

## Article

# Containment Control of Multi-Agent Systems with Prescribed Performance by Using Constraint Allocation and Residual Compensation

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## Abstract

This paper investigates prescribed-performance containment control of nonlinear multi-agent systems with multiple leaders under a fixed directed graph. To reconcile follower-specific physical containment envelopes, multi-leader targets determined by the communication graph, and the virtual errors available for distributed feedback, the inverse-Laplacian transfer principle is used to construct sufficient virtual-error bounds. Nonlinear evaluation and convex averaging also generate a possibly nonvanishing nonlinear averaging residual at the containment target. The controller therefore couples the allocated barrier feedback with a directional robust term that dominates this residual in the Lyapunov estimate. For every maximal Filippov solution, a barrier Lyapunov analysis based on an improper integral establishes forward completeness, invariance of the allocated virtual and prescribed physical envelopes, and asymptotic physical containment. Finally, simulation studies across multiple configurations and two direct comparisons with existing methods verify the theoretical results.

**Keywords:** nonlinear multi-agent systems; containment control; prescribed performance; barrier Lyapunov function

## 1. Introduction

Across robot teams, fleets of unmanned vehicles, and distributed sensor networks, local information exchange provides the basis for coordinated decision making and collective action. Containment control drives multiple followers toward the convex hull spanned by multiple leaders [1,2], extending consensus-based coordination [3,4] to multi-reference tasks. Existing graph-based laws have addressed nonlinear dynamics, external disturbances, and actuator faults [5–7]. Input delays, actuator saturation, and state constraints have also been incorporated into containment-control designs [8–10]. Zhang et al. [11] studied event- and self-triggered containment under switching topologies. Du et al. [12] considered dynamically evolving topologies with parameter uncertainty. These studies clarify convergence under varied operating conditions, whereas follower-specific transient tolerances for physical containment errors remain less developed.

Prescribed-performance control (PPC) turns such a tolerance into a trajectory-level constraint: the selected error must remain inside a designer-specified envelope from the initial



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time onward [13,14]. Nonlinear multi-agent studies combine PPC with stochastic pure-feedback, disturbance-observer, event-triggered, and state-constrained designs [15–18]. Together, these studies establish a substantial foundation for prescribed-performance coordination. Nevertheless, the constrained coordinate differs across these formulations. It may be a physical displacement from a leader-weighted target, a Laplacian neighborhood signal, a backstepping variable, or an estimator error. A follower-specific physical error is the natural quantity for specifying containment accuracy, whereas a Laplacian error is the quantity available to distributed feedback. Because the follower Laplacian couples the agent blocks, componentwise physical envelopes cannot be imposed unchanged on the virtual coordinates.

Recent results address complementary aspects of this mismatch. Luo and Liu [19] obtain a zero physical-error limit for a linear single-leader network, whereas Wu et al. [20] derive explicit Laplacian-error estimates for several leaders with an adjustable terminal tolerance. For nonlinear multi-leader systems, performance bounds are commonly assigned to graph-weighted or estimator-based signals. The designs in [16–18] yield semiglobal ultimate bounds or prescribed residual sets. Li et al. [21] further drive the containment error to zero by the prescribed time while keeping the constrained coordinates inside preset regions. Ni et al. [22] transfer an output requirement to a neighborhood condition under switching communication for a single-input–single-output (SISO) model with one leader and a nonzero prescribed residual. Prescribed-time and event-triggered containment designs address timing and communication requirements under different coordinates or plant classes [23–25]. Within the representative methods summarized above, physical-error performance specifications are treated mainly in linear or single-leader settings, whereas nonlinear multi-leader designs typically constrain auxiliary graph-based signals; their conclusions range from ultimate boundedness to asymptotic containment.

The ELM formation design in [26] and the funnel-based Euler–Lagrange tracking method in [27] provide adjacent uncertainty-treatment context, but their plants and objectives do not support a containment comparison on a common plant. The numerical comparisons in Sections 5.3 and 5.6 therefore complement the formulation-level evidence in Table 1. Building on Ref. [19]: the full-row-rank setting, inverse-Laplacian transfer principle, and continuous performance-dependent feedback follow that work. The present construction extends this foundation to follower-specific physical envelopes relative to convex combinations of leader states determined by the communication graph. It also isolates the resulting nonlinear averaging residual and integrates a directional robust term with the allocated barrier feedback. In contrast to the auxiliary graph coordinates or nonzero residual sets used in the closest nonlinear PPC formulations summarized in Table 1, Theorem 1 establishes physical-envelope invariance and asymptotic physical containment for every maximal Filippov solution of the stated model.

This paper addresses these coupled requirements for homogeneous nonlinear agents over a fixed directed graph. Relative to the cited foundations, its contributions are threefold.

1. *Allocation of follower-specific physical constraints:* The inverse-Laplacian transfer principle is adapted to physical containment envelopes defined relative to convex combinations of leader states determined by the communication graph. This produces sufficient bounds for the distributed virtual errors.
2. *Prescribed-performance feedback with residual compensation:* The nonlinear averaging residual is explicitly isolated and bounded. A directional robust term is combined with the allocated barrier feedback to dominate the residual in the Lyapunov estimate under a stated gain condition.
3. *Guarantee for every maximal Filippov solution:* A barrier Lyapunov analysis based on an improper integral establishes forward completeness, invariance of the allocated

virtual and prescribed physical envelopes, and asymptotic physical containment for the stated nonlinear multi-leader model.

Table 1 compares representative methods by prescribed coordinate, physical–feedback error relation, treatment of nonlinear effects, and established result.

**Table 1.** Comparison with closely related prescribed-performance containment methods.

| Work         | Setting                                                           | Prescribed Coordinate                           | Constraint Transfer and Nonlinear Treatment                                                                                                                                                                                                                                                    | Established Result                                                                          |
|--------------|-------------------------------------------------------------------|-------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------|
| [16,17]      | Nonlinear; multiple leaders                                       | Local neighborhood containment error            | Adaptive or observer-based uncertainty treatment. The stated formulations do not explicitly map follower-specific physical constraints to virtual-error bounds or isolate the nonlinear averaging residual                                                                                     | Prescribed boundary or residual; SGUUB                                                      |
| [18]         | Nonlinear nonstrict-feedback; multiple leaders; state constraints | Local neighborhood containment error            | PPC constrains the local neighborhood containment error, while separate barrier Lyapunov functions enforce the full-state constraints. RBF approximation and disturbance observers address nonlinear uncertainty and disturbances; the nonlinear averaging residual is not explicitly isolated | SGUUB; stated neighborhood-error bound and state-constraint satisfaction                    |
| [21]         | Nonlinear; multiple leaders                                       | Local containment and normalized virtual errors | Prescribed-time mapping to preset regions; the nonlinear averaging residual is not explicitly isolated in the stated formulation                                                                                                                                                               | Containment error tends to zero; constrained coordinates remain in preset regions           |
| [19]         | Linear; single leader                                             | Physical consensus error                        | Inverse-Laplacian physical–virtual transfer and continuous performance-dependent feedback; the nonlinear residual is not applicable                                                                                                                                                            | Invariant time-varying envelope and asymptotic zero consensus error                         |
| [20]         | Linear; multiple leaders with bounded unknown inputs              | Containment-error bound                         | Explicit graph-based estimate; the nonlinear residual is not applicable                                                                                                                                                                                                                        | With the hybrid trigger, exponential convergence to an adjustable residual                  |
| [22]         | Nonlinear SISO strict-feedback; single leader; switching graph    | Output error via a neighborhood-error condition | Output-to-neighborhood performance transfer; the multi-leader residual is not applicable                                                                                                                                                                                                       | SGUUB with a user-prescribed nonzero residual and rate                                      |
| Present work | Nonlinear full-state; multiple leaders                            | Follower-specific physical containment error    | Ref. [19]’s transfer principle is adapted to multi-leader physical envelopes; the nonlinear averaging residual is isolated and robustly dominated                                                                                                                                              | For every maximal Filippov solution: envelope invariance and asymptotic zero physical error |

*Abbreviations:* RBF, radial basis function; SGUUB, semiglobal uniform ultimate boundedness; SISO, single-input–single-output. Statements describe the cited formulations and do not imply absence of effects that those formulations do not isolate.

Section 2 formulates the model and error coordinates. Sections 3 and 4 develop the protocol and its analysis, respectively. Section 5 reports the numerical study, and Section 6 concludes the paper.

## 2. Mathematical Model and Problem Formulation

**Notation 1.** Throughout,  $\|\cdot\|$  is the Euclidean vector norm or induced matrix 2-norm. The symbols  $(\cdot)^T$ ,  $\otimes$ ,  $\text{col}(\cdot)$ , and  $\text{diag}\{\cdot\}$  denote transposition, the Kronecker product, column stacking, and formation of a diagonal matrix, respectively. The symbol  $I_k$  is the identity matrix of dimension  $k$ , and  $\mathbf{1}_k$  is the corresponding vector of ones. For a symmetric matrix  $X$ ,  $X \succ 0$  denotes positive definiteness

and  $\lambda_{\min}(X)$  its smallest eigenvalue. The symbol  $0_{r \times s}$  denotes the  $r \times s$  zero matrix,  $\text{im}(B)$  denotes the column space of  $B$ , “a.e.” means almost everywhere, and  $\text{ess sup}$  is the essential supremum. The spaces  $L^2([0, \infty))$  and  $L^\infty([0, \infty))$  contain square-integrable and essentially bounded signals, respectively. The variable  $t \in [0, \infty)$  denotes continuous time. We omit time arguments when they are unambiguous.

## 2.1. Graph and Agent Dynamics

We consider a weighted directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  with  $\mathcal{V} = \{1, \dots, N\}$  and  $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ . An edge  $(j, i) \in \mathcal{E}$  indicates that agent  $i$  continuously receives  $x_j$ . The edge has weight  $a_{ij} > 0$ . For an absent edge,  $a_{ij} = 0$ . We set  $\Delta = \text{diag}\{\delta_1, \dots, \delta_N\}$ ,  $\delta_i = \sum_{j=1}^N a_{ij}$ , and  $\mathcal{L} = \Delta - \mathcal{A}$ . Thus,  $\delta_i$ ,  $\Delta$ , and  $\mathcal{L}$  are the weighted in-degree of agent  $i$ , the in-degree matrix, and the graph Laplacian, respectively. The network contains  $M$  followers,  $\mathcal{F} = \{1, \dots, M\}$ , and  $N - M$  leaders,  $\mathcal{R} = \{M + 1, \dots, N\}$ . We set  $R := N - M$ . The followers and leaders have the following homogeneous dynamics:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + f(x_i(t)), \quad i \in \mathcal{F}, \quad (1)$$

$$\dot{x}_l(t) = Ax_l(t) + f(x_l(t)), \quad l \in \mathcal{R}, \quad (2)$$

Here,  $N$  and  $M$  are the total number of agents and the number of followers, while  $n$  and  $m$  are the state and input dimensions. The variables satisfy  $x_i \in \mathbb{R}^n$  and  $u_i \in \mathbb{R}^m$ . The known matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are agent-level matrices. The mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is common to all agents, but its instantaneous value is not required by the feedback law. We define  $x = \text{col}(x_1, \dots, x_N) \in \mathbb{R}^{Nn}$ ,  $u_F = \text{col}(u_1, \dots, u_M) \in \mathbb{R}^{Mm}$ , and  $E_F = \text{col}(I_M, 0_{(N-M) \times M}) \in \mathbb{R}^{N \times M}$ . The stacked dynamics are

$$\dot{x} = (I_N \otimes A)x + (E_F \otimes B)u_F + \text{col}(f(x_1), \dots, f(x_N)).$$

Thus, the aggregate drift and follower-input matrices have dimensions  $Nn \times Nn$  and  $Nn \times Mm$ . Only the  $M$  follower inputs appear because the leaders are autonomous. No edge enters a leader. Ordering the followers before the leaders gives  $\mathcal{L} = \begin{bmatrix} L_1 & L_2 \\ 0 & 0 \end{bmatrix}$ , where  $L_1 \in \mathbb{R}^{M \times M}$  and  $L_2 \in \mathbb{R}^{M \times (N-M)}$ . The blocks  $L_1$  and  $L_2$  collect the follower-follower and leader-to-follower couplings, respectively.

**Definition 1** ([28]). A real square matrix  $S = [s_{ij}]$  is a nonsingular  $M$ -matrix if  $s_{ij} \leq 0$  for  $i \neq j$  and every eigenvalue of  $S$  has a positive real part.

**Assumption 1** ([29]). Each follower is reachable from at least one leader through a directed path in  $\mathcal{G}$ .

**Assumption 2** ([19]). The input matrix  $B$  has full row rank. Equivalently,  $BB^T \succ 0$ .

**Assumption 3** ([30]). The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz: for every compact set  $\Omega \subset \mathbb{R}^n$  there exists  $L_\Omega > 0$  such that  $\|f(x) - f(y)\| \leq L_\Omega \|x - y\|$ ,  $x, y \in \Omega$ .

**Assumption 4** ([31]). There exists a known constant  $\bar{f} \geq 0$  such that  $\|f(x)\| \leq \bar{f}$ ,  $x \in \mathbb{R}^n$ .

**Remark 1.** Assumption 1 makes  $L_1$  a nonsingular  $M$ -matrix and  $\Pi = -L_1^{-1}L_2$  a matrix of convex leader weights. Assumption 2 allows  $B$  to be rectangular but requires  $\text{im}(B) = \mathbb{R}^n$ . Hence  $BB^T \succ 0$ , which provides the coercive term in (33). The present proof therefore applies to full-row-rank input matrices.

The two assumptions on  $f$  have different roles. Local Lipschitz continuity provides the regularity required for solution existence. The known bound  $\bar{f}$  controls the nonlinear residual and enters the offline gain conditions. A related scalar time-varying local-regularity condition appears in Ref. [30]. Peng [31] similarly assumes a known uniform bound on a nonlinear term. In practice, one can check full row rank through  $\sigma_{\min}(B) > 0$  and obtain  $\bar{f}$  from analytical component bounds or known saturation levels. Underactuated dynamics require a different factorization or additional structure.

## 2.2. Physical and Virtual Containment Errors

Under Assumption 1,  $L_1$  is a nonsingular  $M$ -matrix [29]. Hence  $L_1^{-1} \geq 0$ . Since  $-L_2 \geq 0$ ,  $\Pi := -L_1^{-1}L_2 \geq 0$ . The Laplacian row-sum identity  $L_1\mathbf{1}_M + L_2\mathbf{1}_R = 0$  further gives  $\Pi\mathbf{1}_R = \mathbf{1}_M$ . We write  $\Pi = [\pi_{il}]_{i \in \mathcal{F}, l \in \mathcal{R}}$ . Its  $i$ th row gives the convex combination of leader states assigned to follower  $i$ . We define  $x_F = \text{col}(x_1, \dots, x_M)$  and  $x_R = \text{col}(x_{M+1}, \dots, x_N)$ . The physical containment error is

$$\zeta = x_F + (L_1^{-1}L_2 \otimes I_n)x_R, \quad \zeta_i = x_i - \sum_{l=M+1}^N \pi_{il}x_l. \quad (3)$$

Thus,  $\zeta_i$  is the displacement from the convex combination determined by the  $i$ th row of  $\Pi$ . It is not the Euclidean distance to the leader hull.

The virtual containment error is

$$\zeta_v = (L_1 \otimes I_n)\zeta = (L_1 \otimes I_n)x_F + (L_2 \otimes I_n)x_R \quad (4)$$

Its  $i$ th block is

$$\zeta_{vi} = \sum_{j=1}^N a_{ij}(x_i - x_j).$$

$\zeta_{vi}$  can be formed from relative states received along incoming edges. Since  $L_1$  is nonsingular,

$$\zeta = (L_1^{-1} \otimes I_n)\zeta_v, \quad (5)$$

and hence  $\zeta_v(t) \rightarrow 0$  exactly when  $\zeta(t) \rightarrow 0$ . Because  $\Pi$  is row-stochastic, the corresponding follower targets remain in the moving leader convex hull.

## 2.3. Prescribed-Performance Specification

Following Ref. [19], we prescribe an envelope  $\psi_i : [0, \infty) \rightarrow (0, \infty)$  for the physical containment error of each follower  $i \in \mathcal{F}$ . The allocation introduced below requires  $\psi_i(t)$  to be locally absolutely continuous and nonincreasing. We also assume that

$$0 < \underline{\psi}_i \leq \psi_i(t) \leq \bar{\psi}_i, \quad \dot{\psi}_i(t) \leq 0 \text{ a.e.}, \quad \text{ess sup}_{t \geq 0} |\dot{\psi}_i(t)| < \infty, \quad (6)$$

where  $\underline{\psi}_i$  and  $\bar{\psi}_i$  are positive constants. These conditions keep the minimum-based allocation locally absolutely continuous. The usual smooth exponential envelopes satisfy them.

## 2.4. Constraint Transfer from Physical to Virtual Errors

The feedback law is constructed from  $\zeta_v$ , whereas the prescribed performance envelopes are assigned to the physical errors  $\zeta$ . To connect these coordinates, we adapt

the inverse-Laplacian construction in [19] from the single-leader setting to the present multi-leader containment problem. We write

$$L_1^{-1} = [\bar{\ell}_{ij}], \quad \bar{\ell}_i = \sum_{j=1}^M |\bar{\ell}_{ij}|, \quad \mathcal{I}_j = \{i \in \mathcal{F} : \bar{\ell}_{ij} \neq 0\}.$$

Since  $L_1^{-1}$  is nonsingular, none of its columns is zero, and hence  $\mathcal{I}_j \neq \emptyset$  for every  $j \in \mathcal{F}$ . Therefore, the following allocations are well defined:

$$\begin{aligned} \underline{\alpha}_j &:= \min_{i \in \mathcal{I}_j} \frac{\psi_i}{\bar{\ell}_i}, & \bar{\alpha}_j &:= \min_{i \in \mathcal{I}_j} \frac{\bar{\psi}_i}{\bar{\ell}_i}, \\ \alpha_j(t) &:= \min_{i \in \mathcal{I}_j} \frac{\psi_i(t)}{\bar{\ell}_i}, & \bar{r}_\alpha &:= \max_{j \in \mathcal{F}} \operatorname{ess\,sup}_{t \geq 0} \left( -\frac{\dot{\alpha}_j(t)}{\alpha_j(t)} \right). \end{aligned} \quad (7)$$

Here,  $\alpha_j(t)$  is the allocated envelope for virtual-error block  $j$ ,  $\underline{\alpha}_j$  and  $\bar{\alpha}_j$  are its uniform lower and upper bounds, and  $\bar{r}_\alpha$  is a uniform upper bound on its relative decay rate.

For each  $j \in \mathcal{F}$ ,  $\alpha_j(t)$  is the pointwise minimum of finitely many positive, locally absolutely continuous, and nonincreasing functions. It is therefore positive, nonincreasing, and locally absolutely continuous. Moreover, the positive lower bounds and essentially bounded derivatives in (6) imply

$$0 < \underline{\alpha}_j \leq \alpha_j(t) \leq \bar{\alpha}_j, \quad \bar{r}_\alpha < \infty.$$

We define the open virtual-performance region

$$\Omega_\alpha(t) = \left\{ \xi_v \in \mathbb{R}^{Mn} : \|\xi_{vj}\| < \alpha_j(t), j \in \mathcal{F} \right\}. \quad (8)$$

Accordingly, a closed-loop initial condition is called admissible if  $\xi_v(0) \in \Omega_\alpha(0)$ . For any  $\xi_v(t) \in \Omega_\alpha(t)$ , (5) and (7) give, for every  $i \in \mathcal{F}$ ,

$$\begin{aligned} \|\xi_i(t)\| &\leq \sum_{j: \bar{\ell}_{ij} \neq 0} |\bar{\ell}_{ij}| \|\xi_{vj}(t)\| \\ &< \sum_{j: \bar{\ell}_{ij} \neq 0} |\bar{\ell}_{ij}| \alpha_j(t) \\ &\leq \sum_{j: \bar{\ell}_{ij} \neq 0} |\bar{\ell}_{ij}| \frac{\psi_i(t)}{\bar{\ell}_i} = \psi_i(t). \end{aligned}$$

Thus, positive invariance of the virtual-performance region is sufficient to enforce the physical-error envelope prescribed for each follower.

**Problem 1.** *The objective is to design a full-state feedback law with distributed online implementation for the physical-error envelopes  $\{\psi_i(t)\}_{i \in \mathcal{F}}$ . For every admissible initial condition, all resulting closed-loop trajectories must be forward complete and satisfy*

$$\|\xi_i(t)\| < \psi_i(t), \quad t \geq 0,$$

and

$$\lim_{t \rightarrow \infty} \xi_i(t) = 0, \quad i \in \mathcal{F}.$$

**Remark 2.** *The allocation in (7) is sufficient but generally conservative. The triangle inequality and the minimum over physical constraints replace a coupled relation with independent virtual-*

error bounds. The resulting envelopes are computed offline, so the controller requires no online allocation optimization.

For each physical-error block, we define the utilization ratio

$$\gamma_i^{\text{alloc}}(t) := \frac{\sum_{j=1}^M |\bar{\ell}_{ij}| \alpha_j(t)}{\psi_i(t)} = \sup_{\|z_j\| \leq \alpha_j(t)} \frac{\left\| \sum_{j=1}^M \bar{\ell}_{ij} z_j \right\|}{\psi_i(t)} \leq 1.$$

For any unit vector  $v$ , the choice  $z_j = \text{sign}(\bar{\ell}_{ij}) \alpha_j(t) v$  for every nonzero  $\bar{\ell}_{ij}$  attains the supremum. Thus,  $\gamma_i^{\text{alloc}}(t)$  is the largest fraction of the  $i$ th physical envelope occupied by the product of the allocated virtual-error balls. At a fixed time, the common dilation allowed by this certificate is  $1/\gamma_{\max}^{\text{alloc}}(t)$ , where  $\gamma_{\max}^{\text{alloc}}(t) = \max_i \gamma_i^{\text{alloc}}(t)$ . A dilation valid for all time is  $[\text{ess sup}_{t \geq 0} \gamma_{\max}^{\text{alloc}}(t)]^{-1}$ .

The envelope design can also enforce initial feasibility. We define  $\mathcal{J}_i = \{j : \bar{\ell}_{ij} \neq 0\}$  and choose a normalized margin  $\sigma_0 \in (0, 1)$ . The condition  $\max_j \|\xi_{vj}(0)\| / \alpha_j(0) \leq \sigma_0$  is ensured by

$$\psi_i(0) \geq \max \left\{ \psi_{i,0}^{\text{task}}, \frac{\bar{\ell}_i}{\sigma_0} \max_{j \in \mathcal{J}_i} \|\xi_{vj}(0)\| \right\},$$

where  $\psi_{i,0}^{\text{task}}$  is the initial bound required by the task. For every  $i \in \mathcal{I}_j$ , this inequality gives  $\psi_i(0) / \bar{\ell}_i \geq \|\xi_{vj}(0)\| / \sigma_0$ . Taking the minimum over  $i \in \mathcal{I}_j$  proves the stated margin. If the initial envelope can be chosen, the allocation and gains are recomputed offline after this choice. If the envelope is a fixed safety limit, the same inequality serves as a feasibility test. A coupled offline allocation may admit directions excluded by the separable construction.

### 3. Nonlinear Residual and Distributed Control

The controller is developed in two steps. We first identify the nonlinear residual in the transformed dynamics. We then combine barrier feedback with a directional term that dominates this residual.

#### 3.1. Nonlinear Residual

Equation (3) yields the physical-error dynamics

$$\dot{\xi}_i(t) = A \xi_i(t) + B u_i(t) + \Delta_{f,i}(t), \quad (9)$$

where  $\Delta_{f,i}(t) = f(x_i(t)) - \sum_{l=M+1}^N \pi_{il} f(x_l(t))$ .

Since  $\Pi$  is nonnegative and row-stochastic, Assumption 4 gives

$$\|\Delta_{f,i}(t)\| \leq \|f(x_i(t))\| + \sum_{l=M+1}^N \pi_{il} \|f(x_l(t))\| \leq 2\bar{f}, \quad \|\Delta_f(t)\| \leq 2\sqrt{M}\bar{f} =: \delta_f, \quad (10)$$

Here,  $\Delta_f = \text{col}(\Delta_{f,1}, \dots, \Delta_{f,M})$  is the stacked nonlinear residual and  $\delta_f = 2\sqrt{M}\bar{f}$  is its uniform bound.

**Remark 3.** Along any trajectory,  $\Delta_{f,i} = 0$  if and only if

$$f(x_i) = \sum_{l=M+1}^N \pi_{il} f(x_l).$$

At exact physical containment,  $x_i = \bar{x}_i := \sum_l \pi_{il} x_l$ , and the residual becomes

$$\Delta_{f,i}^0 = f(\bar{x}_i) - \sum_{l=M+1}^N \pi_{il} f(x_l).$$

This residual vanishes when  $f$  preserves the relevant weighted average. Affine  $f$ , a single active leader, and coincident leader coordinates on which  $f$  depends are examples. For nonlinear  $f$  and distinct leaders, the residual can remain nonzero at the target. The continuous component of (12) vanishes at  $\xi_v = 0$  and cannot keep that manifold invariant when  $\Delta_f^0 \neq 0$  because  $\dot{\xi}_v = (L_1 \otimes I_n)\Delta_f^0 \neq 0$ . The directional term does not cancel the unknown residual pointwise. Under (17), it provides sufficient set-valued control authority to dominate the residual in the Lyapunov estimate.

### 3.2. Distributed Barrier Controller with Robust Compensation

The directional term is interpreted in the Filippov sense [32]. For  $z \in \mathbb{R}^n$ , we define the single-valued vector sign function and its Filippov regularization by

$$\text{sgn}(z) = \begin{cases} \frac{z}{\|z\|}, & z \neq 0, \\ 0, & z = 0, \end{cases} \quad \text{Sgn}(z) := \begin{cases} \frac{z}{\|z\|}, & z \neq 0, \\ \{v \in \mathbb{R}^n : \|v\| \leq 1\}, & z = 0. \end{cases}$$

Here,  $\text{sgn}$  denotes the single-valued vector function implemented in the controller, whereas  $\text{Sgn}$  denotes its set-valued Filippov regularization. Closing the loop produces a differential inclusion. A Filippov solution is an absolutely continuous trajectory that satisfies this inclusion almost everywhere. The proof below treats existence and possible nonuniqueness on the switching set explicitly.

For follower  $i$ , the positive constants  $c_i$ ,  $\mu_i$ , and  $\nu_i$  are, respectively, the overall gain, the baseline gain of the continuous term, and the gain used for residual compensation. Within the admissible region  $\xi_{vi}^T(t)\xi_{vi}(t) < \alpha_i^2(t)$ , we define the performance-dependent function

$$\varphi_i(\xi_{vi}^T(t)\xi_{vi}(t), \alpha_i(t)) = \mu_i \left[ \alpha_i^2(t) - \xi_{vi}^T(t)\xi_{vi}(t) \right] + \frac{\alpha_i^2(t)}{\alpha_i^2(t) - \xi_{vi}^T(t)\xi_{vi}(t)}. \quad (11)$$

Follower  $i$  applies the input

$$u_i(t) = -c_i B^T \left[ \frac{\varphi_i(\xi_{vi}^T(t)\xi_{vi}(t), \alpha_i(t))}{\alpha_i^2(t) - \xi_{vi}^T(t)\xi_{vi}(t)} \xi_{vi}(t) + \nu_i \text{sgn}(\xi_{vi}(t)) \right], \quad i \in \mathcal{F}. \quad (12)$$

The continuous part retains the performance-dependent structure of Ref. [19]. Expanding (11) gives

$$\frac{\varphi_i(\xi_{vi}^T \xi_{vi}, \alpha_i)}{\alpha_i^2 - \xi_{vi}^T \xi_{vi}} = \mu_i + \frac{\alpha_i^2}{(\alpha_i^2 - \xi_{vi}^T \xi_{vi})^2}. \quad (13)$$

The coefficient multiplying the virtual error grows as the error approaches its boundary. The directional term compensates for the nonlinear averaging residual under the gain conditions stated below. The allocation and gain selection are completed offline. Online, follower  $i$  implements (12) using only the locally available virtual error  $\xi_{vi}(t)$  and the stored envelope  $\alpha_i(t)$ .

## 4. Stability and Performance Analysis

We construct a matrix certificate for the graph from the linear system

$$L_1^T \zeta = \mathbf{1}_M, \quad D = \text{diag}(\zeta), \quad Q = DL_1 + L_1^T D. \quad (14)$$

*Topology certificate:* Assumption 1 makes  $L_1$  a nonsingular  $M$ -matrix with  $L_1^{-1} \geq 0$  [29]. Hence  $\zeta = L_1^{-T} \mathbf{1}_M > 0$  and  $D \succ 0$ . We define  $b = L_1 \mathbf{1}_M = -L_2 \mathbf{1}_R \geq 0$ . Then

$$Q \mathbf{1}_M = Db + \mathbf{1}_M > 0.$$

The matrix  $Q$  is symmetric, and its off-diagonal entries are nonpositive. Each row is therefore strictly diagonally dominant with a positive diagonal entry. It follows that  $Q \succ 0$ .

**Theorem 1.** *The nonlinear multi-agent system (1) and (2) satisfies Assumptions 1–4 and is controlled by (12). The envelope  $\psi_i(t)$  satisfies (6),  $\alpha_i(t)$  is defined by (7), and the matrices  $D = \text{diag}(d_1, \dots, d_M)$  and  $Q$  are defined by (14). The analysis uses the constants*

$$\begin{aligned} \lambda_* &= \lambda_{\min}(Q)\lambda_{\min}(BB^T), & g_{\mu,*} &= \min_{i \in \mathcal{F}}(c_i \mu_i), & g_{v,*} &= \min_{i \in \mathcal{F}}(c_i \nu_i), \\ d_{\max} &= \max_{i \in \mathcal{F}} d_i, & a_* &= d_{\max}(\|A\| + \bar{r}_\alpha), & b_* &= 2\|DL_1\|\delta_f. \end{aligned} \quad (15)$$

The initial condition is admissible:

$$\|\xi_{vi}(0)\| < \alpha_i(0), \quad i \in \mathcal{F}, \quad (16)$$

The gains also satisfy

$$g_{\mu,*} > \frac{2a_*}{\lambda_*}, \quad g_{v,*} > \frac{b_*}{\lambda_* - 2a_*/g_{\mu,*}}. \quad (17)$$

Under these conditions, the closed-loop system admits at least one maximal Filippov solution. Every maximal solution is forward complete and satisfies

$$\|\xi_{vi}(t)\| < \alpha_i(t), \quad \|\xi_i(t)\| < \psi_i(t), \quad t \geq 0, \quad i \in \mathcal{F}. \quad (18)$$

Moreover,

$$\begin{aligned} \lim_{t \rightarrow \infty} \xi_v(t) &= 0, \\ \lim_{t \rightarrow \infty} \left[ x_i(t) - \sum_{l=M+1}^N \pi_{il} x_l(t) \right] &= 0, \quad i \in \mathcal{F}. \end{aligned} \quad (19)$$

**Proof.** *Existence and closed-loop inclusion:* We define

$$\omega_i(t, z) := \frac{\varphi_i(z^T z, \alpha_i(t))}{\alpha_i^2(t) - z^T z}, \quad q_i^c(t, z) := c_i \omega_i(t, z) z,$$

$q^c(t, \xi_v) = \text{col}(q_1^c(t, \xi_{v1}), \dots, q_M^c(t, \xi_{vM}))$ ,  $C_v = \text{diag}(c_1 \nu_1, \dots, c_M \nu_M)$ , and  $\mathcal{K}_{\text{sgn}}(\xi_v) := \times_{i=1}^M \text{Sgn}(\xi_{vi})$ . In the coordinates  $(\xi_v, x_R)$ , the regularized closed loop is

$$\begin{aligned} \dot{\xi}_v &\in (I_M \otimes A)\xi_v - (L_1 \otimes BB^T)[q^c(t, \xi_v) + (C_v \otimes I_n)\mathcal{K}_{\text{sgn}}(\xi_v)] + (L_1 \otimes I_n)\Delta_f(\xi_v, x_R), \\ \dot{x}_R &= (I_R \otimes A)x_R + \text{col}(f(x_{M+1}), \dots, f(x_N)). \end{aligned} \quad (20)$$

Here,  $\Delta_f(\xi_v, x_R)$  denotes the residual in (10) expressed through the inverse coordinate transformation. Because  $L_1$  is nonsingular, the map  $(x_F, x_R) \mapsto (\xi_v, x_R)$  is an invertible linear coordinate change. The directional map is block-separable, so its Filippov regularization is the Cartesian product  $\mathcal{K}_{\text{sgn}}(\xi_v)$ . On every compact subset of the open performance region, the remaining terms are measurable in time, locally bounded, and continuous in the state. The set-valued right-hand side of (20) is upper semicontinuous with nonempty, compact, convex values. It therefore satisfies the time-varying existence conditions in Proposition S1 of Ref. [32]. A locally absolutely continuous Filippov solution exists from every admissible initial condition. Each trajectory admits a measurable selection  $\eta_i(t) \in \text{Sgn}(\xi_{vi}(t))$  realizing its right-hand side almost everywhere. We fix an arbitrary solution on its maximal interval  $[0, T_{\max})$ .  $\square$

#### 4.1. Barrier Coordinates and Lyapunov Function

We retain  $D$ ,  $Q$ , and the constants in (15). For  $i \in \mathcal{F}$ , we define

$$\mathcal{B}_i(t) = \ln \frac{\alpha_i^2(t)}{\alpha_i^2(t) - \xi_{vi}^T(t) \xi_{vi}(t)}, \|\xi_{vi}(t)\| < \alpha_i(t). \quad (21)$$

$\mathcal{B}_i(t)$  is nonnegative, with  $\mathcal{B}_i(t) \rightarrow +\infty$  as  $\alpha_i^2(t) - \|\xi_{vi}(t)\|^2 \rightarrow 0^+$ . Equivalently,

$$\xi_{vi}^T(t) \xi_{vi}(t) = \alpha_i^2(t) (1 - e^{-\mathcal{B}_i(t)}). \quad (22)$$

Equation (22) gives the following expression for the performance-dependent function:  $\varphi_i = \mu_i \alpha_i^2 e^{-\mathcal{B}_i} + e^{\mathcal{B}_i}$ . To place the directional compensation in the same Lyapunov construction, we define for  $s > 0$

$$\vartheta_i(s, \alpha_i) = \mu_i \alpha_i^2 e^{-s} + e^s + \nu_i \alpha_i \frac{e^{-s}}{\sqrt{1 - e^{-s}}}. \quad (23)$$

As  $s \downarrow 0$ , the last term is  $O(s^{-1/2})$  and is therefore integrable. Accordingly, the integral below is understood as an improper integral at the lower endpoint  $s = 0$  for  $\mathcal{B}_i > 0$ , and is defined as zero when  $\mathcal{B}_i = 0$ .

We choose

$$V(t) = \sum_{i=1}^M d_i c_i \int_0^{\mathcal{B}_i(t)} \vartheta_i(s, \alpha_i(t)) ds. \quad (24)$$

Evaluating the integral gives

$$V(t) = \sum_{i=1}^M d_i c_i \left[ \mu_i \alpha_i^2(t) (1 - e^{-\mathcal{B}_i(t)}) + e^{\mathcal{B}_i(t)} - 1 + 2\nu_i \alpha_i(t) \sqrt{1 - e^{-\mathcal{B}_i(t)}} \right]. \quad (25)$$

On every compact subinterval on which the trajectory remains in the open performance region,  $\alpha_i$  and  $\xi_{vi}$  are absolutely continuous and the quantity  $\alpha_i^2 - \|\xi_{vi}\|^2$  is bounded away from zero. Consequently, the rational expression for  $V$  in (25) is absolutely continuous, and  $\dot{V}$  exists almost everywhere. The function  $V$  is positive definite in  $\xi_v$  and diverges when any  $\|\xi_{vi}\|$  approaches  $\alpha_i(t)$ . For every nonzero  $\xi_{vi}$ , the definitions give

$$\frac{\vartheta_i(\mathcal{B}_i, \alpha_i)}{\alpha_i^2 - \xi_{vi}^T \xi_{vi}} = \frac{\varphi_i(\xi_{vi}^T \xi_{vi}, \alpha_i)}{\alpha_i^2 - \xi_{vi}^T \xi_{vi}} + \frac{\nu_i}{\|\xi_{vi}\|}, \quad (26)$$

which shows that the gradient of  $V$  contains the same continuous and directional coefficients as the controller in (12).

*Dissipation estimate:* We use the measurable selection  $\eta_i(t)$  fixed above. For  $\xi_{vi} \neq 0$ ,  $\eta_i = \xi_{vi} / \|\xi_{vi}\|$ , whereas  $\|\eta_i\| \leq 1$  when  $\xi_{vi} = 0$ . This auxiliary variable is used only in the proof.

For compactness, we define the auxiliary vector  $q_i(t) \in \mathbb{R}^n$  by

$$q_i(t) = c_i \left[ \frac{\varphi_i(\xi_{vi}^T(t) \xi_{vi}(t), \alpha_i(t))}{\alpha_i^2(t) - \xi_{vi}^T(t) \xi_{vi}(t)} \xi_{vi}(t) + \nu_i \eta_i(t) \right], \quad i \in \mathcal{F}. \quad (27)$$

Then  $u_i = -B^T q_i$  and  $q = \text{col}(q_1, \dots, q_M)$ . This definition introduces no additional controller state. We write (25) as  $V(t) = \sum_{i=1}^M V_i(t)$ , where the  $i$ th summand can be written as

$$V_i = d_i c_i \left[ \mu_i \|\xi_{vi}\|^2 + \frac{\|\xi_{vi}\|^2}{\alpha_i^2 - \|\xi_{vi}\|^2} + 2\nu_i \|\xi_{vi}\| \right].$$

For every nonzero block  $\xi_{vi}$ , differentiation yields the following identity almost everywhere:

$$\dot{V}_i = 2d_i q_i^T \dot{\xi}_{vi} - 2d_i c_i \frac{\alpha_i \dot{\alpha}_i}{(\alpha_i^2 - \|\xi_{vi}\|^2)^2} \|\xi_{vi}\|^2. \quad (28)$$

We treat zero blocks in the almost-everywhere sense and define  $\mathcal{Z}_i = \{t \geq 0 : \xi_{vi}(t) = 0\}$ . Absolute continuity gives  $\dot{\xi}_{vi} = 0$  almost everywhere on  $\mathcal{Z}_i$ . Also  $V_i = 0$  on that set, so  $\dot{V}_i = 0$  there almost everywhere. Assigning zero to the second term of (28) at  $\xi_{vi} = 0$  therefore preserves the identity almost everywhere. From the definition of  $\bar{r}_\alpha$ ,

$$-\frac{\dot{\alpha}_i(t)}{\alpha_i(t)} \leq \bar{r}_\alpha \text{ a.e.}$$

Furthermore,

$$q_i^T \xi_{vi} = c_i \left[ \mu_i + \frac{\alpha_i^2}{(\alpha_i^2 - \|\xi_{vi}\|^2)^2} \right] \|\xi_{vi}\|^2 + c_i v_i \|\xi_{vi}\| \geq c_i \frac{\alpha_i^2}{(\alpha_i^2 - \|\xi_{vi}\|^2)^2} \|\xi_{vi}\|^2. \quad (29)$$

The performance-envelope term then satisfies

$$-2d_i c_i \frac{\alpha_i \dot{\alpha}_i}{(\alpha_i^2 - \|\xi_{vi}\|^2)^2} \|\xi_{vi}\|^2 \leq 2d_i \bar{r}_\alpha q_i^T \xi_{vi}. \quad (30)$$

The stacked virtual-error dynamics are obtained from  $\xi_v = (L_1 \otimes I_n) \xi$  as

$$\dot{\xi}_v = (I_M \otimes A) \xi_v - (L_1 \otimes BB^T) q + (L_1 \otimes I_n) \Delta_f. \quad (31)$$

Substituting (31) into (28), summing over all followers, and using (30) yield

$$\dot{V} \leq 2q^T (D \otimes A) \xi_v + 2\bar{r}_\alpha q^T (D \otimes I_n) \xi_v - 2q^T (DL_1 \otimes BB^T) q + 2q^T (DL_1 \otimes I_n) \Delta_f. \quad (32)$$

We next estimate the terms in (32). By the definition of  $Q$  and the symmetry of  $BB^T$ , the control-dependent term satisfies

$$-2q^T (DL_1 \otimes BB^T) q = -q^T (Q \otimes BB^T) q \leq -\lambda_* \|q\|^2. \quad (33)$$

For every nonzero virtual-error block, both terms in  $q_i$  point in the direction of  $\xi_{vi}$ . It follows that

$$\|q\| \geq g_{\mu,*} \|\xi_v\|. \quad (34)$$

Hence, the terms containing  $\xi_v$  can be estimated as

$$2q^T (D \otimes A) \xi_v + 2\bar{r}_\alpha q^T (D \otimes I_n) \xi_v \leq 2a_* \|q\| \|\xi_v\| \leq \frac{2a_*}{g_{\mu,*}} \|q\|^2. \quad (35)$$

The residual estimate  $\|\Delta_f(t)\| \leq \delta_f$  also gives

$$2q^T (DL_1 \otimes I_n) \Delta_f \leq b_* \|q\|. \quad (36)$$

Equations (33)–(36) give

$$\dot{V} \leq -\left(\lambda_* - \frac{2a_*}{g_{\mu,*}}\right) \|q\|^2 + b_* \|q\|.$$

The last term is absorbed by the discontinuous directional component. Whenever  $\xi_v(t) \neq 0$ , at least one virtual-error block is nonzero, and therefore  $\|q(t)\| \geq g_{v,*}$ . Thus

$$b_* \|q\| \leq \frac{b_*}{g_{v,*}} \|q\|^2.$$

Under the gain condition (17), we define

$$d_* = \lambda_* - \frac{2a_*}{g_{\mu,*}} - \frac{b_*}{g_{v,*}} > 0. \quad (37)$$

Together with (34), this gives for almost every  $t$ , such that  $\xi_v(t) \neq 0$ ,

$$\dot{V} \leq -d_* \|q\|^2 \leq -\kappa \|\xi_v\|^2, \kappa = d_* g_{\mu,*}^2 > 0.$$

On the set  $\mathcal{Z} = \{t \geq 0 : \xi_v(t) = 0\}$ , absolute continuity gives  $\dot{\xi}_v(t) = 0$  almost everywhere. At every differentiability point accumulated by values in  $\mathcal{Z}$ , this follows from the corresponding zero-valued difference quotients. The isolated points of  $\mathcal{Z}$  are at most countable. Since  $V(t) = 0$  on the same set,  $\dot{V}(t) = 0$  there almost everywhere. The dissipation inequality therefore also holds on  $\mathcal{Z}$ . Every Filippov solution in the admissible region satisfies

$$\dot{V}(t) \leq -\kappa \|\xi_v(t)\|^2 = -\kappa \sum_{i=1}^M \|\xi_{vi}(t)\|^2 \quad \text{a.e.} \quad (38)$$

#### 4.2. Invariance, Forward Completeness, and Convergence

We now prove invariance and forward completeness before establishing convergence. The fixed solution's initial condition satisfies (16), while (38) gives  $0 \leq V(t) \leq V(0)$ ,  $t \in [0, T_{\max})$ . Every summand in (25) is nonnegative. In particular,  $d_i c_i (e^{\mathcal{B}_i(t)} - 1) \leq V(t) \leq V(0)$ . Thus

$$0 \leq \mathcal{B}_i(t) \leq \mathcal{B}_i^* = \ln \left( 1 + \frac{V(0)}{d_i c_i} \right), \quad i \in \mathcal{F}. \quad (39)$$

Equation (22) then gives

$$\|\xi_{vi}(t)\| \leq \chi_i^* \alpha_i(t) < \alpha_i(t), \quad \chi_i^* = \sqrt{1 - e^{-\mathcal{B}_i^*}} < 1. \quad (40)$$

Hence no virtual error reaches its boundary in finite time. In addition,

$$\alpha_i^2(t) - \|\xi_{vi}(t)\|^2 = \alpha_i^2(t) e^{-\mathcal{B}_i(t)} \geq \underline{\alpha}_i^2 e^{-\mathcal{B}_i^*} > 0. \quad (41)$$

The controller denominators are uniformly separated from zero on  $[0, T_{\max})$ .

Bounds (40)–(41) and  $\|\eta_i(t)\| \leq 1$  keep  $q(t)$  bounded. Assumption 4 gives  $\|\Delta_f(t)\| \leq \delta_f$ , so the right-hand side of (31) is bounded on each finite interval.

The leader dynamics satisfy

$$\|\dot{x}_R(t)\| \leq \|A\| \|x_R(t)\| + \sqrt{N-M} \bar{f}.$$

Gronwall's inequality applied to this bound rules out finite-time escape of  $x_R(t)$ . Equation (5) then bounds  $\zeta(t)$ , and (3) excludes finite escape of the follower states. Suppose that  $T_{\max} < \infty$ . The preceding bounds make the full set-valued right-hand side uniformly bounded on  $[0, T_{\max})$ . The trajectory is therefore Lipschitz and has a limit as  $t \uparrow T_{\max}$ . The state bounds and the uniform margin in (41) place this limit inside the open admissible

domain. Local existence from this endpoint extends the trajectory beyond  $T_{\max}$ , which contradicts maximality. Therefore,

$$T_{\max} = \infty.$$

The strict virtual bound and the constraint relation in Section 2.4 now imply the physical inequality in (18) for all time. To establish convergence, we integrate (38) over  $[0, T]$ :

$$\kappa \int_0^T \|\xi_v(t)\|^2 dt \leq V(0) - V(T) \leq V(0).$$

The bound holds for every  $T > 0$ , so

$$\int_0^\infty \|\xi_v(t)\|^2 dt < \infty, \quad \xi_v \in L^2([0, \infty)). \quad (42)$$

Equation (31), together with the bounds on  $\xi_v$ ,  $q$ , and  $\Delta_f$ , shows that  $\dot{\xi}_v \in L^\infty([0, \infty))$ . Because  $\xi_v$  is also bounded,  $g(t) = \|\xi_v(t)\|^2$  satisfies  $g \in L^1([0, \infty))$  and  $|\dot{g}| \leq 2\|\xi_v\|\|\dot{\xi}_v\|$  almost everywhere. Hence  $g$  is uniformly continuous. Barbalat's lemma [33] gives  $\lim_{t \rightarrow \infty} \xi_v(t) = 0$ .

Nonsingularity of  $L_1$  in (5) then gives  $\lim_{t \rightarrow \infty} \xi(t) = 0$ . This gives the follower-wise limit in (19). Since  $\pi_{il} \geq 0$  and  $\sum_{l=M+1}^N \pi_{il} = 1$ , the reference point  $\sum_{l=M+1}^N \pi_{il} x_l(t)$  lies in the leader convex hull. The prescribed transient bounds and asymptotic containment hold simultaneously. The maximal Filippov solution fixed at the start of the proof was arbitrary. These conclusions therefore hold for every maximal solution, and the argument does not assume uniqueness.  $\square$

**Remark 4.** Once the envelopes and topology are fixed, we choose offline margins  $\gamma$  and  $\beta$  such that

$$\gamma > \frac{2a_*}{\lambda_*}, \quad \beta > \frac{b_*}{\lambda_* - 2a_*/\gamma}.$$

We then impose  $c_i \mu_i \geq \gamma$  and  $c_i v_i \geq \beta$  for every follower. The implemented input can be written as

$$u_i = -B^T \left[ c_i \mu_i \xi_{vi} + c_i \frac{\alpha_i^2}{(\alpha_i^2 - \|\xi_{vi}\|^2)^2} \xi_{vi} + c_i v_i \operatorname{sgn}(\xi_{vi}) \right].$$

Holding the products  $c_i \mu_i$  and  $c_i v_i$  fixed preserves the sufficient gain conditions. A smaller  $c_i$ , with proportional increases in  $\mu_i$  and  $v_i$ , reduces only the barrier coefficient that becomes large close to the boundary. Section 5.4 illustrates this effect.

#### 4.3. Offline Design Cost and Online Scalability

We use  $\mathcal{E}_F = \{(j, i) \in \mathcal{E} : i \in \mathcal{F}\}$  for the edges terminating at followers, and  $d_i^{\text{in}}$  for follower  $i$ 's in-degree. For a fixed graph, one factorization of  $L_1$  is reused to solve  $L_1 \Pi = -L_2$  and  $L_1^T \zeta = \mathbf{1}_M$ . The dense offline cost is  $\mathcal{O}(M^3 + M^2 R)$ , with  $\mathcal{O}(M^2 + MR)$  stored entries. The envelope allocation,  $D$ ,  $Q$ , spectral constants, and gains are also computed offline. During operation, the agents perform no matrix inversion, eigenvalue calculation, or gain adaptation. Follower  $i$  requires  $\mathcal{O}(nd_i^{\text{in}} + nm)$  arithmetic and receives  $nd_i^{\text{in}}$  state scalars per update. The network-wide online cost is  $\mathcal{O}(n|\mathcal{E}_F| + Mnm)$ . Thus, for fixed agent dimensions and bounded in-degree, online computation and communication scale linearly with  $M$ , whereas a dense graph gives the expected quadratic worst case. Gain scaling remains explicit through  $\delta_f = 2\sqrt{M\bar{f}}$ ,  $\lambda_*$ , and  $\|DL_1\|$ . The constructive verification of  $Q \succ 0$  is provided immediately before Theorem 1.

## 5. Numerical Simulation and Discussion

### 5.1. Simulation Setup

The example is an engineering-motivated simulation of the outer-loop motion of fully actuated planar robots. It evaluates the distributed containment mechanism at the kinematic level. It does not represent platform-specific hardware validation. The state  $x_i = [p_{x,i}, p_{y,i}]^T$  is the position, and the input  $u_i = [v_{x,i}^c, v_{y,i}^c]^T$  is an inertial-frame velocity correction. Three leaders define a moving convex region. Seven followers use locally received relative positions to approach the graph-determined points in that region.

The simulation uses seven followers and three leaders with  $A = \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix}$ ,  $B = I_2$ ,  $f(x_i(t)) = \begin{bmatrix} 0 \\ \sin(p_{x,i}^2(t)) \end{bmatrix}$ . Here,  $(N, M, R) = (10, 7, 3)$  and  $n = m = 2$ . The term  $Ax_i$  produces a weakly unstable drift in the  $p_y$  direction. The term  $f(x_i)$  adds a bounded nonlinear component in the same direction. The identity input matrix maps the command directly to the position rate. The nonlinear field is locally Lipschitz and bounded by one, so the example satisfies Assumptions 3 and 4 with  $\bar{f} = 1$ . The leaders follow closed-form trajectories: their first coordinates remain constant while the second coordinates evolve, producing a translating and expanding leader hull.

The fixed communication graph shown in Figure 1 satisfies Assumption 1. Its numerical Laplacian blocks are

$$L_1 = \begin{bmatrix} 1.8 & -0.8 & 0 & 0 & 0 & 0 & 0 \\ -0.6 & 1.6 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & -0.6 & 1.2 & -0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.7 & 1.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 1.9 & -0.4 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & 1.6 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & -0.8 & 1.6 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0 & -0.4 & -0.5 \\ 0 & -0.4 & 0 \\ -0.4 & 0 & -0.4 \end{bmatrix}. \quad (43)$$

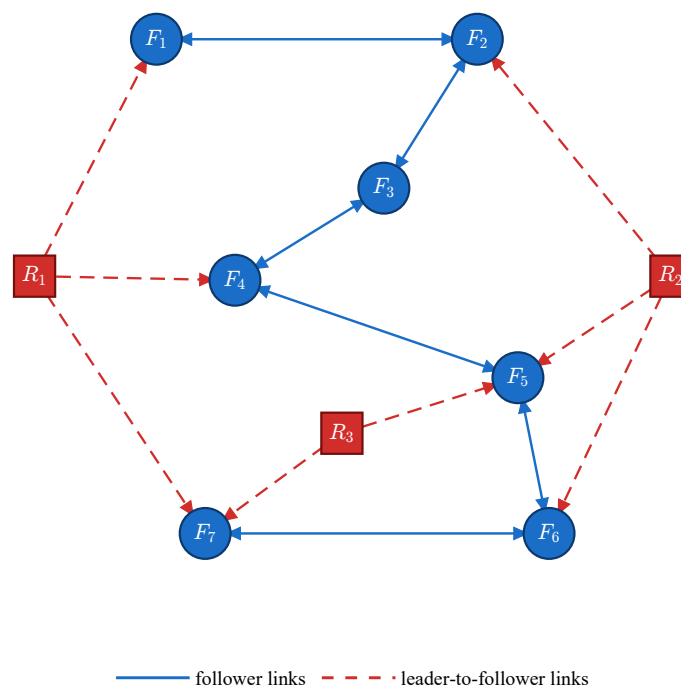


**Figure 1.** Directed graph used in the simulation. Solid blue arrows connect followers. Dashed red arrows carry leader states to followers. The numerical weights are encoded by  $L_1$  and  $L_2$ .

The physical envelopes are  $\psi_i(t) = s_i \rho(t)$ , where the common decay profile is  $\rho(t) = (\rho_0 - \rho_\infty)e^{-rt} + \rho_\infty$  with nominal parameters  $(\rho_0, \rho_\infty, r) = (4, 0.2, 0.3)$ , and the positive dimensionless coefficients  $s_i$  set an individual scale for each follower:  $[s_1, \dots, s_7] = [1.00, 0.95, 1.05, 0.90, 1.10, 0.98, 0.92]$ . For the chosen topology,  $L_1^{-1}$  is elementwise positive, so that  $\bar{\ell}_{ij} > 0$  for all  $i, j \in \mathcal{F}$  and  $\mathcal{L}_j = \mathcal{F}$  for every  $j \in \mathcal{F}$ . Consequently, the allocation in (7) reduces to

$$\alpha_j(t) = \rho(t) \min_{i \in \mathcal{F}} \frac{s_i}{\bar{\ell}_i} \simeq 0.3447 \rho(t) =: \alpha(t), j \in \mathcal{F}.$$

Thus, all seven virtual-error blocks share one allocated envelope, while  $s_i$  preserves the separate physical bound for each follower.

Write  $X_R^0 = [x_{M+1}(0), \dots, x_N(0)]$  and  $X_F^0 = [x_1(0), \dots, x_M(0)]$ , so that states are arranged in columns. The nominal leader states and the seed follower positions are

$$X_R^0 = \begin{bmatrix} 0 & \sqrt{\pi/2} & \sqrt{\pi} \\ -1 & 0 & 1 \end{bmatrix}, \quad Z^0 = X_R^0 \Pi^T, \quad (44)$$

$$\hat{X}_F = \begin{bmatrix} -0.25923 & 0.07837 & 0.38702 & 0.37090 & 0.71762 & 0.94891 & 1.45130 \\ -0.75250 & 0.71160 & 1.76110 & 1.24970 & 1.05380 & -0.26684 & -0.40315 \end{bmatrix}. \quad (45)$$

The displayed decimal seed values define the initialization inputs. For a chosen direction matrix  $H = [h_1, \dots, h_M]$ , define

$$\kappa_\alpha = \min_i \frac{s_i}{\sum_j |(L_1^{-1})_{ij}|}, \quad \beta(H) = \max_j \frac{\|[HL_1^T]_j\|}{\kappa_\alpha \rho_0}, \quad (46)$$

$$X_F^0 = Z^0 + \frac{\sigma}{\beta(H)} H, \quad r_v^{\max}(0) = \sigma,$$

where  $[Y]_j$  denotes column  $j$  of matrix  $Y$ . The nominal case uses  $H = \hat{X}_F - Z^0$  and  $\sigma = 0.95$ . The 0.70 initial-ratio block in Table 2 uses the same  $H$  with  $\sigma = 0.70$ . Thus, (46) specifies every follower initial state without rounding the final scaled positions.

**Table 2.** Comparison on the same nonlinear system. For each initial ratio, all rows use the same plant, graph, leader trajectories, initial states, envelopes, solver, step size, horizon, and evaluation grid. The runs with an initial ratio of 0.70 use states obtained by scaling the common direction of the initial errors.

| Controller                                                            | $r_v^{\max}(0)$ | $T_{10^{-2}}(s)$ | Largest Follower $E_{i,RMS}$ | Steady $U_{RMS}$ | Peak $u_{pk}$ | $J_u$ |
|-----------------------------------------------------------------------|-----------------|------------------|------------------------------|------------------|---------------|-------|
| Proposed boundary-layer controller ( $v_i = 22, \varepsilon = 0.01$ ) | 0.95            | 0.050            | $1.623 \times 10^{-4}$       | 0.3668           | 202.1         | 511.4 |
| Wen et al. [34] (boundary-layer realization)                          | 0.95            | 1.817            | $7.120 \times 10^{-4}$       | 0.3666           | 6.372         | 65.20 |
| Ref. [19]-derived continuous component ( $v_i = 0$ )                  | 0.95            | 8.506            | $1.899 \times 10^{-3}$       | 0.3668           | 158.1         | 101.0 |
| Proposed boundary-layer controller ( $v_i = 22, \varepsilon = 0.01$ ) | 0.70            | 0.040            | $1.623 \times 10^{-4}$       | 0.3668           | 57.56         | 356.0 |
| Wen et al. [34] (boundary-layer realization)                          | 0.70            | 2.564            | $1.546 \times 10^{-3}$       | 0.3667           | 5.415         | 49.45 |
| Ref. [19]-derived continuous component ( $v_i = 0$ )                  | 0.70            | 8.506            | $1.899 \times 10^{-3}$       | 0.3668           | 13.56         | 57.23 |

Follower 3 determines the common allocation, for which  $\gamma_{\max}^{\text{alloc}} = 1$ . Along the selected initial-error direction, the physical and allocated allowable scaling factors are 1.5758 and 1.0526. The separate allocation therefore reduces the allowable radius by 33.2%. With  $\sigma_0 = 0.95$ , the rule in Remark 2 gives  $\rho(0) = 4.0$ . The chosen initial states place every

follower outside the initial leader hull. At  $t = 0$ , the largest normalized physical and virtual errors are 0.6346 and 0.9500, respectively. Follower 1 has the largest initial virtual-error ratio.

We set  $c_i = 2$ ,  $\mu_i = 5$ , and  $v_i = 22$  for every follower. Using the constructive rule (14),  $\zeta = L_1^{-T} \mathbf{1}_7$  determines  $D$ . The constants in (15) are  $\lambda_* = 1.952$ ,  $a_* = 0.9387$ ,  $b_* = 69.31$ , and  $\bar{r}_\alpha = 0.285$ . The two gain checks are  $c_i \mu_i = 10 > 0.9620$ ,  $c_i v_i = 44 > 39.29$ , thereby verifying both sufficient gain inequalities for this parameter set.

The simulations use MATLAB R2024b with double-precision arithmetic. Unless stated otherwise, the model is integrated over  $T = 25$  s by a fixed-step fourth-order Runge–Kutta (RK4) method with  $h = 10^{-4}$  s. Section 5.4 specifies the sign and boundary-layer realizations used in the reported results.

## 5.2. Containment Snapshots and Constraint Diagnostics

Figure 2 shows four snapshots of the expanding leader hull. Initially, every follower lies outside the hull. Gray segments join each follower to the reference point determined by its row of  $\Pi$ . By  $t = 5$  s, each follower marker nearly overlaps its reference marker inside the moving hull. The followers need not meet at one point because the rows of  $\Pi$  define different reference points.

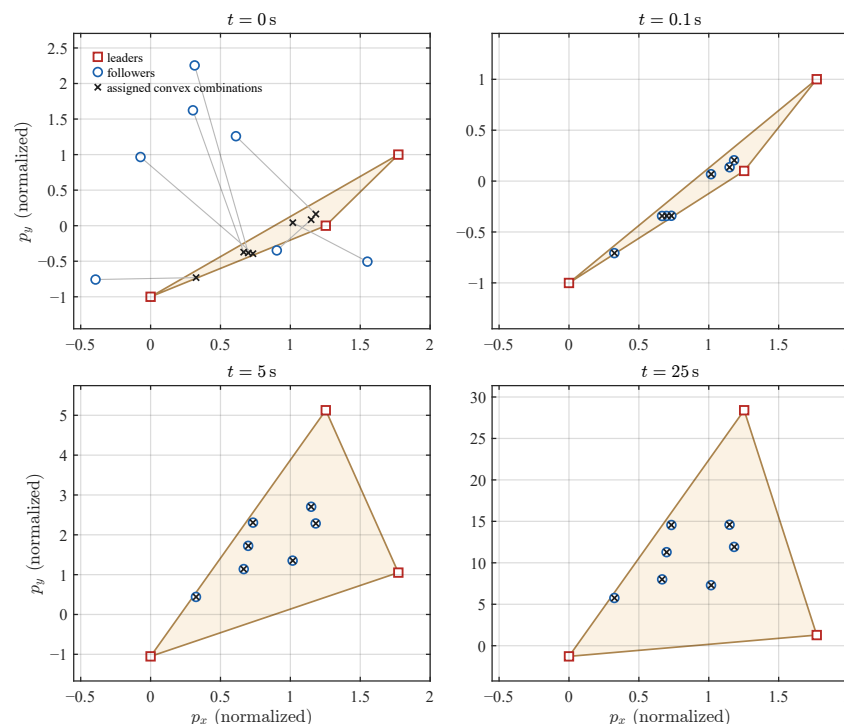


**Figure 2.** Evolution of the moving containment configuration in the normalized  $p_x$ – $p_y$  plane. Squares, circles, and crosses denote leaders, followers, and the convex combinations of leader states determined by the graph, respectively. Gray segments represent physical containment errors.

We use the normalized ratios

$$r_{a,i}(t) = \frac{\|\zeta_i(t)\|}{\psi_i(t)}, r_{v,i}(t) = \frac{\|\zeta_{vi}(t)\|}{\alpha_i(t)}.$$

Figure 3 plots their pointwise maxima,

$$r_a^{\max}(t) = \max_{i \in \mathcal{F}} r_{a,i}(t), r_v^{\max}(t) = \max_{i \in \mathcal{F}} r_{v,i}(t).$$

The initial condition gives  $r_v^{\max}(0) = 0.95$ . As shown in Figure 3, the virtual-error ratio decreases rapidly during the first 0.05 s. All displayed ratios remain below one. Section 5.4 examines the numerical realizations and step-size sensitivity.

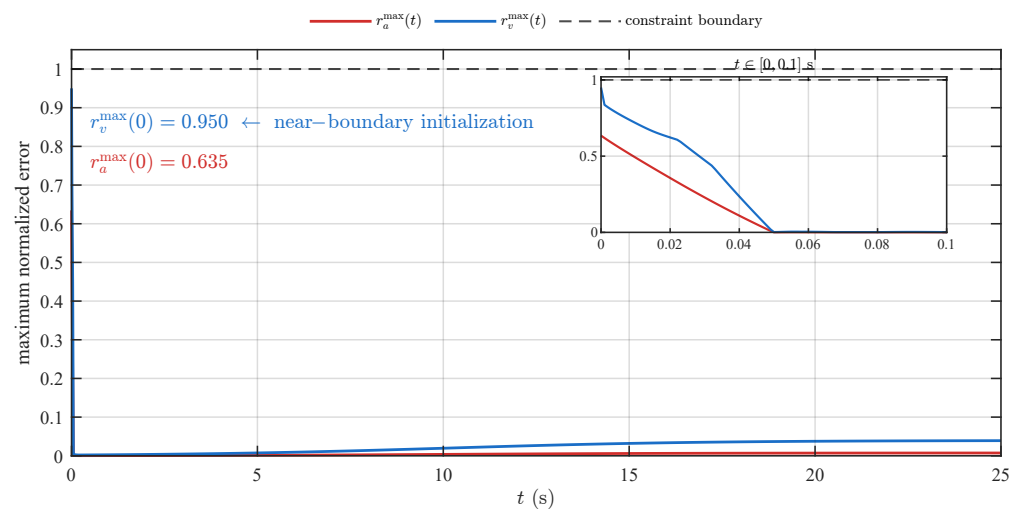


**Figure 3.** Maximum normalized physical and virtual containment-error ratios,  $r_a^{\max}(t)$  and  $r_v^{\max}(t)$ , respectively. The inset resolves  $t \in [0, 0.1]$  s, and the dashed unit line is the prescribed boundary.

### 5.3. Comparisons on the Same Nonlinear System

We compare the proposed controller with a finite-horizon implementation based on Equations (14) and (39) of Wen et al. [34]. We also set  $v_i = 0$  while retaining the continuous term from Ref. [19]. This ablation isolates the effect of residual compensation.

For Wen et al. [34], the local disagreement equals  $\xi_{vi}$ . We implement Equations (14) and (39) with  $P_W = K = I_2$ ,  $(\alpha_W, \beta_W, \nu_W, c_W, d_0) = (0.25, 1.52, 200, 0.5, 25)$ , componentwise saturation with width 0.01, and  $\min_i \phi_i^{\text{db}} = 1$ . The exact feature  $\Phi_i(x_i) = \sin(x_{i1}^2)$  represents the simulated nonlinearity without basis-approximation error. With zero initial adaptive matrices, the comparator contains 28 adaptive scalar states. The choice  $(\chi, \theta) = (0.05, 0.02)$  satisfies the cited gain and LMI conditions. Because the common test case does not satisfy the bounded-leader assumption in Ref. [34], we compare only the implemented trajectories over the common finite horizon.

The second comparator is the continuous component of Ref. [19] in the common virtual-error coordinate, obtained from (12) by setting  $v_i = 0$ . The proposed boundary-layer realization uses  $v_i = 22$  and  $\text{sgn}_\varepsilon(z) = z / \max\{\|z\|, \varepsilon\}$  with  $\varepsilon = 0.01$ .

All three runs use the same nonlinear plant, graph, leader trajectories, initial states, step size, and  $T = 25$  s horizon. Every run is evaluated against the same physical and virtual envelopes. The proposed controller and the ablation use those envelopes directly. The Wen comparator retains the cited control and adaptation structure with the parameter choices stated above. The ablation also keeps the allocation and continuous gains  $c_i = 2$  and  $\mu_i = 5$ . It is needed because the nonlinear residual remains nonzero at exact containment:

$$R_0(t) = \max_{i \in \mathcal{F}} \left\| f \left( \sum_{l=M+1}^N \pi_{il} x_l(t) \right) - \sum_{l=M+1}^N \pi_{il} f(x_l(t)) \right\| \equiv 0.6049.$$

This value is constant because  $f$  depends only on the leaders' fixed first coordinates. Thus, nonlinear forcing persists at the containment target.

We define

$$E_{\max}(t) = \max_{i \in \mathcal{F}} \|\xi_i(t)\|, \quad u_{\max}(t) = \max_{i \in \mathcal{F}} \|u_i(t)\|, \quad u_{\text{pk}} = \max_{0 \leq t \leq T} u_{\max}(t).$$

For the  $N_s$  equally spaced samples  $\{t_k\}_{k=1}^{N_s} \subset [20, 25]$  s, the common root-mean-square (RMS) measures are

$$E_{i,\text{RMS}} = \left( \frac{1}{N_s} \sum_{k=1}^{N_s} \|\tilde{g}_i(t_k)\|^2 \right)^{1/2}, \quad U_{\text{RMS}} = \left( \frac{1}{MN_s} \sum_{i=1}^M \sum_{k=1}^{N_s} \|u_i(t_k)\|^2 \right)^{1/2}.$$

The sustained settling time is

$$T_{10^{-2}} = \inf \left\{ t : E_{\max}(\tau) < 10^{-2} \text{ for every } \tau \in [t, T] \right\}.$$

We approximate the settling time on a  $10^{-3}$  s sampling grid. It is the first sample after which the condition holds at every remaining sample. Control energy is  $J_u = \int_0^T \sum_{i=1}^M \|u_i(t)\|^2 dt$  and is accumulated with the RK4 stage weights.

All normalized errors remain below one over the common finite horizon. The proposed controller reaches the threshold first and gives the smallest RMS error over the final 5 s. It uses more transient control effort than the Wen comparator. The results show a trade-off among convergence speed, accuracy, and control effort. The  $v_i = 0$  case isolates the directional term and is not an external benchmark.

#### 5.4. Discontinuous and Boundary-Layer Implementations

We compare the fixed-step sign implementation with the boundary-layer approximation  $\text{sgn}_\varepsilon(z) = z / \max\{\|z\|, \varepsilon\}$  at  $\varepsilon = 0.01$ . In the fixed-step realization, we use  $\text{sgn}(z) = z/\|z\|$  for  $\|z\| > 10^{-12}$  and zero otherwise. This fixed-step sign implementation produces a deterministic numerical approximation of the ideal discontinuous closed loop and does not establish uniqueness of its Filippov solutions. The smoothed controller is locally Lipschitz within the performance region and has locally unique Carathéodory trajectories. Theorem 1 does not directly cover this approximation, although both implementations satisfy the envelopes numerically over the reported horizon.

Figures 2 and 3 use the fixed-step sign realization. Table 2 and Figure 4 use the boundary layer. Figure 5 compares the two implementations. Its first panel shows all command norms and  $u_{\max}$ . The initial condition close to the boundary produces  $u_{pk} = 202.1$ . The second panel shows the RK4-stage values of  $u_{7,2}$  during the final 2 ms. The boundary layer removes visible switching and leaves a small error neighborhood.

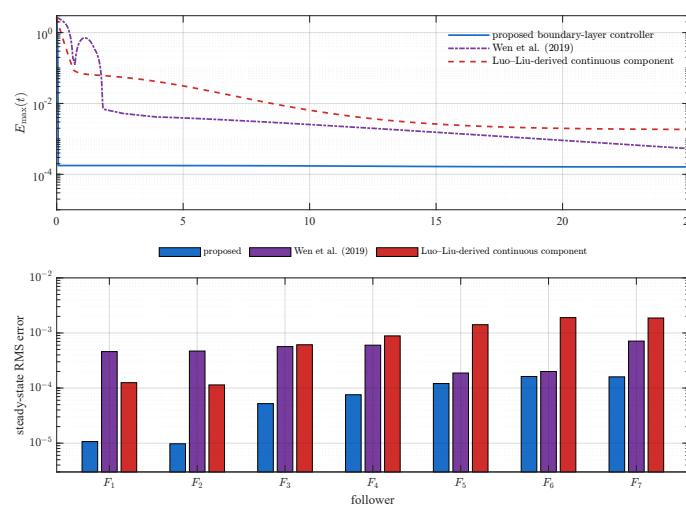


**Figure 4.** Comparison on the same nonlinear system. The comparator curves use the method of Wen et al. [34] and the continuous component derived from Luo and Liu [19]. The upper panel shows the largest physical containment error, and the lower panel shows follower-wise RMS containment errors over the final 5 s on a logarithmic scale.

For  $\varepsilon \in \{0.005, 0.01, 0.02\}$ , the largest follower-wise RMS errors on  $[20, 25]$  s are  $8.48 \times 10^{-5}$ ,  $1.62 \times 10^{-4}$ , and  $2.99 \times 10^{-4}$ , respectively. These values show the trade-off between smoothness and accuracy as the layer widens. Their common normalized maximum, 0.6346, occurs at the shared initial condition. Halving the integration step from  $10^{-4}$  to  $5 \times 10^{-5}$  s leaves that maximum unchanged and lowers the fixed-step RMS floor from  $1.35 \times 10^{-3}$  to  $6.57 \times 10^{-4}$ . This reduction indicates that the fixed-step floor is caused mainly by time discretization.

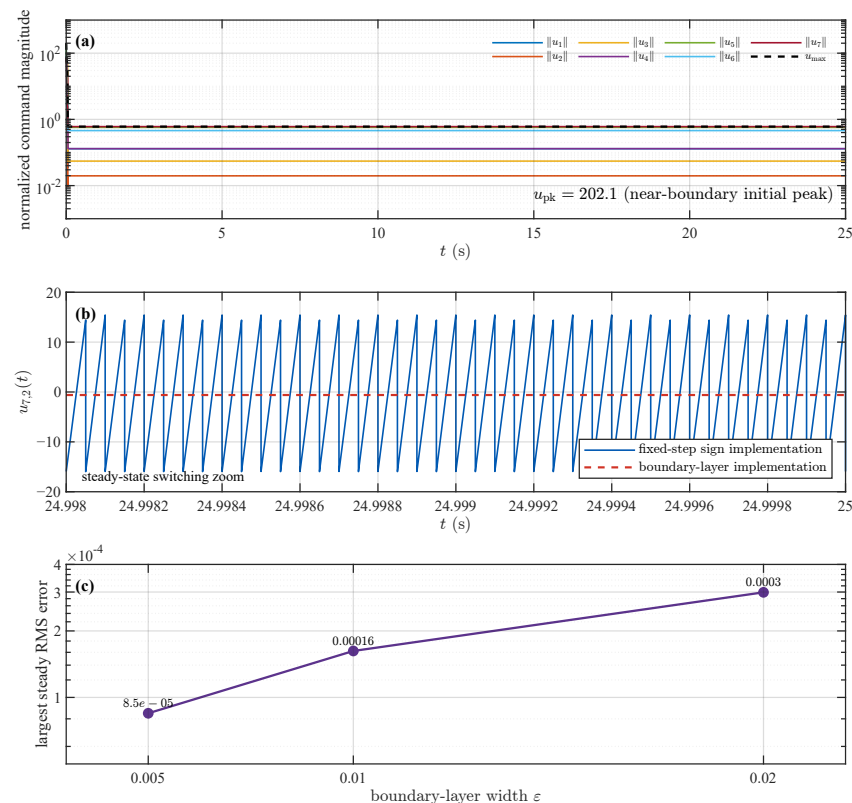


**Figure 5.** Control-input responses and boundary-layer diagnostics. Panel (a) shows the full-horizon normalized command norms for all seven followers and their pointwise maximum  $u_{\max}(t)$ . Panel (b) shows the final 2 ms RK4-stage values of the second command component  $u_{7,2}(t)$  of follower 7. Panel (c) shows the final 5 s follower-wise RMS error for three boundary-layer widths  $\varepsilon$ .

For follower 1 at  $t = 0$ , the baseline, barrier, and directional terms have magnitudes 13.10, 145.0, and 44.00, respectively. The barrier term therefore contributes most of the peak. We keep the gain products  $(c\mu, c\nu) = (10, 44)$  fixed and change  $(c, \mu, \nu)$  from  $(2, 5, 22)$  to  $(0.2, 50, 220)$ . The boundary-layer peak then falls from 202.1 to 71.59. The constraint maxima and steady input RMS remain unchanged, while  $T_{10-2}$  changes only from 0.050 to 0.052 s. These results illustrate the gain factorization in Remark 4.

### 5.5. Validation Across Configurations

Table 3 applies a structured set of changes to the nominal configuration. Unless stated otherwise, each case uses the setup in Section 5.1, the boundary layer  $\varepsilon = 0.01$ ,  $h = 10^{-4}$  s, and  $T = 25$  s. The notation  $\text{mod}(a, b)$  denotes the remainder after dividing  $a$  by  $b$ . Cases S1 and S2 use an asymmetric ring. Follower  $i$  receives information from its predecessor and successor with weights  $0.20 + 0.02 \text{ mod}(i - 1, 3)$  and  $0.30 - 0.02 \text{ mod}(i - 1, 2)$ , respectively. It is pinned to leader  $M + 1 + \text{mod}(i - 1, R)$  with weight 1.5. Case S7 replaces the ring

by the dense directed weights  $a_{ij} = 0.08 + 0.01 \bmod(i + 2j, 3)$  for  $i \neq j$ , with the same leader-pinning rule. Case S3 uses

$$f_{\text{alt}}(x) = \begin{bmatrix} 0.6 \tanh(x_1) \\ 0.8 \sin(x_1 x_2) \end{bmatrix}, \quad \|f_{\text{alt}}(x)\| \leq 1,$$

which is bounded, cross-coupled, and locally Lipschitz.

Case S4 changes the initial-error directions and sets  $r_v^{\max}(0) = 0.70$ . Case S5 uses  $(\rho_0, \rho_\infty, r) = (4, 0.1, 0.45)$ , and S6 uses  $(c, \mu, \nu) = (2, 1, 36)$ . For every case,  $\Pi$ , the constraint allocation,  $D$ , the spectral constants, and the gain thresholds are recomputed, and both initial performance conditions are checked before integration.

The initial states are constructed by (46). For S1 and S7,  $X_R^0$  and  $\hat{X}_F$  retain their nominal values, while  $\Pi$ ,  $Z^0$ ,  $H = \hat{X}_F - Z^0$ , and  $\kappa_\alpha$  are recomputed for the new graph before setting  $\sigma = 0.95$ . For S2, let  $\theta_i = 2\pi(i - 1)/12$ . The four leader columns, individual envelope scales, and direction columns are

$$[X_R^0]_q = \begin{bmatrix} \sqrt{(q-1)\pi/3} \\ -1 + 2(q-1)/3 \end{bmatrix}, \quad q = 1, \dots, 4, \quad (47)$$

$$s_i = 1 + 0.10 \sin(0.7i), \quad h_i = (0.75 + 0.20 \cos \theta_i) \begin{bmatrix} \cos(\theta_i + 0.17) \\ \sin(1.7\theta_i + 0.31) \end{bmatrix},$$

for  $i = 1, \dots, 12$ , with  $\sigma = 0.95$ . For S4, the nominal graph, leader states, and scales are retained; with  $\theta_i = 2\pi(i - 1)/7$ , its new direction is

$$h_i = (0.70 + 0.25 \sin(\theta_i + 0.2)) \begin{bmatrix} \cos(\theta_i + 0.43) \\ \sin(1.9\theta_i - 0.22) \end{bmatrix}, \quad i = 1, \dots, 7, \quad (48)$$

and (46) uses  $\sigma = 0.70$ . S0, S3, S5, and S6 use the nominal initial states; S3 applies  $f_{\text{alt}}$  to both followers and leaders.

**Table 3.** Validation of the proposed boundary-layer controller. The RMS quantities use the final 5 s.

| Case | Configuration Change                              | (M, R)  | Horizon Maxima<br>( $r_a^{\max}, r_v^{\max}$ ) | $T_{10^{-2}}$ (s) | Largest<br>$E_{i,\text{RMS}}$ | Steady<br>$U_{\text{RMS}}$ | Peak $u_{\text{pk}}$ |
|------|---------------------------------------------------|---------|------------------------------------------------|-------------------|-------------------------------|----------------------------|----------------------|
| S0   | Nominal configuration                             | (7, 3)  | (0.6346, 0.9500)                               | 0.050             | $1.62 \times 10^{-4}$         | 0.3668                     | 202.1                |
| S1   | Asymmetric ring topology                          | (7, 3)  | (0.8238, 0.9500)                               | 0.044             | $8.56 \times 10^{-5}$         | 0.3938                     | 132.3                |
| S2   | Larger ring network                               | (12, 4) | (0.6965, 0.9500)                               | 0.046             | $7.57 \times 10^{-5}$         | 0.3250                     | 132.3                |
| S3   | Cross-coupled $f_{\text{alt}}$                    | (7, 3)  | (0.6346, 0.9500)                               | 0.051             | $4.17 \times 10^{-4}$         | 0.6657                     | 202.1                |
| S4   | New initial directions,<br>$r_v^{\max}(0) = 0.70$ | (7, 3)  | (0.1776, 0.7000)                               | 0.014             | $1.62 \times 10^{-4}$         | 0.3668                     | 57.56                |
| S5   | Faster, tighter envelope                          | (7, 3)  | (0.6346, 0.9500)                               | 0.050             | $1.28 \times 10^{-4}$         | 0.3668                     | 202.1                |
| S6   | Gains $(c, \mu, \nu) = (2, 1, 36)$                | (7, 3)  | (0.6346, 0.9500)                               | 0.035             | $1.03 \times 10^{-4}$         | 0.3668                     | 219.6                |
| S7   | Dense directed topology                           | (7, 3)  | (0.7360, 0.9500)                               | 0.040             | $8.59 \times 10^{-5}$         | 0.3960                     | 132.3                |

Both normalized error ratios remain below one in all eight cases. This result persists under the listed changes in topology, network size, nonlinearity, initialization, envelope, and gains.

### 5.6. Comparison on a Common Linear System

We next compare the methods on a vector single-integrator system, which is a special case of the model considered here. We use the continuously evaluated feedback map of Wu et al. [20] and the prescribed-time control law of Wang et al. [23]. All methods use  $A = 0$ ,  $f = 0$ , and  $B = I_2$ , with the same static leaders, graph, initial states, error measures,

and diagnostic envelopes. For Wu et al., we evaluate Equation (3) continuously with  $P = I_2$ ,  $K = -I_2$ , and  $(c_1, c_2, \varrho) = (2, 0.1, 0.01)$ . This comparison does not include the hybrid triggering layer. For Wang et al., we use Equation (11) with  $(k, c, h_p, T_p) = (1, 3.3, 3, 1)$ . We propagate the fixed-graph linear error analytically to its limiting zero at  $T_p$ . This avoids direct evaluation of the singular gain. Since  $\delta_f = b_* = 0$  in this specialization, we also evaluate a 24-point gain grid for the proposed method. Table 4 reports three representative settings for the proposed method. The first keeps the gain products used in the nonlinear example. The other two have control effort comparable to Wang et al. and peak input comparable to Wu et al., respectively. The full numerical dataset contains results for all 24 gain settings. The common graph and initial states are given by (43)–(46) with  $\sigma = 0.95$ ; the leader states remain fixed in this linear test. The nominal physical and allocated envelopes are retained. All proposed settings use the boundary-layer realization with  $\varepsilon = 0.01$ ,  $h = 10^{-4}$  s, and  $T = 5$  s. The complete gain grid is

$$c \in \{0.005, 0.01, 0.02, 0.05, 0.1, 0.2\}, c\mu \in \{1, 2, 5, 10\}, \quad cv = 10^{-3}, \quad (49)$$

with  $\mu = (c\mu)/c$  and  $v = 10^{-3}/c$  for each of the 24 combinations. The first proposed setting in Table 4 additionally retains the nonlinear gain products.

**Table 4.** Common linear multi-leader comparison. All metrics use a common  $10^{-4}$  s grid. The RMS quantities use  $[4, 5]$  s, and  $J_u = \int_0^5 \sum_i \|u_i(t)\|^2 dt$  is evaluated by trapezoidal quadrature on that grid. Values below  $10^{-12}$  are reported as the numerical precision floor. The dagger denotes the exact zero reached after  $T_p$  by the limiting solution of Wang et al.

| Controller                                                                                               | $T_{10^{-2}}$ (s) | Largest $E_{i,\text{RMS}}$ | Steady $U_{\text{RMS}}$ | Peak $u_{\text{pk}}$ | $J_u$ |
|----------------------------------------------------------------------------------------------------------|-------------------|----------------------------|-------------------------|----------------------|-------|
| Proposed, gain products from the nonlinear example<br>( $c, c\mu, cv$ ) = (0.2, 10, 44)                  | 0.0521            | $< 10^{-12}$               | $< 10^{-12}$            | 71.59                | 462.7 |
| Proposed, setting with control effort comparable to Wang et al.<br>( $c, c\mu, cv$ ) = (0.01, 10, 0.001) | 1.270             | $2.187 \times 10^{-8}$     | $5.387 \times 10^{-8}$  | 13.82                | 53.55 |
| Proposed, setting with peak input comparable to Wu et al.<br>( $c, c\mu, cv$ ) = (0.02, 1, 0.001)        | $> 5$             | 0.3114                     | 0.08529                 | 2.760                | 5.723 |
| Wu et al. [20], continuously updated feedback map                                                        | 3.362             | $1.112 \times 10^{-4}$     | $3.237 \times 10^{-4}$  | 2.720                | 11.71 |
| Wang et al. [23], $T_p = 1$ s                                                                            | 0.7042            | 0 <sup>†</sup>             | 0 <sup>†</sup>          | 14.28                | 62.48 |

Figure 6 shows the corresponding time histories of the maximum physical containment error and maximum command norm.

All runs satisfy the common diagnostic envelopes. The first proposed setting gives the fastest response and the largest control energy. With  $(c, c\mu, cv) = (0.01, 10, 0.001)$ , the proposed controller settles in 1.270 s, compared with 0.7042 s for Wang et al. At a peak input comparable to Wu et al., the proposed controller does not reach the threshold within 5 s. These results show a clear trade-off between settling time and control effort. The Wu curve represents continuous evaluation of the published feedback map and does not include the hybrid triggering mechanism.

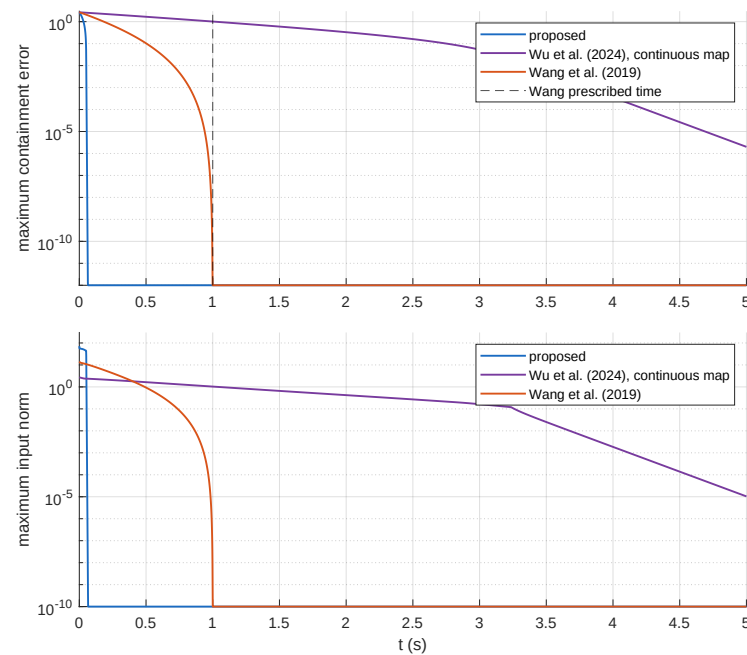


**Figure 6.** Common linear multi-leader comparison. The upper panel shows the maximum physical containment error, and the lower panel shows the maximum command norm. The curve for Wu et al. [20] uses continuous state updates of the published feedback map. The vertical line marks the prescribed time  $T_p = 1$  s used by Wang et al. [23]. The proposed curve uses the first proposed setting in Table 4.

## 6. Conclusions

This paper developed a prescribed-performance containment controller for nonlinear multi-agent systems with multiple leaders. The inverse-Laplacian allocation converts separately prescribed physical-error envelopes into bounds on the distributed virtual errors. A directional robust term compensates for the residual created by nonlinear averaging. For every maximal Filippov solution, the closed loop is forward complete, both sets of envelopes remain invariant, and each physical containment error converges to zero.

The numerical tests changed the topology, network size, nonlinearity, initial states, envelopes, and gains. Both normalized error ratios remained below one over every reported horizon. Comparisons on common systems showed a trade-off between settling speed and control effort. Changing the gain factorization also reduced the peak input without changing the gain products used in the theorem.

The analysis assumes a fixed directed graph, homogeneous full-state dynamics, full-row-rank actuation, and a known bound on the nonlinear term. Future work will address underactuated systems, switching graphs, and actuator saturation. A distributed residual estimator may also use the KAN architecture described in Refs. [35,36]. Such an extension requires locally available inputs and a certified estimation-error bound before it can be incorporated into the stability proof. In addition, in the presence of multiple uncertainties and nonlinear couplings, the proposed scheme will provide key technical support for the evolution of microgrids toward high reliability and high autonomy.

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