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Revisiting the Lognormal Modelling of Shadowing Effects during Wireless Communications by Means of the α-μ/α-μ Composite Distribution

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Abstract: Properly modeling the shadowing effects during wireless transmissions is crucial to perform the network quality assessment. From a mathematical point of view, using composite distributions allows one to combine both fast fading and slow fading stochastic phenomena. Numerous statistical distributions have been used to account for the fast fading effects. On the other hand, even though several studies indicate the adequacy of the Lognormal distributon (LNd) as a shadowing model, they also reveal this distribution renders some analytic tractability issues. Past works include the combination of Rayleigh and Weibull distributions with LNd. Due to the difficulty inherent to obtaining closed form expressions for the probability density functions involved, other authors approximated LNd as a Gamma distribution, creating Nakagami-m/Gamma and Rayleigh/Gamma composite distributions. In order to better mimic the LNd, approximations using the inverse Gamma and the inverse Nakagami-m distributions have also been considered. Although all these alternatives were discussed, it is still an open question how to effectively use the LNd in the compound models and still get closed-form results. We present a novel understanding on how the α-μ distribution can be reduced to a LNd by a limiting procedure, overcoming the analytic intractability inherent to Lognormal fading processes. Interestingly, new closed-form and series representations for the PDF and CDF of the composite distributions are derived. We build computational codes to evaluate all the expression hereby derived as well as model real field trial results by the equations developed. The accuracy of the codes and of the model are remarkable.

Keywords: composite fading; channel modelling; α-μ-α-μ

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1. Introduction

In the path from the transmitter to the receiver, besides the propagation loss, the mobile-radio signals can be blocked by physical obstructions—shadowing—and suffer multiple reflections, scattering and diffraction—multipath fading. In this context, a composite fading model, i.e., those describing the combined effects of small fading and fast fading, arises in several opportunities and, therefore, is considered to be of great importance [1].

Composite distributions have been suggested in the technical literature to model this compound phenomenon [2–4]. The most common composite models used in the literature are the Rayleighl-Lognormal, composed by the fast fading modeled by Rayleigh distribution and shadowing modeled by Lognormal distribution. Modeling the slow fading by a lognormal process is known to render difficult analytic tractability.

In order to avoid the mathematical difficulties inherent to the Lognormal distribution, authors such as [5] opted for the Gamma distribution as substitute, due to its rather convenient algebraic representation. This approach allowed the composition of Nakagami-m/Gamma and Rayleigh/Gamma distribution to obtain fading models in closed-form [6].

The shadowing phenomenon has been also modelled by inverse distribution models, as inverse Gamma and inverse Nakagami-m. These have shown to better approximate the Lognormal distribution while compared to the direct Gamma distribution. This way, the literature review indicates that using better approximations to the Lognormal distribution for shadowing modelling is of interest [7]. In a recent paper [8], the Weibull/Lognormal composite fading was studied, once again reinforcing the importance of this distribution.

As indicated, analytic tractability issues prevent researchers from using the Lognormal distribution as a shadowing model, mainly because the evaluation of the probability density functions of the composite distributions require specific computational codes. Besides, closed-form expressions are not available, which also impacts practical engineers to use such distributions. Motivated by the opportunity to allow researchers to actually model the shadowing effect by Lognormal distributions, by exploring a limiting procedure, this work is committed to further discuss the applicability of the α-μ distribution as a surrogate Lognormal shadowing fading model.

Furthermore, besides exploring the connection between α-μ and Lognormal random variables, in the present article, we derive alternative expressions for the PDF and CDF of composite α-μ/α-μ distribution. We obtain new closed-form results which complement previous studies and also provide their simple fast converging series representations. In order to enhance the usability of the expressions hereby proposed, we provide computational codes for their numerical evaluation.

This article is organized as follows: Section 2 revisits the α-μ distribution and its relation to Lognormal random variables. In Section 3, the model formulations used to obtain the product and composite distributions of α-μ random variables are presented. Section 4 presents a discussion about the numerical implementations of the expressions proposed. In Section 5, real world field trials are modelled by the new formulas, indicating a remarkable adequacy. Finally, in Section 6 final conclusions are presented.

2. The α-μ Distribution

Even though other distributions have been recently discussed as models for fast and slow fading processes [9], the α-μ distribution is still of great importance due to its simplicity easy of use. The α-μ fading model [10] has been proposed in order to provide a more realistic analysis of the propagated signal [11]. In several propagation environments, the α-μ model better accommodates the statistical variations of the propagated signal [12]. In this sense, the α-μ/α-μ composite distribution offer a flexible and powerful model that can be used to model the wireless channel in a remarkable accurate form [1].

The literature seems to have overlooked the relation between the α-μ and the generalized Gamma distribution (GGD). This relation is, in general, only briefly indicated, as seen in [13,14], for example. In the work of [15], approximate expressions have been proposed to model the composition of GGD and Lognormal distributions, but no mention has been made to the α-μ distribution. On the other hand, ref. [1] studied the composition of α-μ distributions but the relation to the GGD continued to be a neglected.

As shall be subsequently indicated, α-μ distributions are nothing but reparametrized GGDs. The latter distribution has been extensively studied [16] and some of its properties are of great interest to the modelling of wireless transmission systems. For example, by a limiting procedure the GGD can be transformed into a Lognormal distribution [16]. Therefore, the α-μ distribution can also be transformed into a Lognormal distribution, which indicates its correctness as a candidate for a shadowing distribution.

In a recent paper [1], mathematical expressions for the product and composition of α-μ random variables were derived. Their result was given in terms of ${}_pF_q$ hypergeometric functions for particular values of the parameters involved. Furthermore, ref. [15] obtained approximate expressions to model the composition of GGD and lognormal distributions.

A complete paper on the α-μ composite distribution was also recently published [14]. In their work, the author presented general and simple closed-form expressions for the PDF and CDF of this composite fading distribution. The expressions were presented in terms of the well known Fox H-function, and are valid for arbitrary values of the fading parameters. The author also presented simplified expressions when the ratio of the nonlinearity parameters α are integers.

In order to build our main results, some mathematical definitions are needed.

Mathematical Definitions

Let the random variable $R \geq 0$ designate the signal envelope following the α-μ distribution. The PDF $f_R(r)$ of R is expressed as [10]

$$f_R(r) = \frac{\alpha \mu^{\mu}}{\Gamma(\mu)} \frac{r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu}} \exp\left[-\mu\left(\frac{r}{\hat{r}}\right)^{\alpha}\right], \tag{1}$$

in which $\alpha > 0$ denotes the nonlinearity parameter, $\mu > 0$ is related to the number of multipath clusters, $\Gamma(\cdot)$ is the gamma function, $\hat{r} \triangleq \sqrt[\alpha]{E\{R^{\alpha}\}} = \frac{\bar{r}\sqrt[\alpha]{\mu}\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{\alpha}\right)}$, $\bar{r} = E\{R\}$ and $E\cdot$ denotes the expectation operator.

The α-μ distribution includes as special cases other important distributions [1,14]. Besides, as previously indicated, the α-μ distribution can be also a reparametrization of the GGD. Let T be a generalized Gamma random variable, then its PDF $f_T(t)$ can be expressed as [16]:

$$f_T(t) = \frac{\beta}{\Gamma(k)\theta}\left(\frac{t}{\theta}\right)^{k\beta-1} \exp\left(-\left(\frac{t}{\theta}\right)^{\beta}\right), \tag{2}$$

in which $\theta > 0$ is a scale parameter and $\beta, k > 0$ are shape parameters.

By directly comparing Equations (1) and (2), it is easy to see that $k = \mu$, $\beta = \alpha$ and $\theta = \hat{r}\mu^{-1/\alpha}$. On the other hand, ref. [16] reveals that when $k = \lambda^{-2}$, $\beta = \sigma^{-1}\lambda$ and $\theta = \exp\left(\mu_{LN} - \lambda^{-1}\sigma \ln(\lambda^{-2})\right)$, by making $\lambda \to 0$, the GGD reduces to a Lognormal distribution with location parameter μ_{LN} and scale parameter σ. Thus, for an α-μ distribution, by making $\mu = \lambda^{-2}$, $\alpha = \sigma^{-1}\lambda$ and $\hat{r} = \lambda^{-2\sigma/\lambda} \exp\left(\mu_{LN} - \lambda^{-1}\sigma \ln(\lambda^{-2})\right)$ and letting $\lambda \to 0$, the α-μ distribution also reduces to a Lognormal one.

This way, the composition of α-μ random variables have, as a special case, the composition between an α-μ and a Lognormal random variable. As will be demonstrated in the next section, there exist general closed form expressions for the composition of α-μ random variables, which finally properly addressed the analytic tractability issues regarding Lognormal shadowing models.

3. Composite Fading Model

It is known that in order to build the composition of two random variables (RVs), one may obtain the product of such RVs and then simply take the mean value of one of the Rvs as 1. Thus, at first, we shall obtain the product of two α-μ RVs.

Let X and Y be two statistically independent α-μ random variables RVs, with average values $\bar{x} = E\{X\}$ and $\bar{y} = E\{Y\}$, and $W \triangleq X \times Y$ their product. Using basic statistical procedures concerning the transformation of random variables, the PDF of W can be calculated as [1,14]:

$$f_W(w; \bar{x}, \bar{y}) = \int_0^{\infty} \frac{1}{y} f_X\left(\frac{w}{y}; \bar{x}\right) f_Y(y; \bar{y}) dy, \tag{3}$$

in which $f_X(.)$ and $f_Y(.)$ denote the PDFs of X and Y, respectively. One may obtain the CDF of the distribution by simply integrating (3).

By using (1) with appropriate subscripts, the PDFs of X and Y will be written in terms of the parameters α_x, μ_x, $\bar{x}$ and α_y, μ_y, $\bar{y}$, respectively. Thus, the product PDF α-μ-α-μ can be expressed by considering (3). After some algebraic manipulations, the following expression is obtained

$$f_W(w) = \frac{\alpha_x w^{\alpha_y \mu_y - 1}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y \mu_y}} \int_0^{\infty} t^{\frac{\alpha_x \mu_x - \alpha_y \mu_y}{\alpha_y} - 1} \exp\left(-t^{\frac{\alpha_x}{\alpha_y}} - \frac{w^{\alpha_y}}{(u_x u_y)^{\alpha_y} t}\right) dt, \tag{4}$$

in which $u_x = \bar{x}\Gamma(\mu_x)/\Gamma(\mu_x + \alpha_x^{-1})$ and $u_y = \bar{y}\Gamma(\mu_y)/\Gamma(\mu_y + \alpha_y^{-1})$.

One may notice that (4) is an alternative representation of Equation (12) in [1]. Furthermore, the authors of [1] indicated that no closed-form representation of (4) was available, which is not true, as indicated by [14]. The integral in the right hand side of Equation (4) can be expressed in terms of the Kratzel function [17] $Z_\rho^\nu(x)$ as

$$f_W(w) = \frac{\alpha_x w^{\alpha_y\mu_y - 1}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y\mu_y}} Z_{\frac{\alpha_x}{\alpha_y}}^{\frac{\alpha_x\mu_x - \alpha_y\mu_y}{\alpha_y}}\left(\frac{w^{\alpha_y}}{(u_x u_y)^{\alpha_y}}\right). \tag{5}$$

in which [17] $Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1}\exp(-t^\rho - x/t)\mathrm{d}t$. On the other hand, the Kratzel function can be also expressed in terms of the H-function as [17]

$$f_W(w) = \frac{\alpha_y w^{\alpha_y\mu_y - 1}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y\mu_y}} H_{0,2}^{2,0}\left[\frac{w^{\alpha_y}}{(u_x u_y)^{\alpha_y}} \,\middle|\, \begin{matrix} () \\ (0,1), \left(\mu_x - \frac{\alpha_y\mu_y}{\alpha_x}, \frac{\alpha_y}{\alpha_x}\right) \end{matrix}\right], \tag{6}$$

in which the H-function (see [18]) is defined as a contour complex integral which contain Gamma functions in their integrands. This special function has been extensively applied to study communication problems, as seen on recent works [9,19,20]. In (6), empty parentheses indicates there are no parameters of the specific type.

Thus, the PDF of the product of α-μ random variables can be analytically expressed in closed-forms by using well-known special functions, as indicated in [14] and in the present work. In fact, the representations in (5) and (6) are equivalent to the one presented in [14]. Regarding the composite CDF of the α-μ/α-μ product distribution, it can be readily calculated by integrating (6). Thus, for the CDF one obtains:

$$\begin{aligned} F_W(w) = {} & \frac{w^{\alpha_y\mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y\mu_y}} \times \\ & H_{1,3}^{2,1}\left[\frac{w^{\alpha_y}}{(u_x u_y)^{\alpha_y}} \,\middle|\, \begin{matrix} (1-\mu_y, 1) \\ (0,1), \left(\mu_x - \frac{\alpha_y\mu_y}{\alpha_x}, \frac{\alpha_y}{\alpha_x}\right), (-\mu_y, 1) \end{matrix}\right]. \end{aligned} \tag{7}$$

The result in (7) easily follows from (6) by considering the contour integral representation of the H-function. If one still wants to consider the real integral representation in (4), the following formula is obtained for the CDF of the composite distribution:

$$F_W(w) = \frac{\alpha_x}{\Gamma(\mu_x)\Gamma(\mu_y)\alpha_y}\int_0^\infty t^{\frac{\alpha_x\mu_x}{\alpha_y} - 1}\exp\left(-t^{\frac{\alpha_x}{\alpha_y}}\right)\gamma\left(\mu_y, \frac{w^{\alpha_y}}{(u_x u_y)^{\alpha_y} t}\right)\mathrm{d}t \tag{8}$$

in which $\gamma(s, x)$ stands for the lower incomplete Gamma function [18].

All the equations above do not rely on any restrictions on the parameters involved, as the results of [1] did. The results above have been previously presented in an alternative version in [14]. Besides, except for (8), the expression are all in closed forms in terms of well-known special functions.

Simplified Results When $\alpha_x/\alpha_y = p/q$

In order to obtain their results, the authors in [1] considered that $\alpha_x/\alpha_y = p/q$, in which $p \geq 1$ and $q \geq 1$ are co-prime integers. Their results were in terms of two summations of hypergeometric functions of the type ${}_1F_{p+q}$ for the PDF and of the type ${}_2F_{p+q+1}$ for the CDF. The author in [14], on the other hand, considered the cases when $\alpha_x/\alpha_y = m$, where m is a non-null natural number. The summations presented in [1] can be hard to implement and the cases where $\alpha_x/\alpha_y = m$ reported by [14] are not general enough. We can generalize the results presented by [14] in terms of Meijer-G function to account for cases where $\alpha_x/\alpha_y = p/q$, in which $p \geq 1$ and $q \geq 1$ are co-prime integers.

By definition, the Meijer-G function is a special case of the H-function. Thus, by considering the representation in (6) and using the multiplication theorem of the Gamma function [18], when $\alpha_x/\alpha_y = p/q$, the PDF of the product of α-μ random variables is expressed in (9).

$$f_W(w) = \frac{\alpha_x w^{\alpha_y \mu_y - 1} q^{1/2+\mu_x - \frac{q\mu_y}{p}}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y \mu_y} p^{1/2}(2\pi)^{\frac{p+q-2}{2}}} \times$$
$$G^{p+q,0}_{0,p+q}\left[\frac{w^{\alpha_y p} p^{-p} q^{-q}}{(u_x u_y)^{\alpha_y p}} \middle| \begin{matrix} () \\ \left(0, \frac{1}{p}, \dots, \frac{p-1}{p}, \frac{\mu_x}{q} - \frac{\mu_y}{p}, \frac{1+\mu_x}{q} - \frac{\mu_y}{p}, \dots, \frac{q-1+\mu_x}{q} - \frac{\mu_y}{p}\right)\end{matrix}\right] \quad (9)$$

The same procedure can be carried out to express the CDF of the product of α-μ random variables. The resulting expression is presented in (10).

$$F_W(w) = \frac{w^{\alpha_y \mu_y} q^{\mu_x - \frac{q\mu_y}{p} - 1/2}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y \mu_y} p^{1/2}(2\pi)^{\frac{p+q-2}{2}}} \times$$
$$G^{p+q,1}_{1,p+q+1}\left[\frac{w^{\alpha_y p} p^{-p} q^{-q}}{(u_x u_y)^{\alpha_y p}} \middle| \begin{matrix} \left(1 - \frac{\mu_y}{p}\right) \\ \left(0, \frac{1}{p}, \dots, \frac{p-1}{p}, \frac{\mu_x}{q} - \frac{\mu_y}{p}, \frac{1+\mu_x}{q} - \frac{\mu_y}{p}, \dots, \frac{q-1+\mu_x}{q} - \frac{\mu_y}{p}, -\frac{\mu_y}{p}\right)\end{matrix}\right] \quad (10)$$

In order to obtain the composite PDF and CDF, one may simply set $\bar{x} = 1$ in (5) or (6) or (9) and (7) or (8) or (10), respectively. This will only affect u_x.

All the equations hereby presented are in terms of integrals. These formulas can be easily evaluated as shall be seen in the next section. The numerical codes needed to evaluate the new solutions are presented in order to enhance their usability. Besides, series representations are also presented to provide quick and accurate evaluations.

4. Numerical Evaluation of the Expressions

4.1. Integral Representations

As previously indicated, the PDF of the composite distribution is expressible in terms of both the Kratzel function as well as the H-function. When $\alpha_x/\alpha_y = p/q$, the results are also expressible in terms of the Meijer-G function. The most direct way to evaluate the PDF and CDF of the composite distributions is to numerically calculate the integrals involved.

For the PDF, if one uses (5), the real integral in (4) needs to be evaluated. This is straightforward to be achieved by softwares such as Mathematica. On the other hand, if one wants to use (6), the contour integral definition of the H-function presented in [18] shall be considered. Contour integrals are easily evaluated in math softwares such as Wolfram Mathematica. On the other hand, one has to pay close attention to the contour chosen. By the definition of the H-function in (6), the contour L can considered as the line from $c - i\infty$ to $c + i\infty$ for any $c > 0$. Furthermore, if $\alpha_x/\alpha_y = p/q$, the Meijer-G function can be used to perform the calculations. This function is implemented in numerous mathematical software. In the present paper, its Mathematica implementation is considered.

Regarding the CDF, if the real integral is to be considered, (8) can be evaluated by means of standard mathematical softwares. However, if the contour integral is chosen, the contour used to evaluate the H-function in (7) must be taken as the line from $c - i\infty$ to $c + i\infty$ for any $0, < c < \mu_y$. This choice is important to separate the poles of the Gamma functions in the numerator of the definition of the H-function. Furthermore, if $\alpha_x/\alpha_y = p/q$, the Meijer-G function can again be used to perform the calculations.

Numerical experiments have shown that both the real integrals and the Meijer-G functions behave nicely for all the possible values of the parameters involved. On the other hand, directly calculating the contour integral may not be the best choice, as high precision must be set to get correct results.

A set of parameters has been chosen to illustrate the applicability of the expressions hereby developed. These are presented in Table 1. In this Table, the rational approximations to α_x/α_y present the rational number with smallest denominator that lies within 0.003 of the real ratio. Numerical experiments have shown that this tolerance is enough to correctly approximate the expressions for practical applications.

Table 1. Random variables considered for numerical analyses.

Set	$X(\alpha_x, \mu_x, \bar{x})$	$Y(\alpha_y, \mu_y, \bar{y})$	$\frac{\alpha_x}{\alpha_y}$
S1	$X(1.279, 4.011, 1)$	$Y(3.486, 4.981, 3.581)$	$\frac{7}{19}$
S2	$X(3.195, 3.598, 1)$	$Y(3.723, 0.767, 4.069)$	$\frac{6}{7}$
S3	$X(3.327, 0.373, 1)$	$Y(3.151, 4.829, 0.915)$	$\frac{19}{18}$
S4	$X(2.415, 3.321, 1)$	$Y(0.318, 100, 2.810)$	$\frac{91}{12}$

One shall notice that for the set $S4$, the shadowing distribution given by the RV Y is approximately Lognormal. It has been obtained by letting $\lambda = 0.10$, $\sigma = \pi/10$ and $\mu_{LN} = 1$ in the parametrization of the α-μ distribution discussed on Section 2. To illustrate how the approximation behaves, Figure 1 has been plotted. This figure presents the difference between the PDFs of a Lognormal RV, $f_{LN}(r)$, and an α-μ RV, $f_R(r)$, with $\sigma = \pi/10$ and $\mu_{LN} = 1$.

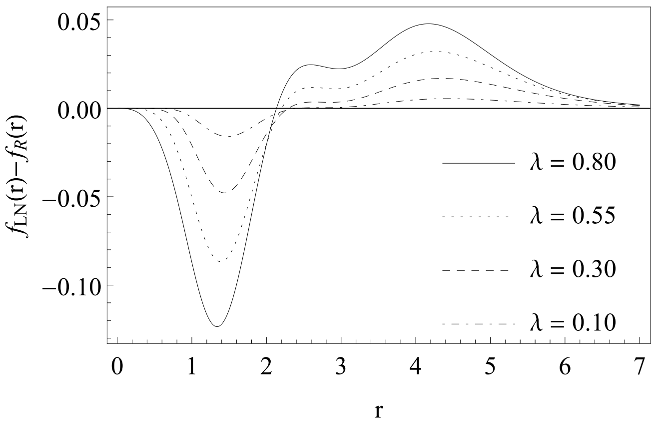

Figure 1. Comparing the Lognormal distribution and the corresponding α-μ RV.

It can be seen from Figure 1 that, for practical purposes, it is sufficient to take $\lambda = 0.10$ to get a good agreement to the Lognormal distribution.

For the sets presented in Table 1, the PDFs and CDFs of the composite distributions are plotted in Figure 2. The plots and calculations have been performed by the real integral representations as these are the most stable numerical implementation found. For the set $S4$, both the H-function contour integral and the Meijer-G function built-in implementations do not behave well. This comes from the fact that Gamma functions of large arguments appear, making the calculations lose precision.

Now that the possible integral evaluations have been discussed, it is of interest to present series expansions of the function involved. This shall be discussed in the next subsection.

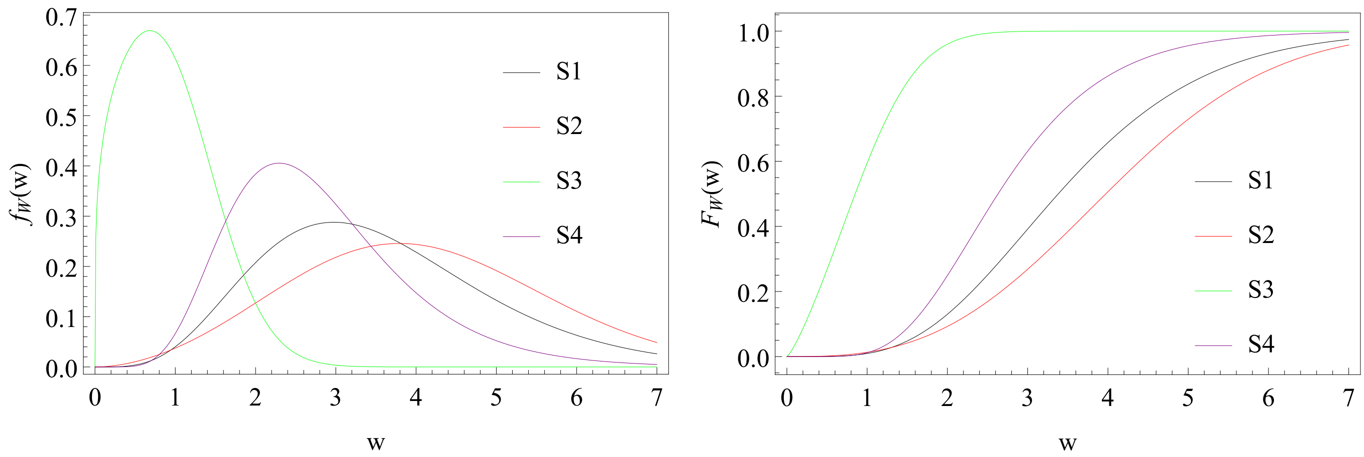

Figure 2. PDFs and CDFs Plotted Using the Integral Representations Provided

4.2. Series Representations

In [17], the following series representation was presented for the Kratzel's function when $l \neq \nu + \rho m$ for any $l, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$

$$Z_\rho^\nu(x) = \frac{1}{\rho}\sum_{m=0}^{\infty}\Gamma\left(\frac{\nu - m}{\rho}\right)\frac{(-x)^m}{m!} + x^\nu \sum_{m=0}^{\infty}\Gamma(-\nu - \rho m)\frac{(-x^\rho)^m}{m!}. \tag{11}$$

The restrictions on ν and ρ are easily surpassed when any of these numbers is an irrational number or when the number of decimal digits of the numbers involved is high enough such that the remaining terms of the summation are negligible. The series in (11) can also be obtained for the H-function representation in (6) by using the residue theorem to evaluate the contour integral [18] (with simple poles). Thus, a series representation for the PDF of the composite α-μ/α-μ distribution can be given in (12).

$$\begin{aligned} f_W(w) = {} & \frac{\alpha_x w^{\alpha_y \mu_y - 1}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y \mu_y}} \times \\ & \left(\frac{\alpha_y}{\alpha_x}\sum_{m=0}^{\infty}\frac{(-1)^m \Gamma\left(\mu_x - \frac{\alpha_y(\mu_y+m)}{\alpha_x}\right)}{m!}\left(\frac{w}{u_x u_y}\right)^{\alpha_y m} + \right. \\ & \left. \sum_{m=0}^{\infty}\frac{(-1)^m \Gamma\left(\mu_y - \frac{\alpha_x(\mu_x+m)}{\alpha_y}\right)}{m!}\left(\frac{w}{u_x u_y}\right)^{\alpha_x(\mu_x+m) - \alpha_y \mu_y}\right) \end{aligned} \tag{12}$$

On the other hand, considering (12), simple integration provides that the CDF of the composite distribution is expressed in (13).

$$\begin{aligned} F_W(w) = {} & \frac{w^{\alpha_y \mu_y}}{\Gamma(\mu_x)\Gamma(\mu_y)(u_x u_y)^{\alpha_y \mu_y}} \times \\ & \left(\sum_{m=0}^{\infty}\frac{\Gamma\left(\mu_x - \frac{\alpha_y(\mu_y+m)}{\alpha_x}\right)}{(\mu_y + m)}\frac{(-1)^m}{m!}\left(\frac{w}{u_x u_y}\right)^{\alpha_y m} + \right. \\ & \left. \sum_{m=0}^{\infty}\frac{\Gamma\left(\mu_y - \frac{\alpha_x(\mu_x+m)}{\alpha_y}\right)}{(\mu_x + m)}\frac{(-1)^m}{m!}\left(\frac{w}{u_x u_y}\right)^{\alpha_x(\mu_x+m) - \alpha_y \mu_y}\right) \end{aligned} \tag{13}$$

Both (12) and (13) are way simpler than the series presented in [1]. Besides, the only possible restriction to the usage of the series in (12) and (13) is that all the poles of the Gamma functions involved are simple. This is achieved whenever: $\alpha_y l \neq \alpha_x \mu_x - \alpha_y \mu_y + \alpha_x m$ for any $l, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. In general, when the parameters α_x,μ_x,α_y,μ_y are all different and any of them is an irrational number, the restrictions are satisfied.

It should be noticed that when these parameters are considered with 2 or 3 decimal digits, for example, the restrictions above will only fail for l, m of about 100 and 1000, respectively. In these cases, the residues over the remaining poles will become negligible. Therefore, this is not quite a drawback, as, in general, estimation procedures will produce rational numbers with as many decimal digits as desired for the parameters. Besides, the series converge for every value of $w > 0$.

Numerical experiments show that the series obtained nicely converge to their limiting value. In general, by taking about 30 terms is sufficient for practical purposes, as Table 2 indicates.

Table 2. Convergence analyses at $w = 2$.

(**a**) Equation (12)				
Set	**Exact Value**	**10 Terms**	**30 Terms**	**100 Terms**
S1	0.207465	0.204785	0.207465	0.207465
S2	0.127178	0.127178	0.127178	0.127178
S3	0.126769	0.275901	0.126769	0.126769
S4*	0.383627	8.42×10^7	0.391774	0.391728

(**b**) Equation (13)				
Set	**Exact Value**	**10 Terms**	**30 Terms**	**100 Terms**
S1	0.130012	0.129598	0.130012	0.130012
S2	0.0926511	0.0926511	0.0926511	0.0926511
S3	0.959844	0.96768	0.959844	0.959844
S4*	0.248891	6.85×10^6	0.272829	0.272827

It can be seen in Table 2 that the series behaves well for all the sets, except $S4$. When the limiting procedure to get a Lognormal RV is carried out, the number of significant figures which must be considered to properly evaluate the summation increases rapidly, as the series involved contain gamma functions. Therefore, the last computable value using Mathematica's regular precision (53 binary digits of precision) is when $\lambda = 0.3$. Regarding the CDF, the same behavior observed for the series PDF is observed for the series CDF. Once again, for $S4$, from $\lambda = 0.3$ on, the number of significant figures is increasingly high.

Figure 3 presents the codes used to implement the special functions and series used in the present paper.

From Figure 3, we can see that:

- The function $KratzelF[x, \nu, \rho]$ implements the numerical integral definition of the Kratzel function given right below (5).
- $KratzelG[x, \nu, p, q]$ implements the Meijer G-function alternative representation of the Kratzel function used to obtain (9) whenever $\rho = p/q$.
- $H2002F[x, \nu, \rho]$ implements the H-function alternative definition of the Kratzel function used to obtain (6).
- $KratzelSF[x, \nu, \rho, mmax]$ implements the series representation of the Kratzel function given in (11). Here, $mmax$ represents the number of terms to be considered for the summation.
- $H2113F[x, \nu, \rho, \mu]$ implements the H-function used in (7).
- $H2113SF[x, \nu, \rho, \mu, mmax]$ implements the series expression of the H function used in (13) after integrating (12).Here, $mmax$ represents the number of terms to be considered for the summation.
- $H2113G[x, \nu, p, q, \mu]$ implements the Meijer G-function simplification of the H function used in (7) when $\rho = p/q$ and presented in (10).

```
KratzelF[x_, ν_, ρ_] :=
 NIntegrate[t^(ν-1) * Exp[-t^ρ - x / t], {t, 0, Infinity}]
KratzelG[x_, ν_, p_, q_] :=
 (q^(1/2+ν*q/p) / (p^(1/2) * (2 * Pi)^((p+q-2)/2))) *
  MeijerG[{{}, {}},
   {Join[Table[j / p, {j, 0, p - 1}], Table[ν / p + j / q, {j, 0, q - 1}]],
    {}}, x^p * p^-p * q^-q]
H2002F[z_, ν_, ρ_] :=
 Chop[(2 * Pi * I * ρ)^-1 * NIntegrate[Gamma[s] * Gamma[(s + ν) / ρ] * z^-s,
    {s, 1 / 2 - I * 1000, 1 / 2 + I * 1000}]]
KratzelSF[z_, ν_, ρ_, mmax_] :=
 ρ^-1 * Sum[Gamma[(ν - k) / ρ] * (-z)^k / Gamma[k + 1], {k, 0, mmax}] +
  z^ν * Sum[Gamma[-ν - ρ * k] * (-z^ρ)^k / Gamma[k + 1], {k, 0, mmax}]
H2113F[z_, ν_, ρ_, μ_] :=
 Chop[(2 * Pi * I)^-1 * NIntegrate[Gamma[s] * Gamma[(s + ν) / ρ] * z^-s / (μ - s),
    {s, μ / 2 - I * 2000, μ / 2 + I * 2000}, MaxRecursion → 50]]
H2113SF[z_, ν_, ρ_, μ_, mmax_] :=
 Sum[Gamma[(ν - k) / ρ] * (-z)^k / ((μ + k) * Gamma[k + 1]), {k, 0, mmax}] +
  ρ * z^ν * Sum[Gamma[-ν - ρ * k] * (-z^ρ)^k / ((μ + k * ρ + ν) * Gamma[k + 1]),
    {k, 0, mmax}]
H2113G[x_, ν_, p_, q_, μ_] :=
 (q^(ν*q/p-1/2) / (p^(1/2) * (2 * Pi)^((p+q-2)/2))) *
  MeijerG[{{1 - μ / p}, {}},
   {Join[Table[j / p, {j, 0, p - 1}], Table[ν / p + j / q, {j, 0, q - 1}]],
    {-μ / p}}, x^p * p^-p * q^-q]
```

Figure 3. Algorithms used to implement the special functions used in the present paper.

On the other hand, we can see how the functions presented in Figure 3 were used to implement the PDF computational codes in Figure 4.

- The function $PDFKratzel[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the PDF of the composed distribution when the numerical integral definition of the Kratzel function is used.
- $PDFH2002[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the PDF of the composed distribution when the H-function alternative definition of the Kratzel function is used.
- $PDFG[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the PDF of the composed distribution when $\alpha x/\alpha y = p/q$ and the Meijer G-function alternative definition of the Kratzel function is used. In order to obtain the rational approximations to α_x/α_y, we automatically calculate the rational number with smallest denominator that lies within 0.003 of the real ratio.
- $PDFSeries[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the series expression of the PDF by considering $mmax$ terms into the summation.

Furthermore, by using the functions presented in Figure 3, we show how to implement the CDF computational codes in Figure 5.

- The function $CDFHInt[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the CDF of the composed distribution when the numerical integral definition of the Kratzel function is used.
- $CDFH2113[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the CDF of the composed distribution when the numerical integral definition of the H-function is used.
- $CDFG[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar]$ implements the CDF of the composed distribution when $\alpha x/\alpha y = p/q$ and the Meijer G-function alternative definition of the Kratzel function is used. In order to obtain the rational approximations to α_x/α_y, we automatically calculate the rational number with smallest denominator that lies within 0.003 of the real ratio.
- $CDFSeries[w, \alpha x, \mu x, xbar, \alpha y, \mu y, ybar, mmax]$ implements the series expression of the CDF by considering $mmax$ terms into the summation.

```
PDFKratzel[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  ((αx * w^(αy*μy-1)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   KratzelF[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, αx / αy]]
PDFH2002[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  ((αy * w^(αy*μy-1)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   H2002F[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, αx / αy]]
PDFG[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1],
   p = Numerator[Rationalize[αx / αy, 0.003]],
   q = Denominator[Rationalize[αx / αy, 0.003]]},
  ((αx * w^(αy*μy-1)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   KratzelG[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, p, q]]
PDFSeries[w_, αx_, μx_, xbar_, αy_, μy_, ybar_, mmax_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  ((αx * w^(αy*μy-1)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   KratzelSF[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, αx / αy, mmax]]
```

Figure 4. Algorithms used to implement the PDFs used in the present paper.

```
CDFHInt[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  1 - ((αx) / (Gamma[μx] * Gamma[μy] * αy)) *
    NIntegrate[t^((αx*μx/αy)-1) * Exp[-t^(αx/αy)] *
      (Gamma[μy, (w / (ux * uy))^αy / t]), {t, 0, Infinity}]]
CDFH2113[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  ((w^(αy*μy)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   H2113F[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, αx / αy, μy]]
CDFG[w_, αx_, μx_, xbar_, αy_, μy_, ybar_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1],
   p = Numerator[Rationalize[αx / αy, 0.003]],
   q = Denominator[Rationalize[αx / αy, 0.003]]},
  ((w^(αy*μy)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   H2113G[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, p, q, μy]]
CDFSeries[w_, αx_, μx_, xbar_, αy_, μy_, ybar_, mmax_] :=
 Module[{ux = xbar * Gamma[μx] / Gamma[μx + αx^-1],
   uy = ybar * Gamma[μy] / Gamma[μy + αy^-1]},
  ((w^(αy*μy)) / (Gamma[μx] * Gamma[μy] * (ux * uy)^(αy*μy))) *
   H2113SF[(w / (ux * uy))^αy, (αx * μx - αy * μy) / αy, αx / αy, μy, mmax]]
```

Figure 5. Algorithms used to implement the CDFs used in the present paper.

5. Application to Experimental Data

Literature presents the results of a series of outdoors field trials which were conducted at the University of Brasília (UnB) and at the University of Campinas (Unicamp), Brazil, in order to obtain the empirical PDFs of the composite multipath/shadowing phenomena [21]. In short, the experiments consisted in placing a transmitter on the rooftop of a building and moving the receiver through the campus of each university [21]. The mobile reception equipment was especially assembled for this purpose and consisted of a vertically polarized omnidirectional receiving antenna, a low noise amplifier, a spectrum analyzer, a data acquisition equipment and a notebook computer [21].

Regarding the transmission, it consisted of a continuous wave tone at 2500 MHz and 780 MHz at UnB and at 1800 MHz at Unicamp. The spectrum analyzer was set to zero span and centered at the desired frequency, and its video output used as the input of the data acquisition equipment with a sampling interval of 300 samples per second. From the collected data, the short term, the long term and the path loss were isolated, then a composite envelope was made by adding the short and the long terms.

Figure 6 reveals the remarkable accuracy of the model. Actually, the best fit distributions were $X(2.31, 3.41, 0.95)$ and $Y(1.4, 90, 1.13)$ for the 1800 MHz case and $X(3.2, 4.5, 0.91)$ and $Y(0.22, 115, 1.255)$ for the 780 MHz. By looking at the estimated values, it can be seen that a nearly Lognormal distribution was fitted for the shadowing distributions, as the equivalent parameters were $\lambda = 0.105409$, $\sigma = 0.0752923$, $\mu_{LN} = 0.123351$ for Y at 1800 MHz and $\lambda = 0.0932505$, $\sigma = 0.423866$ and $\mu_{LN} = 0.157875$ for Y at 780 MHz. One may notice that Figure 1 reveals that for $\lambda \approx 0.10$, the α-μ surrogate Lognormal approximation is already pretty close to the actual Lognormal distribution.

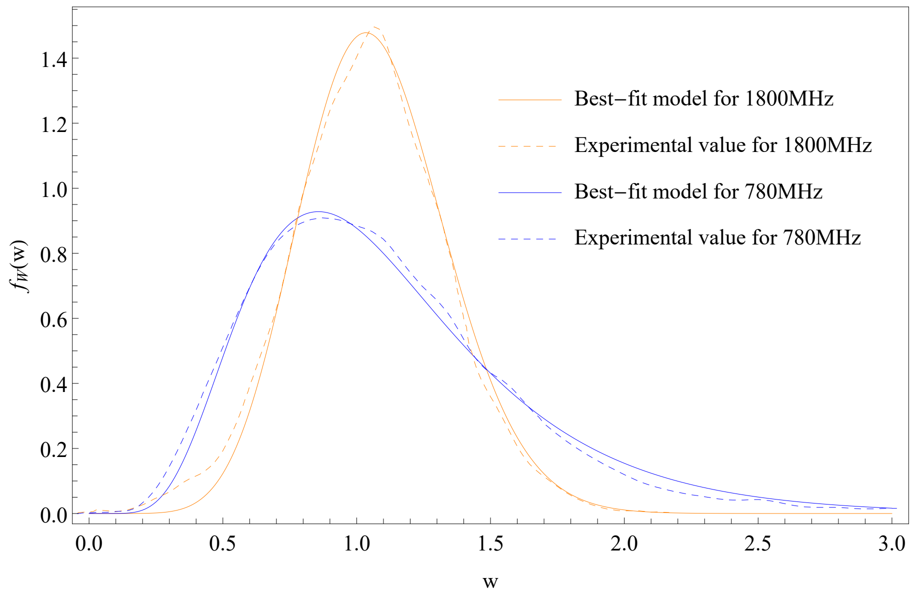

Figure 6. Empirical and Fitted PDFs for field trials.

6. Conclusions

While studying wireless communication systems, it is known that correctly modeling the shadowing effect is crucial to perform the network quality assessment. Besides, literature reveals that composite distributions have shown to be good alternatives to combine both fast fading and slow fading stochastic phenomena.

In the present paper, we discuss the applicability of the α-μ distribution as a surrogate Lognormal shadowing model. At first, we revisited α-μ distributions and showed their connection to Generalized Gamma distributions. Furthermore, by pointing out this link, we could prove that, by a limiting procedure, α-μ distributions can be reduced to Lognormal distributions.

We also revisited the analytical mathematical formulas of the combination of α-μ distributions as both fast and slow fading models. This issue has been explored in the past by [1], which present some intricate expressions. Besides, the equations in [1] rely on considering special values for the parameters involved, which restricts the applicability to practical situations. Furthermore, those authors did not indicate that the α-μ distribution can be converted to a Lognormal one by a limiting procedure. The author in [14] presented the exact expressions for the PDF and CDF of the α-μ composite distribution but only presented results in terms of the Meijer-G function when the ratio of α's was a non-null natural number. We extended their results to cases when such ratio is a rational number p/q, were p and q are coprime integers.

We also obtained new closed-form and series representations for the PDF and CDF of the composite distributions. In order to enhance the applicability of the expression hereby derived, we present the Mathematica codes used to evaluate them. Real field trials were also modelled using the expressions and codes presented, revealing a remarkable modelling accuracy. By using the codes and expressions presented, we believe practitioners will be able to better model Lognormal shadowing effects.

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