the material to deform, equality must hold in Equation (37), i.e., $\sum_{i=1}^{3} (\sigma_i - \sigma)^2 = k^2\sigma^2$. The constitutive relation proposed by him is:

$$\boldsymbol{T} = \sigma\left[\frac{k}{|D|}\boldsymbol{D} + 1\right] \tag{38}$$

where $\boldsymbol{D}$ is now defined by $D_{ij} = -\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, where $|\boldsymbol{A}|$ indicates the magnitude of a tensor $\boldsymbol{A}$ and is given by $|\boldsymbol{A}| = \sqrt{\mathbf{A}.\mathbf{A}} = \sqrt{tr(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A})}$. This is meaningful only if $|\boldsymbol{D}| \neq 0$, i.e., the material must actually be deforming. Later, Tardos [14] showed that Schaffer's model can be described in the form

$$T_{ij} = p(\delta_{ij} + \sqrt{2}sin\phi \frac{D_{ij}}{|D_{ij}|}) \tag{39}$$

where ϕ is the internal angle of friction. This equation is somewhat similar to non-Newtonian fluid models proposed by Oldroyd [109] to describe the Bingham solid (see [110]):

$$T_{ij} = p\delta_{ij} + (\mu + \frac{\tau_0}{|D_{ij}|})D_{ij} \tag{40}$$

Tardos [14] indicates that this constitutive relation differs from (38) "by the solid having, in addition to the constant yield strength τ_0 (instead of the stress dependent form $p\ sin\phi$), a shear viscosity μ". There are many other non-Newtonian models which have been proposed; we will not, as mentioned in the Introduction, look at the latest developments of these topics.

4.3. Micromechanical Approach

The micromechanical approach is a multi-scale approach where the macro-level constitutive equations are based on the micro-level properties. Here the constitutive equations are mathematical formulations whose constants are obtained from experimental tests. For granular materials, it is not possible to give a deterministic description of the micro-structure and as a result a statistical approximation is needed. Therefore, the RVE (representative volume element) can be used as a statistically homogenous cell.

A systematic approach for developing constitutive relations for granular materials based on microstructures was given by Mehrabadi et al. [111]. In this approach, a theoretical micro-mechanical model relating the overall nominal stress rate to the overall velocity gradient in terms of the local micro-mechanical variables (contact forces, distribution of contact normal and contact area) is formulated. The evolution and characterization of the microstructure by means of the distribution of contact normals, and the contact areas have been adequately formulated. Each orientation in the distribution of contact normals corresponds to a class of contacts. As the granular sample deforms, the distribution of the contact normals changes and this characterizes a corresponding change of the fabric, or microstructure. The deformation is now viewed as an average of the local deformations associated with each class of contacts. A Taylor-averaging method is used, i.e., it is assumed that, as a first order approximation, the concentration tensor used in the self-consistent method is the identity tensor, which leads to:

$$l^a_{ij} = L_{ij}, (a = 1, ..., Q) \tag{41}$$

where l^a_{ij}, $(a = 1, ..., Q)$ designates the components of the velocity gradient associated with a typical class of contacts and L_{ij} are components of the overall velocity gradient. The contact force for a specific class of contacts is written in the form

$$f^a_i = \varepsilon^a \tau^a_{ij} m^a_j,\ (a = 1, \ldots, Q). \tag{42}$$

where ε^a is a parameter representing a measure of the contact area, τ^a_{ij} are components of a local stress associated with a class of contacts, and m^a_j is the unit branch vector. The local stress τ^a_{ij} is related to the local kinematic measures l^a_{ij}, where $(a = 1, \ldots, Q)$, through a local constitutive equation. Since the number of contacts in each class is influenced by the magnitude of the contact force as well as by the local volumetric change, the density of contacts for each class is assumed to be:

$$\frac{M_\alpha}{M} = \frac{1}{\alpha}\varepsilon^2 e^{\beta \hat{f}}, \tag{43}$$

where

$$\alpha = \sum_{a=1}^{Q} \varepsilon^2 e^{\beta \hat{f}}, \tag{44}$$

where β is a macroscopic constant, and $\hat{f}$ is a non-dimensional quantity related to the magnitude of the contact force for class a, i.e., $\hat{f}^a$, as follows:

$$\hat{f} = M_0 l^0 f / (-\frac{1}{2} tr\boldsymbol{\sigma}). \tag{45}$$

where M_0 is the number of contacts per unit volume and l^0 is the branch vector; both quantities are defined in the reference configuration. Following the procedure outlined in [22], which is based on the theory of virtual work, the expression of the macroscopic nominal stress $\boldsymbol{N}$ in terms of the contact forces and the contact normal at the microscopic level may be written as:

$$N_{ij} = \sum_{a=1}^{Q} l_i^{0a} f_j^a = \sum_{a=1}^{Q} l^{0a} m^{0a} f_j^a. \tag{46}$$

The macroscopic Cauchy stress, $\boldsymbol{\sigma}$, is obtained by the relation:

$$\sigma_{ij} = \frac{1}{detG} G_{ik} N_{kj} \tag{47}$$

where $\boldsymbol{G}$, the average deformation gradient, is related to the local deformation gradient, $\boldsymbol{g}$, by:

$$G_{ij} = \frac{1}{V_0} \int_{V_0} g_{ij} dV_0 \tag{48}$$

Chang and Ma [112] proposed a micromechanical model which treats the translation and the rotation of discrete particles as a statistically homogeneous system. In their study, the strain field is considered to be nonlinear, and the effect of particle rotation is taken into account. The displacement and rotation of particles are represented by continuum fields; therefore, the granular materials can be represented by an equivalent continuum that resembles micro-polar materials. Additional assumptions (for example, elastic material, truncation of series, restriction to periodic media, etc.) need be made to obtain closed-form expressions for the higher-order terms [113].

Under a homogenization framework, Cambou et al. [114] proposed a model for the non-reversible behavior of granular materials, adopting a non-linear elastic law (Hertz–Mindlin) which leads to a rather complex non-linear anisotropic global behavior. With very few parameters, this model seems to be able to describe a wide range of loading paths with good agreement with the experiment. The micromechanical study of granular materials is initially dedicated to the understanding of the fundamental mechanisms that influence the complex global behavior of such materials. The motivation of this approach is that starting from micromechanical observations, a satisfactory constitutive model for the equivalent continuum could be derived [115]. However, due to a lack of information at the micro level, the micro-mechanical models developed so far [22,112,113,115] are difficult to calibrate in order to quantitatively capture the observed behaviors of granular materials.

4.4. Hypoplastic Models

In most constitutive models where the yield stress is important, the total stress tensor is usually decomposed into two parts: a yield stress (sometimes referred to as the equilibrium or static stress) and a viscous part (often referred to as the dynamic part of the stress), as in Equation (27); there is usually a discontinuity in the form of the function. Hypoplasticity models, on the other hand, provide a smooth function to describe the behavior of the material.

Hypoplasticity is a generalization of hypoelasticity theory which was introduced by Truesdell [116]. Truesdell (see [42], p.404) defines a hypoelastic material as one whose constitutive relation may be written in the form (see also [116–119])

$$\overset{\circ}{\boldsymbol{T}} = \boldsymbol{H}(\boldsymbol{T})[\boldsymbol{D}] \tag{49}$$

where

$$\overset{\circ}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{W} \tag{50}$$

is the co-rotational stress rate, $\boldsymbol{W}$ is the spin tensor, and $\boldsymbol{D}$ is the symmetric part of the velocity gradient. The tensor function $\boldsymbol{H}(\boldsymbol{T})[\boldsymbol{D}]$ is linear in $\boldsymbol{D}$ and isotropic in $\boldsymbol{T}$ and $\boldsymbol{D}$. It is possible to use other (frame-invariant) time derivatives such as the convected stress rate

$$\overset{*}{\boldsymbol{T}} = \dot{\boldsymbol{T}} + \boldsymbol{L}^T\boldsymbol{T} + \boldsymbol{T}\boldsymbol{L} = \overset{\circ}{\boldsymbol{T}} + \boldsymbol{D}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{D} \tag{51}$$

If we apply the representation theorems [120,121] to Equation (49), we obtain the following polynomial expression:

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{T})[\boldsymbol{D}] = {} & [\alpha_1 tr\boldsymbol{D} + \alpha_2 tr(\mathbf{TD}) + \alpha_3 tr(\boldsymbol{T}^2\boldsymbol{D})]\mathbf{1} + [\alpha_4 tr\boldsymbol{D} + \alpha_5 tr(\mathbf{TD}) + \alpha_6 tr(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T} + [\alpha_7 tr\boldsymbol{D} + \alpha_8 tr(\mathbf{TD}) \\ & + \alpha_9 tr(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T}^2 + \alpha_{10}\boldsymbol{D} + \alpha_{11}(\mathbf{DT} + \mathbf{TD}) + \alpha_{12}(\mathbf{DT}^2 + \boldsymbol{T}^2\boldsymbol{D})\end{aligned} \tag{52}$$

where α_1 through α_{12} are polynomials in the principal invariants $\mathrm{I_T}$, $\mathrm{II_T}$, and $\mathrm{III_T}$. We can obtain special types of hypoelastic materials by restricting the response function $\boldsymbol{H}(\boldsymbol{T})[\,]$ upon $\boldsymbol{T}$. For example, if the response is a polynomial of degree n in the components of $\boldsymbol{T}$, then the material is said to be of grade n. Therefore, the constitutive relation for a hypoelastic material of grade 1, is ([42], p. 405):

$$\dot{S} = \mathbf{WS} - \mathbf{SW} + (\frac{\lambda}{2\mu} + \gamma_0 I_s)(tr\boldsymbol{D})\mathbf{1} + (1 + \gamma_1 I_s)\boldsymbol{D} + \gamma_2(tr\boldsymbol{D})\boldsymbol{S} + \gamma_3(tr(\mathbf{SD}))\mathbf{1} + \frac{1}{2}\gamma_4(\mathbf{SD} + \mathbf{DS}) \tag{53}$$

where $\boldsymbol{S} = \frac{\boldsymbol{T}}{2\mu}$ and $\lambda, \mu, \gamma_0 - \gamma_4$ are material constants. The basic idea of hypoplasticity is similar to the above formulation; it was developed by Kolymbas [122], who dropped the requirement that the function $\dot{\boldsymbol{T}} = \boldsymbol{h}(\mathbf{T},\boldsymbol{D})$ be linear in $\boldsymbol{D}$. Improved versions of hypoplasticity have since been proposed [19,123–129]. Here, we will briefly review the basic equations of the hypoplasticity model of Wu et al. [19]. The general form of the equation can be written as

$$\overset{\circ}{\boldsymbol{T}} = \boldsymbol{H}[\mathbf{T},\mathbf{D}] + \boldsymbol{N}(\boldsymbol{T})\|\boldsymbol{D}\|, \tag{54}$$

where $\boldsymbol{H}$ is linear in $\boldsymbol{D}$, $\|\boldsymbol{D}\| = \sqrt{tr\boldsymbol{D}^2}$ stands for the norm of $\boldsymbol{D}$, and it is clear that the latter term is non-linear in $\boldsymbol{D}$. Wu and Bauer [122] multiplied the non-linear term $\boldsymbol{N}(\boldsymbol{T})\|\boldsymbol{D}\|$ by the factor $\mathrm{I_e}$ which depends on the void ratio e and becomes equal to 1 when $e = e_{crit}$.

$$\overset{\circ}{\boldsymbol{T}} = C_1(tr\boldsymbol{T})\boldsymbol{D} + C_2\frac{tr(\boldsymbol{T}\boldsymbol{D})}{tr\boldsymbol{T}}\boldsymbol{T} + I_e\left(C_3\frac{\boldsymbol{T}^2}{tr\boldsymbol{T}} + C_4\frac{\boldsymbol{T}^{*2}}{tr\boldsymbol{T}}\right)\|\boldsymbol{D}\|, \tag{55}$$

where

$$I_e = (1-a)\frac{e - e_{min}}{e_{crit} - e_{min} + a}. \tag{56}$$

is assumed to fulfill the following conditions:

$$\mathrm{I_e}|_{\mathrm{e}=\mathrm{e_{crit}}} = 1 \text{ and } \mathrm{I_e}|_{\mathrm{e}=\mathrm{e_{min}}} = \mathrm{a}. \tag{57}$$

To account for the effects of pressure, they assumed that the critical void ratio and the parameter a depend on the stress level $tr\boldsymbol{T}$, i.e.,

$$e_{crit} = p_1 + p_2\, exp(p_3(tr\boldsymbol{T})),$$

$$a = q_1 + q_2\, exp(q_3(tr\boldsymbol{T})). \tag{58}$$

The parameters $p_i, q_i (i = 1, 2, 3)$ are determined by fitting the experimental data. In Equation (55), C_i (i = 1, …, 4) are dimensionless constants and the deviatoric stress $\boldsymbol{T}^*$ is defined as

$$\boldsymbol{T}^* = \boldsymbol{T} - \frac{1}{3}(tr\boldsymbol{T})\mathbf{1}. \tag{59}$$

The general concepts used in elasto-plastic theory, such as yield surface, plastic potential, and the decomposition of deformation into elastic and plastic parts [130,131], are not used in developing the hypoplastic constitutive equations.

If the critical state coincides with some prescribed curves [127,128,132], then the proposed constitutive equation is written as:

$$\overset{\circ}{\boldsymbol{T}} = f_b(tr\boldsymbol{T}, e) f_e(tr\boldsymbol{T}, e)\left[\boldsymbol{L}(\hat{\boldsymbol{T}}, \boldsymbol{D}) + f_d(tr\boldsymbol{T}, e)\boldsymbol{N}(\hat{\boldsymbol{T}})\|\boldsymbol{D}\|\right] \tag{60}$$

The tensorial parts $\boldsymbol{L}$ and $\boldsymbol{N}\|D\|$ depend on $\boldsymbol{D}$ and the normalized stress tensor $\hat{\boldsymbol{T}}$,

$$\hat{\boldsymbol{T}} = \frac{\boldsymbol{T}}{tr\boldsymbol{T}} \tag{61}$$

The basic concept of (60) is to separate the density dependence and the pressure dependence from the tensorial parts of this constitutive equation. Density dependence and pressure dependence are described by f_b, f_e, f_d only, which are scalar functions of the current void ratio e and the stress level $tr\boldsymbol{T}$.

Using the general concept of hypoplasticity, different aspects of the mechanical behavior of granular materials have been studied as stated in the above references, for example, shear banding, cyclic loading, rate dependence, and cohesion. Critical states are used to better describe the influence of pressure levels and density on the materials' behavior. The attempts in these papers have been to show that the hypoplastic models can describe the dependence of the materials' behavior on pressure and density with a constitutive equation.

The continuum mechanics models (especially the hypoplastic models) are phenomenological and are primarily concerned with mathematical modeling of the observed phenomenon without detailed attention to their fundamental physical significance. In granular materials, forces are transferred through contacts between the particles. This discrete nature makes the constitutive relationship of granular materials more complex and additional laboratory tests are necessary to understand the mechanical behaviors before the physical processes in these materials can be modeled. The classical double shearing model actually incorporates some physical aspects but still not enough to, for example, determine the different shearing rates along the two slip planes, which is needed to describe the non-coaxiality in the classical double shearing model.

4.5. Double Shearing Model

Drucker and Prager [17] presented one of the earliest theories for the dilatant behavior of granular materials. The dilatant double shearing model of Mehrabadi and Cowin [133] is based on the kinematic assumption that the deformation of granular materials consists of two simple dilatant shear deformations (or, "pure stretches", Hayes [134]) along the stress characteristics. As the stress and the velocity characteristics are coincidental, these sets of characteristics can also be considered as slip lines. The two constitutive equations based on this postulate are an extension of the theory developed in [135] for incompressible granular materials. Incorporating the effect of elastic deformation and plastic work hardening, these equations become similar to a set of rate-type double slip constitutive equations developed in [84] for single crystals. Spencer [136] extended the planar incompressible double shearing model of [135] to three dimensions by employing a three-dimensional generalization of the Mohr–Coulomb yield condition presented in [137], who proposed three yield conditions based on the relative magnitude of the principal stresses. Anand

and Gu [62] employed a similar yield condition using an elastic–plastic set of constitutive relations that are identical to those of the elastic–plastic dilatant double shearing model proposed in [84] except that they take the shear rates on the two slip systems to be equal. One consequence of this assumption is that the plastic spin vanishes.

Based on the works of Mandel [138] and de Josselin de Jong [139], Spencer [135] developed a properly invariant, planar, general theory for the deformation of incompressible granular materials, which is called the double shearing model. In this model, the two basic kinematic equations are the velocity equations:

$$D_{11} + D_{22} = 0 \tag{62}$$

$$sin2\psi(D_{11} - D_{22}) - 2D_{12}cos2\psi - 2sin\phi(W_{12} + \dot{\psi}) = 0. \tag{63}$$

where ϕ is the angle of internal friction and the superimposed dot denotes the material time rate. Equation (62) is the incompressibility condition, and Equation (63) relates to the strain-rate tensor components, D_{11}, D_{22}, D_{12}, the spin component, W_{12}, and the angle ψ, which represents the inclination of the algebraically greater principal stress direction. A complete theory consists of Equations (62) and (63), the Coulomb yield condition, the stress equation of motion, and the relations among the components of the velocity fields D_{11}, D_{22}, D_{12}, and W_{12}.

Mehrabadi and Cowin [133] extended Spencer's velocity Equations (62) and (63) to dilatant materials. Their model is based on the kinematic postulation that the deformation of granular materials consists of two simple dilatant shearing deformations along the stress characteristics. The stress and velocity characteristics are coincident in this model; the two sets of characteristics can also be taken as slip lines. Their modified velocity equations are:

$$sin\nu_d[(D_{11} - D_{22})cos2\psi + 2D_{12}sin2\psi] - cos(\phi - \nu_d)(D_{11} + D_{22}) = 0, \tag{64}$$

$$cos\nu_d[sin2\psi(D_{11} - D_{22}) - 2D_{12}cos2\psi] - 2sin(\phi - \nu_d)(W_{12} + \dot{\psi}) = 0 \tag{65}$$

where ν_d is the angle of dilatancy, and $tan\nu_d$ is defined as the ratio of the velocity normal to a slip line to the velocity along the slip line at a point. In the special case where the angle of dilatancy is zero, Equations (64) and (65) reduce to Spencer's [135] velocity Equations (62) and (63), respectively.

Mehrabadi and Cowin [140] established a connection between pre-failure and post-failure flow models of dilatant granular materials by proposing a more general model (the modified rigid–plastic yield–vertex model) which incorporates the perfect plastic model [133] and the rigid–plastic yield–vertex model [83] as special cases. Employing the double shearing model, Mehrabadi and Cowin [140] also derived the condition under which the localization of deformation into shear bands becomes possible. Anand [141] discussed the "modified rigid–plastic yield–vertex" flow rule. By introducing isotropic hardening (softening) into the constitutive equations via the usual consideration of scalar hardening variables in the yield function, he derived the flow rule directly from the basic assumptions of the double shearing model. In 1955, Shield proposed a generalization of the Mohr–Coulomb yield condition to three dimensions and used it to develop an associated flow rule. Based on Shield's yield condition, while discarding the associated flow rule, Spencer [136] extended the double shearing model from two dimensions to three dimensions.

The models mentioned so far are rigid–plastic models and assume a non-work hardening (Mohr–Coulomb) yield condition. In addition, the shearing rates are not determined by the yield function and the elastic deformation is also not taken into account. Nemat-Nasser, et al. [84] develop a set of rate type double slip constitutive equations incorporating the effect of elastic deformation, dilatancy, and pressure sensitivity. In their model, the total velocity gradient, $\boldsymbol{D}+\boldsymbol{W}$, is obtained by plastic distortion giving rise to $\boldsymbol{D}^p+\boldsymbol{W}^p$, followed by elastic distortion with an elastic stretching rate, $\boldsymbol{D}^e$, and an elastic spin $\boldsymbol{W}^e$. Therefore,

$$\boldsymbol{D}=\boldsymbol{D}^e+\boldsymbol{D}^p, \tag{66}$$

$$\boldsymbol{W}=\boldsymbol{W}^e+\boldsymbol{W}^p. \tag{67}$$

The plastic part in the above equations is viewed to stem from the plastic slip rates, $\dot{\gamma}^{\alpha}$, and the plastic dilation rates, $\dot{\delta}^{\alpha}$, over both active slip lines. Then, the plastic constituents are:

$$\boldsymbol{D}^p=\dot{\gamma}^{\alpha}\boldsymbol{p}^{\alpha}, \tag{68}$$

$$\boldsymbol{W}^p=\dot{\gamma}^{\alpha}\boldsymbol{\omega}^{\alpha}, \tag{69}$$

$$p_{ij}^{\alpha}=\frac{1}{2}(s_i^{\alpha}n_j^{\alpha}+s_j^{\alpha}n_i^{\alpha})+tan\nu_d n_i^{\alpha}n_j^{\alpha} \tag{70}$$

$$\omega_{ij}^{\alpha}=\frac{1}{2}(s_i^{\alpha}n_j^{\alpha}-s_j^{\alpha}n_i^{\alpha}).\ (\text{no sum on } \alpha) \tag{71}$$

where $\boldsymbol{n}^{\alpha}$ is the unit vector normal to the slip plane, $\boldsymbol{s}^{\alpha}$ is the unit vector in the direction of the slip plane, $tan\nu_d$ is the dilatancy parameter, and it is assumed to be the same for all α. The elastic part in Equations (66) and (67) gives rise to an objective rate co-rotational stress with the corresponding spin, according to:

$$\overset{\nabla}{\boldsymbol{T}}=\boldsymbol{C}:\boldsymbol{D}^e \tag{72}$$

where

$$\overset{\nabla}{\boldsymbol{T}}=\dot{\boldsymbol{T}}-\boldsymbol{W}^e\boldsymbol{T}+\boldsymbol{T}\boldsymbol{W}^e \tag{73}$$

For a typical active slip α, the shearing rate is assumed to be governed by the following constitutive relation:

$$\dot{\tau}^{\alpha}+tan\eta\dot{T}^{\alpha}=h^{\alpha\beta}\dot{\gamma}^{\beta} \tag{74}$$

where

$$\tau^{\alpha}=T_{ij}s_i^{\alpha}n_j^{\alpha}, \tag{75}$$

and

$$T^{\alpha}=T_{ij}n_i^{\alpha}n_j^{\alpha},\ (\text{no sum on } \alpha) \tag{76}$$

are the shear and the normal stresses transmitted over the α^{th} slip plane, $h^{\alpha\beta}(\alpha,\beta=1,2)$ is a symmetric work hardening matrix, and $tan\eta$ is a pressure sensitivity parameter. For the evolution of the unit vector $\boldsymbol{n}^{\alpha}$ and $\boldsymbol{s}^{\alpha}$, it assumed that:

$$\dot{\boldsymbol{n}}^{\alpha}=\boldsymbol{W}^e\boldsymbol{n}^{\alpha}, \tag{77}$$

$$\dot{s}^{\alpha}=\boldsymbol{W}^e\boldsymbol{s}^{\alpha}. \tag{78}$$

Hence, the final expression for the stress and the strain relation can be written as:

$$\overset{\nabla}{\boldsymbol{T}}=\boldsymbol{L}:\boldsymbol{D}+\boldsymbol{M}^{\alpha\beta}(\boldsymbol{T}\omega^{\alpha}-\boldsymbol{\omega}^{\alpha}\boldsymbol{T}-\boldsymbol{L}:\boldsymbol{p}^{\alpha})(\boldsymbol{q}^{\beta}:\boldsymbol{L}:\boldsymbol{D}) \tag{79}$$

where $M^{\alpha\beta}$ is the inverse of matrix $h^{\alpha\beta}+\boldsymbol{q}^{\alpha}:\boldsymbol{L}:\boldsymbol{p}^{\beta}$. However, the work hardening parameters and the pressure sensitivity parameter in (73) need to be specified to solve boundary value problems. In addition, an evolution equation for the dilatancy parameter $tan\nu_d$ is also needed to complete the model.

Anand and Gu [62] developed a set of constitutive relations that are similar to those of the elastic–plastic dilatant double shearing model proposed by Nemat-Nasser et al. [84]. They propose plausible equations describing the variations in the work hardening parameter, the dilatancy parameter, and the solid volume fraction. However, their model is for amorphous isotropic material only, and it is assumed that there are no preferred directions other than the principal directions of stress in such materials. Such an assumption leads to the conclusion that shearing rates along the two symmetrically disposed slip systems with respect to the maximum principal stress direction are equal, which means that the flow rule is coaxial. For amorphous isotropic materials, this assumption may be plausible, but in general, granular materials have a microstructure, which affects their macro-mechanical response. Therefore, it is crucial to consider the effects of anisotropy (inherent anisotropy and induced anisotropy) in developing an appropriate constitutive model for granular materials.

Nemat-Nasser [77] has also proposed a constitutive model for frictional granular materials based on micromechanical considerations. In Nemat-Nasser's model, fabric is used to define an anisotropy tensor (back stress) and is not related to the shearing rates along the two slip directions. Therefore, the two shearing rates along the two slip systems are still undetermined. Nemat-Nasser introduces a non-coaxial material parameter, trying to bypass the determination of the two different shearing rates. However, this 'non-coaxial material parameter' is not a constant and there is no practical way for its determination and its evolution for a specific material. We believe that the relative amplitude of the shearing rates along the two slip directions is closely related to the non-coaxiality and thus to the material anisotropy. Therefore, it seems natural to determine the relation between shearing rates along the two slip systems based on microstructure (fabric) considerations.

Zhu et al. [82] incorporated the effects of fabric and its evolution into the dilatant double shearing model [133] for granular materials in order to capture the anisotropic behavior and the complex response of granular materials in cyclic shear loading and sand pile settling. In this approach, based on the consideration of fabric tensor, the shearing rates along the two slip directions are unequal. This leads to non-coaxiality between the principal axes of stress and strain rates, which is more appropriate to model initial and induced anisotropy for granular materials. In this approach, a fabric-dependent elasticity tensor is developed with the orthotropic axes of the elasticity tensor aligned to the principal axes of the fabric tensor [142].

The fabric tensor is denoted as a second-rank tensor Φ. The evolution equation for the fabric tensor depends on the elastic rate of deformation [78]:

$$\overset{\nabla}{\Phi} = \lambda(D^e\Phi + \Phi D^e\Phi) \tag{80}$$

where

$$\overset{\nabla}{\Phi} = \dot{\Phi} - W^e\Phi + \Phi W^e \tag{81}$$

is the objective co-rotational rate of the fabric tensor with respect to the elastic spin, and λ is a material constant. The relationship between the co-rotational stress rate $\overset{\circ}{\boldsymbol{T}}$ and the elastic rate of deformation $\boldsymbol{D^e}$ can be written as:

$$\begin{pmatrix} \overset{\circ}{\mathbf{T}}_{11} \\ \overset{\circ}{\mathbf{T}}_{22} \\ \overset{\circ}{\mathbf{T}}_{33} \\ \overset{\circ}{\mathbf{T}}_{12} \end{pmatrix} = \frac{E}{\Delta} \begin{pmatrix} 1/t_2 - v^2 & v + v^2 & v/t_2 + v^2 & 0 \\ v - v^2 & 1/t_1 - v^2 & 1/t_1 + v^2 & 0 \\ v/t_2 - v^2 & v/t_1 - v^2 & 1 - v^2 & 0 \\ 0 & 0 & 0 & \frac{t_{12}\Delta}{1+v} \end{pmatrix} \begin{pmatrix} D^e_{11} \\ D^e_{22} \\ 0 \\ D^e_{12} \end{pmatrix} \tag{82}$$

where E is Young's modulus, v is the Poisson ratio, and

$$\Delta = \left(1/t_1 - v^2\right)\left(1/t_2 - v^2\right) - v^2(1+v)^2 \tag{83}$$

$$t_i = \frac{r_i + a}{1 + a}, \ (i = 1,\ 2) \tag{84}$$

$$t_{12} = \frac{r_{12} + a}{1 + a} \tag{85}$$

and $r_i(i = 1, 2)$ is the ith principal value of the fabric tensor, Φ, and a is a material parameter. This model [78] can capture the strength anisotropy and the complex response of granular materials under the cyclic shear loading condition. In addition, this model can capture the gradual concentration of the contact normals towards the maximum compressive principal stress direction.

5. Conclusions

In this paper, we have reviewed and mentioned some of the basic concepts, such as dilatancy, cohesion, yield criterion, shear banding, etc., which can be used to derive constitutive relations for granular materials, especially in the slow flow regime. A summary of a few different approaches using continuum mechanics as the basis for derivation is given; these approaches include the formulations for constitutive relations using density (volume fraction) gradient theories, non-Newtonian fluid models, the micromechanical approach, hypoplastic models, and the double shearing model. A few early examples of these constitutive relations are also provided.

It is almost impossible to come up with a single constitutive relation for the stress tensor of a granular medium which would be applicable for all situations and geometries, where all or most of the fundamental concepts mentioned in this paper are incorporated in that model. The few brief remarks which are presented here can hopefully be used as guidelines. For certain applications, perhaps one or more of these concepts can be used to obtain an equation for the stress tensor. Whether we use an explicit approach, for example, the equations given in Section 4.2, or an implicit approach, as shown with the hypoplastic model in Section 4.4, the governing equations will be non-linear, and before solving any meaningful problem, additional boundary conditions are needed. That is, since most of these constitutive relations are non-linear, possibly due to the presence of density gradients or normal stress effects, and may contain discontinuities such as the yield stress, the computational scheme will be more complicated and as a result more time consuming. The need for additional boundary conditions for the higher order models remains a challenging issue in this area [143,144]. It is also to be expected that the numerical scheme will be more complicated and more time consuming compared to the typical schemes used in CFD applications. Finally, even though we have not discussed the damping ratio, it is another important parameter which can influence the response of granular materials [145,146].

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Nomenclature

ρ_0	the reference density of the material
ρ	the current density
$\boldsymbol{F}$	the deformation gradient
div	the divergence operator
$\mathbf{v}$	the velocity vector
$grad$	the gradient operator
$\boldsymbol{b}$	the body force vector
$\boldsymbol{T}$ or $\boldsymbol{\sigma}$	the Cauchy stress tensor
S and T	the shear stress and the normal stress, respectively, acting on a plane at a point
c	the coefficient of cohesion
b_0	the coefficient of static friction
ϕ	the internal angle of friction
$\boldsymbol{n}$	the normal direction
v_s	the volume of solid particles
v_v	the total volume of the voids
e	the void ratio
φ	volume fraction $\varphi = \frac{v_s}{v_s+v_v} = \frac{1}{1+e}$
∇	the gradient operator
Δ	the Laplacian operator
$\otimes$	the outer (dyadic) product of two vectors
$\boldsymbol{D}$	the symmetric part of the velocity gradient
μ	the viscosity
$\mathbf{1}$	the identity tensor
P	pressure
$\boldsymbol{W}$	the spin tensor
l_{ij}^a	the components of the velocity gradient associated with a typical class of contacts
L_{ij}	components of the overall velocity gradient.
ε^a	a parameter representing a measure of the contact area
τ_{ij}^a	components of local stress associated with a class of contacts
m_j^a	the unit branch vector
$\hat{f}$	a non-dimensional quantity related to the magnitude of the contact force
M_0	the number of contacts per unit volume
l^0	the branch vector
ψ	the angle representing the inclination of the algebraically greater principal stress direction
ν_d	the angle of dilatancy
Φ	the fabric tensor
E	Young's modulus
v	the Poisson ratio

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