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Dispersive Optical Solitons for Stochastic Fokas-Lenells Equation with Multiplicative White Noise

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Abstract: For the first time, we study the Fokas–Lenells equation in polarization preserving fibers with multiplicative white noise in Itô sense. Four integration algorithms are applied, namely, the method of modified simple equation (MMSE), the method of sine-cosine (MSC), the method of Jacobi elliptic equation (MJEE) and ansätze involving hyperbolic functions. Jacobi-elliptic function solutions, bright, dark, singular, combo dark-bright and combo bright-dark solitons are presented.

Keywords: stochastic F L equation; modified simple equation method; sine-cosine method; Jacobi-elliptic function expansion method; ansätze method

1. Introduction

Nonlinear differential equations (NLDEs) play a very important role in scientific fields and engineering such as optical fibers, the heat flow, plasma physics, solid-state physics, chemical kinematics, the proliferation of shallow water waves, fluid mechanics, quantum mechanics, wave proliferation phenomena, etc. One of the fundamental physical problems for these models is to obtain their traveling wave solutions. As a consequence, the search for mathematical methods to create exact solutions of NLDEs is an important and essential activity in nonlinear sciences. In recent years, many articles have studied optical solitons' form in telecommunications industry. These soliton molecules form the information transporter across intercontinental distances around the world. Lastly, the nonlinear Schrödinger's equation (NLSE) has been discussed with the help of many models [1–38]. The aspect of stochasticity is one of the features that is less touched upon and there are hardly any papers that have debated this point [3–9]. The Fokas–Lenells equation (FLE) appears as a model which appoints nonlinear pulse propagation in optical fibers. The FLE is a completely integrable equation which has arisen as an integrable generalization of the NLSE using bi-Hamiltonian methods [10]. On the other hand, the FLE models have the propagation of nonlinear light pulses in monomode optical fibers when certain higher-order nonlinear effects are considered in optics field [11]. The complete integrability of the FLE has been presented by using the inverse scattering transform (IST) method [12]. In the main, a Lax pair and a few conservation laws connected to it have been obtained using the bi-Hamiltonian structure and the multi-soliton solutions have been derived by using the dressing method [13]. One more main characteristic of the FLE is that it is the first negative flow of the integrable hierarchy of the derivative NLSE [14].

In the present article, we will study the FLE with multiplicative white noise in the Itô sense. Our results are presented after a comprehensive analysis obtained in this article.

2. Governing Model

The dimensionless structure of the stochastic perturbed FLE in polarization preserving fiber with multiplicative white noise in the Itô sense is written, for the first time, as:



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$$iq_t + a_1 q_{xx} + a_2 q_{xt} + |q|^2(bq + icq_x) + \sigma(q - ia_2 q_x) \frac{dW(t)}{dt} = i \left[\alpha q_x + \lambda (|q|^2 q)_x + \mu (|q|^2)_x q \right], \quad (1)$$

where $q(x, t)$ is a complex-valued function that represents the wave profile, while $a_1, a_2, b, c, \sigma, \alpha, \lambda, \mu$ are real-valued constants and $i = \sqrt{-1}$. The first term in Equation (1) is the linear temporal evolution, a_1 is the coefficient of chromatic dispersion (CD), a_2 is the coefficient of spatio-temporal dispersion (STD), b is the coefficient of self-phase modulation (SPM), c is the coefficient of nonlinear dispersion term, σ is the coefficient of the strength of noise, the Wiener process is denoted by $W(t)$, while $dW(t)/dt$ represents the white noise. Also, the term $dW(t)/dt$ is the time derivative of the standard Wiener process $W(t)$ which is called a Brownian motion and has the following properties [7]: (i) $W(t), t \geq 0$, is a continuous function of t , (ii) For $s < t$, $W(t) - W(s)$ is independent of increments. (iii) $W(t) - W(s)$ has a normal distribution with mean zero and variance $(t - s)$.

Next, α is the coefficient of inter-modulation dispersion (IMD), λ is the coefficient of self-steepening (SS) term, and finally μ is the coefficient of higher-order nonlinear dispersion term. If $\sigma = 0$, Equation (1) reduces to the familiar FLE which is studied in [1,2,37]. The authors [37] studied Equation (1) with variable coefficients and $\sigma = 0$. The motivation of adding the stochastic term $\sigma(q - ia_2 q_x) \frac{dW(t)}{dt}$ to Equation (1) is to formulate the stochastic FLE with noise or fluctuations depending on the time, which has been recognized in many areas via physics, engineering, chemistry and so on. This stochastic term has been constructed with the help of the two terms iq_t and $a_2 q_{xt}$. Therefore, in general, the stochastic model means that the model of differential equations should contain the white noise term ($\sigma \neq 0$). The physical importance of the stochastic FL Equation (1) is to find its traveling wave stochastic solutions which appoint the nonlinear pulse propagations in optical fibers.

The aim of this article is to use the method of MMSE in Section 3, the method of MSC in Section 4, the method of MJEE in Section 5 and the ansatzes involving hyperbolic functions in Section 6 to find the bright, dark, singular soliton solutions, as well as the Jacobi elliptic function solutions of Equation (1). Some numerical simulations are obtained in Section 7. Finally, conclusions are illustrated in Section 8.

3. On Solving Equation (1) by MMSE

In order to solve the stochastic Equation (1), we use a wave transformation involving the noise coefficient σ and the Wiener process $W(t)$ in the form:

$$q(x, t) = \phi(\xi) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (2)$$

where the transformation $\xi = x - vt$ is used. Here, κ, w, v , are real constants, such that κ represents the wave number, w represents the frequency and v represents the soliton velocity. The function $\phi(\xi)$ is real function which represents the amplitude part. When we put Equation (2) into Equation (1), we obtain the ordinary differential equation (ODE):

$$[a_1 - a_2 v] \phi'' + Y \phi + [b + \kappa(c - \lambda)] \phi^3 = 0, \quad (3)$$

and the soliton velocity,

$$v = \frac{Y}{(a_2 \kappa - 1)}, \quad a_2 \kappa \neq 1 \quad (4)$$

as well as the constraint condition,

$$c - 3\lambda - 2\mu = 0, \quad (5)$$

where $Y = [(w - \sigma^2)(a_2\kappa - 1) - a_1\kappa^2 - \alpha\kappa]$ and $'' = \frac{d^2}{d\xi^2}$. We have the balance number $N = 1$ by balancing ϕ'' with the ϕ^3 in Equation (3). According to the method of MSE [15–20], the solution of Equation (3) is written as:

$$\phi(\xi) = A_0 + A_1 \left[\frac{\psi'(\xi)}{\psi(\xi)} \right], \quad (6)$$

where $\psi(\xi)$ is a new function of ξ , and A_0, A_1 are constants to be determined later, provided $A_1 \neq 0, \psi(\xi) \neq 0$ and $\psi'(\xi) \neq 0$.

Inserting Equation (6) into Equation (3), and collecting all the coefficients of $\psi^{-i}(\xi)$ ($i = 0, 1, 2, 3$), we obtain the equations:

$$\psi^0 : A_0 Y + A_0^3 [b + \kappa(c - \lambda)] = 0, \quad (7)$$

$$\psi^{-1} : A_1 \psi''' [a_1 - a_2 v] + A_1 \psi Y + 3A_0^2 A_1 \psi' [b + \kappa(c - \lambda)] = 0, \quad (8)$$

$$\psi^{-2} : -3A_1 \psi' \psi'' [a_1 - a_2 v] + 3A_0 A_1^2 \psi'^2 [b + \kappa(c - \lambda)] = 0, \quad (9)$$

$$\psi^{-3} : 2A_1 \psi'^3 [a_1 - a_2 v] + A_1^3 \psi'^3 [b + \kappa(c - \lambda)] = 0. \quad (10)$$

By solving Equations (7) and (10), we obtain:

$$A_0 = 0, \quad A_0 = \pm \sqrt{-\frac{Y}{[b + \kappa(c - \lambda)]}}, \quad A_1 = \pm \sqrt{-\frac{2[a_1 - a_2 v]}{[b + \kappa(c - \lambda)]}}, \quad (11)$$

provided $[b + \kappa(c - \lambda)]Y < 0$ and $[b + \kappa(c - \lambda)][a_1 - a_2 v] < 0$.

By solving Equations (8) and (9), we conclude that $A_0 = 0$ is rejected. Therefore, $A_0 \neq 0$. Now, Equation (9) reduces to the ODE :

$$[a_1 - a_2 v] \psi'' - A_0 A_1 [b + \kappa(c - \lambda)] \psi' = 0, \quad (12)$$

which has the solution

$$\psi'(\xi) = \xi_0 \exp \left[\frac{A_0 A_1 [b + \kappa(c - \lambda)]}{[a_1 - a_2 v]} \xi \right], \quad (13)$$

where $\xi_0 \neq 0$ is a constant. From Equation (11) and Equation (13), we can show that Equation (8) is valid. Hence, we have the results:

$$\psi(\xi) = \frac{\xi_0 [a_1 - a_2 v]}{A_0 A_1 [b + \kappa(c - \lambda)]} \exp \left[\frac{A_0 A_1 [b + \kappa(c - \lambda)]}{[a_1 - a_2 v]} \xi \right] + \xi_1, \quad (14)$$

where ξ_1 is a nonzero constant of integration. Now, the exact solution of Equation (1) has the form:

$$q(x, t) = \left\{ A_0 + A_1 \frac{\xi_0 \exp \left[\frac{A_0 A_1 [b + \kappa(c - \lambda)]}{[a_1 - a_2 v]} (x - vt) \right]}{\xi_1 + \frac{\xi_0 [a_1 - a_2 v]}{A_0 A_1 [b + \kappa(c - \lambda)]} \exp \left[\frac{A_0 A_1 [b + \kappa(c - \lambda)]}{[a_1 - a_2 v]} (x - vt) \right]} \right\} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (15)$$

In particular, if we set,

$$\xi_1 = \frac{\xi_0 [a_1 - a_2 v]}{A_0 A_1 [b + \kappa(c - \lambda)]}, \quad (16)$$

we have the dark soliton solution:

$$q(x, t) = \pm \sqrt{-\frac{Y}{[b + \kappa(c - \lambda)]}} \tanh \left[\sqrt{\frac{Y}{2[a_1 - a_2 v]}} (x - vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (17)$$

while, if we set,

$$\xi_1 = -\frac{\xi_0[a_1 - a_2v]}{A_0A_1[b + \kappa(c - \lambda)]}, \quad (18)$$

we have the singular soliton solution:

$$q(x, t) = \pm \sqrt{-\frac{Y}{[b + \kappa(c - \lambda)]}} \coth \left[\sqrt{\frac{Y}{2[a_1 - a_2v]}} (x - vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (19)$$

provided,

$$[b + \kappa(c - \lambda)]Y < 0, [a_1 - a_2v]Y > 0. \quad (20)$$

On comparing our above results (17) and (19) with the results (19) and (20) obtained in [37], we deduce that they are equivalent when $\sigma = 0$.

4. On Solving Equation (1) by MSC

To apply this method according to [21–25], assume that Equation (3) has the sine-solution form:

$$\phi(\xi) = \begin{cases} \lambda_1 \sin^{\beta_1}(\mu_1 \xi), & \text{if } |\xi| < \frac{\pi}{\mu_1}, \\ 0, & \text{otherwise.} \end{cases} \quad (21)$$

Substituting Equation (21) into Equation (3), we obtain:

$$\begin{aligned} & [a_1 - a_2v] \left[\lambda_1 \mu_1^2 \beta_1 (\beta_1 - 1) \sin^{\beta_1-2}(\mu_1 \xi) - \lambda_1 \mu_1^2 \beta_1^2 \sin^{\beta_1}(\mu_1 \xi) \right] \\ & + Y \lambda_1 \sin^{\beta_1}(\mu_1 \xi) + [b + \kappa(c - \lambda)] \lambda_1^3 \sin^{3\beta_1}(\mu_1 \xi) = 0. \end{aligned} \quad (22)$$

From (22), we deduce that $\beta_1 - 2 = 3\beta_1$ which leads $\beta_1 = -1$. Consequently, we have the results:

$$\mu_1^2 = \frac{Y}{[a_1 - a_2v]}, \quad \lambda_1^2 = -\frac{2Y}{[b + \kappa(c - \lambda)]}. \quad (23)$$

Now, the periodic solution of Equation (1) is:

$$q(x, t) = \pm \sqrt{-\frac{2Y}{[b + \kappa(c - \lambda)]}} \csc \left[\sqrt{\frac{Y}{[a_1 - a_2v]}} (x - vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (24)$$

provided $[b + \kappa(c - \lambda)]Y < 0, [a_1 - a_2v]Y > 0$.

Since $\csc(ix) = -i \operatorname{csch} x$, then the singular soliton solution of Equation (1) is written as:

$$q(x, t) = \pm \sqrt{\frac{2Y}{[b + \kappa(c - \lambda)]}} \operatorname{csch} \left[\sqrt{-\frac{Y}{[a_1 - a_2v]}} (x - vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (25)$$

provided $Y[b + \kappa(c - \lambda)] > 0, [a_1 - a_2v]Y < 0$.

In parallel, if we allow that Equation (3) has the cosine-solution:

$$\phi(\xi) = \begin{cases} \lambda_1 \cos^{\beta_1}(\mu_1 \xi), & \text{if } |\xi| < \frac{\pi}{2\mu_1}, \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

Putting Equation (26) into Equation (3), we obtain

$$\begin{aligned} & [a_1 - a_2v] \left[-\mu_1^2 \beta_1^2 \lambda_1 \cos^{\beta_1}(\mu_1 \xi) + \lambda_1 \mu_1^2 \beta_1 (\beta_1 - 1) \cos^{\beta_1-2}(\mu_1 \xi) \right] \\ & + Y \lambda_1 \cos^{\beta_1}(\mu_1 \xi) + [b + \kappa(c - \lambda)] \lambda_1^3 \cos^{3\beta_1}(\mu_1 \xi) = 0. \end{aligned} \quad (27)$$

From Equation (27), we deduce that $\beta_1 - 2 = 3\beta_1$, which leads $\beta_1 = -1$. Therefore, we have the solutions:

$$q(x, t) = \pm \sqrt{-\frac{2Y}{[b+\kappa(c-\lambda)]}} \sec \left[\sqrt{\frac{Y}{[a_1-a_2v]}} (x-vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (28)$$

with conditions $[b + \kappa(c - \lambda)]Y < 0$, $[a_1 - a_2v]Y > 0$.

Since, $\sec(ix) = \operatorname{sech}x$, we have the bright soliton solution:

$$q(x, t) = \pm \sqrt{-\frac{2Y}{[b+\kappa(c-\lambda)]}} \operatorname{sech} \left[\sqrt{\frac{Y}{[a_1-a_2v]}} (x-vt) \right] \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (29)$$

provided $[b + \kappa(c - \lambda)]Y < 0$, $[a_1 - a_2v]Y < 0$.

5. On Solving Equation (1) by MJEE

If we multiply Equation (3) by $\phi'(\xi)$ and integrate, we have the JEE as:

$$\phi'^2(\xi) = l_0 + l_2 \phi^2(\xi) + l_4 \phi^4(\xi), \quad (30)$$

where,

$$l_0 = \frac{2c_1}{[a_1 - a_2v]}, l_2 = -\frac{Y}{[a_1 - a_2v]}, l_4 = -\frac{[b + \kappa(c - \lambda)]}{2[a_1 - a_2v]}, \quad (31)$$

and c_1 is the integration constant, $[a_1 - a_2v] \neq 0$. It is noted [26–30] that Equation (30) has the Jacobi-elliptic solutions in the forms:

(1) If $l_0 = 1, l_2 = -(1 + m^2), l_4 = m^2, 0 < m < 1$, then,

$$\phi(\xi) = \operatorname{sn}(\xi) \text{ or } \phi(\xi) = \operatorname{cd}(\xi). \quad (32)$$

Then, Equation (1) has the JEE solution:

$$\begin{aligned} q(x, t) &= \operatorname{sn}(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \\ \text{or} \\ q(x, t) &= \operatorname{cd}(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \end{aligned} \quad (33)$$

where,

$$\begin{aligned} c_1 &= \frac{1}{2}(a_1 - a_2v), \\ Y &= (1 + m^2)(a_1 - a_2v), \\ b + \kappa(c - \lambda) &= -2m^2(a_1 - a_2v), \end{aligned} \quad (34)$$

and consequently, we obtain

$$Y = -\frac{(1 + m^2)}{2m^2} [b + \kappa(c - \lambda)].$$

Particularly, if $m \rightarrow 1$, we get,

$$q(x, t) = \tanh(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (35)$$

Note that the solution Equation (35) is equivalent to the solution Equation (17) under the conditions of Equation (34).

(2) If $l_0 = m^2, l_2 = -(1 + m^2), l_4 = 1, 0 < m < 1$, then,

$$\phi(\xi) = \operatorname{ns}(\xi) \text{ or } \phi(\xi) = \operatorname{dc}(\xi). \quad (36)$$

Then, we obtain the JEE solution for Equation (1),

$$\begin{aligned} q(x, t) &= \operatorname{ns}(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \\ \text{or} \\ q(x, t) &= \operatorname{dc}(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \end{aligned} \quad (37)$$

where,

$$\begin{aligned} c_1 &= \frac{1}{2}m^2(a_1 - a_2v), \\ Y &= (1 + m^2)(a_1 - a_2v), \\ b + \kappa(c - \lambda) &= -2(a_1 - a_2v), \end{aligned} \quad (38)$$

and consequently, we have,

$$Y = -\frac{(1 + m^2)}{2}[b + \kappa(c - \lambda)].$$

Particularly, if $m \rightarrow 1$, we obtain,

$$q(x, t) = \coth(x - vt) \exp i[-\kappa x + wt + \sigma W(t) - \sigma^2 t]. \quad (39)$$

Note that the solution in Equation (39) is equivalent to the solution in Equation (19) under the conditions in Equation (38).

(3) If $l_0 = 1 - m^2, l_2 = 2m^2 - 1, l_4 = -m^2, 0 < m < 1$, then,

$$\phi(\xi) = \text{cn}(\xi). \quad (40)$$

Now, we have the JEE solution for Equation (1),

$$q(x, t) = \text{cn}(x - vt) \exp i[-\kappa x + wt + \sigma W(t) - \sigma^2 t], \quad (41)$$

where,

$$\begin{aligned} c_1 &= \frac{1}{2}(1 - m^2)(a_1 - a_2v), \\ Y &= -(2m^2 - 1)(a_1 - a_2v), \\ b + \kappa(c - \lambda) &= 2m^2(a_1 - a_2v), \end{aligned} \quad (42)$$

and consequently, we have,

$$Y = -\frac{(2m^2 - 1)}{2m^2}[b + \kappa(c - \lambda)].$$

Particularly, if $m \rightarrow 1$, we obtain,

$$q(x, t) = \text{sech}(x - vt) \exp i[-\kappa x + wt + \sigma W(t) - \sigma^2 t] \quad (43)$$

Note that the solution of Equation (43) is equivalent to the solution of Equation (29) under the conditions of Equation (42).

(4) If $l_0 = -m^2(1 - m^2), l_2 = 2m^2 - 1, l_4 = 1, 0 < m < 1$, then,

$$\phi(\xi) = \text{ds}(\xi). \quad (44)$$

Consequently, we have the JEE solution for Equation (1),

$$q(x, t) = \text{ds}(x - vt) \exp i[-\kappa x + wt + \sigma W(t) - \sigma^2 t], \quad (45)$$

where,

$$\begin{aligned} c_1 &= -\frac{m^2}{2}(1 - m^2)(a_1 - a_2v), \\ Y &= -(2m^2 - 1)(a_1 - a_2v), \\ b + \kappa(c - \lambda) &= -2(a_1 - a_2v), \end{aligned} \quad (46)$$

and we have,

$$Y = \frac{1}{2}(2m^2 - 1)[b + \kappa(c - \lambda)].$$

Particularly, if $m \rightarrow 1$, we obtain

$$q(x, t) = \operatorname{csch}(x - vt) \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (47)$$

Note that the solution of Equation (47) is equivalent to the solution of Equation (25) under the conditions of Equation (46).

(5) If $l_0 = \frac{1}{4}, l_2 = \frac{1}{2}(1 - 2m^2), l_4 = \frac{1}{4}, 0 < m < 1$, then,

$$\phi(\xi) = \frac{\operatorname{sn}(\xi)}{1 \pm \operatorname{cn}(\xi)}. \quad (48)$$

Now, we have the JEE solution for the Equation (1),

$$q(x, t) = \frac{\operatorname{sn}(x - vt)}{1 \pm \operatorname{cn}(x - vt)} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (49)$$

where,

$$\begin{aligned} c_1 &= \frac{1}{8}(a_1 - a_2 v), \\ Y &= -\frac{1}{2}(1 - 2m^2)(a_1 - a_2 v), \\ b + \kappa(c - \lambda) &= -\frac{1}{2}(a_1 - a_2 v), \end{aligned} \quad (50)$$

and consequently, we have,

$$Y = (1 - 2m^2)[b + \kappa(c - \lambda)].$$

Particularly, if $m \rightarrow 1$, we obtain the combo dark-bright soliton solutions:

$$q(x, t) = \frac{\tanh(x - vt)}{1 \pm \operatorname{sech}(x - vt)} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (51)$$

(6) If $l_0 = \frac{1-m^2}{4}, l_2 = \frac{1+m^2}{2}, l_4 = \frac{1-m^2}{4}, 0 < m < 1$, then,

$$\phi(\xi) = \frac{\operatorname{cn}(\xi)}{1 \pm \operatorname{sn}(\xi)}. \quad (52)$$

Then, we have the JEE solution for Equation (1),

$$q(x, t) = \frac{\operatorname{cn}(x - vt)}{1 \pm \operatorname{sn}(x - vt)} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right], \quad (53)$$

where

$$\begin{aligned} c_1 &= \frac{1}{8}(1 - m^2)(a_1 - a_2 v), \\ Y &= -\frac{1}{2}(1 + m^2)(a_1 - a_2 v), \\ b + \kappa(c - \lambda) &= -\frac{1}{2}(1 - m^2)(a_1 - a_2 v). \end{aligned} \quad (54)$$

Particularly, if $m \rightarrow 1$, we obtain the combo bright-dark soliton solutions:

$$q(x, t) = \frac{\operatorname{sech}(x - vt)}{1 \pm \tanh(x - vt)} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (55)$$

Finally, there are many other Jacobi elliptic solutions which are omitted here for simplicity.

6. Ansatz Involving Hyperbolic Functions

To this aim, we first write Equation (3) in the simple form,

$$A\phi'' + Y\phi + C\phi^3 = 0, \quad (56)$$

where,

$$\begin{aligned} A &= a_1 - a_2 v, \\ C &= b + \kappa(c - \lambda). \end{aligned} \quad (57)$$

Along these lines, the main steps of the proposed ansatz have been presented according to the ansatz involving the hyperbolic functions method [31].

6.1. Combo Bright-Dark Solitons

We assume the ansatz,

$$\phi(\xi) = \frac{\alpha_1 \operatorname{sech}(\mu_1 \xi)}{1 + \lambda_1 \tanh(\mu_1 \xi)}. \quad (58)$$

where $\alpha_1, \lambda_1, \mu_1$ are parameters to be determined. Now, we obtain

$$\phi''(\xi) = \frac{\alpha_1 \mu_1^2 (2\lambda_1^2 - 1) \operatorname{sech}(\mu_1 \xi) + 2\alpha_1 \lambda_1 \mu_1^2 \operatorname{sech}(\mu_1 \xi) \tanh(\mu_1 \xi) + \alpha_1 \mu_1^2 (2 - \lambda_1^2) \operatorname{sech}(\mu_1 \xi) \tanh^2(\mu_1 \xi)}{(1 + \lambda_1 \tanh(\mu_1 \xi))^3}. \quad (59)$$

Substituting Equations (58) and (59) into Equation (56), combining all the coefficients of $\operatorname{sech}^p(\xi) \tanh^q(\xi)$ ($p = 1, q = 0, 1, 2$), we obtain the set of equations:

$$\begin{aligned} A\alpha_1 \mu_1^2 (2\lambda_1^2 - 1) + Y\alpha_1 + C\alpha_1^3 &= 0, \\ 2A\alpha_1 \lambda_1 \mu_1^2 + 2Y\alpha_1 \lambda_1 &= 0, \\ A\alpha_1 \mu_1^2 (2 - \lambda_1^2) + Y\alpha_1 \lambda_1^2 - C\alpha_1^3 &= 0. \end{aligned} \quad (60)$$

By resolving the Equation (60), we have the results:

$$\mu_1^2 = \frac{-Y}{A}, AY < 0, \lambda_1^2 = \frac{2Y + C\alpha_1^2}{2Y} > 0, \alpha_1 \neq 0.$$

Now, we obtain

$$q(x, t) = \left\{ \frac{\alpha_1 \operatorname{sech} \left[\sqrt{\frac{-Y}{A}} (x - vt) \right]}{1 \pm \sqrt{\frac{2Y + C\alpha_1^2}{2Y}} \tanh \left[\sqrt{\frac{-Y}{A}} (x - vt) \right]} \right\} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (61)$$

which represent the combo-bright-dark soliton solutions and are equivalent to the solutions Equation (55) of Section 5, if $A = -Y$, $C = 0$ and $\alpha_1 = 1$.

6.2. Combo Dark-Bright Solitons

We assume the ansatz

$$\phi(\xi) = \frac{\alpha_1 \tanh(\mu_1 \xi)}{1 + \lambda_1 \operatorname{sech}(\mu_1 \xi)}, \quad (62)$$

where $\alpha_1, \lambda_1, \mu_1$ are parameters to be determined. Now, we obtain

$$\phi''(\xi) = \frac{\alpha_1 \mu_1^2 (\lambda_1^2 - 2) \operatorname{sech}^2(\mu_1 \xi) \tanh(\mu_1 \xi) - \alpha_1 \lambda_1 \mu_1^2 \operatorname{sech}(\mu_1 \xi) \tanh(\mu_1 \xi)}{(1 + \lambda_1 \operatorname{sech}(\mu_1 \xi))^3}. \quad (63)$$

Substituting Equations (62) and (63) into Equation (56), combining all the coefficients of $\tanh^p(\xi) \operatorname{sech}^q(\xi)$ ($p = 1, q = 0, 1, 2$), we obtain the algebraic equations:

$$\begin{aligned} Y\alpha_1 + C\alpha_1^3 &= 0, \\ -A\alpha_1 \lambda_1 \mu_1^2 + 2Y\alpha_1 \lambda_1 &= 0, \\ A\alpha_1 \mu_1^2 (\lambda_1^2 - 2) + Y\alpha_1 \lambda_1^2 - C\alpha_1^3 &= 0. \end{aligned} \quad (64)$$

Solving the algebraic Equation (64), we obtain the results:

$$\mu_1^2 = \frac{2Y}{A}, AY > 0, \alpha_1^2 = \frac{-Y}{C}, YC < 0, \lambda_1^2 = 1.$$

Now, Equation (1) has the combo dark-bright soliton solutions:

$$q(x,t) = \pm \sqrt{\frac{-Y}{C}} \left\{ \frac{\tanh \left[\sqrt{\frac{2Y}{A}}(x-vt) \right]}{1 \pm \operatorname{sech} \left[\sqrt{\frac{2Y}{A}}(x-vt) \right]} \right\} \exp i \left[-\kappa x + wt + \sigma W(t) - \sigma^2 t \right]. \quad (65)$$

which are equivalent to the solutions Equation (51) of Section 5, if $A = 2Y$ and $Y = -C$.

7. Numerical Simulations

In this section, we present the graphs of some solutions for Equation (1). Let us now examine Figures 1–15, as it illustrates some of our solutions obtained in this paper. To this aim, we select some special values of the obtained parameters.

Figure 1: The numerical simulations of the solutions (17) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values

$a_1 = 1, a_2 = 1, b = 1, \sigma = 0, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 3, -5 \leq x, t \leq 5$.

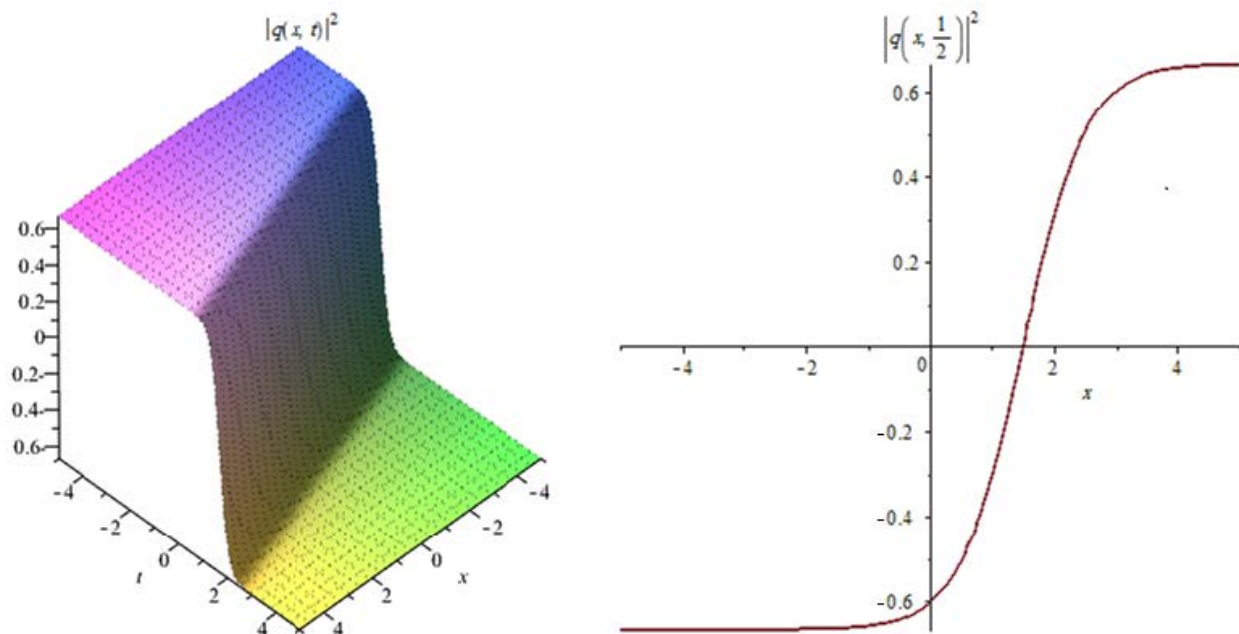


Figure 1. The profile of the dark soliton solutions (17).

Figure 2: The numerical simulations of the solutions (17) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 1, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 4, -5 \leq x, t \leq 5$.

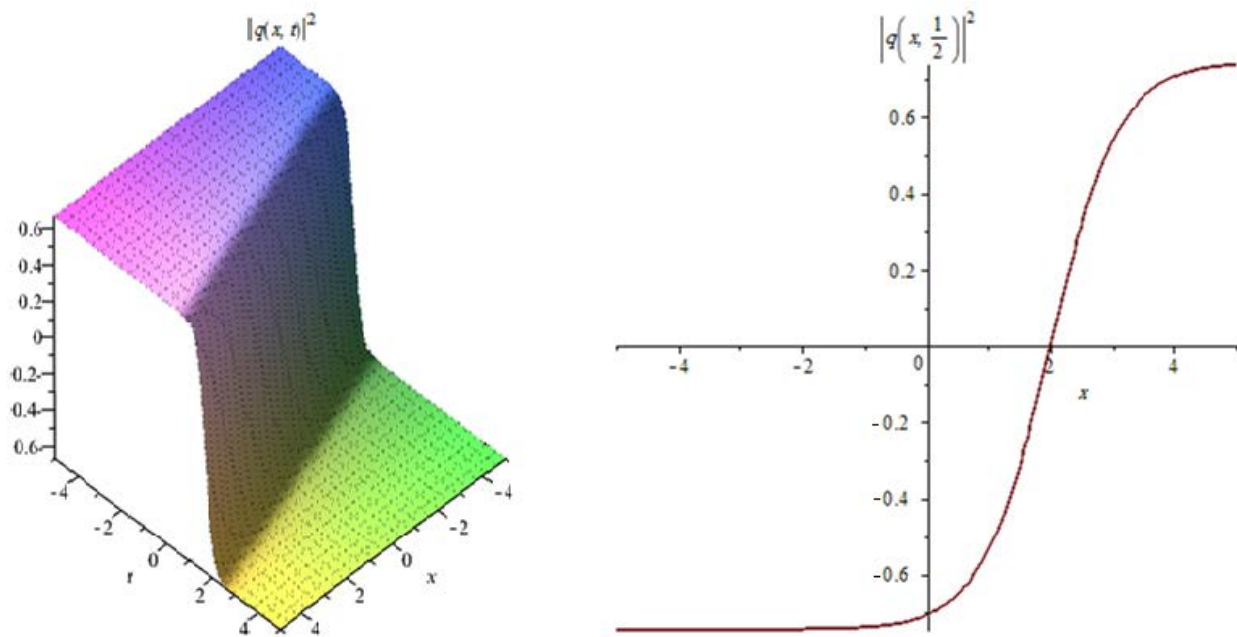


Figure 2. The profile of the dark soliton solutions (17).

Figure 3: The numerical simulations of the solutions (17) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 2, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 8, -5 \leq x, t \leq 5$.

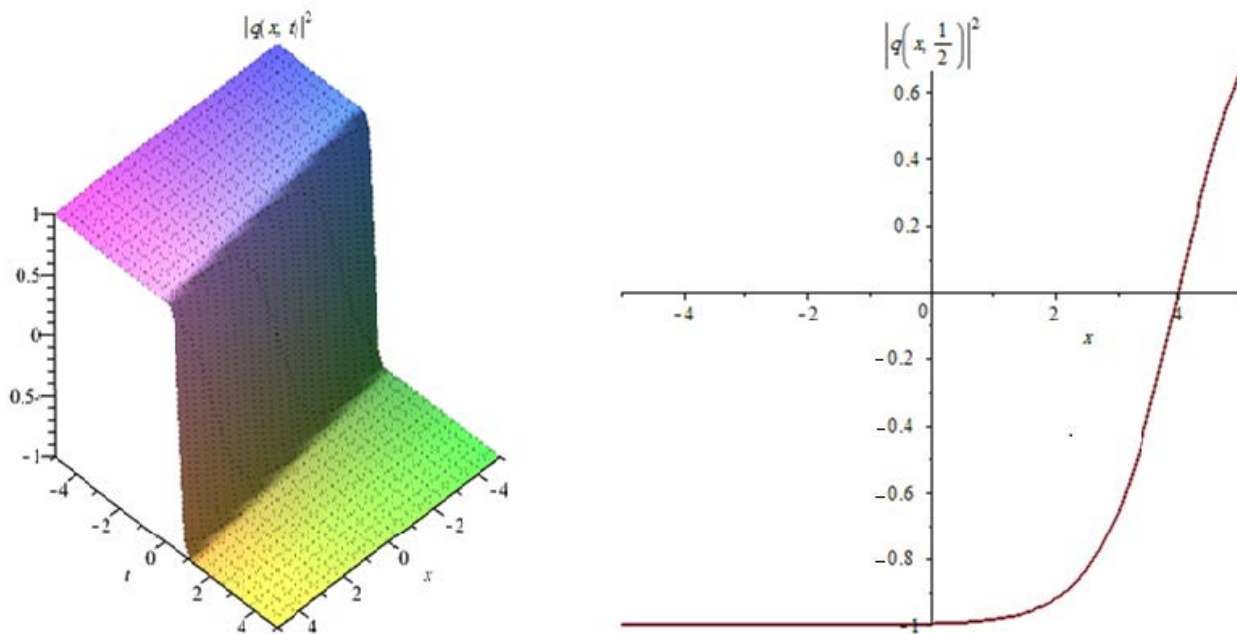


Figure 3. Shows the profile of the dark soliton solutions (17).

Figure 4: The numerical simulations of the solutions (19) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 0, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 3, -5 \leq x, t \leq 5$.

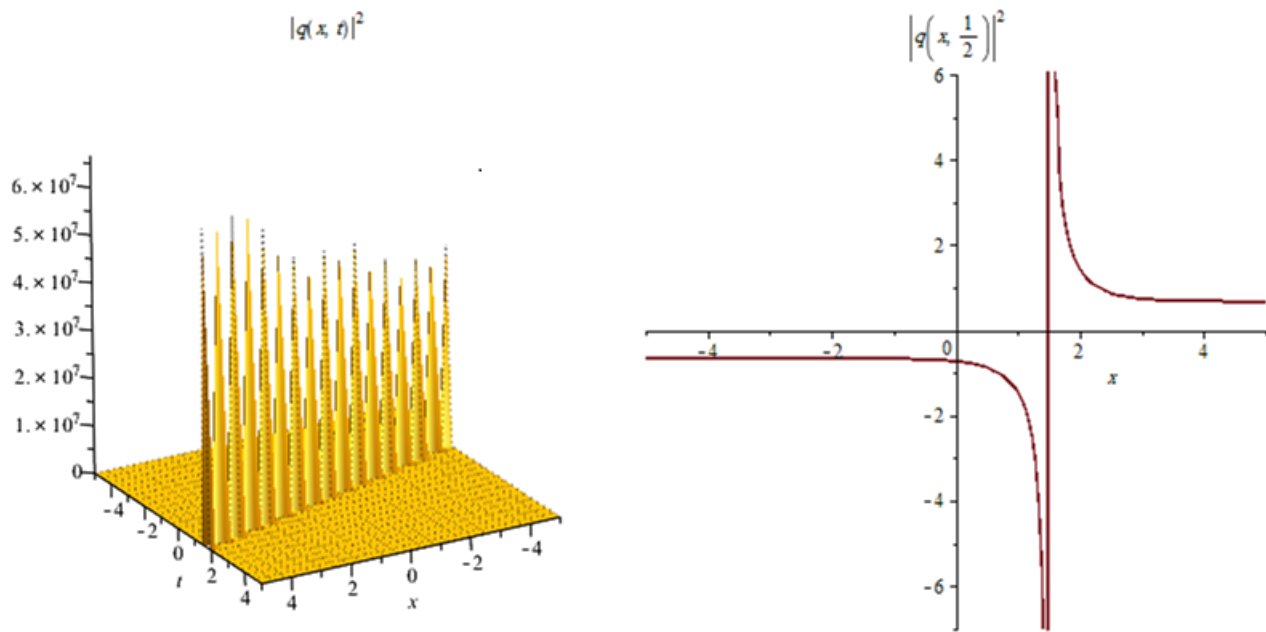


Figure 4. Shows the profile of the singular soliton solutions (19).

Figure 5: The numerical simulations of the solutions (19) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 1, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 4, -5 \leq x, t \leq 5$.

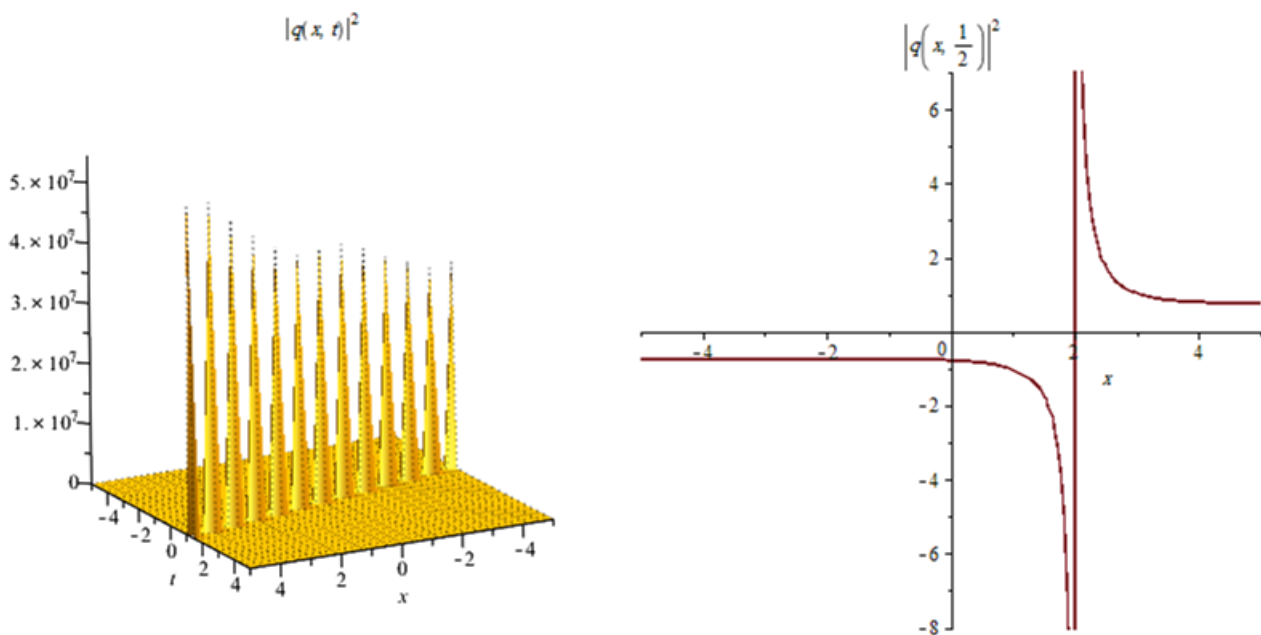


Figure 5. Shows the profile of the singular soliton solutions (19).

Figure 6: The numerical simulations of the solutions (19) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 2, \alpha = 1, \kappa = 2, w = 2, \lambda = 1, \mu = 1, c = 5, v = 8, -5 \leq x, t \leq 5$.

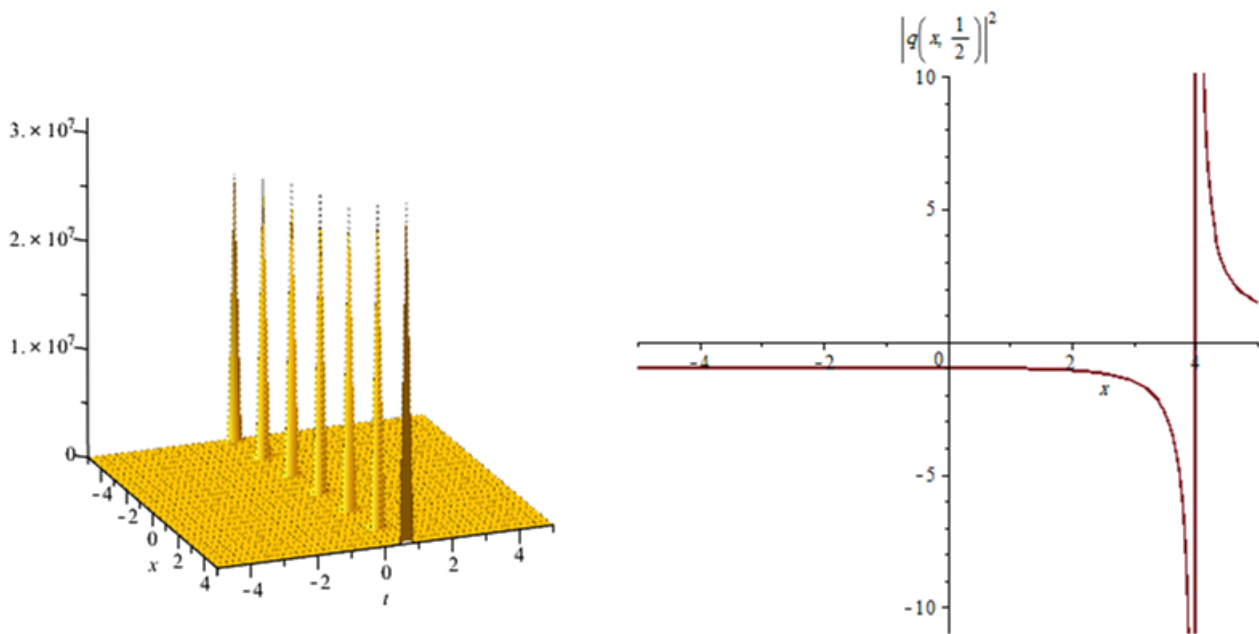


Figure 6. Shows the profile of the singular soliton solutions (19).

Figure 7: The numerical simulations of the solutions (29) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 0, \alpha = 1, \kappa = \frac{1}{2}, w = 2, \lambda = 1, \mu = 1, c = 5, v = -2, -5 \leq x, t \leq 5$.

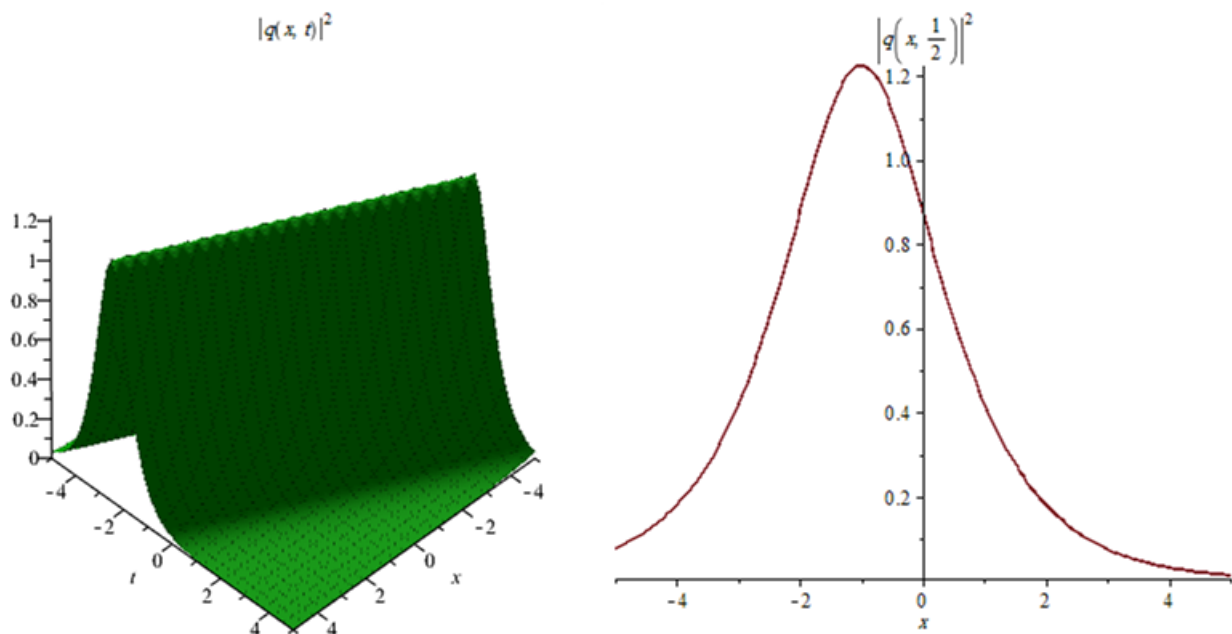


Figure 7. Shows the profile of the bright soliton solutions (29).

Figure 8: The numerical simulations of the solutions (29) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 4, \alpha = 4, \kappa = \frac{1}{4}, w = 16, \lambda = 2, \mu = 2, c = 10, v = -6, -5 \leq x, t \leq 5$.

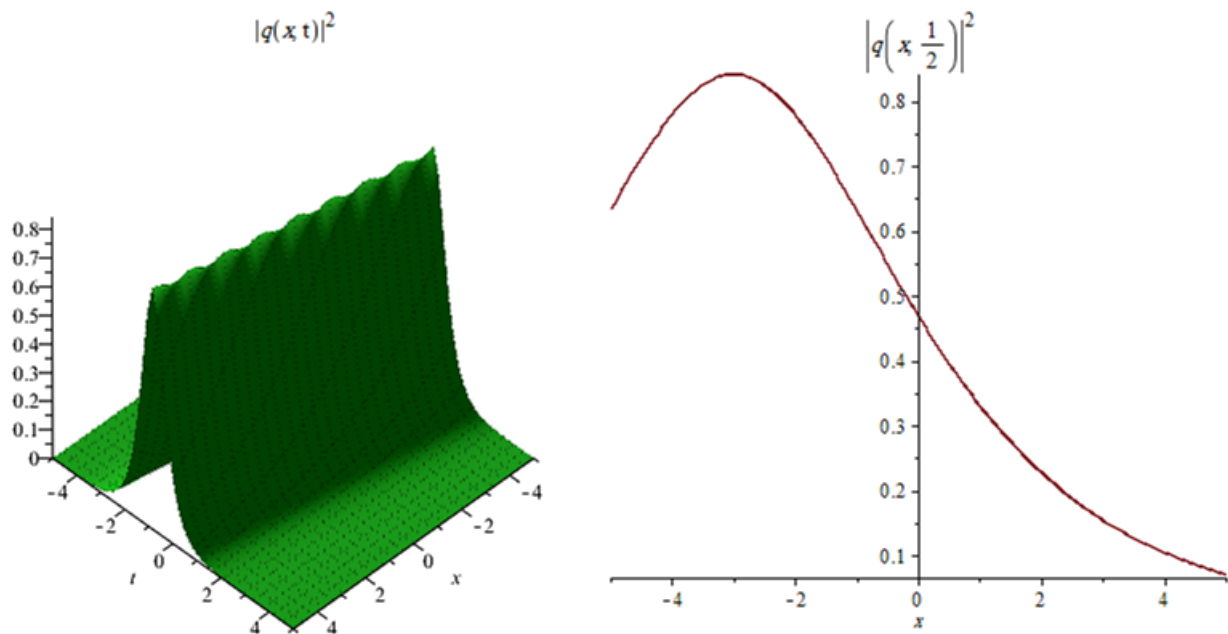


Figure 8. Shows the profile of the bright soliton solutions (29).

Figure 9: The numerical simulations of the solutions (29) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = 1, \sigma = 2, \alpha = 2, \kappa = \frac{1}{2}, w = 2, \lambda = 1, \mu = 1, c = 5, v = -10, -5 \leq x, t \leq 5$.

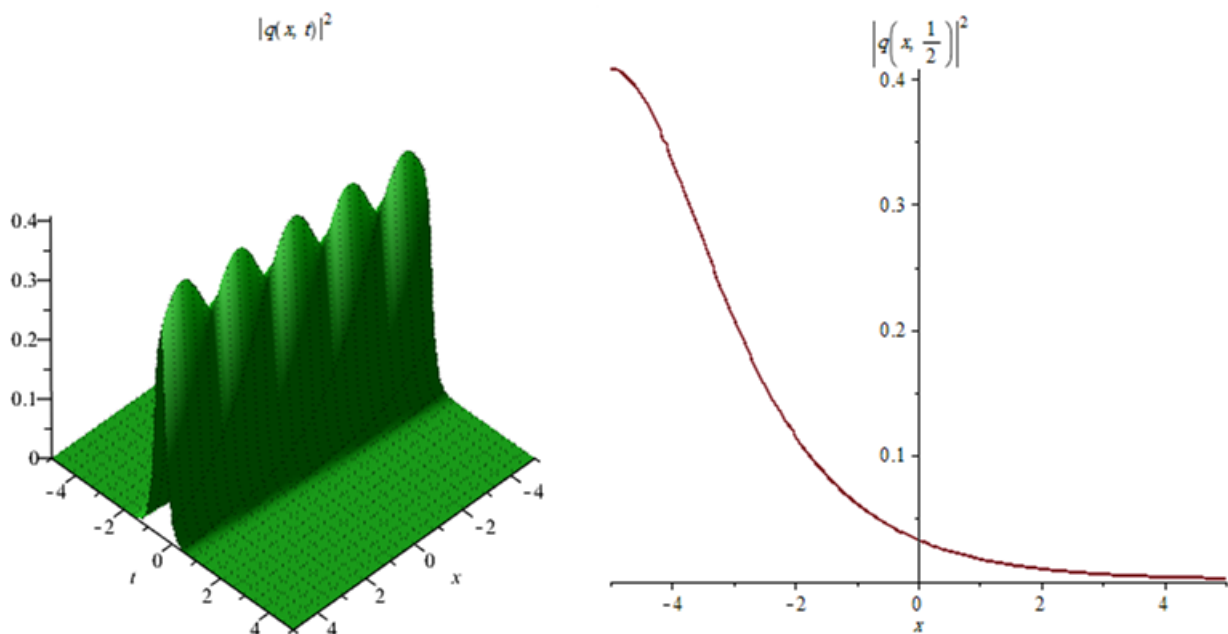


Figure 9. Shows the profile of the bright soliton solutions (29).

Figure 10: The numerical simulations of the solutions (51) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 4, a_2 = 2, b = -16, \sigma = 0, \alpha = 1, \kappa = 2, w = 10, \lambda = 2, \mu = 2, c = 10, v = 2, -5 \leq x, t \leq 5$.

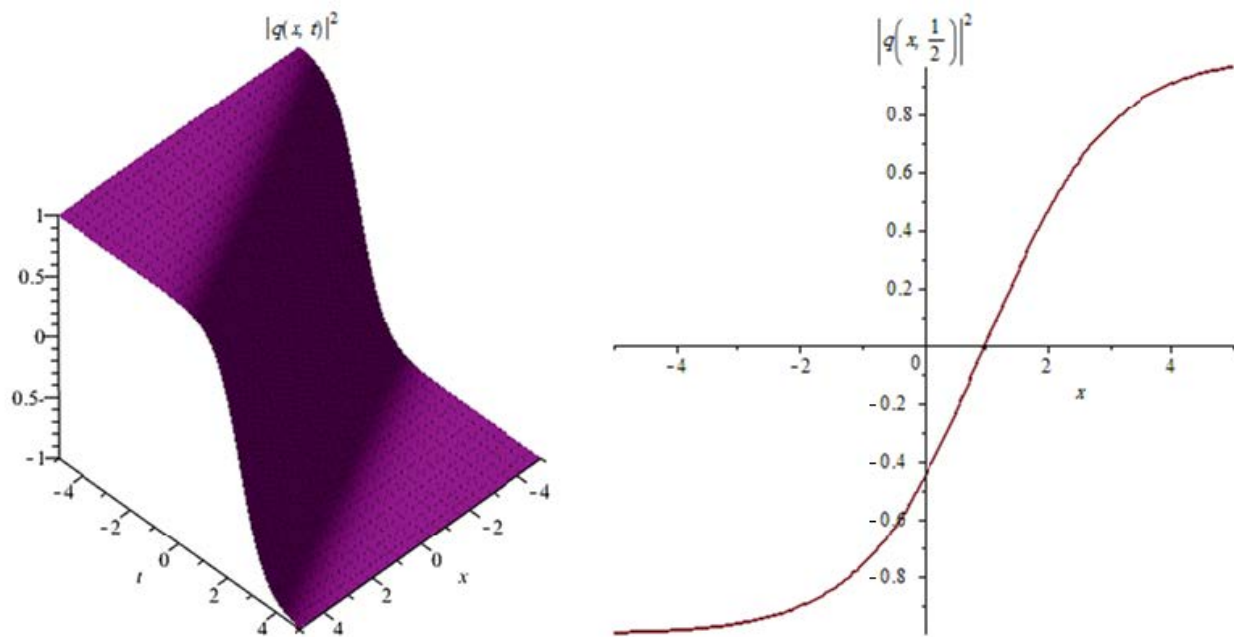


Figure 10. The profile of the combination of dark-bright soliton solutions (51).

Figure 11: The numerical simulations of the solutions (51) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 4, a_2 = 1, b = -21, \sigma = 1, \alpha = 1, \kappa = 2, w = 24, \lambda = 2, \mu = 2, c = 10, v = -6, -5 \leq x, t \leq 5$.

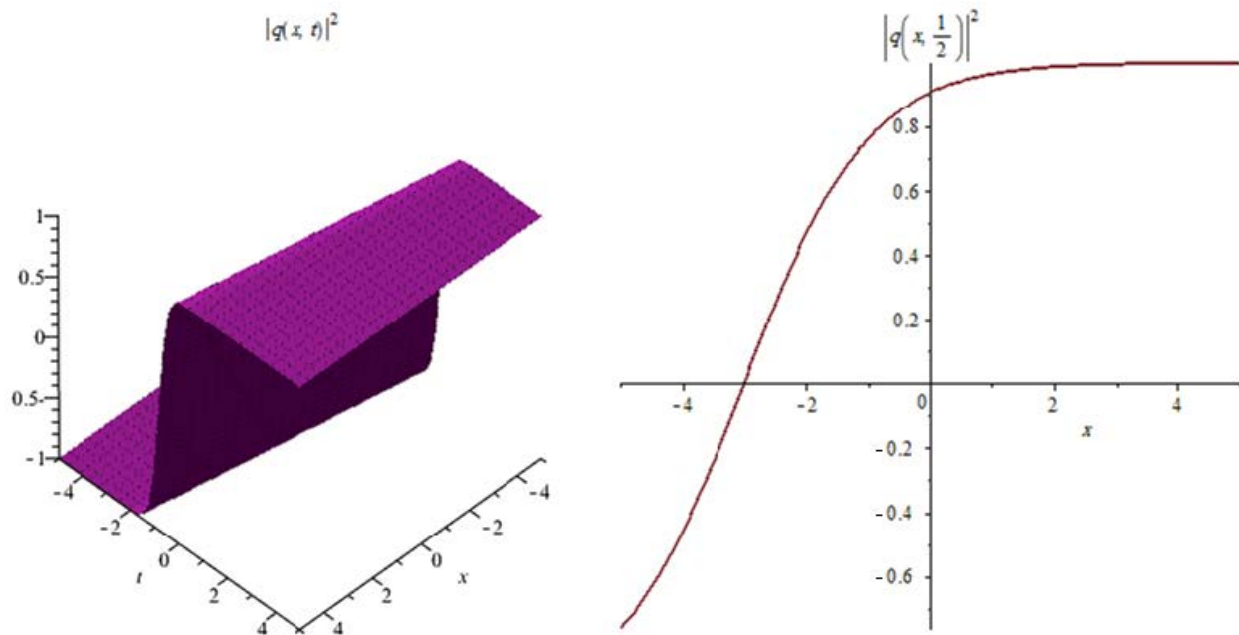


Figure 11. The profile of the combination of dark-bright soliton solutions (51).

Figure 12: The numerical simulations of the solutions (51) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 6, a_2 = 1, b = -16, \sigma = 2, \alpha = 1, \kappa = 2, w = 30, \lambda = 2, \mu = 2, c = 10, v = 6, -5 \leq x, t \leq 5$.

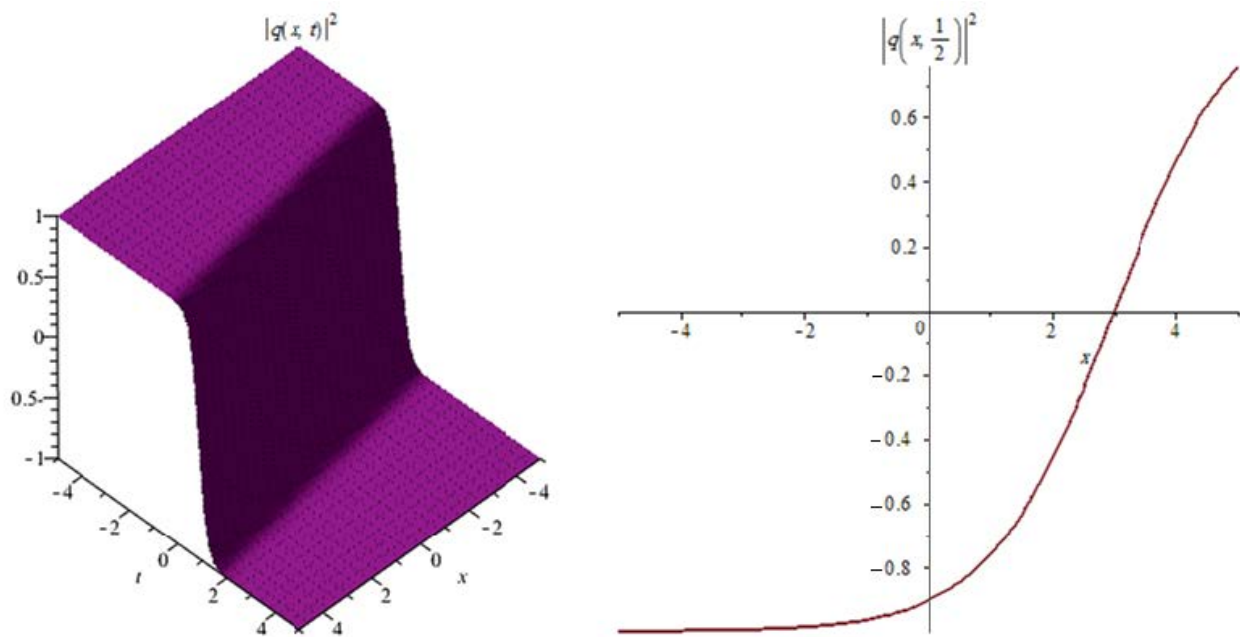


Figure 12. The profile of the combination of dark-bright soliton solutions (51).

Figure 13: The numerical simulations of the solutions (55) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = -16, \sigma = 0, \alpha = 2, \kappa = 2, w = 2, \lambda = 2, \mu = 2, c = 10, v = -7, -5 \leq x, t \leq 5$.

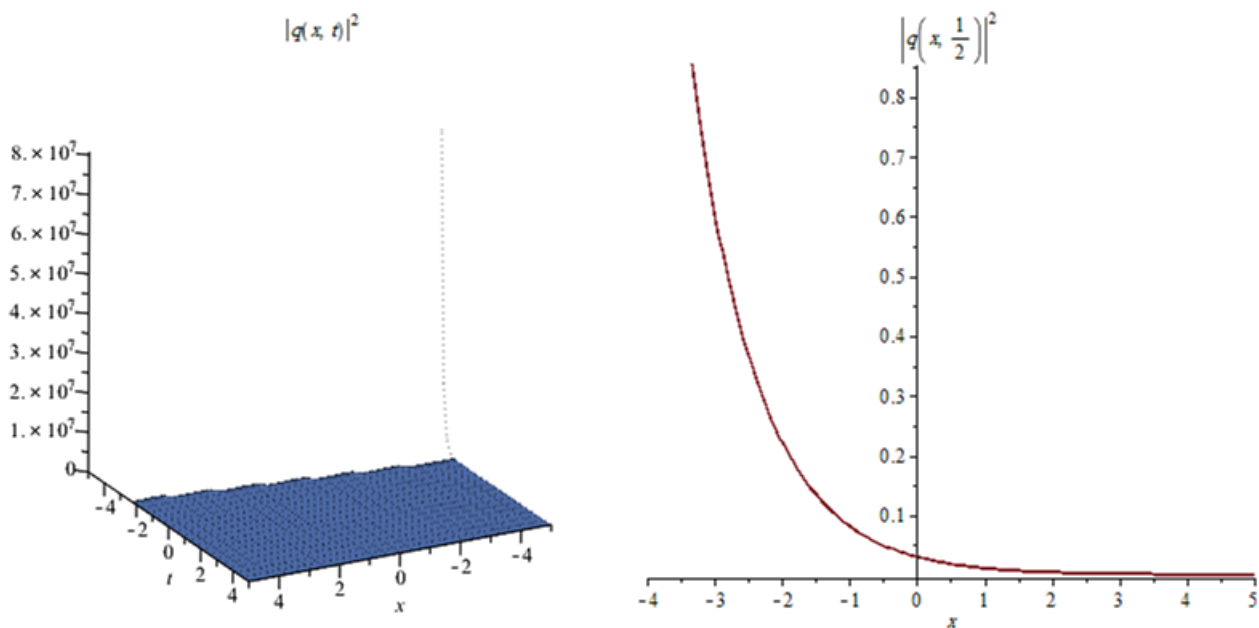


Figure 13. The profile of the combination of bright-dark soliton solutions (55).

Figure 14: The numerical simulations of the solutions (55) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = -16, \sigma = 1, \alpha = \frac{-5}{3}, \kappa = 2, w = 2, \lambda = 2, \mu = 2, c = 10, v = \frac{4}{3}, -5 \leq x, t \leq 5$.

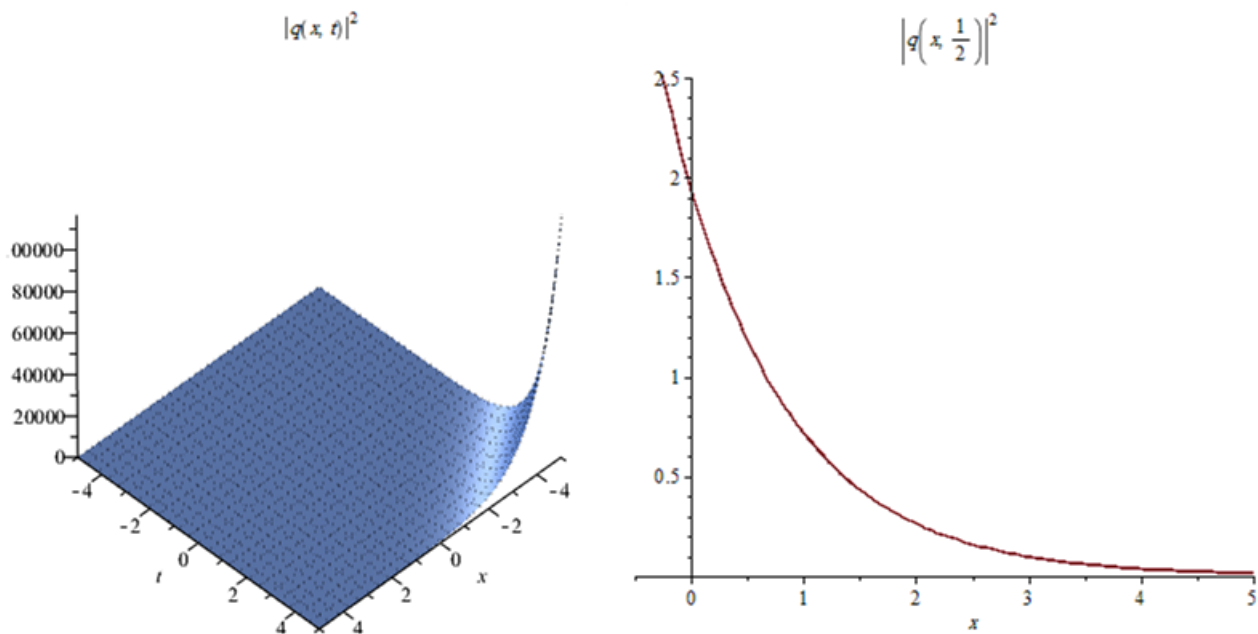


Figure 14. The profile of the combination of bright-dark soliton solutions (55).

Figure 15: The numerical simulations of the solutions (55) 3D and 2D (with $t = \frac{1}{2}$) with the parameter values $a_1 = 1, a_2 = 1, b = -16, \sigma = 2, \alpha = 2, \kappa = 2, w = 2, \lambda = 2, \mu = 2, c = 10, v = -9, c_1 = 0, -5 \leq x, t \leq 5$.

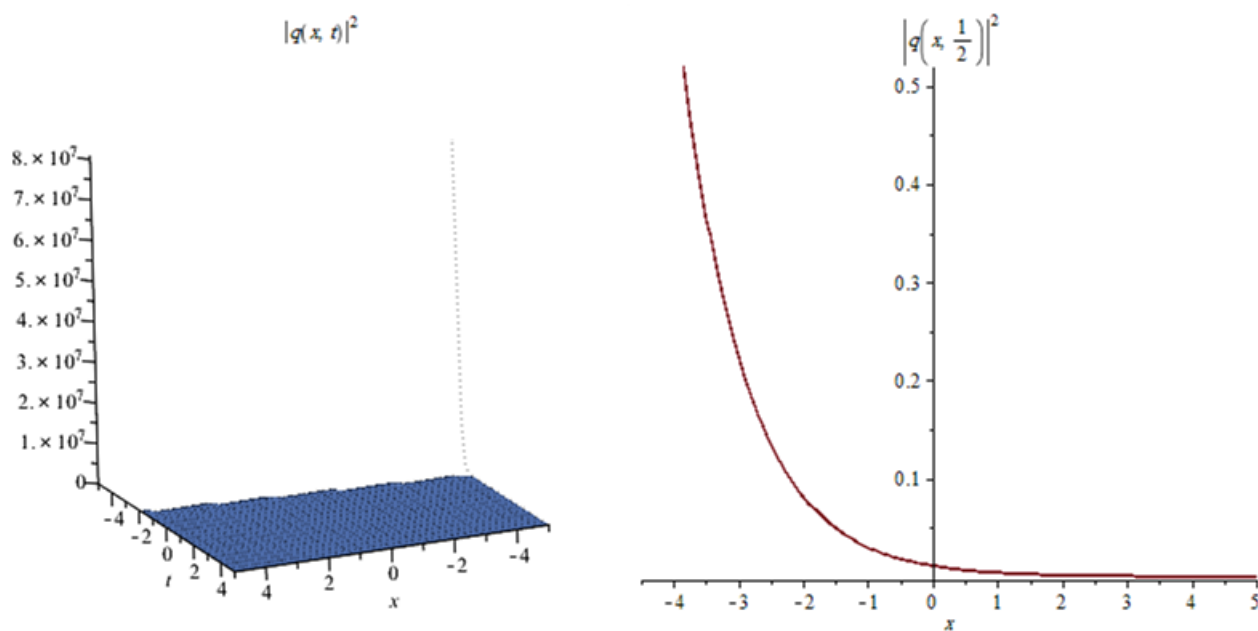


Figure 15. The profile of the combination of bright-dark soliton solutions (55).

Let us now explain the effect of multiplicative white noise in the obtained solutions as follows:

In Figures 1, 4, 7, 10 and 13 when the noise $\sigma = 0$, we note that the surface is less planer. But in Figures 2, 3, 5, 6, 8, 9, 11 and 12 when the noise σ increases ($\sigma = 1, 2, 4$), we note that the surface becomes more planer after small transit behaviors. This means the multiplicative noise effects on the solutions and it makes the solutions stable.

8. Conclusions

In this article, we have obtained the solutions of the stochastic FLE in the presence of multiplicative white noise in the Itô sense. The modified simple equation method, the sine-cosine method, the Jacobi-elliptic function expansion method and the ansatz method are applied. Dark solitons, bright solitons, singular solitons, combo dark-bright solitons, combo bright-dark solitons, as well as Jacobi-elliptic solutions are given. Without noise ($\sigma = 0$) the authors [1,2,37] studied a number of methods to get the exact solutions of FL equation while the stochastic FL Equation (1) is not yet studied. So, on comparing our stochastic solutions ($\sigma \neq 0$) obtained in our present article with the non-stochastic solutions ($\sigma = 0$) obtained in [1,2,37] we deduce that the stochastic solutions are more general than the non-stochastic solutions. Finally, in future, this work will be extended in birefringent fibers, in fiber Bragg gratings and in magneto-optic waveguides. Also, we will study the stochastic FL Equation (1) with variable coefficients [37] when $\sigma \neq 0$, to get stochastic solutions.

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