

Article

Power Flow of Electric Dipole Radiation Extinction in an Absorbing Host Medium

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Abstract

We have studied the extinction power flow for a dipole in a laser beam, and embedded in a dissipating medium. The power flows along the field lines of the Poynting vector. We have shown that near the particle, the field lines form closed loops, which start and end at the location of the dipole. A closed-form expression for these loops has been derived, and we have shown how the orientation direction of a loop is determined by the permittivities and permeabilities of the host medium and the particle. It is also shown that the spatial extent of these loops is determined by singularities in the flow pattern. It is shown that the extent of the loop structure near the dipole diminishes strongly when there is dissipation in the medium. This is due to the appearance of singularities very close to the particle, which are due to the damping. At greater distances, flow lines run off to the far field or they come in from the far field. Most flow lines change from incoming to outgoing, or vice versa, so they turn around somewhere in the flow field. Singularities, points where the Poynting vector vanishes, appear on the coordinate axes. At these points, field lines split. Off the axes, singularities appear as the centers of vortices. Near a vortex, energy swirls around the singular point. Field lines can come out of the center of a vortex or end there.

Keywords: Poynting vector; power flow; vortices; extinction; dipole radiation

1. Introduction

When a small particle is irradiated by a laser beam, some of the incident radiation is scattered off the particle, and this radiation is observable in the far field. This radiation carries information about the properties of the particle, and as such this is a valuable diagnostic tool for experimentalists. Properties of aerosols, like shape and composition, can be extracted from the information embedded in the scattered radiation [1]. In most studies, the particle is assumed to be spherical, which has the great advantage that this is theoretically an exactly solvable problem, first presented by Mie [2]. Building on this known solution, it has been proposed to construct metamaterials from collections of Mie particles [3–5]. Another interesting phenomenon associated with Mie particles is the appearance of photonic jets in the forward direction [6,7]. Of particular interest are the Mie resonances in the particle. For very specific values of the parameters in the problem, sharp scattering resonances appear [8–12], which can be utilized to enhance either the absorption in the particle or the scattered power. Comprehensive reviews of Mie scattering can be found in [13–15], and a review of the Mie coefficients was presented in [16].

The schematic setup of the system is shown in Figure 1. The laser beam with angular frequency ω propagates into the positive z direction, and is incident upon a spherical particle with radius R . The (relative) permittivity of the particle is ϵ_p and the (relative)



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permeability is μ_p , both of which may be complex. We shall use the free-space wave number $k_0 = \omega/c$ to adopt dimensionless spatial variables, so $\bar{x} = k_0 x$, etc. The overbar indicates the dimensionless quantity. The radius of the particle is R , and so the dimensionless radius is $\bar{R} = k_0 R$. We shall simply refer to $\bar{R}$ as the radius of the particle. The particle is embedded in a medium with parameters ε_1 and μ_1 . Any imaginary part in ε or μ gives rise to absorption in the particle or the medium. For causality reasons, it has to hold that in a passive medium

$$\text{Im}\varepsilon \geq 0, \quad \text{Im}\mu \geq 0, \quad (1)$$

both for the particle and the medium. The index of refraction follows from $n^2 = \varepsilon\mu$, and this parameter also cannot have a negative imaginary part. Therefore, the root has to be taken as

$$n = \sqrt{\varepsilon}\sqrt{\mu}, \quad (2)$$

both for the particle and the medium.

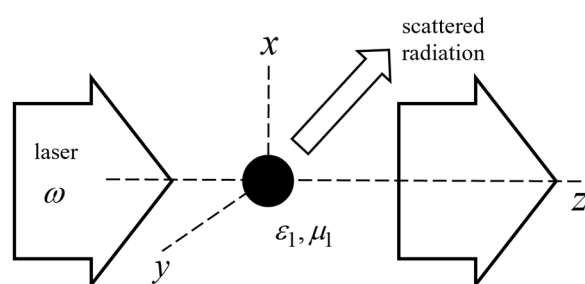


Figure 1. Shown is the setup for a spherical particle in a laser beam.

The complex amplitude of the electric field of the laser beam will be indicated by $\mathbf{E}_L(\mathbf{r})$, and for the scattered field we have $\mathbf{E}_{sc}(\mathbf{r})$. The total electric field outside the particle is then

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_L(\mathbf{r}) + \mathbf{E}_{sc}(\mathbf{r}). \quad (3)$$

For the magnetic field we have similarly

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_L(\mathbf{r}) + \mathbf{B}_{sc}(\mathbf{r}). \quad (4)$$

We shall not be concerned with the fields inside the particle. Energy, or power, flows along the field lines of the (time-averaged) Poynting vector, defined as

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_0} \text{Re} \frac{1}{\mu_1} \mathbf{E}(\mathbf{r})^* \times \mathbf{B}(\mathbf{r}). \quad (5)$$

With Equations (3) and (4) this gives three distinct contributions,

$$\mathbf{S}(\mathbf{r}) = \mathbf{S}_L(\mathbf{r}) + \mathbf{S}_{sc}(\mathbf{r}) + \mathbf{S}_\times(\mathbf{r}), \quad (6)$$

with

$$\mathbf{S}_L(\mathbf{r}) = \frac{1}{2\mu_0} \text{Re} \frac{1}{\mu_1} \mathbf{E}_L(\mathbf{r})^* \times \mathbf{B}_L(\mathbf{r}), \quad (7)$$

$$\mathbf{S}_{sc}(\mathbf{r}) = \frac{1}{2\mu_0} \text{Re} \frac{1}{\mu_1} \mathbf{E}_{sc}(\mathbf{r})^* \times \mathbf{B}_{sc}(\mathbf{r}), \quad (8)$$

$$\mathbf{S}_\times(\mathbf{r}) = \frac{1}{2\mu_0} \text{Re} \frac{1}{\mu_1} [\mathbf{E}_L(\mathbf{r})^* \times \mathbf{B}_{sc}(\mathbf{r}) + \mathbf{E}_{sc}(\mathbf{r})^* \times \mathbf{B}_L(\mathbf{r})]. \quad (9)$$

The contribution $\mathbf{S}_L(\mathbf{r})$ to the Poynting vector represents the freely propagating laser beam, and $\mathbf{S}_{sc}(\mathbf{r})$ accounts for the power flow of the scattered radiation. Of interest in this paper is the cross term $\mathbf{S}_\times(\mathbf{r})$. This term is a result of interference between the laser field and the scattered field. It represents the energy transport from the laser beam towards the particle, a process which is usually referred to as *extinction*. This energy is partially reflected as scattered radiation, and is partially absorbed by the particle, leading to heating.

2. Incident Field and Polarizability

The incident laser beam is monochromatic, polarized radiation, propagating into the positive z direction. The complex amplitudes of the electric and magnetic fields are

$$\mathbf{E}_L(\mathbf{r}) = E_0 \mathbf{u} e^{in_1 \bar{z}}, \quad (10)$$

$$\mathbf{B}_L(\mathbf{r}) = \frac{n_1}{c} E_0 (\mathbf{e}_z \times \mathbf{u}) e^{in_1 \bar{z}}. \quad (11)$$

The, in general complex, polarization vector $\mathbf{u}$ lies in the xy plane and is normalized as $\mathbf{u} \cdot \mathbf{u} = 1$. For $\mathbf{u}$ real, the field is linearly polarized, and for $\mathbf{u}$ a spherical unit vector, the polarization is circular. For any other choice of $\mathbf{u}$, the polarization is elliptical. The electric field itself is

$$\mathbf{E}_L(\mathbf{r}, t) = \text{Re} \mathbf{E}_L(\mathbf{r}) e^{-i\omega t}. \quad (12)$$

For the index of refraction of the host medium, we set

$$n_1 = n_0 + i\gamma, \quad n_0 \text{ real}, \quad \gamma \geq 0. \quad (13)$$

We shall exclude the possibility of $n_0 = 0$. The electric field then takes the form

$$\mathbf{E}_L(\mathbf{r}, t) = e^{-\gamma \bar{z}} \text{Re} E_0 \mathbf{u} e^{i(n_0 \bar{z} - \omega t)}. \quad (14)$$

The phase velocity of this plane wave is c/n_0 . For $n_0 > 0$, the wave travels into the positive z direction, and for $n_0 < 0$, the wave pattern travels into the negative z direction. The amplitude decays into the positive z direction as $\exp(-\gamma \bar{z})$.

In order to eliminate irrelevant overall constants, we introduce dimensionless representations of the electric and magnetic fields as

$$\mathbf{E}(\mathbf{r}) = E_0 \mathcal{E}(\mathbf{r}), \quad (15)$$

$$\mathbf{B}(\mathbf{r}) = n_1 \frac{E_0}{c} \mathcal{B}(\mathbf{r}). \quad (16)$$

For the laser field we then get

$$\mathcal{E}_L(\mathbf{r}) = \mathbf{u} e^{in_1 \bar{z}}, \quad (17)$$

$$\mathcal{B}_L(\mathbf{r}) = (\mathbf{e}_z \times \mathbf{u}) e^{in_1 \bar{z}}. \quad (18)$$

For the Poynting vector we write

$$\mathbf{S}(\mathbf{r}) = I_0 \boldsymbol{\sigma}(\mathbf{r}), \quad (19)$$

with

$$I_0 = \frac{1}{2} c \epsilon_0 |E_0|^2, \quad (20)$$

as the intensity of the laser beam in free space. The dimensionless Poynting vector then becomes

$$\boldsymbol{\sigma}(\mathbf{r}) = \text{Re} \frac{n_1}{\mu_1} \mathcal{E}(\mathbf{r})^* \times \mathcal{B}(\mathbf{r}), \quad (21)$$

in terms of the dimensionless fields.

The Poynting vector $\sigma_L(\mathbf{r})$ for the laser beam is found to be

$$\sigma_L(\mathbf{r}) = \mathbf{e}_z e^{-2\gamma\bar{z}} \text{Re}(n_1/\mu_1). \quad (22)$$

It is easy to show that $\text{Re}(n_1/\mu_1) \geq 0$, so the energy propagates into the positive z direction. The factor $\exp(-2\gamma\bar{z})$ indicates that the beam intensity weakens upon propagation, which is due to damping in the host medium. The field lines of $\sigma_L(\mathbf{r})$ are straight lines, parallel to the z axis, in the positive z direction.

It is assumed that the particle is small compared to an optical wavelength, and so only the electric dipole moment will contribute significantly to the scattered radiation. Let the complex amplitude of the induced dipole moment be $\mathbf{d}$. This dipole moment is determined by the incident field as

$$\mathbf{d} = \alpha_e \mathbf{E}_L(0), \quad (23)$$

with α_e the electric polarizability of the particle. Usually, this relation is assumed, for a lack of better. For a Mie particle, however, it can be shown that this holds without approximation [16]. With Equation (10), this becomes $\mathbf{d} = \alpha_e E_0 \mathbf{u}$. The polarizability α_e is most conveniently expressed in terms of the electric polarizability volume $\mathcal{V}_e$, defined as

$$\mathcal{V}_e = \mu_1 n_1 \frac{k_0^3}{4\pi \epsilon_0} \alpha_e. \quad (24)$$

The quantity $\mathcal{V}_e$ is dimensionless. It can be shown [17,18] that without dissipation in the medium and the particle, $\mathcal{V}_e$ lies on a universal circle in the complex plane, the *polarizability circle*. It is centered around $3i/4$ and has a radius of $3/4$. The maximum value of $\mathcal{V}_e$ is therefore $\mathcal{V}_e = 3i/2$. Moreover, $\mathcal{V}_e$ is directly proportional to the electric dipole Mie scattering coefficient $a_1(e)$ as

$$\mathcal{V}_e = \frac{3i}{2} a_1(e). \quad (25)$$

The maximum value of $\mathcal{V}_e$ corresponds to $a_1(e) = 1$, which is at a Mie resonance. The Mie coefficient $a_1(e)$ can be expressed in terms of the particle parameters $\bar{R}$, ϵ_p and μ_p , and it also depends on the parameters ϵ_1 and μ_1 of the embedding medium. The rather lengthy expression is given in Appendix A. For small $\bar{R}$, we have approximately

$$\mathcal{V}_e = \frac{\epsilon_p - \epsilon_1}{\epsilon_p + 2\epsilon_1} (n_1 \bar{R})^3, \quad \bar{R} \text{ small}, \quad (26)$$

provided that ϵ_p is not too close to $-2\epsilon_1$. The electric quadrupole Mie coefficient is proportional to $\bar{R}^5$, and so is the magnetic dipole Mie coefficient (for $\mu_1 = \mu_p = 1$). In the illustrations below, we take $\bar{R} = 0.1$, so the neglected terms are of the order of 1% of the electric dipole contribution. This justifies the electric dipole approximation for small particles.

3. Scattered Radiation

The electric and magnetic fields of the scattered radiation are the common electric dipole fields. In their most compact form, they are given by

$$\mathcal{E}_{sc}(\mathbf{r}) = \mathcal{V}_e \frac{1}{q} [f(q)\mathbf{u} - g(q)(\hat{\mathbf{r}} \cdot \mathbf{u})\hat{\mathbf{r}}] e^{iq}, \quad (27)$$

$$\mathcal{B}_{sc}(\mathbf{r}) = \mathcal{V}_e \frac{1}{q} (\hat{\mathbf{r}} \times \mathbf{u}) \left(1 + \frac{i}{q}\right) e^{iq}, \quad (28)$$

with $q = n_1 \bar{r}$ for the radial variable. When there is damping in the host medium, this variable is complex. We have set

$$f(z) = 1 + \frac{i}{z} \left(1 + \frac{i}{z} \right), \quad (29)$$

$$g(z) = 1 + \frac{3i}{z} \left(1 + \frac{i}{z} \right), \quad (30)$$

needed for $z = q$. In (dimensionless) spherical coordinates $(\bar{r}, \theta, \phi)$, the angle dependence only enters through the unit vector $\hat{\mathbf{r}}$, pointing from the origin of coordinates to the field point. The properties of the particle $(\bar{R}, \varepsilon_p, \mu_p)$ only enter through the overall constant $\mathcal{V}_e$.

The Poynting vector $\sigma_{sc}(\mathbf{r})$ follows from Equation (21). We obtain

$$\begin{aligned} \sigma_{sc}(\mathbf{r}) = & |\mathcal{V}_{e,o}|^2 \frac{1}{\bar{r}^2} e^{-2\gamma \bar{r}} \langle \zeta(\bar{r}) [1 - (\mathbf{u}^* \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})] \hat{\mathbf{r}}. \\ & - 2\zeta(\bar{r}) \{ (\mathbf{u}^* \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \text{Im} \varepsilon_1 - \text{Im}[\varepsilon_1 (\mathbf{u}^* \cdot \hat{\mathbf{r}}) \mathbf{u}] \} \rangle \end{aligned} \quad (31)$$

The polarizability constant $\mathcal{V}_{e,o}$, introduced here, is defined as

$$\mathcal{V}_{e,o} = \frac{k_o^3}{4\pi \varepsilon_o} \alpha_e, \quad (32)$$

and compared to Equation (24), we have $\mathcal{V}_e = n_1 \mu_1 \mathcal{V}_{e,o}$. The $\mathcal{V}_{e,o}$ is identical in form to the polarizability volume in free space, although it should be noted that α_e depends on the permittivities and permeabilities of the particle and the embedding medium. The auxiliary functions $\zeta(\bar{r})$ and $\xi(\bar{r})$ are defined as

$$\zeta(\bar{r}) = \frac{1}{\bar{r}} \frac{|\mu_1|}{|\varepsilon_1|} \left[1 + \frac{1}{|n_1|^2 \bar{r}} \left(2\gamma + \frac{1}{\bar{r}} \right) \right], \quad (33)$$

$$\xi(\bar{r}) = \text{Re}(n_1 \mu_1^*) + \frac{1}{\bar{r}} \text{Im} \mu_1 + \zeta(\bar{r}) \text{Im} \varepsilon_1. \quad (34)$$

Both functions are real and positive. Their rather complicated form is due to possible damping in the host material. Without damping, we have, for instance, $\xi(\bar{r}) = n_1 \mu_1$.

Although the study of $\sigma_{sc}(\mathbf{r})$ is not the topic of this paper, let us review its main features. This will illustrate how different the extinction power flow $\sigma_{\times}(\mathbf{r})$ is. First, we notice that $\sigma_{sc}(\mathbf{r})$ only contains information about the particle properties through the overall constant $|\mathcal{V}_{e,o}|^2$. Since field lines of a vector field are determined only by the direction of the field at each point, and not its magnitude, the flow line pattern of $\sigma_{sc}(\mathbf{r})$ is independent of any of the particle parameters. Then, far away from the particle ($\bar{r}$ large), we have $\zeta(\bar{r}) \approx 0$, and Equation (31) simplifies to

$$\sigma_{sc}(\mathbf{r}) \approx |\mathcal{V}_{e,o}|^2 \text{Re}(n_1 \mu_1^*) \frac{1}{\bar{r}^2} e^{-2\gamma \bar{r}} [1 - (\mathbf{u}^* \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})] \hat{\mathbf{r}}. \quad (35)$$

The Poynting vector is approximately proportional to $\hat{\mathbf{r}}$, and therefore the field lines run radially outwards.

Now let us consider the simplest case of a linear dipole in free space, oscillating along the x axis. Equation (31) becomes

$$\sigma_{sc}(\mathbf{r}) = |\mathcal{V}_{e,o}|^2 \frac{1}{\bar{r}^2} \sin^2 \beta, \quad (36)$$

with β the angle between the dipole axis (the x axis) and the direction of the field point $\hat{\mathbf{r}}$. The Poynting vector is radially outward at all distances. Its magnitude is maximum along the z axis and zero along the dipole axis. The field lines are depicted in Figure 2.

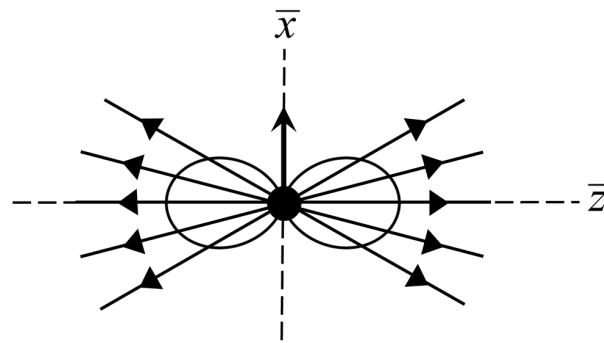


Figure 2. Shown are the field lines of the Poynting vector of the scattered radiation in free space. The dipole oscillates along the $\bar{x}$ axis, and the field line pattern is rotationally symmetric around the $\bar{x}$ axis. The arrow indicates the direction of the dipole moment. The curve is the power per unit solid angle.

When there is absorption in the host medium, the magnitude of the Poynting vector diminishes along a field line, and this is accounted for by the overall factor $\exp(-2\gamma\bar{r})$ in Equation (31). Interestingly, the damping also makes the field lines curve towards the dipole axis [19,20], which is illustrated in Figure 3. Some field lines end at the dipole axis, which is a singular line of the flow field.

For a circular or elliptical dipole, the dipole moment rotates in the xy plane. The field lines swirl around the z axis in the near field, and without damping, they lie on a cone with the z axis as its center line. This has the appearance of a vortex in the neighborhood of the dipole [21,22]. Far away, a field line stays on the cone, and straightens out to a radially outgoing curve, as in Equation (35). When there is absorption in the host medium, the field line pattern in the near field resembles a funnel, and field lines do not lie on a cone [23].

Intimately related to the Poynting vector is the power outflow through a volume. We shall take this volume as a sphere around the dipole, with radius r_0 . This power is, in general

$$P = \oint \mathbf{S}(\mathbf{r}) \cdot \hat{\mathbf{r}} dA. \quad (37)$$

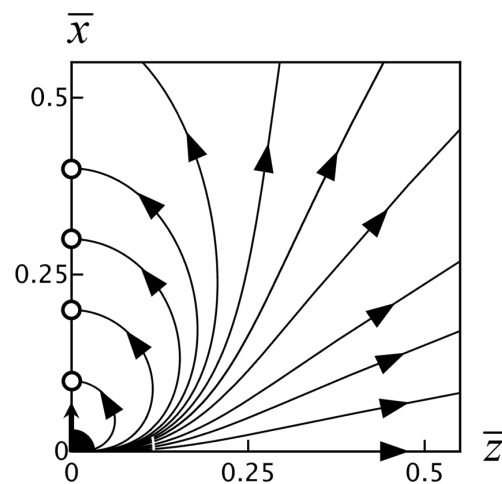


Figure 3. The figure shows the field lines of the scattered power for a linear dipole, oscillating along the $\bar{x}$ axis and embedded in an absorbing medium. The field line pattern is reflection symmetric in the $\bar{z}$ axis and rotationally symmetric around the $\bar{x}$. The $\bar{x}$ axis is a singular line, and white circles indicate a singular point.

We introduce the dimensionless power $\mathcal{P}$ as

$$P = \frac{I_0}{k_0^2} \mathcal{P}. \quad (38)$$

Here, I_0/k_0^2 equals the laser power flowing through a square wavelength in free space, divided by $4\pi^2$. The power per unit solid angle becomes

$$\frac{d\mathcal{P}}{d\Omega} = \bar{r}_0^2 \boldsymbol{\sigma}(\mathbf{r}) \cdot \hat{\mathbf{r}}, \quad (39)$$

and the total power outflow is

$$\mathcal{P} = \oint \frac{d\mathcal{P}}{d\Omega} d\Omega. \quad (40)$$

Here, the integration runs over the unit sphere.

With Equation (31) we find for the scattered radiation the remarkably simple result

$$\frac{d\mathcal{P}_{\text{sc}}}{d\Omega} = |\mathcal{V}_{\text{e,o}}|^2 e^{-2\gamma\bar{r}_0} \zeta(\bar{r}_0) [1 - (\mathbf{u}^* \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})]. \quad (41)$$

The entire term $\{ \dots \}$ in Equation (31) does not contribute to the scattered power per unit solid angle. The angular dependence is accounted for by the factor $1 - (\mathbf{u}^* \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})$, which is independent of the dissipation in the host medium, and is the same as it would be for a dipole in free space. When looking at Figure 3, we see that the field lines curve, and some never make it to the far field. Apparently, this curving does not affect $d\mathcal{P}_{\text{sc}}/d\Omega$. When we increase the radius $\bar{r}_0$ of the integration sphere, then more power is absorbed by the medium, and this attenuation is reflected in the factor $\exp(-2\gamma\bar{r}_0)$, as could be expected. An additional $\bar{r}_0$ dependence enters through the function $\zeta(\bar{r}_0)$. For large $\bar{r}_0$ this function levels off to

$$\zeta(\bar{r}_0) \approx \text{Re} n_1 \mu_1^*, \quad \bar{r}_0 \text{ large}, \quad (42)$$

but for small $\bar{r}_0$ we find

$$\zeta(\bar{r}_0) \approx \frac{1}{\bar{r}_0^3} \frac{1}{|\varepsilon_1|^2} \text{Im} \varepsilon_1, \quad \bar{r}_0 \text{ small}. \quad (43)$$

This function, and thereby $d\mathcal{P}_{\text{sc}}/d\Omega$, diverge in the limit $\bar{r}_0 \rightarrow 0$ when $\text{Im} \varepsilon_1 > 0$. One could expect this in general, because both the electric and magnetic field diverge in this limit. When there is no dissipation in the host medium, there is a delicate cancellation of diverging terms, and we find $\zeta(\bar{r}_0) = n_1 \mu_1$ for all $\bar{r}_0$. Then also $\gamma = 0$, and $d\mathcal{P}_{\text{sc}}/d\Omega$ is independent of $\bar{r}_0$. For a realistic Mie particle, the above only holds for $\bar{r}_0 > \bar{R}$, but there is still the fact that $d\mathcal{P}_{\text{sc}}/d\Omega$ can become very large for $\bar{r}_0 \approx \bar{R}$. This implies that the scattered power can exceed the scattered power for a dipole in free space, due to dissipation in the medium. The total power flowing out of the sphere is

$$\mathcal{P}_{\text{sc}} = \frac{8\pi}{3} |\mathcal{V}_{\text{e,o}}|^2 e^{-2\gamma\bar{r}_0} \zeta(\bar{r}_0). \quad (44)$$

We notice that $\mathcal{P}_{\text{sc}}$ is non-negative and that $d\mathcal{P}_{\text{sc}}/d\Omega$ is non-negative at all angles. So, energy only flows outwards. Without damping in the medium, Equation (44) simplifies to

$$\mathcal{P}_{\text{sc}} = \frac{8\pi}{3} n_1 \mu_1 |\mathcal{V}_{\text{e,o}}|^2. \quad (45)$$

4. Extinction Poynting Vector

The Poynting vector of the cross term is found to be

$$\begin{aligned} \sigma_{\times}(\mathbf{r}) = & \frac{1}{\bar{r}} \text{Re} \left[\mathcal{V}_{e,o} n_1 e^{i(q-q^* \cos \theta)} \left\langle \left(1 + \frac{i}{q} \right) [\hat{\mathbf{r}} - (\hat{\mathbf{r}} \cdot \mathbf{u}^*) \mathbf{u}] \right. \right. \\ & \left. \left. + \frac{n_1^* \mu_1}{n_1 \mu_1^*} \{ f(q) \mathbf{e}_z - g(q) (\hat{\mathbf{r}} \cdot \mathbf{u}) [(\hat{\mathbf{r}} \cdot \mathbf{u}^*) \mathbf{e}_z - (\hat{\mathbf{r}} \cdot \mathbf{e}_z) \mathbf{u}^*] \} \right\rangle \right], \end{aligned} \quad (46)$$

with $q = n_1 \bar{r}$. With Equation (39), we find for the power per unit solid angle through a sphere with radius $\bar{r}_o$

$$\begin{aligned} \frac{d\mathcal{P}_{\times}}{d\Omega} = & \bar{r}_o \text{Re} \left[\mathcal{V}_{e,o} n_1 e^{i(q_o - q_o^* \cos \theta)} \left\langle \left(1 + \frac{i}{q_o} \right) [1 - (\hat{\mathbf{r}} \cdot \mathbf{u}^*)(\hat{\mathbf{r}} \cdot \mathbf{u})] \right. \right. \\ & \left. \left. + \frac{n_1^* \mu_1}{n_1 \mu_1^*} f(q_o) (\mathbf{e}_z \cdot \hat{\mathbf{r}}) \right\rangle \right], \end{aligned} \quad (47)$$

with $q_o = n_1 \bar{r}_o$. The term with $g(q)$ in Equation (46) does not contribute to $d\mathcal{P}_{\times}/d\Omega$. The total extinction power follows by integrating the right-hand side of Equation (47) over the unit sphere. We obtain

$$\begin{aligned} \mathcal{P}_{\times} = & \pi \bar{r}_o \text{Re} \left[\mathcal{V}_{e,o} n_1 e^{iq_o} \left\langle \left(1 + \frac{i}{q_o} \right) [\chi_0(q_o^*) + \chi_2(q_o^*)] \right. \right. \\ & \left. \left. + 2 \frac{n_1^* \mu_1}{n_1 \mu_1^*} f(q_o) \chi_1(q_o^*) \right\rangle \right] \end{aligned} \quad (48)$$

Here we introduced the auxiliary functions

$$\chi_n(z) = \int_{-1}^1 dt e^{-izt} t^n, \quad n = 0, 1, 2, \dots, \quad (49)$$

needed for $n = 0, 1$ and 2 , and evaluated at $z = q_o^* = n_1^* \bar{r}_o$. Some details about the functions $\chi_n(z)$ are given in Appendix B.

In order to highlight the effect of damping in the host material, let us consider the case without damping. Then we have $\varepsilon_1 > 0, \mu_1 > 0, n_1 > 0$ and q_o real. Equation (47) becomes

$$\mathcal{P}_{\times} = \pi q_o \text{Re} \left[\mathcal{V}_{e,o} e^{iq_o} \left\langle \left(1 + \frac{i}{q_o} \right) [\chi_0(q_o) + \chi_2(q_o)] + 2f(q_o) \chi_1(q_o) \right\rangle \right]. \quad (50)$$

The q_o dependence is $q_o \exp(iq_o) \langle \dots \rangle$. With the explicit expressions for the functions $\chi_n(z)$ in Appendix B, we find that this is

$$q_o e^{iq_o} \left\langle \left(1 + \frac{i}{q_o} \right) [\chi_0(q_o) + \chi_2(q_o)] + 2f(q_o) \chi_1(q_o) \right\rangle = 4i. \quad (51)$$

The extinction power from Equation (47) then simplifies to

$$\mathcal{P}_{\times} = -4\pi \text{Im} \mathcal{V}_{e,o}. \quad (52)$$

In particular, the dependence on $\bar{r}_o$ drops out, as expected when there is no absorption in the medium. When there is also no absorption in the particle, the electric polarizability $\mathcal{V}_e$ from Equation (25) lies on the polarizability circle [17], and we have

$$|\mathcal{V}_e|^2 = \frac{3}{2} \text{Im} \mathcal{V}_e. \quad (53)$$

In terms of $\mathcal{V}_{e,o}$ this becomes

$$n_1 \mu_1 |\mathcal{V}_{e,o}|^2 = \frac{3}{2} \text{Im} \mathcal{V}_{e,o}. \quad (54)$$

The scattered power from Equation (45) can then be written as

$$\mathcal{P}_{sc} = 4\pi \text{Im} \mathcal{V}_{e,o}, \quad (55)$$

and we see that $\mathcal{P}_x = -\mathcal{P}_{sc}$. Power is extracted from the laser beam, and this power is emitted by the particle as scattered radiation. It should be noted that the powers per unit solid angle, e.g., $d\mathcal{P}_x/d\Omega$ and $d\mathcal{P}_{sc}/d\Omega$ are not related in any simple way for the case of no absorption in the medium and the particle. This implies that extinction radiation is incident on the particle with a different angular distribution than under which it is scattered.

5. Linear Polarization

We would like to visualize the extinction power flow with a flow pattern of the Poynting vector. Such flow lines are in general 3D curves, which makes a visualization next-to-impossible. An exception is when the laser is linearly polarized and flow lines in a symmetry plane are considered. For the scattered radiation this gives the pattern shown in Figure 3. We shall now assume a linear polarization along the x axis, so that $\mathbf{u} = \mathbf{e}_x$. We set $\hat{\mathbf{r}} \cdot \mathbf{e}_x = \cos \beta$, with $0 \leq \beta \leq \pi$, so that β is the angle between the positive x axis and the direction $\hat{\mathbf{r}}$ of the field point. This is similar to $\hat{\mathbf{r}} \cdot \mathbf{e}_z = \cos \theta$ with θ the polar angle with the z axis. Expression (46) for $\sigma_x(\mathbf{r})$ becomes

$$\begin{aligned} \sigma_x(\mathbf{r}) = \frac{1}{r} \text{Re} \left[\mathcal{V}_{e,o} n_1 e^{i(q-q^* \cos \theta)} \left\langle \left(1 + \frac{i}{q} \right) [\hat{\mathbf{r}} - \mathbf{e}_x \cos \beta] \right. \right. \\ \left. \left. + \frac{n_1^* \mu_1}{n_1 \mu_1^*} \{f(q) \mathbf{e}_z - g(q) \cos \beta [\mathbf{e}_z \cos \beta - \mathbf{e}_x \cos \theta]\} \right\rangle \right]. \end{aligned} \quad (56)$$

This hardly seems like a simplification, but we note the following. Upon reflection in the zy plane, β goes over in $\pi - \beta$, and θ remains the same. We see from Equation (56) that the $\hat{\mathbf{r}}$ and $\mathbf{e}_z$ components remain the same, and that the $\mathbf{e}_x$ component picks up a minus sign. Vector $\hat{\mathbf{r}}$ follows the reflection. Therefore, $\sigma_x(\mathbf{r})$ is reflection-symmetric in the zy plane. For reflection of a field point in the zx plane, both β and θ remain the same. So the $\mathbf{e}_x, \mathbf{e}_z$ and $\hat{\mathbf{r}}$ components remain the same, but vector $\hat{\mathbf{r}}$ follows the reflection. Therefore, $\sigma_x(\mathbf{r})$ is reflection-symmetric in the zx plane.

6. Poynting Vector in the $\bar{z} \bar{x}$ Plane

Since $\sigma_x(\mathbf{r})$ is dimensionless, and will be drawn in a dimensionless coordinate system, we shall consider the $\bar{z} \bar{x}$ plane, rather than the zx plane. For a field point in the $\bar{z} \bar{x}$ plane, angle β can be eliminated in favor of angle θ with

$$\cos \beta = \text{sgn}(\bar{x}) \sin \theta. \quad (57)$$

We then have

$$\hat{\mathbf{r}} - \mathbf{e}_x \cos \beta = \mathbf{e}_z \cos \theta, \quad (58)$$

$$\cos \beta [\mathbf{e}_z \cos \beta - \mathbf{e}_x \cos \theta] = -\mathbf{e}_\theta \sin \theta. \quad (59)$$

Expression (56) becomes

$$\begin{aligned} \sigma_{\times}(\mathbf{r}) = \frac{1}{\bar{r}} \operatorname{Re} \left[\mathcal{V}_{e,\rho} n_1 e^{i(q-q^* \cos \theta)} \left\langle \left(1 + \frac{i}{q} \right) \mathbf{e}_z \cos \theta \right. \right. \\ \left. \left. + \frac{n_1^* \mu_1}{n_1 \mu_1^*} [f(q) \mathbf{e}_z + g(q) \mathbf{e}_\theta \sin \theta] \right\rangle \right]. \end{aligned} \quad (60)$$

Vector $\sigma_{\times}(\mathbf{r})$ lies in the $\bar{z}$ – $\bar{x}$ plane for every point in the $\bar{z}$ – $\bar{x}$ plane, and therefore field lines of $\sigma_{\times}(\mathbf{r})$ are 2D curves in the $\bar{z}$ – $\bar{x}$ plane. Vector $\sigma_{\times}(\mathbf{r})$ is reflection-symmetric in the $\bar{z}$ axis, and so is the field line pattern.

Let $\mathbf{r}(t)$ be a parametrization of a field line of $\sigma_{\times}(\mathbf{r})$, with t a dummy variable. In dimensionless coordinates, this is $k_o \mathbf{r}(t)$. At each point of a field line, $\sigma_{\times}(\mathbf{r})$ is on the tangent line, which can be expressed as

$$k_o \frac{d\mathbf{r}}{dt} = \sigma_{\times}(\mathbf{r}). \quad (61)$$

Given an initial point in the $\bar{z}$ – $\bar{x}$ plane, Equation (61) can be solved for the field line that goes through this point. In Cartesian coordinates $(\bar{z}, \bar{x})$, this equation reads

$$\frac{d\bar{x}}{dt} = \sigma_{\times}(\mathbf{r}) \cdot \mathbf{e}_x, \quad (62)$$

$$\frac{d\bar{y}}{dt} = \sigma_{\times}(\mathbf{r}) \cdot \mathbf{e}_y, \quad (63)$$

and in spherical coordinates $(\bar{r}, \theta)$ we have

$$\frac{d\bar{r}}{dt} = \sigma_{\times}(\mathbf{r}) \cdot \hat{\mathbf{r}}, \quad (64)$$

$$\bar{r} \frac{d\theta}{dt} = \sigma_{\times}(\mathbf{r}) \cdot \mathbf{e}_\theta. \quad (65)$$

With $\sigma_{\times}(\mathbf{r})$ given by Equation (60), it is obvious that solving for the field lines would in general require a numerical approach. Before continuing with the numerical solution for the field lines, let us first discuss some general properties of $\sigma_{\times}(\mathbf{r})$.

7. Poynting Vector on the $\bar{x}$ Axis

On the $\bar{x}$ axis, we have $\theta = \pi/2$, $\mathbf{e}_\theta = -\mathbf{e}_z$ and with $n_1 = n_o + i\gamma$, Equation (60) simplifies to

$$\sigma_{\times}(\mathbf{r}) = \mathbf{e}_z \frac{2}{\bar{r}^2} e^{-\gamma \bar{r}} \operatorname{Im} \left[\mathcal{V}_{e,\rho} e^{in_o \bar{r}} \frac{n_1^* \mu_1}{n_1 \mu_1^*} \left(1 + \frac{i}{n_1 \bar{r}} \right) \right], \quad (66)$$

with $\bar{r} = |\bar{x}|$. Here we have used Equations (29) and (30) to eliminate the appearance of the functions $f(q)$ and $g(q)$. Most notable is that $\sigma_{\times}(\mathbf{r})$ is proportional to $\mathbf{e}_z$, and therefore the Poynting vector is parallel to the $\bar{z}$ axis. The overall factor $\exp(-\gamma \bar{r})$ accounts for the attenuation due to damping with increasing $\bar{r}$. The crucial $\bar{r}$ dependence enters through the factor $\exp(in_o \bar{r})$. With increasing $\bar{r}$, this factor rotates over the unit circle in the complex plane. At some point, the factor in square brackets is rotated to the real axis, and this gives $\sigma_{\times}(\mathbf{r}) = 0$. A point where the Poynting vector is zero, its direction is undetermined, and this corresponds to a singular point of the flow field. When $\bar{r}$ increases more, the factor $\exp(in_o \bar{r})$ eventually rotates the expression over 180° , and $\sigma_{\times}(\mathbf{r})$ is zero again. Therefore, there are singular points on the $\bar{x}$ axis, and every next point is due to a rotation of $\exp(in_o \bar{r})$ over approximately 180° . The other $\bar{r}$ dependence is smooth compared to this rotation. Consequently, there are singular points on the $\bar{x}$ axis which are approximately

$$\Delta \bar{x} = \frac{\pi}{n_o} \quad (67)$$

apart. This holds for field points not too close to $\bar{x} = 0$, otherwise the $1/\bar{r}$ factors become large. In terms of x , this is half a wavelength, with the wavelength in the medium given by $\lambda = 2\pi/n_o k_o$.

For small $\bar{r}$, the Poynting vector is approximately

$$\sigma_{\times}(\mathbf{r}) \approx -\mathbf{e}_z \frac{2\kappa}{\bar{r}^3}, \quad (68)$$

in terms of the parameter

$$\kappa = -\text{Re} \left(\mathcal{V}_{e,o} \frac{n_1^*}{\epsilon_1 \mu_1^*} \right). \quad (69)$$

For $\kappa > 0$, the Poynting vector is in the negative $\bar{z}$ direction, and for $\kappa < 0$ it is in the positive $\bar{z}$ direction for $\bar{r}$ small.

The Poynting vector on the $\bar{x}$ axis is illustrated in Figure 4 for parameters given in the caption. For all figures we shall take $\mu_1 = \mu_p = 1$, and $\bar{R} = 0.1$ for the particle radius.

8. Poynting Vector on the $\bar{z}$ Axis

Let us first consider the negative $\bar{z}$ axis, for which we have $\sin \theta = 0$ and $\cos \theta = -1$. Then $q - q^* \cos \theta = 2n_o \bar{r}$. Equation (60) simplifies to

$$\sigma_{\times}(\mathbf{r}) = \mathbf{e}_z \frac{1}{\bar{r}} \text{Re} \left[\mathcal{V}_{e,o} n_1 e^{2in_o \bar{r}} \left\langle \frac{n_1^* \mu_1}{n_1 \mu_1^*} f(n_1 \bar{r}) - 1 - \frac{i}{n_1 \bar{r}} \right\rangle \right]. \quad (70)$$

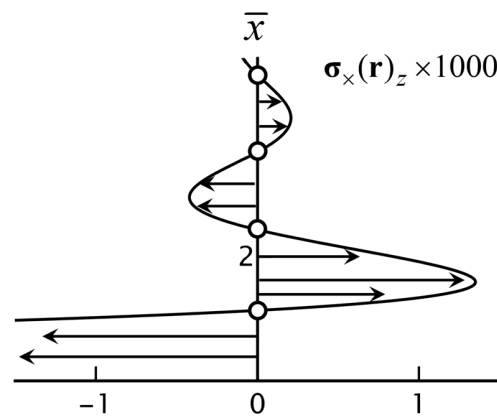


Figure 4. Shown are the Poynting vectors on the $\bar{x}$ axis. They point either in the positive or the negative $\bar{z}$ direction. The white circles are singular points, and the Poynting vector changes direction across each singular point. The parameters are $\epsilon_1 = 6.5 + 0.01i$ and $\epsilon_p = 2 + 0.3i$. The value of n_o is 2.55, and with Equation (67) this gives $\Delta \bar{x} = 1.23$ for the approximate distance between the singularities. The small value of $\sigma_{\times}(\mathbf{r})_z$ is due to the small value of the polarizability volume, which is $\mathcal{V}_{e,o} = (-1.89 + 0.163i) \times 10^{-3}$ for these parameters. We have $\kappa = 7.42 \times 10^{-4}$, which is positive. Therefore, for small $\bar{r}$, the Poynting vector is in the negative $\bar{z}$ direction.

As compared to Equation (66) for the $\bar{x}$ axis, the exponential damping factor is absent. The Poynting vector $\sigma_{\times}(\mathbf{r})$ is proportional to $\mathbf{e}_z$, and is therefore along the $\bar{z}$ axis. Here, the factor $\exp(2in_o \bar{r})$ rotates around the unit circle in the complex plane. Therefore, there are singular points on the negative $\bar{z}$ axis, and they are approximately spaced by

$$\Delta \bar{z} = \frac{\pi}{2n_o}, \quad (71)$$

following the same argument as in the previous section. The Poynting vector changes direction across a singularity. The spacing of the singular points on the $\bar{x}$ axis is approximately twice as large as the spacing on the $\bar{z}$ axis.

For small $\bar{r}$ we have

$$\sigma_{\times}(\mathbf{r}) \approx \mathbf{e}_z \frac{\kappa}{\bar{r}^3}. \quad (72)$$

For $\kappa > 0$, the Poynting vector is in the positive $\bar{z}$ direction, and for $\kappa < 0$ it is in the negative $\bar{z}$ direction. Interestingly, this is opposite to the direction of the Poynting vector on the $\bar{x}$ axis for small $\bar{x}$. The reason for that will become clear in Section 9. Figure 5 shows the directions of the Poynting vector on the negative $\bar{z}$ axis for the parameters given in the caption.

The situation on the positive $\bar{z}$ axis is entirely different. We have $\sin \theta = 0$ and $\cos \theta = 1$. Then $q - q^* \cos \theta = 2i\gamma\bar{r}$. Equation (60) becomes

$$\sigma_{\times}(\mathbf{r}) = \mathbf{e}_z \frac{1}{\bar{r}} e^{-2\gamma\bar{r}} \operatorname{Re} \left[\mathcal{V}_{e,o} n_1 \left\langle 1 + \frac{i}{n_1 \bar{r}} + \frac{n_1^* \mu_1}{n_1 \mu_1^*} f(n_1 \bar{r}) \right\rangle \right]. \quad (73)$$

The Poynting vector is in the positive or negative $\bar{z}$ direction. Here we have the exponential damping with $\exp(-2\gamma\bar{r})$, but the most important feature is that there is no rotating factor, as we had for the $\bar{x}$ axis and the negative $\bar{z}$ axis. Expression (73) can be further simplified by introducing the parameters.

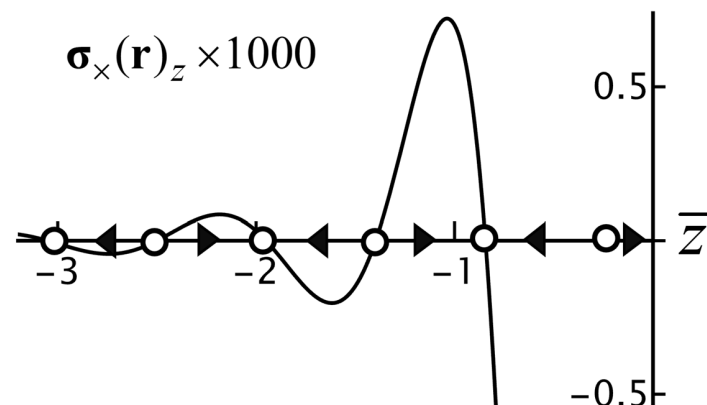


Figure 5. The curve represents the $\bar{z}$ component of the Poynting vector on the negative $\bar{z}$ axis. Close to the origin, the curve takes another swing in the upward direction, but this has been left out for clarity of the presentation. The intersections of the curve with the $\bar{z}$ axis are singular points, indicated by white circles. The singularity closest to the origin is located at $\bar{z} = -0.29$. The parameters are $\varepsilon_1 = 8 + 0.01i$ and $\varepsilon_p = 2 + 0.3i$. The value of n_o is 2.55, and with Equation (71) this gives $\Delta \bar{z} = 0.56$ for the approximate separation between the singularities. We have $\kappa = 9.07 \times 10^{-4}$, which is positive. With Equation (72) we then see that close to the origin the Poynting vector is into the positive $\bar{z}$ direction, as indicated by the arrow on the axis. Since the Poynting vector changes direction across each singularity, this determines the directions of the Poynting vector in between the singularities to the left.

$$\kappa' = -\operatorname{Im} \left[\mathcal{V}_{e,o} \left(1 + \frac{n_1^* \mu_1}{n_1 \mu_1^*} \right) \right], \quad (74)$$

$$\kappa'' = \operatorname{Re} \left[\mathcal{V}_{e,o} n_1 \left(1 + \frac{n_1^* \mu_1}{n_1 \mu_1^*} \right) \right], \quad (75)$$

and with κ from Equation (69) we then find

$$\sigma_{\times}(\mathbf{r}) = \mathbf{e}_z \frac{1}{\bar{z}} e^{-2\gamma\bar{z}} \left(\frac{\kappa}{\bar{z}^2} + \frac{\kappa'}{\bar{z}} + \kappa'' \right). \quad (76)$$

For $\bar{z}$ small, this becomes

$$\sigma_{\times}(\mathbf{r}) \approx \mathbf{e}_z \frac{\kappa}{\bar{z}^3}. \quad (77)$$

This Equation is identical to Equation (72) for the negative $\bar{z}$ axis, and the direction of the Poynting vector is again determined by the sign of κ . This implies that a field line which approaches the origin from the left, as in Figure 5, continues going to the right after passing through the origin (which is a singular point). The Poynting vector on the $\bar{z}$ axis changes direction across a singular point, except across the singularity at the origin. The positive and negative parts of the $\bar{z}$ axis are field lines of the Poynting vector, interrupted by singularities. This implies that a field line in the upper half of the $\bar{z}$ $\bar{x}$ plane stays in the upper half of the plane, because field lines cannot cross.

For $\bar{z}$ large, Equation (76) becomes

$$\sigma_{\times}(\mathbf{r}) \approx \mathbf{e}_z \frac{\kappa''}{\bar{z}} e^{-2\gamma\bar{z}}. \quad (78)$$

This shows that far away the direction of the Poynting vector is determined by the sign of κ'' . So, if κ and κ'' have opposite signs, there must be a singularity on the positive $\bar{z}$ axis.

A singularity of the Poynting vector is a point where $\sigma_{\times}(\mathbf{r}) = 0$. With Equation (76), it has to hold that

$$\kappa'' \bar{z}^2 + \kappa' \bar{z} + \kappa = 0, \quad (79)$$

for $\bar{z}$ to be a singularity. This quadratic equation has zero, one or two solutions on the positive $\bar{z}$ axis. By considering the behavior of the Poynting vector for small and large $\bar{z}$, we conclude that when κ and κ'' are of opposite sign, there has to be one, and only one, singularity on the positive $\bar{z}$ axis. When κ and κ'' are of equal sign, there can be zero or two singularities on the positive $\bar{z}$ axis.

Let us analyze this in more detail. The solutions of Equation (79) are

$$\bar{z}_{\pm} = \frac{1}{2\kappa''} \left[-\kappa' \pm \sqrt{(\kappa')^2 - 4\kappa\kappa'} \right]. \quad (80)$$

These solutions satisfy

$$\bar{z}_+ \bar{z}_- = \frac{\kappa}{\kappa''}. \quad (81)$$

When κ and κ'' are of opposite sign, then $(\kappa')^2 - 4\kappa\kappa'$ is positive, and so there are two real solutions. With Equation (81) we see that $\bar{z}_+ \bar{z}_-$ is negative, and therefore there is one positive and one negative solution. With $\kappa\kappa'' < 0$, we have $\left\{ (\kappa')^2 - 4\kappa\kappa' \right\}^{1/2} > |\kappa'|$. It then follows that $\text{sgn}[-\kappa' \pm \left\{ (\kappa')^2 - 4\kappa\kappa' \right\}^{1/2}] = \pm 1$. So we find that $\text{sgn}(\bar{z}_{\pm}) = \pm \text{sgn}(\kappa'')$, or $\text{sgn}(\bar{z}_{\pm}) = \mp \text{sgn}(\kappa)$. Therefore, for $\kappa < 0$, the positive solution is $\bar{z}_+$, and for $\kappa > 0$ the $\bar{z}_-$ is the positive solution.

When κ and κ'' are of equal sign, then for $4\kappa\kappa'' > (\kappa')^2$ both $\bar{z}_+$ and $\bar{z}_-$ are complex, and each other's complex conjugate. For $4\kappa\kappa'' < (\kappa')^2$, we have $0 < \left\{ (\kappa')^2 - 4\kappa\kappa' \right\}^{1/2} < |\kappa'|$, and there are two real solutions. Then $\text{sgn}[-\kappa' \pm \left\{ (\kappa')^2 - 4\kappa\kappa' \right\}^{1/2}] = -\text{sgn}(\kappa')$, and $\text{sgn}(\bar{z}_{\pm}) = -\text{sgn}(\kappa)\text{sgn}(\kappa')$. So, for κ and κ' of opposite sign, both $\bar{z}_+$ and $\bar{z}_-$ are positive, and so we have two solutions on the positive $\bar{z}$ axis. For κ and κ' of the same sign, both $\bar{z}_+$ and $\bar{z}_-$ are negative, and there are no solutions on the positive $\bar{z}$ axis.

Figure 6 shows the $\bar{z}$ component of $\sigma_{\times}(\mathbf{r})$ for a case when there is one solution on the positive $\bar{z}$ axis. Another example is $\varepsilon_1 = 2 + 3.5i$, $\varepsilon_p = -6 + 2.9i$, for which all κ' s are negative. We find that $(\kappa')^2 < 4\kappa\kappa''$, and so both $\bar{z}_+$ and $\bar{z}_-$ are complex. For $\varepsilon_1 = 2 + 2.9 \times i$, $\varepsilon_p = 6$, all κ' s are positive, and we have $(\kappa')^2 > 4\kappa\kappa''$. Since the signs of κ

and κ' are the same, there are no solutions on the positive $\bar{z}$ axis. For most host materials, the damping is relatively small. Let us consider the case $\varepsilon_1 > 0$, $\mu_1 > 0$, for which n_1 is positive. We then have $\kappa = -(\text{Re}\mathcal{V}_{e,o})/n_1$, $\kappa' = -2\text{Im}\mathcal{V}_{e,o}$ and $\kappa'' = -2n_1^2\kappa$. The signs of κ and κ'' are opposite, so there is one solution on the positive $\bar{z}$ axis. For $\kappa > 0$ this is $\bar{z}_-$ and for $\kappa < 0$ this is $\bar{z}_+$. This shows that the example shown in Figure 7 is the most common. If we further assume that the particle is small, and that there is no dissipation in the particle ($\varepsilon_p > 0$), then it follows from Equation (26) that $\mathcal{V}_{e,o}$ is real. Then $\kappa' \approx 0$, and we have $\bar{z}_{\pm} \approx \mp \text{sgn}(\kappa)/n_1\sqrt{2}$. Therefore, the singularity is approximately located at

$$\bar{z} = \frac{1}{n_1\sqrt{2}}. \quad (82)$$

For the parameters of Figure 6, this gives $\bar{z} = 0.35$.

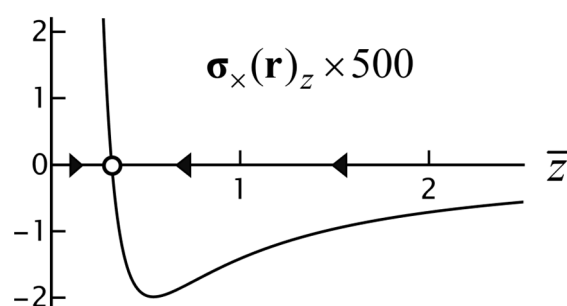


Figure 6. The curve represents the $\bar{z}$ component of the Poynting vector on the positive $\bar{z}$ axis. The parameters are $\varepsilon_1 = 4 + 0.1i$ and $\varepsilon_p = 2 + 0.3i$. For these values we have $\kappa > 0$, $\kappa' < 0$ and $\kappa'' < 0$. So, the field line comes out of the origin and out of infinity, and they meet at the singular point. Since $\kappa'' < 0$, the positive solution is $\bar{z}_-$, which is $\bar{z}_- = 0.32$, and $\bar{z}_+$ is negative. Since the above only holds for $|\bar{z}| > \bar{R}$, the diverging behavior for $\bar{z} \rightarrow 0$ is of no concern.

9. Small $\bar{r}$

Equation (61) for the field lines, with $\sigma_{\times}(\mathbf{r})$ given by Equation (60), is too complicated to solve analytically, so numerical methods will be used. An interesting exception is for field lines in the neighborhood of the dipole. For $\bar{r}$ small, Equation (60) simplifies to

$$\sigma_{\times}(\mathbf{r}) = \frac{\kappa}{\bar{r}^3}(\mathbf{e}_z + 3\mathbf{e}_{\theta} \sin \theta). \quad (83)$$

In this section, we shall write an equal sign, instead of $\approx$. The only parameter dependence comes in through κ . On the $\bar{x}$ axis, we have $\sin \theta = 1$ and $\mathbf{e}_{\theta} = -\mathbf{e}_z$, and so Equation (83) is the same as Equation (68). For $\kappa > 0$, the Poynting vector is in the negative $\bar{z}$ direction, and for $\kappa < 0$, the Poynting vector is in the positive $\bar{z}$ direction. On the $\bar{z}$ axis, we have $\sin \theta = 0$, and Equation (83) is the same as Equations (72) and (77). For $\kappa > 0$, the Poynting vector is in the positive $\bar{z}$ direction, and for $\kappa < 0$, the Poynting vector is in the negative $\bar{z}$ direction.

We shall use spherical coordinates $(\bar{r}, \theta)$, for which the field line equations are given by Equations (64) and (65). Upon eliminating t , we get

$$\frac{1}{\bar{r}} \frac{d\bar{r}}{d\theta} = \frac{\sigma_{\times} \cdot \hat{\mathbf{r}}}{\sigma_{\times} \cdot \mathbf{e}_{\theta}}. \quad (84)$$

From Equation (83) we have $\sigma_{\times} \cdot \hat{\mathbf{r}} = (\kappa/\bar{r}^3) \cos \theta$ and $\sigma_{\times} \cdot \mathbf{e}_{\theta} = 2(\kappa/\bar{r}^3) \sin \theta$, and Equation (84) becomes

$$\frac{2}{\bar{r}} \frac{d\bar{r}}{d\theta} = \cot \theta. \quad (85)$$

This Equation is separable, and the solution is

$$\bar{r} = C\sqrt{\sin \theta}, \quad 0 \leq \theta \leq \pi. \quad (86)$$

For $\theta = 0$, we have $\bar{r} = 0$, so this is the origin of coordinates. When θ increases, so does $\bar{r}$, until $\theta = \pi/2$. Then $\bar{r}$ has its maximum value, and since $\theta = \pi/2$, this represents a point on the $\bar{x}$ axis. Let us indicate this $\bar{x}$ value by $\bar{x}_0$. Then the integration constant is $C = |\bar{x}_0|$. So, the parameter representation of this field line is

$$\bar{r} = |\bar{x}_0|\sqrt{\sin \theta}, \quad 0 \leq \theta \leq \pi. \quad (87)$$

For $\theta \rightarrow \pi$, we have $\bar{r} \rightarrow 0$, which is the origin of coordinates. Consequently, these field lines form loops. We call these ‘field lines of the second kind’, and an example is shown in Figure 7. It is interesting to notice that the solution (87) is independent of any of the parameters in the problem, and as such these are universal curves. The directions of these field lines cannot be seen from Equation (87), but we know from the arguments in the beginning of this section that for $\kappa > 0$ the rotation is counterclockwise and for $\kappa < 0$ it is clockwise in the upper half of the plane. Due to the reflection symmetry in the $\bar{z}$ axis, the rotation direction is reversed in the lower half of the plane. One field line goes through a given point $(\bar{z}_0, \bar{x}_0)$ in the $\bar{z}\bar{x}$ plane. The approximate field line from Equation (87) is the field line through the point $(0, \bar{x}_0)$. The exact field line through this point is also shown in Figure 7.

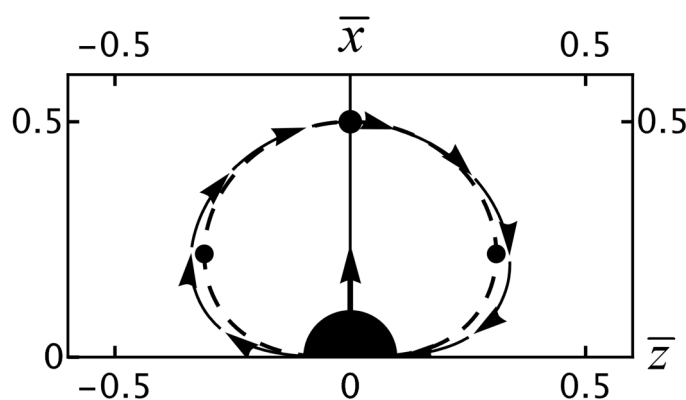


Figure 7. The curve with the arrows is the exact field line through the point $\bar{x}_0 = 0.5$ on the $\bar{x}$ axis, and the dashed curve is the approximation for small $\bar{r}$ through the same point. The black disk represents, to scale, the particle with radius $\bar{R} = 0.1$. The extremes on the curve are located at $\bar{z}_m = \pm 0.31, \bar{x}_m = 0.22$.

The Cartesian coordinates of a point on a field line are

$$\bar{z} = \bar{r} \cos \theta, \quad (88)$$

$$\bar{x} = \text{sgn}(\bar{x})\bar{r} \sin \theta, \quad (89)$$

with $\bar{r}$ given by Equation (87). We also have $\text{sgn}(\bar{x}) = \text{sgn}(\bar{x}_0)$, because a field line cannot cross the $\bar{z}$ axis. So we have

$$\bar{z} = |\bar{x}_0| \cos \theta \sqrt{\sin \theta}, \quad (90)$$

$$\bar{x} = \bar{x}_0 (\sin \theta)^{3/2}. \quad (91)$$

On the curve, $\bar{z}$ has two extrema, $\bar{z}_m$. These occur for angles θ_m , given by $\sin \theta_m = 1/\sqrt{3}$, and we find

$$|\bar{z}_m| = |\bar{x}_0| 3^{-3/4} \sqrt{2} = 0.620 |\bar{x}_0|. \quad (92)$$

The corresponding $\bar{x}$ coordinate is

$$\bar{x}_m = \bar{x}_0 3^{-3/4} = 0.439 \bar{x}_0. \quad (93)$$

These extrema are indicated in Figure 7 by two black dots. For Figures 7–17 we shall take the parameters to be $\varepsilon_1 = 1, \varepsilon_p = 4$, together with the already assumed values of $\mu_1 = 1, \mu_p = 1$ and $\bar{R} = 0.1$. The flow line pattern of $\sigma_{\times}(\mathbf{r})$ in the near field is illustrated in Figure 8. This corresponds to a small dielectric particle in free space, with a radius of about 3% of an optical wavelength in vacuum. It should be noted that the dependence on the radius of the particle on the power flow lines is minimal, as long as the particle radius is relatively small compared to a wavelength. We see from Equation (26) that for $\bar{R}$ relatively small, the $\bar{R}$ dependence only gives an overall factor of $\bar{R}^3$. Since field lines are determined by the direction of $\sigma_{\times}(\mathbf{r})$ at each field point, and not its magnitude, this $\bar{R}$ dependence drops out. However, our results for $\sigma_{\times}(\mathbf{r})$ only hold for the Poynting vector outside the particle, so for $\bar{r} > \bar{R}$, and therefore structures close to the dipole, as in Figure 8, could be covered up (by the black disk, representing the particle). For the chosen parameters we have $\kappa = -5.0 \times 10^{-4}$, so the rotations in the near field are clockwise in the upper half of the $\bar{z}$ $\bar{x}$ plane and counterclockwise in the lower half of the $\bar{z}$ $\bar{x}$ plane. We shall consider the effect of damping in the embedding medium in Section 12.

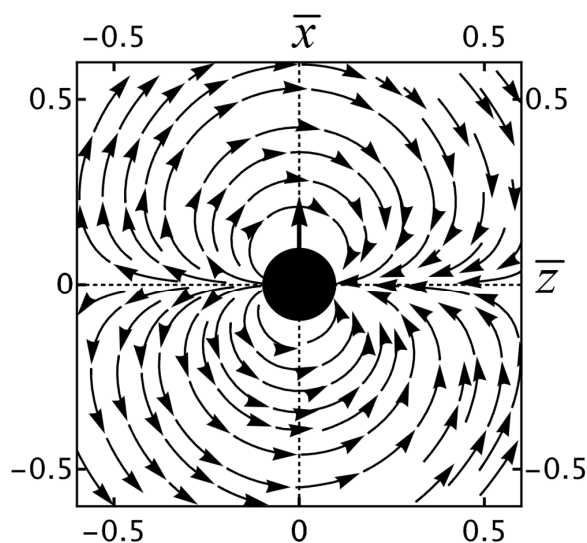


Figure 8. The figure shows the flow pattern close to the dipole. Since κ is negative, the circulation is clockwise for $\bar{x} > 0$ and counterclockwise for $\bar{x} < 0$.

With an increasing value of $\bar{x}_0$, the loops get bigger, and eventually the small- $\bar{r}$ approximation breaks down. This is illustrated in Figure 9. Outside the region of the loops, field lines run away, and these are the ‘field lines of the first kind’. The dashed line is the transition field line. As can be seen from the figure, this field line ends at the singularity $\bar{z}_+$ on the positive $\bar{z}$ axis. At the singularity, the field lines split. Just on the left, field lines run to the left, as part of a loop, and just to the right, they run away into the positive $\bar{z}$ direction. There are other possibilities. For instance, if the singularity on the negative $\bar{z}$ axis would be closer to the dipole than the singularity on the positive $\bar{z}$ axis, the loops would stop at that singularity. Another possibility is a case for which κ and κ'' are both positive. Then there may not be a singularity on the positive $\bar{z}$ axis. Then a singularity on the negative $\bar{z}$ axis

would have to determine the borderline between field lines of different kinds. Yet another possibility is that a singularity on the positive $\bar{x}$ axis is the closest. Then this singularity will determine the extent of the loops.

10. Field Lines

Now that we have discussed some general features of the extinction flow pattern, let us look at the bigger picture. Figure 10 shows the flow pattern of $\sigma_{\times}(\mathbf{r})$ from a wider view. These figures are made by numerically solving Equations (62) and (63). The singularity on the positive $\bar{z}$ axis is located at $\bar{z}_+ = 0.71$. We have $\kappa'' = 1.0 \times 10^{-3}$, which is positive, so the field lines run to the right near the positive $\bar{z}$ axis. If we would extend the picture further to the right, we would see that these field lines keep on going. Near the dipole, we see the small- $\bar{r}$ loops, and since κ is negative, the rotation is clockwise in the upper half of the $\bar{z}\bar{x}$ plane and counterclockwise in the lower half of the plane. On the negative $\bar{z}$ axis we see a row of singularities, which are approximately $\Delta\bar{z} = 1.57$ apart. Field lines that approach these singularities split at these singularities. We also notice four singularities off the $\bar{z}$ axis. These singularities are vortices. Vortices in laser beams have a variety of applications [24–26], although these vortices are of a different kind. An enlargement of the field line pattern near the left-top singularity is shown in Figure 11. The singularity is located at approximately $(-3.4, 1.6)$. Field lines approach the singularity, swing around it in a clockwise direction a few times, and then end at the singularity, which is the center of the vortex. Figure 12 shows a close-up of the second off-axis singularity in the upper half of the $\bar{z}\bar{x}$ plane. The location of this singularity is approximately $(-1.4, 1.3)$. Here, the field lines come out of the center, and rotate counterclockwise. Looking at Figure 10 reveals that some of the field lines that rotate out of this singularity join the vortex to the left of it whereas others run away to the right. If we would extend Figure 10 to the left, the pattern with the on- and off-axis singularities repeats itself.

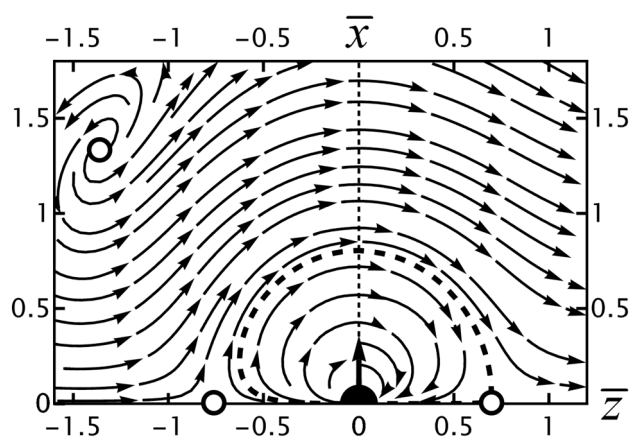


Figure 9. The figure shows the energy flow lines from a wider view than in Figure 9. The dashed curve is the boundary field line between field lines of the first and the second kind. This is the field line that ends at the singularity $\bar{z}_+ = 0.71$ on the positive z axis. The value of κ'' is positive, so far to the right the field lines run into the positive z direction. Two other singularities can also be seen in this picture.

The singularities on the $\bar{x}$ axis cannot be seen on the scale of Figure 10. Figure 13 shows an extension of Figure 10 in the upward direction, and we notice the appearance of a singularity on the $\bar{x}$ axis. The singularities on this axis are twice as far apart as on the $\bar{z}$ axis. We see from the flow pattern in the figure that the singularity on the $\bar{x}$ axis is of the type where field lines split, rather than being the center of a vortex. Figure 14 is a continuation of the right-top corner of Figure 10. Close to the positive $\bar{z}$ axis, the field

lines keep on running to the right, but higher up in the graph, we see field lines coming in from far away. Somewhere midway in the graph they turn around, and then join the outgoing field lines near the $\bar{z}$ axis. Also, at the very top of the figure, we see field lines turning around. If we would extend the graph further outward, we would see that this alternating between incoming and outgoing field lines continues. The origin of this peculiar feature can be found in Equation (60). Consider the factor $\exp(-iq^* \cos \theta)$. For a fixed $\bar{r}, q$ is constant. If we vary θ , then this phase factor rotates in the complex plane, and for a relative small change in θ , this factor can change considerably when $\bar{r}$ is large. If it rotates over about 180° , it picks up a minus sign. This changes the direction of the Poynting vector from incoming to outgoing, or from outgoing to incoming. This alternating also happens in the top-left in Figure 10.

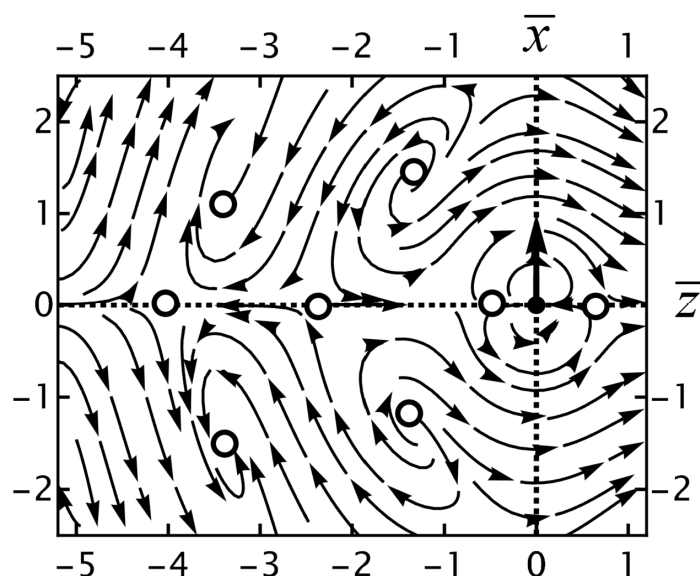


Figure 10. The figure illustrates the energy flow lines of $\sigma_{\times}(\mathbf{r})$ from a wider view than in Figure 9. The off-axis singularities are vortices.

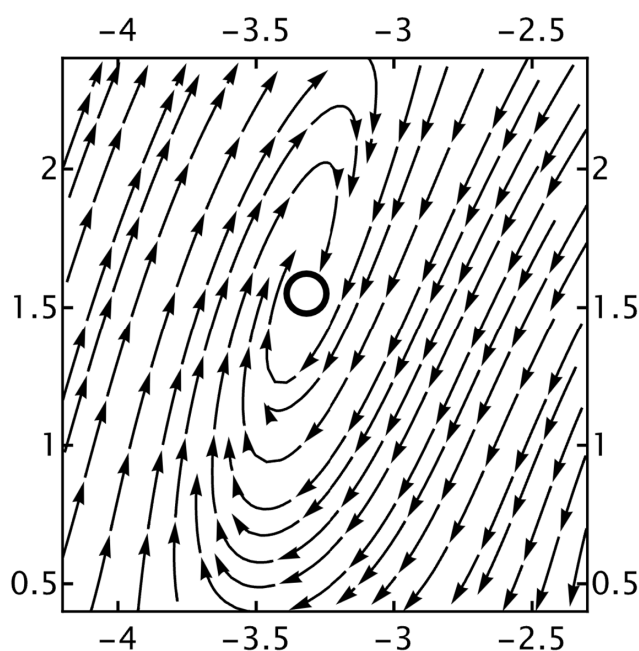


Figure 11. The figure shows an enlargement of the left-top singularity in Figure 10. Here we see clearly that this singularity is a vortex, and that the field lines run towards the singularity. The rotation around the center is clockwise.

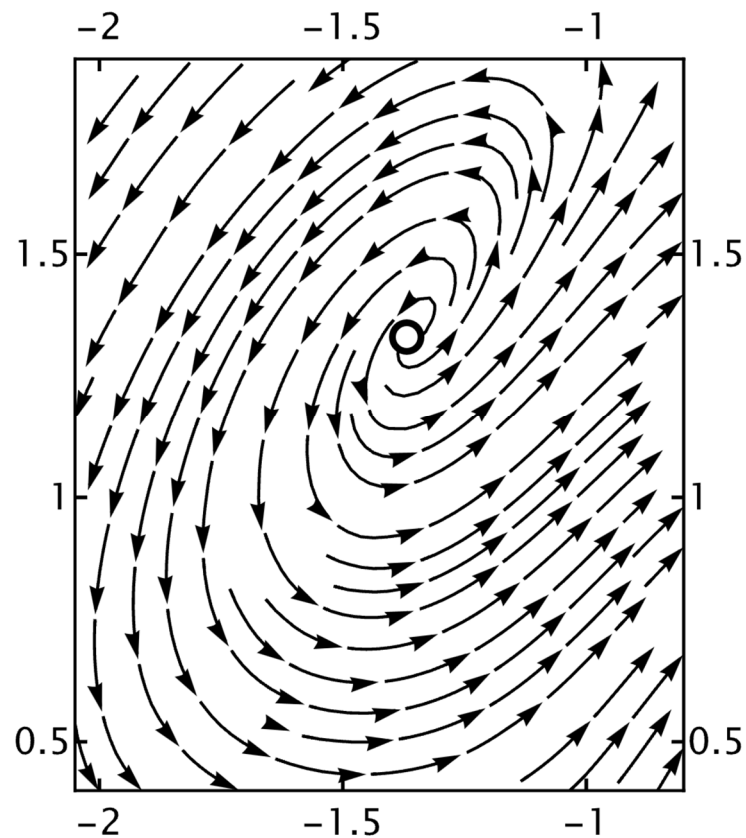


Figure 12. The figure shows an enlargement of the second off-axis singularity in Figure 10. For this singularity, the field lines emanate from its center and rotate counterclockwise.

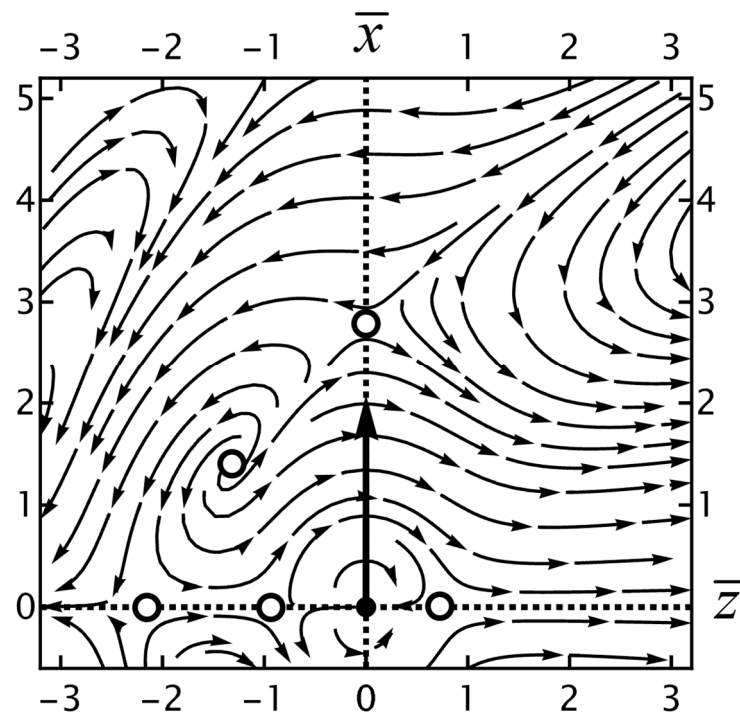


Figure 13. Depicted is the upward extension of Figure 10.

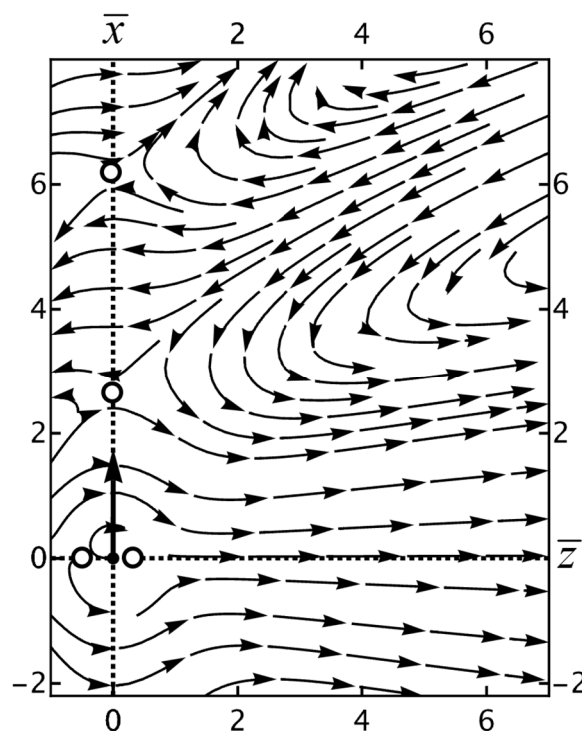


Figure 14. The graph shows an extension of Figure 10, up and to the right. The figure illustrates the alternating incoming and outgoing flowlines.

11. Singularities

As illustrated in the figures above, the locations of the singularities play an important role in the formation of the flow patterns. An interesting way to locate these singularities is shown in Figure 15. For points on the solid contour we have $\sigma_{\times}(\mathbf{r})_z = 0$, so the Poynting vector is in the $\bar{x}$ direction. For points on the dashed contours (here only the axes), the $\bar{x}$ component is zero. Therefore, at intersections between the two types of contours we have a singularity, where $\sigma_{\times}(\mathbf{r}) = 0$. The $\bar{z}$ and the $\bar{x}$ axis are always dashed, because the Poynting vector is in the $\bar{z}$ direction on both axes. Figure 16 shows a wider view of the contours of Figure 15. This corresponds to the flow pattern of Figure 10. Here, the locations of the singularities can be seen much clearer than in Figure 10. Figure 17 shows an even wider view than in Figure 16. The singularities on the $\bar{x}$ axis can be seen on this scale, and they are approximately $\Delta \bar{x} = 3.14$ apart. We also notice that there are no other singularities in the second quadrant, other than the ones showing in Figure 16. The solid and the dashed curves do not intersect in this region (and also not in the first quadrant, not shown here). When we look at the field line on the left in Figure 17, we see that when going up, it hits the dashed curve, so the Poynting vector there is horizontally. Right after that it hits the solid curve, and there the Poynting vector is vertically. In other words, the direction of the Poynting vector suddenly rotates over 90° , and the field line changes from outgoing to incoming.

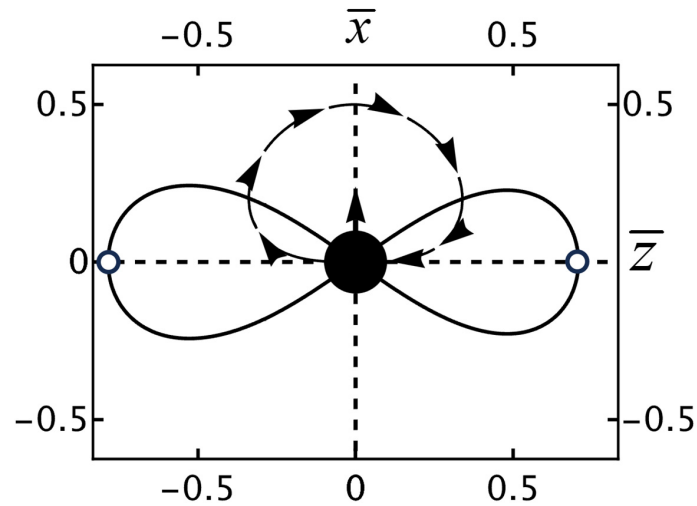


Figure 15. The field line shown here is the same as in Figure 7. The singularity on the positive $\bar{z}$ axis is $\bar{z}_+ = 0.707$. At the two intersections between the field line and the solid curve, the Poynting vector is in the $\bar{x}$ direction. On the left it is going up and on the right it is going down.

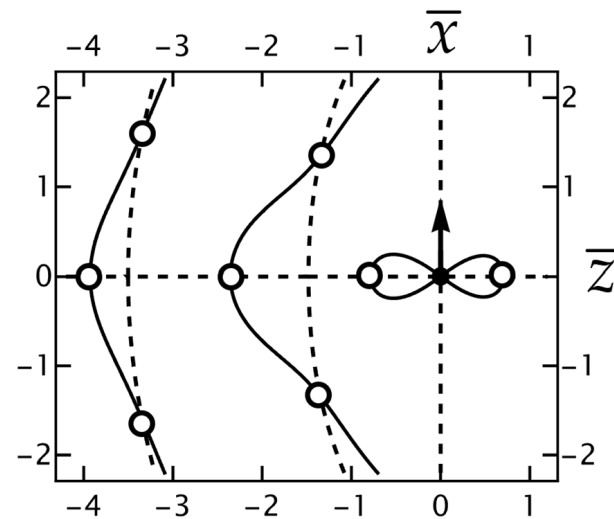


Figure 16. The graph shows a wider view of the curves of Figure 15. Each white circle is a singularity.

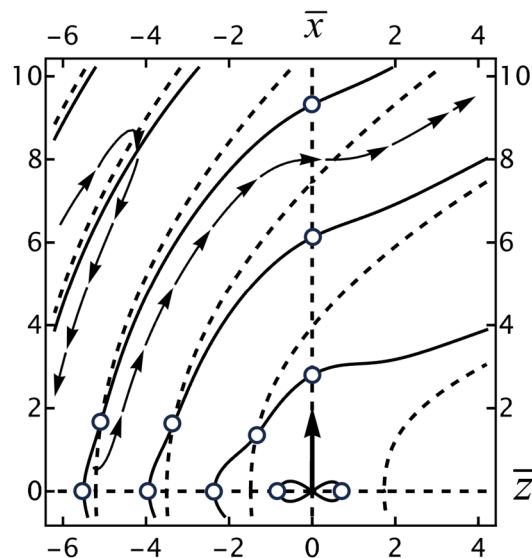


Figure 17. Shown is a wider view of the contour curves of Figure 16. The field line on the left illustrates how field lines can change from outgoing to incoming.

12. Damping in the Host Medium

Let us now consider the effect of absorption in the host medium. Figure 18 shows the flow diagram for $\varepsilon_1 = 2 + 2.9i$, $\varepsilon_p = 6$, and Figure 19 shows the contour lines where $\sigma_{\times}(\mathbf{r})$ is vertical (—) and horizontal (---). As compared the Figures 10 and 16, we notice that the vortices just off the negative $\bar{z}$ axis have disappeared. Some field lines now start at a singularity on the negative $\bar{z}$ axis. They go up, swing around, and end up at an adjacent singularity on the negative $\bar{z}$ axis. Interestingly, a new singularity appears in the first quadrant. As can be seen from the figure, field lines run towards the singularity and end there. It is a vortex singularity. Close to the dipole, there are three solid-curve loops, whereas without damping there are two, as in Figures 15–17, and they are more or less symmetric in left and right. Precise details about the flow pattern near the dipole cannot be resolved on the scale of Figures 18 and 19.

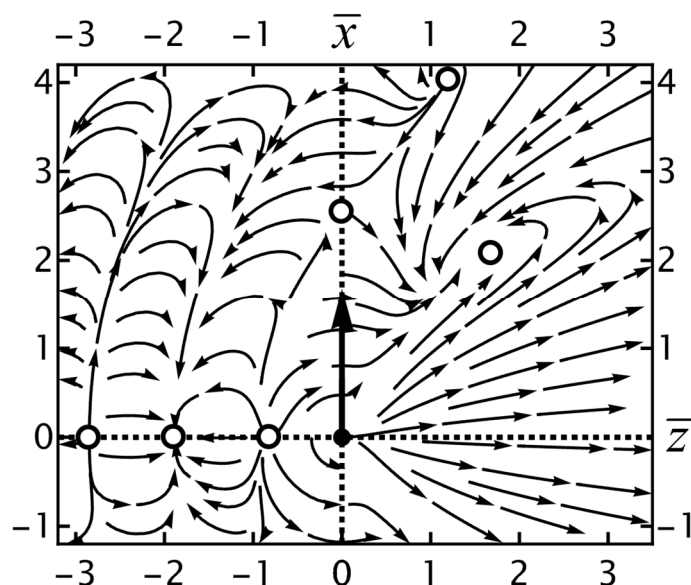


Figure 18. Power flow distribution for $\varepsilon_1 = 2 + 2.9i$ and $\varepsilon_p = 6$. Singularities close to the particle have been suppressed.

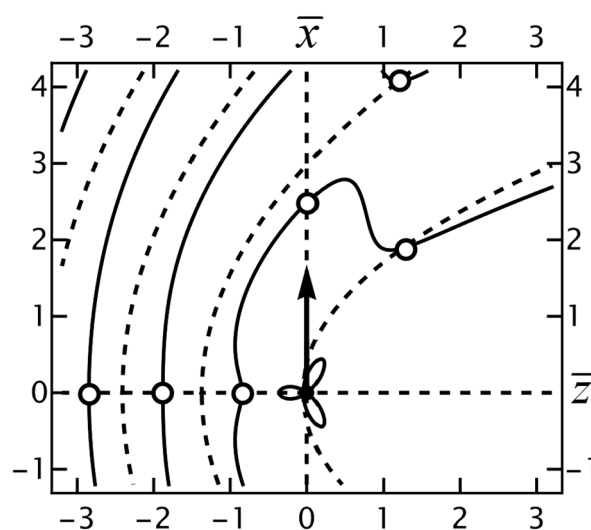


Figure 19. Contour curves for the power flow of Figure 18.

Figure 20 shows an enlargement of the field lines near the particle. Also shown is an expected small $\bar{r}$ approximation, with $\bar{x}_0 = 0.3$. We notice the appearance of a singularity on the $\bar{x}$ axis, at about $\bar{x} = 0.23$, which lies inside the small $\bar{r}$ curve. As explained in Section 9,

the range of the small $\bar{r}$ approximation is determined by the closest singularity, so there can be no singularity inside this curve. The dashed field line in Figure 9 is the ‘largest’ for which the small $\bar{r}$ approximation can reasonably be expected to be accurate. For the parameters of the figure, both $\bar{z}_+$ and $\bar{z}_-$ are negative, so there is no singularity on the positive $\bar{z}$ axis. The singularity on the negative $\bar{z}$ axis is located at about $\bar{z} = -0.35$. The closest singularity here seems to be the one on the $\bar{x}$ axis. Furthermore, the orientation of the small- $\bar{r}$ loops near the dipole are determined by the sign of κ . For the parameters for the figure, we have $\kappa = 5.8 \times 10^{-5}$, which is positive. Therefore, we expect the rotation to be counterclockwise, which is clearly not the case.

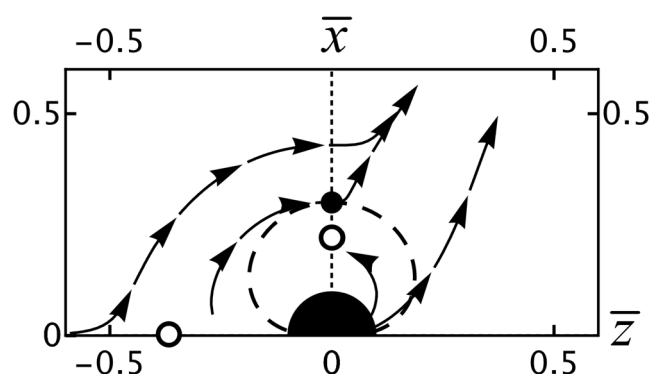


Figure 20. Enlargement of some of the field lines from Figure 18 near the particle. The dashed curve is an expected small $\bar{r}$ approximation that goes through the point $(0, 0.3)$, and the field line in the middle goes through this same point.

Figure 21 shows a closer look at the contour curves from Figure 19 near the origin. The singularities on the negative $\bar{z}$ axis, at $\bar{z} = -0.35$, and on the positive $\bar{x}$ axis, at $\bar{x} = 0.23$, are the same as in Figure 20. Here we see another singularity close to the particle in the upper half of the plane. It is indicated by an arrow, and it is located at about $(-0.042, 0.055)$. We also notice an even smaller loop very close to the dipole.

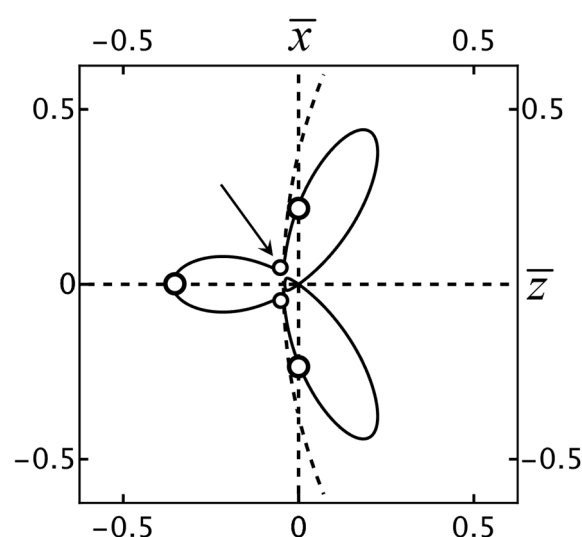


Figure 21. Enlargement of the contours from Figure 19 near the dipole.

Yet another refinement is depicted in Figure 22. The singularity in the top-left corner is the same as the singularity indicated by the arrow in Figure 21. A new singularity appears on the negative $\bar{z}$ axis, at $\bar{z} = -0.039$. This one is clearly the closest to the dipole, so the range of the small $\bar{r}$ approximation is determined by this singularity. On a solid curve close to the dipole, $\sigma_{\times}(\mathbf{r})$ is in the $\bar{x}$ direction, and the small $\bar{r}$ loops can be constructed in

a similar way as in Figure 15. In order for these loops to be actually present, the particle needs to be small compared to the scale in Figure 22.

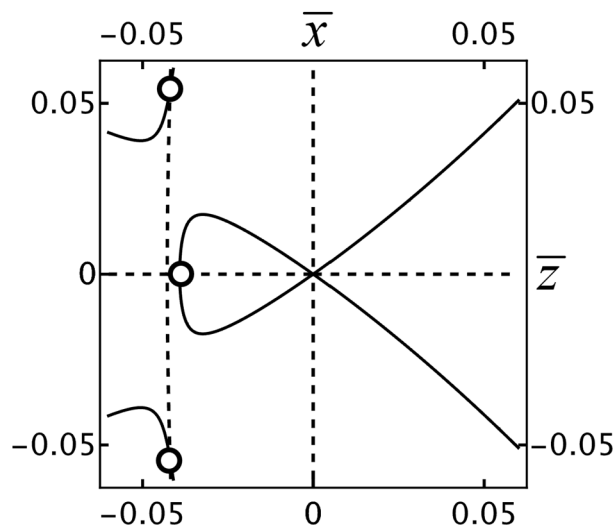


Figure 22. Shown is a closer view of the contour curves from Figure 21. The singularity on the negative $\bar{z}$ axis is located at $\bar{z} = -0.039$, which was measured from the graph.

Apparently, the damping in the medium generates singularities on the negative $\bar{z}$ axis very close to the particle. Let us consider this in more detail. Equation (70) gives the exact $\sigma_{\times}(\mathbf{r})$ on the negative $\bar{z}$ axis, and for $\bar{z}$ small, we have the approximate expression (72). Let us now expand the right-hand side of Equation (70) for $\bar{r}$ small, and keep one more term. We set $\exp(2in_0\bar{r}) = 1 + 2in_0\bar{r} \dots$, and group the terms. This yields

$$\sigma_{\times}(\mathbf{r}) \approx \mathbf{e}_z \frac{1}{\bar{r}^3} (\kappa - \kappa_s \bar{r}), \quad (94)$$

with

$$\kappa_s = -\operatorname{Im} \left[\mathcal{V}_{e,o} \left(1 + \frac{\varepsilon_1^*}{\varepsilon_1} \right) \right], \quad (95)$$

and we have $\bar{z} = -\bar{r}$. Therefore, a singularity will approximately appear at

$$\bar{z}_s = -\frac{\kappa}{\kappa_s}, \quad (96)$$

provided that this $\bar{z}_s$ is small and negative. For the parameters of Figure 22 we have $\kappa = 5.81 \times 10^{-5}$ and $\kappa_s = 1.58 \times 10^{-3}$, so $\bar{z}_s = -0.037$. This is very close to the value of $\bar{z} = -0.039$ that was measured from the graph.

When there is no damping in the medium, we have $\kappa = -(\operatorname{Re} \mathcal{V}_{e,o})/n_1$ (Equation(69)) and $\kappa_s = -2\operatorname{Im} \mathcal{V}_{e,o}$. When there is also no damping in the particle, we have $\operatorname{Im} \mathcal{V}_{e,o} \approx 0$ for a small particle, and so $\bar{z}_s$ is large, and therefore it is not a solution. For instance, for $\varepsilon_1 = 2$ and $\varepsilon_p = 6$ we have $\bar{z}_s = -468$. In other words, the singularity close to the dipole on the negative $\bar{z}$ axis is a result of the absorption in the embedding medium.

13. Conclusions

When a small particle is irradiated by a laser beam, an oscillating electric dipole moment is induced in the particle. The power flow in this system has three distinct contributions. First, we have the laser light, which propagates along straight lines. Then, the oscillating dipole moment emits radiation, which is, from a slightly different point of view, the scattered light. The flow pattern of the scattered power depends on the laser

polarization. For linearly-polarized incident light, the field lines are curved near the particle, and level off to straight lines in the far field, as illustrated in Figure 3. The more elusive extinction power flow is due to interference between the laser field and the scattered radiation. It provides the mechanism by which laser power is transferred to the particle. The energy that ends up at the particle is partially reflected as scattered radiation and partially absorbed by the particle, leading to heating.

We have studied the details of the extinction power flow for the case of linearly-polarized incident light. Power flows along the field lines of the Poynting vector. These flow lines are in general 3D curves, which are very difficult to visualize. However, in the plane spanned by the laser beam propagation direction ($\bar{z}$ axis) and the direction of the dipole moment ($\bar{x}$ axis), any flow line through a point in this plane stays in this plane. It turns out that the extinction flow pattern is far more intricate than the flow pattern of the scattered radiation. A particular interesting feature is the appearance of loops in the optical near field. These are closed loops which start at the dipole and end at the dipole. We have derived an approximate analytical expression for these loops, and we have shown that the extent of these loops is determined by the distance to the closest singularity in the radiation field. The sign of the parameter κ (Equation (69)) determines the rotation direction. As for the far field, it turns out that field lines can run to the far field, as for scattered radiation, but they can also come in from the far field. This phenomenon was traced back to a fast-oscillating interference factor in the expression for the Poynting vector.

Another feature, not present in scattered radiation, is the appearance of singularities in the flow field. At a singularity, the Poynting vector vanishes, which leaves its direction undetermined. Singularities come in two types. At some singularities, a field line bundle splits when coming close to the point, and other singularities are centers of a vortex. Near a vortex, the power swirls around the center, and then either ends up at the center, or the radiation emanates from the center, and then swirls outward. When swirling outward, the flow lines can run off to the far field, or they can join a neighboring vortex at which the power is swirling inwards.

At a singularity on a coordinate axis, field lines split. This means that the Poynting vector changes direction across a singularity. Close to the dipole, and on an axis, the direction of the Poynting vector is determined by the sign of κ , and from this the direction of the Poynting vector in between singularities on an axis follows. The $\bar{z}$ axis is a field line, interrupted by singularities, and on the $\bar{x}$ axis the Poynting vector is in the $\bar{z}$ direction. On the negative $\bar{z}$ axis, the spacing between the singularities is about a quarter wavelength, and on the $\bar{x}$ axis it is about half a wavelength. On the positive $\bar{z}$ axis, there can be at most two singularities, and explicit expressions for their locations has been derived.

When there is absorption in the host medium, some of the off-axis vortices tend to disappear, and new vortices show up in the region $\bar{z} > 0$. The most prominent effect of the damping is that singularities on the negative $\bar{z}$ axis appear, very close to the particle. Since the extent of the near-field loops is determined by the distance between the dipole and the closest singularity, the damping leads to a significant reduction of the extent of the near-field loops.

Finally, it should be mentioned that the $\bar{z}\bar{y}$ plane is also a symmetry plane. A field line through a point in this plane stays in this plane, and so the field lines are 2D curves in this plane. Many features of the flow pattern in this plane are similar to the results in the $\bar{z}\bar{x}$ plane. The $\bar{z}$ axis is still in this plane, so this analysis carries over, and most properties follow from replacing $\bar{x}$ by $\bar{y}$. For any point in this plane, the angle β is 90° , so $\cos \beta = 0$. One prominent difference is that the term proportional to $g(q)$ in Equation (56) drops out. This term was responsible for the loops in the near field. In the $\bar{z}\bar{y}$ symmetry plane,

there are no loops in the near field, and $\sigma_{\times}(\mathbf{r})$ is proportional to $\mathbf{e}_z$ in the neighborhood of the particle.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

The polarizability of the particle can be expressed in terms of the electric dipole Mie scattering coefficient $a_1(e)$. This coefficient is explicitly

$$a_1(e) = \frac{1}{D(e)} e^{-in_1\bar{R}} \times \langle [(n_p\bar{R}) \cos(n_p\bar{R}) - \sin(n_p\bar{R})] \times \left\{ (\hat{\epsilon}_p - 1)[(n_1\bar{R}) \cos(n_1\bar{R}) - \sin(n_1\bar{R})] + \hat{\epsilon}_p(n_1\bar{R})^2 \sin(n_1\bar{R}) \right\} - (n_p\bar{R})^2 \sin(n_p\bar{R})[(n_1\bar{R}) \cos(n_1\bar{R}) - \sin(n_1\bar{R})] \rangle, \quad (\text{A1})$$

with

$$D(e) = [(n_p\bar{R}) \cos(n_p\bar{R}) - \sin(n_p\bar{R})] \times [(\hat{\epsilon}_p - 1)(n_1\bar{R} + i) - i\hat{\epsilon}_p(n_1\bar{R})^2] - (n_1\bar{R} + i)(n_p\bar{R})^2 \sin(n_p\bar{R}). \quad (\text{A2})$$

Here we have set $\hat{\epsilon}_p = \epsilon_p / \epsilon_1$. The indices of refraction n_1 and n_p are given by Equation (2). The permeabilities μ_1 and μ_p only enter through the indices of refraction, whereas the permittivities ϵ_1 and ϵ_p also enter explicitly.

For $\bar{R}$ small, this simplifies to

$$a_1(e) = -\frac{2i}{3} \frac{\hat{\epsilon}_p - 1}{\hat{\epsilon}_p + 2} (n_1\bar{R})^3, \quad (\text{A3})$$

provided that $\hat{\epsilon}_p$ is not close to 2. If that were the case, then Equation (A3) needs to be refined [12].

Appendix B

The functions $\chi_n(z)$, defined by Equation (49) can be evaluated explicitly. We find

$$\chi_0(z) = \frac{1}{z} \left\{ e^{-iz} - e^{iz} \right\}, \quad (\text{A4})$$

$$\chi_1(z) = \frac{1}{z} \left\{ e^{-iz} \left[1 - \frac{i}{z} \right] + e^{iz} \left[1 + \frac{i}{z} \right] \right\}, \quad (\text{A5})$$

$$\chi_2(z) = \frac{1}{z} \left\{ e^{-iz} \left[1 - \frac{2i}{z} - \frac{2}{z^2} \right] - e^{iz} \left[1 + \frac{2i}{z} - \frac{2}{z^2} \right] \right\}. \quad (\text{A6})$$

These expressions, although exact, are not convenient to study the behavior for z small. In Equation (49), we expand $\exp(-izt)$ in a Taylor series around $t = 0$, and integrate term-by-term. This yields for small z

$$\chi_0(z) = 2 - \frac{1}{3}z^2 + \mathcal{O}(z^4), \quad (\text{A7})$$

$$\chi_1(z) = -\frac{2i}{3}z + \frac{i}{15}z^3 + \mathcal{O}(z^5), \quad (\text{A8})$$

$$\chi_2(z) = \frac{2}{3} - \frac{1}{5}z^2 + \mathcal{O}(z^4). \quad (\text{A9})$$

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