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Free Vibration of Single-Walled Carbon Nanotubes Using Nonlocal Truncated Timoshenko-Ehrenfest Beam Theory

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Abstract: Carbon nanotubes with their outstanding mechanical, physical and electrical properties have stimulated a significant amount of scientific and technological research due to their uniqueness compared to conventional materials. As a result, an extensive study on their mechanical properties has been conducted, and the static and dynamic behavior of single- walled and multi-walled carbon nanotubes has been examined using Euler-Bernoulli and Timoshenko beam models. The main objective of this paper is to study the free vibration behaviour of single-walled carbon nanotubes (SWCNT) using the nonlocal truncated Timoshenko beam theory. According to the Hamilton principle, the equation of motion of Timoshenko single-walled carbon nanotubes is calculated taking into account the truncated theory; and the general corresponding boundary conditions are derived. Finally, some numerical examples are performed to evaluate the effects of the nonlocal coefficient and the length of the nanotube. The obtained results are validated by comparing them with those found in the literature, and they show the accuracy and efficiency of the developed model. Particularly, the results demonstrate that the present formulation is highly efficient and capable of satisfactorily describing the behavior of nanobeams.

Keywords: truncated Timoshenko nanotubes; vibration; analytical modelling; computational modelling

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1. Introduction

Carbon nanotubes, with their outstanding mechanical, physical and electrical properties have stimulated extensive research activities in the field of science and technology due to their uniqueness compared to conventional materials. As a result, extensive studies have been conducted to investigate their mechanical properties [1–3] and the static and dynamic behavior of single- and multi-walled carbon nanotubes have been carried out.

In the literature, the main existing approaches to study the behavior of nanostructures can be divided into two classes: One at the atomistic level, and the other at the continuous level. Although the molecular dynamics approach can be considered more suitable for the analysis of nano-sized structures, the complexity of the computational process and the time-consuming calculations [4] have attracted little attention compared to continuum models. Among continuum approaches, beam models have proven to be cost-effective. Using the classical Euler-Bernoulli and Timoshenko beam models, extensive studies have been conducted to investigate the mechanical properties of CNTs and elastic beam models have been used to evaluate static and dynamic problems of carbon nanotubes [5–10].

An important aspect of nanotube modeling is the scale-effect. Although classical continuum approaches are efficient in the mechanical analysis of CNTs, their applicability in identifying small-scale effects on the mechanical behavior of carbon nanotubes is

questionable. Numerous studies have underlined the importance of the scale-effect and have shown that non-local elastic continuum models are more suitable for predicting the structural behavior of nanotubes.

The origins of non-local elasticity theory go back to pioneering works by Eringen [11,12] and Reddy [13]. Using non-local constitutive differential equations, Reddy gives a thorough development of the traditional beam and shear deformation theories in [13] and derives the solutions for, bending, buckling and natural frequency problems for simply-supported beams. Further applications of the nonlocal elasticity theory have been used in the study of the buckling problem and vibration problems in CNTs by applying Euler-Bernoulli beam, plate and shell theories and Timoshenko beam theory [14–27]. The wave dispersion response of elastic nano-sized beams was analytically addressed and the closed-form solution of the phase velocity was determined in [28]. Still based on the nonlocal elasticity beam theory, the effects of the spatial variation of the nonlocal parameter on free frequencies of nanoplates were discussed in [29]. Using nonlocal first-order shear deformation theory with variable nonlocal parameters, the free vibration of the functionally graded doubly curved nanoshells was studied in [30].

The present paper deals with the nonlocal vibrational analysis of the single-walled carbon nanotube (SWCNT). Using the nonlocal elasticity theory, a novel theory for the free vibration analysis of Timoshenko nanotubes is proposed. The main objective of this paper is to investigate the free vibration behaviour of single-walled carbon nanotubes using the nonlocal truncated Timoshenko beam theory. According to Hamilton principle, the equation of motion of Timoshenko single-walled carbon nanotube are calculated and the general corresponding boundary conditions are derived.

The mathematical modeling of CNTs and their solutions play an important role in the field of nanotechnology. As is well-know, it is not always possible to find analytical solutions for all sets of boundary conditions and different geometries. This circumstance has motivated the present research. In addition, since in the literature there are no free vibration solutions for nanotubes with the simplified approach, the authors have chosen to briefly present the key findings and challenges and direct light to possible future research. This paper does not intersect with recent relevant reviews, which reflects its significance to readers.

Recently, a variational model leading to a simplified theory of the dynamic analysis of the Timoshenko beam theory has been proposed, called the *truncated Timoshenko theory* [31]. Starting from the truncated Timoshenko beam theory, the authors in the present paper have developed a similar theory for Timoshenko nanotubes using geometrical and variational methods. Comparing the classical Timoshenko theory with the truncated theory, a significant simplification of the differential equation can be observed. The novelty of the proposed approach is that it shows a perfect analogy between variational and direct methods for the dynamic analysis of beams. The aim of the proposed formulations is to find the truncated Timoshenko equations and the corresponding boundary conditions and to establish their mathematical similarity with the two different approaches. It is shown that the differential equations and the corresponding boundary conditions, used to solve the dynamic problem of the local and nonlocal truncated Timoshenko equations via the variational formulation, have the same form as those obtained via the direct method. In addition, it is shown that the equation is both simpler and more consistent than the appropriate classical Timoshenko equations extended to include nonlocal stress effects.

Finally, the impacts of the nonlocal coefficient and the length of the nanotube are then assessed using a few numerical examples. By comparing the obtained results with those found in the literature, the findings are validated. Moreover, the outcomes demonstrate how effective the simplified approach is at accurately describing the behavior of nanotubes.

2. Theoretical Formulation: Equations of Motion for Nonlocal Timoshenko Beams Model

Eringen Nonlocal Theory Assumptions and Stress Resultants in Nonlocal Theory

According to Eringen [11,12], the nonlocal elasticity theory assumes that the Cauchy stress state, at a given reference point **x**, does not depend only on the strain field at this point, but is considered as a function of the strain field at all points of the body. This assumption leads us to the assertion that, in nonlocal elasticity theory, the stress at a point is determined both by the stress at that point and by its spatial derivatives. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion. For an elastic and isotropic homogeneous body, the nonlocal constitutive behavior is expressed by the following relations:

$$\boldsymbol{\sigma} = \int_V \alpha(|\boldsymbol{x'} - \boldsymbol{x}|, \tau)\boldsymbol{t}(\boldsymbol{x'})\mathrm{d}\mathbf{x'} \tag{1}$$

$$\boldsymbol{t}(\boldsymbol{x}) = \mathbf{C}(\boldsymbol{x}) \otimes \boldsymbol{\epsilon}(\boldsymbol{x}) \tag{2}$$

$$\left(1 - \tau^2 {l_e}^2 \nabla^2\right)\boldsymbol{\sigma} = \boldsymbol{t}\,, \quad \tau = \frac{e_0 l_i}{l_e} \tag{3}$$

where $\boldsymbol{\sigma}$ is the nonlocal stress tensor at point **x**, **t(x)** is the conventional stress tensor at point **x**, $\alpha(|\,\mathbf{x'} - \mathbf{x}|,\tau)$ is the nonlocal coefficient introducing into the constitutive equation the nonlocal effect at the reference point **x** produced by local strain at the source **x′**. The Euclidean distance is $|\mathbf{x'} - \mathbf{x}|$, τ is defined as the scale coefficient that incorporates the small-scale factor, where e_0 is a material constant determined experimentally. In addition, l_i and l_e are the internal and external characteristic lengths, respectively. Finally, **t** at a point **x** is related to the strain $\boldsymbol{\epsilon}$ at the point by the fourth-order elasticity tensor **C** and $\otimes$ denotes the "double-dot product".

Using Equation (3) we can express the resulting stresses in terms of strains. In contrast to the local theory, the nonlocal constitutive relations lead to differential relationship between resulting stresses and strains. In the following, the nonlocal constitutive relations for isotropic and homogeneous beam are described. In particular, for beams the nonlocal constitutive relations in Equation (3) take the following special form:

$$\sigma_{zz} - \mu^2 \frac{\partial^2 \sigma_{zz}}{\partial z^2} = E e_{zz}\,, \quad \mu = (e_0 l_i) \tag{4}$$

where σ_{zz} and e_{zz} are the normal stress and normal strain, respectively, E is elasticity modulus, μ is the small scale parameter that incorporates the small scale effect and z is the coordinate measured from the mid-plane of the nanotube.

$$\sigma_{zy} - \mu^2 \frac{\partial^2 \sigma_{zy}}{\partial z^2} = 2G\, e_{zy} \tag{5}$$

where σ_{zy} and e_{zy} are the transverse shear stress and the transverse shear strain, respectively, G is shear modulus and y is the longitudinal coordinate measured from the left-end of the nanotube.

Multiplying Equation (4) by y and integrating the result over the area A yields:

$$\int_A y\sigma_{zz}\mathrm{dA} - \mu^2 \int_A \frac{\partial^2 (y\sigma_{zz})}{\partial z^2}\mathrm{dA} = \mathrm{E}\int_A y^2 \frac{\partial \phi}{\partial z}\,\mathrm{dA} \tag{6}$$

The integrals with the first member indicate the bending moment, whereas the integral with the second member defines the second moment of area I of the straight section. As a result:

$$M - \mu^2 \frac{\partial^2 M}{\partial z^2} = \mathrm{EI}\frac{\partial \phi}{\partial z} \tag{7}$$

where M is the bending moment and ϕ is the rotation due to bending. In addition, by integrating Equation (5) over the area, one gets:

$$Q - \mu^2 \frac{\partial^2 Q}{\partial z^2} = \mathrm{GA}\kappa\left(\frac{\partial v}{\partial z} + \phi\right) \tag{8}$$

where Q is the shear force, A is the area of the straight section, v the transverse displacement and κ the shear factor.

3. Timoshenko's Non-Local Truncated Theory: Dynamic Analysis

3.1. Equation of Motion for a Truncated Timoshenko Beam: Euler Method

This section presents the equilibrium of the nanotube element using the Timoshenko-Ehrenfest beam model. For example, if we examine the nanotube element in Figure 1, we should impose the equilibrium of the applied loads at the abscissa z: equilibrium at the vertical translation and at the rotation around the center of the right-hand section of all the forces identified on the elementary ashlar, beginning from its equilibrium.

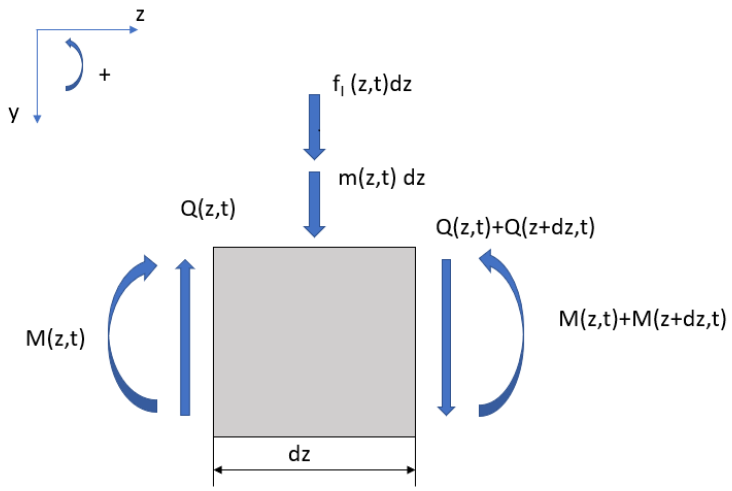

Figure 1. Translational and rotational equilibrium of the elementary ashlar.

The contributions for rotational m and translational inertia f_I are made separately and considering that

$$f_I = -\rho \mathrm{A} \frac{\partial^2 v}{\partial t^2} \tag{9}$$

$$m = -\rho \mathrm{I} \frac{\partial^2 \phi_b}{\partial t^2} \tag{10}$$

where ρ denotes mass density, t denotes the time variable, the rotational inertial term m is independent of the total rotation ϕ and exclusively depends on the flexural rotation $\phi_b = -\frac{\partial v}{\partial z}$:

$$\phi = \phi_b + \psi = -\frac{\partial v}{\partial z} + \psi \tag{11}$$

The equation for the vertical translation equilibrium is:

$$\frac{\partial Q}{\partial z} = \rho \mathrm{A} \frac{\partial^2 v}{\partial t^2} \tag{12}$$

whose first derivative, when substituted in Equation (8), yields:

$$Q = \mathrm{GA}\kappa\left(\frac{\partial v}{\partial z} + \phi\right) + \mu^2 \rho \mathrm{A}\frac{\partial^3 v}{\partial t^2 \partial z} \tag{13}$$

hence, Equation (12) can be completed and expressed as follows:

$$\rho \mathrm{A}\frac{\partial^2 v}{\partial t^2} - \mu^2 \rho \mathrm{A}\frac{\partial^4 v}{\partial t^2 \partial z^2} - \mathrm{GA}\kappa\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial \phi}{\partial z}\right) = 0 \tag{14}$$

On the other hand, the rotation's equilibrium with respect to the center of gravity elementary ashlar's right face is written:

$$\frac{\partial M}{\partial z} = Q - \rho \mathrm{I}\frac{\partial^3 v}{\partial t^2 \partial z} \tag{15}$$

whose first derivative, when replaced in Equation (7), gives:

$$M = \mathrm{EI}\frac{\partial \phi}{\partial z} + \mu^2\left(-\rho \mathrm{I}\frac{\partial^4 v}{\partial t^2 \partial z^2} + \rho \mathrm{A}\frac{\partial^2 v}{\partial t^2}\right) \tag{16}$$

The rotation's equilibrium Equation (15) can be written:

$$\mathrm{EI}\frac{\partial^2 \phi}{\partial z^2} - \mu^2 \rho \mathrm{I}\frac{\partial^5 v}{\partial t^2 \partial z^3} + \rho I\frac{\partial^3 v}{\partial z \partial t^2} - \mathrm{GA}\kappa\left(\frac{\partial v}{\partial z} + \phi\right) = 0 \tag{17}$$

Equations (14) and (17) constitute the set of differential equations of motion for a Timoshenko SWCNT that were obtained from the Timoshenko beam's truncated theory.

3.2. Equations of Motion for a Truncated Timoshenko Beam: Variational Method

In this section, according to the Hamilton principle, the motion equations for a beam system are derived.

For the local theory, the strain energy Π can be expressed as follows:

$$\Pi = \frac{1}{2}\left(\int_0^L M\frac{\partial \phi}{\partial z}dz + \int_0^L Q\left(\frac{\partial v}{\partial z} + \phi\right)dz\right) \tag{18}$$

It is evident from Equations (13) and (16), which are used to determine the shear stress Q and the bending moment M in the case of non-local elasticity, that the terms, defined in this context as additional to the local theory, are constant quantities that do not change with respect to the force; this means that these additional terms do not contribute to the strain energy but instead become the potential energy of the loads P (see Ref. [23]).

The strain energy Π assumes the following form:

$$\Pi = \frac{1}{2}\int_0^L \mathrm{EI}\left(\frac{\partial \phi}{\partial z}\right)^2 dz + \frac{1}{2}\int_0^L \mathrm{GA}\kappa\left(\frac{\partial v}{\partial z} + \phi\right)^2 dz \tag{19}$$

As can be seen, only the local terms are present in the formulation of the strain energy.

The nonlocal potential energy P_d is equal to the work done with changed sign for the corresponding displacement of the inertial forces.

$$P_{\mathrm{d}} = \int_0^L \mu^2\left(-\rho \mathrm{I}\frac{\partial^4 v}{\partial t^2 \partial z^2} + \rho \mathrm{A}\frac{\partial^2 v}{\partial t^2}\right)\frac{\partial \phi}{\partial z} + \mu^2 \rho \mathrm{A}\frac{\partial^3 v}{\partial t^2 \partial z}\left(\frac{\partial v}{\partial z} + \phi\right)\mathrm{dz} \tag{20}$$

As can be observed, only the moment and shear nonlocal contributions accurately reflect the nonlocal potential energy expression in contrast to the local theory.

Moreover, for the corresponding displacement ϕ, the rotational component P_t is expressed in terms of the work of the inertia forces m, with the sign changed:

$$P_t = -\int_0^L m\phi dz = -\int_0^L -\rho \mathrm{I}\frac{\partial^2 \phi_b}{\partial t^2}\phi dz = -\int_0^L -\rho \mathrm{I}\frac{\partial^2}{\partial t^2}\left(-\frac{\partial v}{\partial z}\right)\phi dz \tag{21}$$

Finally, the kinetic energy T is given by:

$$T = \frac{1}{2}\int_0^L \rho \mathrm{A}\left(\frac{\partial v}{\partial t}\right)^2 dz \tag{22}$$

Equations of Motion for a Timoshenko SWCNT: Truncated Theory

According to the Hamilton principle, the equations of motion for a Timoshenko single-walled carbon nanotube are computed using the truncated theory.

Summing the strain energy Equation (19) and the potential energies Equations (20) and (21) minus the kinetic energy Equation (22), we have:

$$\begin{aligned}
&(\Pi + P_{\mathrm{d}} + P_t - T) = \\
&\frac{1}{2}\int_0^L \mathrm{EI}\frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z}\mathrm{dz} + \int_0^L \mu^2\left(\rho \mathrm{A}\frac{\partial^2 v}{\partial t^2} - \rho \mathrm{I}\frac{\partial^4 v}{\partial t^2 \partial z^2}\right)\frac{\partial \phi}{\partial z}\mathrm{dz} + \\
&\int_0^L \mu^2\left(\rho \mathrm{A}\frac{\partial^3 v}{\partial t^2 \partial z}\right)\left(\phi + \frac{\partial v}{\partial z}\right)\mathrm{dz} + \frac{1}{2}\int_0^L \mathrm{GA}\kappa\left(\phi + \frac{\partial v}{\partial z}\right)\left(\phi + \frac{\partial v}{\partial z}\right)\mathrm{dz} \\
&-\frac{1}{2}\int_0^L \rho \mathrm{A}\left(\frac{\partial v}{\partial t}\right)^2 \mathrm{dz} - \int_0^L \rho \mathrm{I}\frac{\partial^3 v}{\partial z \partial t^2}\phi \ \mathrm{dz}
\end{aligned} \tag{23}$$

Applying the Hamilton principle, Equation (23) becomes:

$$\begin{aligned}
&\int_{t_1}^{t_2} \delta(\Pi + P_{\mathrm{d}} + P_t - T)\mathrm{dt} = \\
&\int_{t_1}^{t_2}\left(\int_0^L \mathrm{EI}\frac{\partial \phi}{\partial z}\delta\left(\frac{\partial \phi}{\partial z}\right)\mathrm{dz} + \int_0^L \left(-\mu^2 \rho \mathrm{I}\frac{\partial^4 v}{\partial t^2 \partial z^2} + \mu^2 \rho \mathrm{A}\frac{\partial^2 v}{\partial t^2}\right)\delta\left(\frac{\partial \phi}{\partial z}\right)\mathrm{dz} + \right. \\
&\int_0^L \left(\left(\mu^2 \rho \mathrm{A}\frac{\partial^3 v}{\partial t^2 \partial z}\delta\phi + \mu^2 \rho \mathrm{A}\frac{\partial^3 v}{\partial t^2 \partial z}\right)\delta\left(\frac{\partial v}{\partial z}\right) + \mathrm{GA}\kappa\left(\phi + \frac{\partial v}{\partial z}\right)\delta\phi + \right. \\
&\left.\left. \mathrm{GA}\kappa\left(\phi + \frac{\partial v}{\partial z}\right)\delta\left(\frac{\partial v}{\partial z}\right)\right)\mathrm{dz} - \int_0^L \rho \mathrm{A}\frac{\partial v}{\partial t}\delta\left(\frac{\partial v}{\partial t}\right)\mathrm{dz} - \int_0^L \rho \mathrm{I}\frac{\partial^3 v}{\partial z \partial t^2}\delta\phi \ \mathrm{dz}\right)\mathrm{dt} = 0
\end{aligned} \tag{24}$$

performing integration by parts and collecting all the terms in the previous equations allows us to write:

$$\rho \mathrm{A}\frac{\partial^2 v}{\partial t^2} - \mu^2 \rho \mathrm{A}\frac{\partial^4 v}{\partial t^2 \partial z^2} - \mathrm{GA}\kappa\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial \phi}{\partial z}\right) = 0 \tag{25}$$

$$\mathrm{EI}\frac{\partial^2 \phi}{\partial z^2} - \mu^2 \rho \mathrm{I}\frac{\partial^5 v}{\partial t^2 \partial z^3} + \rho I\frac{\partial^3 v}{\partial z \partial t^2} - \mathrm{GA}\kappa\left(\frac{\partial v}{\partial z} + \phi\right) = 0 \tag{26}$$

The Equations (25) and (26) represent the set of differential equations of motion for a Timoshenko SWCNT and, as can be seen, have the same form as the Equations (14) and (17) obtained by Euler method.

Finally, the corresponding boundary conditions are given by:

$$\left[\left(\mu^2 \rho \mathrm{A}\frac{\partial^2 v}{\partial t^2} - \mu^2 \rho \mathrm{I}\frac{\partial^4 v}{\partial t^2 \partial z^2} + \mathrm{EI}\frac{\partial \phi}{\partial z}\right)\delta\phi\right]_0^L = 0 \tag{27}$$

$$\left[\left(\left(\mu^2\rho\mathrm{A}\frac{\partial^3 v}{\partial t^2\partial z}\right)+\mathrm{GA}\kappa\left(\frac{\partial v}{\partial z}+\phi\right)\right)\delta\mathrm{v}\right]_0^L=0 \tag{28}$$

3.3. The Solution of Differential Equations System

In order to find a solution to the system of differential equations describing the dynamic behavior of the nanotube, we seek periodic solutions of the form:

$$\begin{aligned} v(z,t) &= v(z)e^{\mathrm{i}\omega\mathrm{t}} \\ \phi(z,t) &= \phi(z)e^{\mathrm{i}\omega\mathrm{t}} \end{aligned} \tag{29}$$

where ω is the frequency of natural vibration. On substituting Equation (29) into Equations (25) and (26) we have:

$$\rho\mathrm{A}\omega^2 v-\mu^2\rho\mathrm{A}\omega^2\frac{\partial^2 v}{\partial z^2}+\mathrm{GA}\kappa\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial\phi}{\partial z}\right)=0 \tag{30}$$

$$\mathrm{EI}\frac{\partial^2\phi}{\partial z^2}+\mu^2\rho\mathrm{I}\omega^2\frac{\partial^3 v}{\partial z^3}-\rho\mathrm{I}\omega^2\frac{\partial v}{\partial z}-\mathrm{GA}\kappa\left(\frac{\partial v}{\partial z}+\phi\right)=0 \tag{31}$$

After obtaining $\frac{\partial\phi}{\partial z}$ from Equation (30) and properly substituting it in Equation (31), we arrive at:

$$\begin{aligned} &\left(-\mathrm{EI}+\mu^2\rho\mathrm{I}\omega^2+\mathrm{EI}\frac{\mu^2\rho\mathrm{A}\omega^2}{\mathrm{GA}\kappa}\right)\frac{\partial^4 v}{\partial z^4}+ \\ &\left(-\mathrm{EI}\frac{\rho\mathrm{A}\omega^2}{\mathrm{GA}\kappa}-\mu^2\rho\mathrm{A}\omega^2-\rho\mathrm{I}\omega^2\right)\frac{\partial^2 v}{\partial z^2}+\rho\mathrm{A}\omega^2 v=0 \end{aligned} \tag{32}$$

Equation (32) is the differential equation of motion for a Timoshenko nanotube derived from the fourth order truncated theory in v.

On introducing the following non-dimensional coefficients:

$$\zeta=\frac{z}{L};\ \Omega^2=\frac{\rho\mathrm{A}L^4\omega^2}{\mathrm{EI}};\ \eta^2=\frac{\mu^2}{L^2};\ \beta^2=\frac{\mathrm{EI}}{\mathrm{GA}\kappa L^2};\ \alpha^2=\frac{\mathrm{I}}{\mathrm{A}L^2} \tag{33}$$

with $\zeta\in[0,1]$ the dimensionless counterpart of $z\in[0,L]$, Ω^2 the frequency parameter, η^2 scaling effect parameter, β^2 shear deformation parameter and α^2 slenderness ratio, the governing Equation (32) may be rewritten as:

$$\left(1-\Omega^2\alpha^2\eta^2-\Omega^2\beta^2\eta^2\right)\frac{\partial^4 v}{\partial\zeta^4}+\left(\Omega^2\beta^2+\Omega^2\eta^2+\Omega^2\alpha^2\right)\frac{\partial^2 v}{\partial\zeta^2}-\Omega^2 v=0 \tag{34}$$

The general solution for Equation (34) is given by:

$$v(\zeta)=A_1\cos(a\zeta)+A_2\sin(a\zeta)+A_3\cosh(b\zeta)+A_4\sinh(b\zeta) \tag{35}$$

where:

$$a=\sqrt{\frac{1}{2p}\left(q+\sqrt{q^2+4\mathrm{pr}}\right)};\ b=\sqrt{\frac{1}{2p}\left(-q+\sqrt{q^2+4\mathrm{pr}}\right)} \tag{36}$$

$$\begin{aligned} p &= \left(1-\Omega^2\alpha^2\eta^2-\Omega^2\beta^2\eta^2\right) \\ q &= \left(\Omega^2\beta^2+\Omega^2\eta^2+\Omega^2\alpha^2\right); \\ r &= \Omega^2 \end{aligned} \tag{37}$$

with A_1, A_2, A_3 and A_4 being the integration constants, which are determined using the boundary conditions, and p, q and r are the coefficients of the characteristic polynomial associated with differential Equation (34).

The system of Equations (25) and (26) in dimensionless form is given by:

$$\left(\frac{1}{\beta^2} - \Omega^2\eta^2\right)\frac{\partial^2 v}{\partial\zeta^2} + \Omega^2 v + \frac{1}{\beta^2}\frac{\partial(\phi L)}{\partial\zeta} = 0$$
$$\Omega^2\alpha^2\eta^2\frac{\partial^3 v}{\partial\zeta^3} + \left(-\frac{1}{\beta^2} - \Omega^2\alpha^2\right)\frac{\partial v}{\partial\zeta} + \frac{\partial^2(\phi L)}{\partial\zeta^2} - \frac{1}{\beta^2}(\phi L) = 0 \quad (38)$$

It is possible to derive $\bar{\phi} = \phi\ L$

$$\bar{\phi}(\zeta) = \left(\Omega^2\alpha^2\eta^2\beta^2 + \Omega^2\beta^4\eta^2 - \beta^2\right)\frac{\partial^3 v}{\partial\zeta^3} - \left(1 + \Omega^2\alpha^2\beta^2 + \Omega^2\beta^4\right)\frac{\partial v}{\partial\zeta} \quad (39)$$

3.4. Boundary Conditions

The quantities defined by Equations (27) and (28) lead to the following boundary conditions in dimensionless form:

$$\left[\left(\left(-\frac{1}{\beta^2} + \Omega^2\eta^2\right)\frac{\partial v}{\partial\zeta} - \frac{1}{\beta^2}\bar{\phi}\right)v(\zeta)\right]_0^1 = 0$$
$$\left[\left(-\frac{\partial\bar{\phi}}{\partial\zeta} - \Omega^2\alpha^2\eta^2\frac{\partial^2 v}{\partial\zeta^2} + \Omega^2\eta^2 v\right)\bar{\phi}(\zeta)\right]_0^1 = 0 \quad (40)$$

By substituting the Equations (35) and (39), that were appropriately calculated, in the boundary conditions (40), we obtain a system of four equations in the four unknowns A_i. In order for the system to admit a solution other than the trivial one, the determinant of the coefficient matrix must be set to zero.

The infinite solutions of the transcendental equation obtained by solving the determinant provide the infinite free frequencies of vibration.

3.5. Comparison of the Two Methods

This section compares the boundary conditions and differential equations of motion for the two theories. In particular, we compare the boundary conditions and differential equations of motion for the classical theory (see Equation (18) to Reference [23]) with the truncated theory (see Equation (34) of the current paper).

As demonstrated in Reference [23], the differential equation's characteristic polynomial for a classical SWCNT has the following form:

$$p_C\lambda^4 + q_C\lambda^2 + r_C = 0 \quad (41)$$

with:

$$p_C = \left(1 - \Omega^2\alpha^2\eta^2 - \Omega^2\beta^2\eta^2 + \Omega^4\alpha^2\beta^2\eta^4\right);$$
$$q_C = \left(\Omega^2\beta^2 + \Omega^2\eta^2 + \Omega^2\alpha^2 - 2\Omega^4\alpha^2\beta^2\eta^2\right);$$
$$r_C = \left(\Omega^2 - \Omega^4\alpha^2\beta^2\eta^2\right) \quad (42)$$

where p_C, q_C and r_C are the coefficients of the differential equation's characteristic polynomial. As shown in the present paper, the characteristic polynomial associated with differential Equation (34) based on truncated Timoshenko's theory is represented by:

$$p\lambda^4 + q\lambda^2 - r = 0 \quad (43)$$

with:

$$p = \left(1 - \Omega^2\alpha^2\eta^2 - \Omega^2\beta^2\eta^2\right);$$
$$q = \left(\Omega^2\beta^2 + \Omega^2\eta^2 + \Omega^2\alpha^2\right);$$
$$r = \Omega^2 \tag{44}$$

The comparison of the two Equations (42) and (44) shows that the terms p_C and p differ by the amount $(\alpha^2\beta^2\eta^4\Omega^4)$, q_C and q differ by the amount $(\alpha^2 2\beta^2\eta^4\Omega^4)$ and the terms r_C and r differ by the amount $(\beta^2\eta^4\Omega^4)$. As you can see, they are all terms multiplicative in $\frac{\partial^4 v}{\partial t^4}$.

By examining the free frequencies of vibration of a simply-supported beam at both ends, Timoshenko came to the conclusion that the final multiplicative term may be omitted because it would not affect the results, as was previously noted in Reference [32] in which Elishakoff obtained an equation both more consistent and simpler than the Bresse-Timoshenko equation given by the following expression:

$$\mathrm{EI}\frac{\partial^4 v}{\partial x_3^4} + \rho\mathrm{A}\frac{\partial^2 v}{\partial t^2} - \rho\mathrm{I}\left(1 + \frac{E}{\mathrm{G}\chi}\right)\frac{\partial^4 v}{\partial x_3^2 \partial t^2} + \frac{\rho^2 I}{\mathrm{G}\chi}\frac{\partial^4 v}{\partial t^4} = 0 \tag{45}$$

It can easily be seen from the comparison of the boundary conditions that those relating to displacements and shear remain constant while those relating to rotations and momentum change with respect to a single term. The following equations show how the classical theory and the nonlocal theory for a quantity differ from one another:

$$\left[\left(-\left(1 - \Omega^2\eta^2\alpha^2\right)\frac{\partial\bar{\phi}}{\partial\zeta} + \Omega^2\eta^2 v\right)\bar{\phi}(\zeta)\right]_0^1 = 0 \tag{46}$$

Equation (46) is related to the classical theory, while the following equation relates to the nonlocal theory:

$$\left[\left(-\frac{\partial\bar{\phi}}{\partial\zeta} - \Omega^2\alpha^2\eta^2\frac{\partial^2 v}{\partial\zeta^2} + \Omega^2\eta^2 v\right)\bar{\phi}(\zeta)\right]_0^1 = 0 \tag{47}$$

in which it can be noted that $-\left(\Omega^2\eta^2\alpha^2\right)\frac{\partial\bar{\phi}}{\partial\zeta}$ becomes $-\Omega^2\alpha^2\eta^2\frac{\partial^2 v}{\partial\zeta^2}$.

4. Numerical Examples

In this section, the suggested analytical method is validated in order to assess the impacts of the nonlocal parameter and the length of the nanotube on the free frequencies of vibration. Some numerical examples have been carried out, and the obtained results have been compared with those of papers that have already been published in the literature. All the numerical computations have been performed through in-house software developed in the Mathematica language [33] and the same geometrical features have been used.

For this purpose, Table 1 shows physical and geometrical properties of nanotubes used in Reference [34], which will be used throughout this section. In addition, the following shear factor is introduced:

$$k = \frac{6(1+\nu)\left(1+{c_r}^2\right)^2}{(7+6\nu)(1+{c_r}^2)^2 + (20+12\nu){c_r}^2} \tag{48}$$

where c_r = (d − 2 h)/d. The external diameter d depends directly on the height h in the ratio d/h, and the span L depends on d in the ratio L/d.

Table 1. Geometrical and material properties adopted in numerical experiments.

SWCNT Properties	Symbol	Value	Unit
Cross-sectional area	A	π (d − h) h	m^2
Thickness	h	0.34 $\times 10^{-9}$	m
Moment of inertia	I	$1/8\ \pi\ (hd^3 - 3h^2d^2 + 4h^3d - 2h^4)$	m^4
Density	ρ	2300	kg/m^3
Young's modulus	E	1000	GPa
Poisson's ratio	ν	0.19	-
Shear modulus	G	$\frac{E}{(2+2\nu)}$	GPa

Numerical Comparison between the Two Theories: Conventional and Truncated Theories

A first numerical comparison has been made between the results of the present paper and those obtained by Wu and Lai in Ref. [34]. For different values of the non-dimensional small-scale coefficient $\eta \left[0, \sqrt{2}, 2\right]$, the aspect ratio of L/d [5, 10, 25, 50, 100], with d/h = 3, and for various boundary conditions, the first three nondimensional frequencies Ω_i have been calculated. In Table 2 are quoted the corresponding values and a comparison between the present results with those obtained in [34] has been done. As can be easily observed, for L/d = [5, 10], the results of the classical theory and the truncated theory diverge, whereas they coincide for values higher than L/d = 10. Moreover, they demonstrate that with increasing the ratio L/d, the first three nondimensional frequencies increase and it can be seen that if the nonlocal effect η increases the three first nondimensional frequency value decreases. In particular, in order to assess the impact of the small scale parameter on the frequency parameter of the SWCNT, the frequency parameter ratio as $R(\eta) = \Omega(\eta)/\Omega(\eta = 0)$ is defined. For various boundary conditions the following remarks may be applicable:

- For L/d = 5 and d/h = 3, the lowest frequency parameter ratio of the simply-supported SWCNT with $\eta = \sqrt{2}$, 2 are around 0.75 and 0.63, and they are 0.50, 0.36, and 0.36, 0.26 for the second and third modes, respectively. These findings demonstrate that the impact of the small-scale parameter on the frequency parameter of the SWCNT is greater for higher vibration modes than for lower modes.
- For L/d = 10 and d/h = 3, the nonlocal effect parameter $\eta = \sqrt{2}$ and for four boundary conditions (simply-supported (SS), clamped-clamped (CC), clamped-supported (CS) and clamped-free (CF) the frequency parameter ratios for the first vibration mode of the SWCNT are 0.75, 0.72, 0.73 and 1.03, respectively. According to the results, the SWCNT with SS,CC, and CS boundary conditions exhibit a small scale parameter effect on the frequency parameter that is more important than the SWCNT with CF boundary conditions.

In comparison to Table 3 of Ref. [34], the ratio L/d = 5 has been added and the dimensionless free vibration frequencies are calculated using the algorithm developed in Ref. [23] and referred to as additional constraint conditions other than the clamped-free. These values are crucial because, for larger L/d values, we move toward the Euler-Bernoulli theory, for which the classical Timoshenko and truncated Timoshenko theories will coincide; but for lower ratios, we approach Timoshenko's theory.

Table 2. Three non-dimensional frequency values for $\eta = 0, \sqrt{2}, 2$, d/h = 3 and L/d [5, 10, 25, 50, 100].

η[nm]	d/h	L/d	Ω_i	SS		CC		CS		CF	
				[34]	Present	[34]	Present	[34]	Present	[34]	Present
0	3	5	1st	9.3349	9.3254	18.3188	18.2653	13.6062	13.5807	3.4213	3.4208
			2nd	32.7567	32.4455	42.3973	41.9378	37.7664	37.3703	18.6875	18.6268
			3rd	63.1739	61.5730	71.1007	69.5058	67.2925	65.6741	44.9335	44.3155
		10	1st	9.7254	9.7247	21.1019	21.0951	14.8938	14.8912	3.4915	3.4914
			2nd	37.3394	37.3015	54.4248	54.3228	45.7322	45.6670	21.0258	21.0189
			3rd	79.1305	78.8118	98.9624	98.4371	89.0947	88.6763	55.6261	55.5123
		25	1st	9.8460	9.8460	22.1535	22.1533	15.3303	15.3302	3.5120	3.5120
			2nd	39.1056	39.1044	60.3036	60.2992	49.1981	49.1957	21.8628	21.8625
			3rd	86.9790	86.9659	116.3020	116.2720	101.2204	101.2000	60.5724	60.5679
		50	1st	9.8637	9.8637	22.3177	22.3177	15.3961	15.3961	3.5150	3.5150
			2nd	39.3840	39.3839	61.3211	61.3208	49.7694	49.7692	21.9912	21.9911
			3rd	88.3501	88.3501	119.6965	119.6940	103.4615	103.4600	61.4093	61.4090
		100	1st	9.8681	9.8681	22.3593	22.3593	15.4127	15.4127	3.5158	3.5158
			2nd	39.4547	39.4547	61.5843	61.5843	49.9157	49.9157	22.0236	22.0236
			3rd	88.7067	88.7066	120.5979	120.5980	104.0492	104.0490	61.6248	61.6248
$\sqrt{2}$	3	5	1st	7.0339	7.0314	13.2521	13.1938	9.9602	9.9381	3.5232	3.5225
			2nd	16.3059	16.1509	20.4210	20.0819	18.5642	18.3205	12.6427	12.5757
			3rd	22.5773	22.0042	25.4238	24.5612	24.0527	23.3419	21.2597	20.8030
		10	1st	8.9163	8.9156	19.0006	18.9939	13.4846	13.4823	3.5195	3.5195
			2nd	28.1544	28.1258	39.8407	39.7524	33.9813	33.9290	18.6756	18.6677
			3rd	48.0901	47.8965	58.9143	58.5214	53.5671	53.2866	40.4636	40.3540
		25	1st	9.6999	9.6999	21.7469	21.7466	15.0661	15.0661	3.5167	3.5167
			2nd	36.9279	36.9267	56.4535	56.4493	46.2499	46.2477	21.4257	21.4255
			3rd	77.0842	77.0725	101.9404	101.9080	89.1923	89.1725	56.6912	56.6867
		50	1st	9.8265	9.8265	22.2119	22.2119	15.3284	15.3284	3.5162	3.5162
			2nd	38.7995	38.7994	60.2646	60.2643	48.9689	48.9688	21.8794	21.8794
			3rd	85.4800	85.4792	115.3995	115.3960	99.9168	99.9148	60.3525	60.3522
		100	1st	9.8588	9.8588	22.3330	22.3330	15.3956	15.3956	3.5161	3.5161
			2nd	38.3059	39.3059	61.3136	61.3136	49.7112	49.7112	21.9955	21.9955
			3rd	87.9589	87.9588	119.4686	119.4680	103.1218	103.1210	61.3546	61.3546
2	3	5	1st	5.8830	5.8770	10.8642	10.8026	8.2156	8.1952	3.6507	3.6495
			2nd	12.3183	12.2013	15.3213	15.0436	14.0142	13.8182	9.7649	9.7140
			3rd	16.4993	16.0812	18.6614	17.9761	17.6063	17.0662	16.4645	16.0494
		10	1st	8.2804	8.2798	17.4090	17.4022	12.4063	12.4043	3.5488	3.5489
			2nd	23.5318	23.5079	32.8628	32.7809	28.2456	28.1993	16.8709	16.8630
			3rd	37.6594	37.5078	45.9992	45.6612	41.8802	41.6497	33.4683	33.3679
		25	1st	9.5601	9.5601	21.3611	21.3609	14.8149	14.8148	3.5213	3.5213
			2nd	35.0776	35.0765	53.2455	53.2412	43.7742	43.7720	21.0096	21.0093
			3rd	69.9442	69.9336	91.8434	91.8105	80.6349	80.6157	53.4678	53.4633
		50	1st	9.7897	9.7897	22.1096	22.1096	15.2616	15.2616	3.5173	3.5173
			2nd	38.2403	38.2402	59.2597	59.2593	48.2057	48.2056	21.7690	21.7690
			3rd	82.8719	82.8710	111.5318	111.5270	96.7128	96.7103	59.3479	59.3476
		100	1st	9.8495	9.8495	22.3067	22.3067	15.3787	15.3787	3.5163	3.5164
			2nd	39.1587	39.1587	61.0463	61.0463	49.5092	49.5092	21.9675	21.9675
			3rd	87.2297	87.2297	118.3702	118.3690	102.2186	102.2180	61.0879	61.0879

Figures 2–5 show the nondimensional frequency values versus the ratio h/d, for three different values of the nonlocal parameter η = 0, 1, 2 and various boundary conditions. According to Figures 2–4 in relation to the cases of the simply-supported, clamped-clamped, clamped-supported SWCNT, for a fixed aspect ratio value L/d = 10 and for $2 < d/h < 10$, the non-dimensional frequency values decrease as the nonlocal effect value increases. In addition, when d/h = 10 and higher values of the non-dimensional frequency are found for the clamped-clamped situation, the frequency curves for $\eta = 0$ and $\eta = 1$ approach

and tend to coincide. Ultimately, the frequency increases for $\eta = 1$ and $\eta = 2$ and as the ratio d/h varies, whereas it decreases for $\eta = 0$. The case of the clamped-free single-walled carbon nanotube is explored in Figure 5. For a fixed aspect ratio value L/d = 10 and for $2 < d/h < 10$, the non-dimensional frequency values versus the ratio d/h, for three alternative values of the nonlocal parameter $\eta = 0, 1, 2$, are provided. As can be observed, the nondimensional frequency increases if the nonlocal parameter increases and for the ratio d/h= 10 the frequency curves approach and tend to coincide.

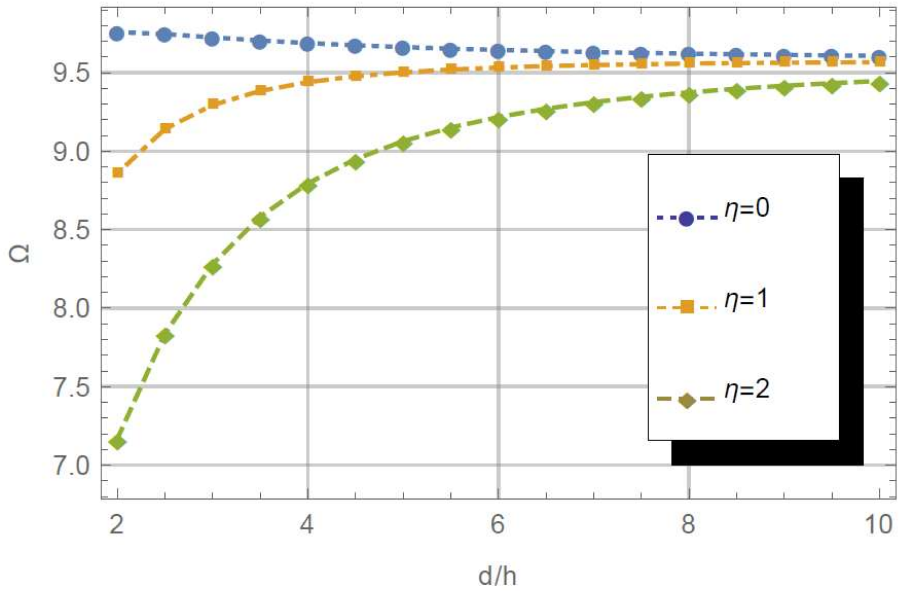

Figure 2. The nondimensional frequency values versus the ratio d/h, for three different values of the nonlocal parameter $\eta = 0, 1, 2$: simply-supported single-walled nanotube case.

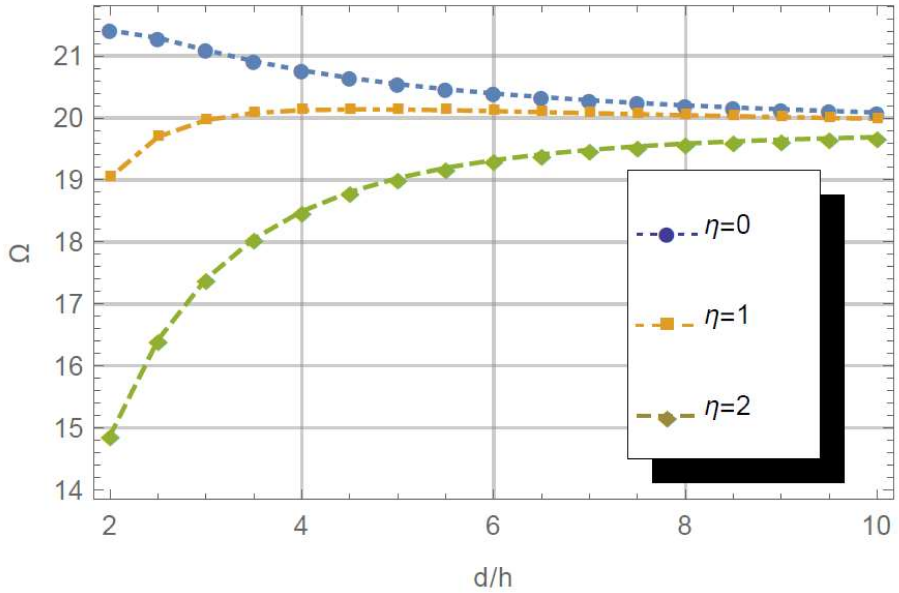

Figure 3. The nondimensional frequency values versus the ratio d/h, for three different values of the nonlocal parameter $\eta = 0, 1, 2$: clamped-clamped single-walled nanotube case.

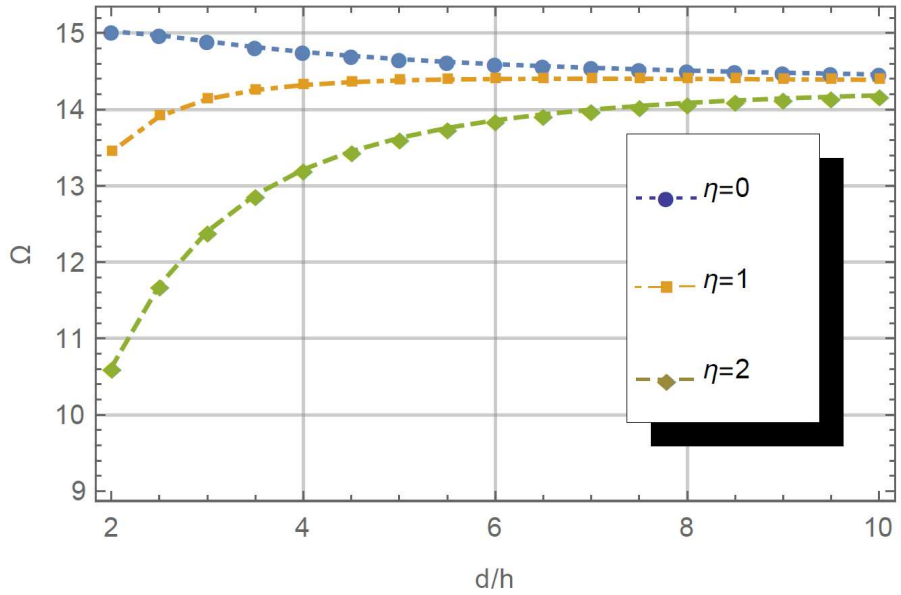

Figure 4. The nondimensional frequency values versus the ratio d/h, for three different values of the nonlocal parameter η = 0, 1, 2: clamped-supported single-walled nanotube case.

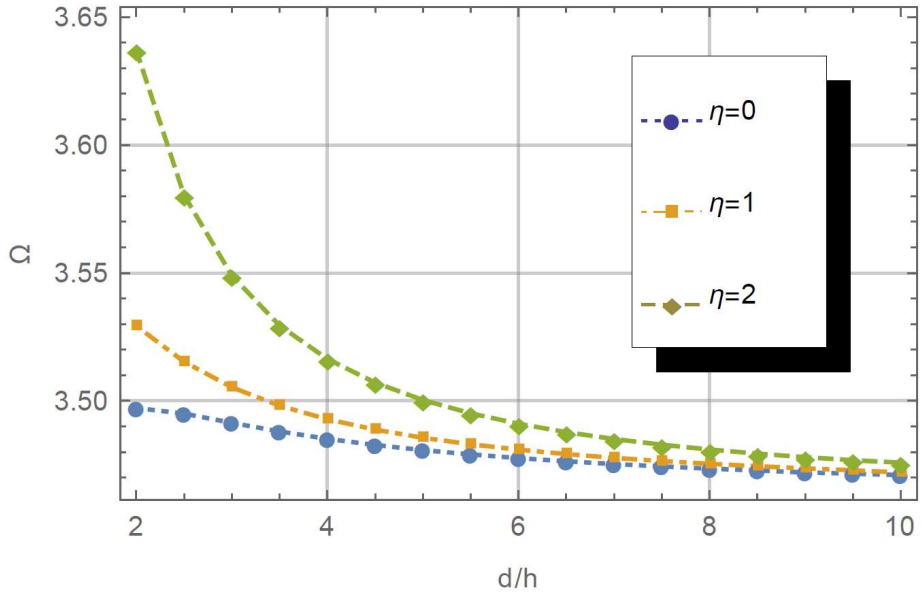

Figure 5. The nondimensional frequency values versus the ratio d/h, for three different values of the nonlocal parameter η = 0, 1, 2: clamped-free single-walled nanotube case.

Table 3 compares the results from the truncated Timoshenko theory and those produced using the conventional method, showing the percentage of errors in each case. This inaccuracy grows with the nonlocal effect η and is greater for the clamped-clamped boundary condition. The simply supported nanotube does not change while η changes. From the first to the third frequency, this error increases.

Table 3. Percentage error between the traditional SWCNT and the result of the truncated theory, for L/d = 5, 10 per ($\eta = 0, \sqrt{2}), 2$.

η[nm]	L/d	Ω_i	SS	CC	CS	CF
			% Error *			
0	5	1st	0.102	0.292	0.187	0.016
		2nd	0.950	1.084	1.049	0.325
		3rd	2.534	2.243	2.405	1.375
	10	1st	0.008	0.032	0.017	0.002
		2nd	0.102	0.187	0.143	0.033
		3rd	0.403	0.531	0.470	0.205
$\sqrt{2}$	5	1st	0.035	0.440	0.222	0.021
		2nd	0.951	1.661	1.313	0.530
		3rd	2.538	3.393	2.955	2.148
	10	1st	0.008	0.035	0.017	0.000
		2nd	0.102	0.222	0.154	0.042
		3rd	0.403	0.667	0.524	0.271
2	5	1st	0.102	0.567	0.248	0.033
		2nd	0.950	1.813	1.399	0.521
		3rd	2.534	3.672	3.068	2.521
	10	1st	0.007	0.039	0.016	0.003
		2nd	0.102	0.249	0.164	0.047
		3rd	0.403	0.735	0.550	0.300

* % error $= \frac{\text{approx - exact}}{\text{exact}} \cdot 100$.

5. Conclusions

In the present paper, the nonlocal truncated Timoshenko single-walled nanotube, with various boundary conditions, has been studied. According to the Eringen's and nonlocal Timoshenko beam theory, a new theory for the free vibration analysis of the Timoshenko nanotube has been presented and the equations of motion and the general corresponding boundary conditions have been derived.

As is well known, it is not always possible to find analytical solutions for all sets of boundary conditions and varied geometries. This circumstance has motivated the present research. The novelty of the proposed approach has been devoted to finding solutions to the eigenvalue problem of single-walled carbon nanotube by employing the truncated theory in those cases for which, to the authors' knowledge, no analytical solutions have been provided in the literature.

Starting from the truncated Timoshenko beam theory, the authors have developed the same theory for Timoshenko nanotubes using the geometrical and variational methods. Comparing the classical Timoshenko theory with the truncated theory, we have shown a considerable simplification of the differential equation. Some numerical examples have shown the effectiveness of the proposed approach, even through a comparison against results in the literature. In particular, the results have demonstrated that the present formulation is very efficient and able to describe the behavior of nanobeams in a satisfactory way.

Finally, it has been explained how the length of the nanotube and nonlocal parameters affect the three initial frequencies of SWCNT. The main points are:

- the first three nondimensional frequencies decrease with increasing of nonlocal effect;
- the first three nondimensional frequencies increase with increasing the ratio L/d;
- the effect of the small scale parameter on the frequency parameter of the SWCNT with SS, CC and CS boundary conditions is more significant than that of the SWCNT with CF boundary conditions.

The present approach can be applied to analyze the stability and dynamics behaviour of single-walled and multi-walled carbon nanotubes (MWCNTs) with different boundary conditions. The first results obtained, and reported in the degree thesis [35], are significant

and in perfect agreement with the results of the literature. In the upcoming publication, which will be released, the effects of surface tension on the natural frequency of nanotubes will also be investigated.

Moreover, the exact nonlocal truncated Timoshenko beam solutions presented herein should be useful to engineers who are designing nano-beams and nanotubes. In addition, the exact solutions serve as reference results for confirming numerical vibration solutions derived from other mathematical models and methods.

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